## Final report for the programme "Analysis on graphs and its applications" at the Isaac Newton Institute for Mathematical Sciences, Jan.-June 2007.

### Basic theme and background information

**Theme**

In the recent years, a lot of attention has been paid to analysis on graphs and graph-like low-dimensional structures. The main (albeit not the only) objects here are the (usual) combinatorial graphs, in which case functions are defined on the set of graph's vertices, "quantum graphs," i.e. graphs considered as singular one-dimensional varieties (CW-complexes) and with functions defined along the edges and fractals (Sierpinski gasket and other self-similar objects). When we used the word "analysis" above, we meant that in all these cases either objects of analytic nature (e.g., discrete Laplace operators, quantum graphs Hamiltonians, fractal Laplace operators, etc.), or problems of analytic nature (e.g., spectra of these operators, heat kernel asymptotics, quantum chaos, etc.) are considered. Thus, discrete and continuous methods blend, producing a unique interesting field.

Why should such objects be studied and why there has been so much activity in this field lately? The motivation comes from many areas of mathematics, sciences (physics, chemistry, computer science) and engineering (nanotechnology, microelectronics, and material science, to name a few). On one hand, these kinds of objects and problems arise currently in a wide range of areas of mathematics: from number theory, to discrete groups, spectral geometry, and mathematical physics. Simultaneously, burgeoning field of meso- and nano-scopic physics and technology strives to use graph (and analogous two-dimensional "open book" or mixed dimension) models to approximate low-dimensional objects such as circuits of quantum wires, thin photonic crystals, thin superconducting structures, etc. At the same time, similar structures have been used to model some chemical compounds, such as electronic states in aromatic molecules, or famous nowadays carbon nanotubes and recently discovered two-dimensional crystals (graphene). Analysis on fractals has also been, at least partially, driven by real world applications, e.g. fractal multi-band (or wide-band) antennas are already commercially available. There are also applications in many other areas. In some other cases, graphs provide toy models, playing with which one hopes to develop some intuition for more complex problems (e.g., quantum chaos, or existence of extended states in Anderson localization models in condensed matter physics).

The program attempted to (and in the view of the organizers succeed in) assemble, essentially for the first time, a diverse group of researchers from various areas of mathematics, physics and chemistry, as well as graduate students and junior scientists, in order to create cross-pollination of ideas and methods and thus develop a joint community and facilitate progress of this important and fascinating new field, rich with challenging problems.

**Organizers** M. Brown (Cardiff), P. Exner (Prague), P. Kuchment (Texas), T. Sunada (Tokyo).


Right to left: T. Sunada, P. Exner, M. Brown, P. Kuchment

**Participants** The program involved around 200 participants from Australia, Austria, Belgium, Brazil, Czech Republic, France, Germany, Holland, Israel, Italy, Japan, New Zealand, Poland, Russia, Sweden, Switzerland, UK and USA. This included world leaders, such as members of leading world scientific Academies (e.g., M. Aizenman, B. Bollobas, E. B. Davies, S. Novikov, H. Stark) and many other leading experts (such as J. Avron, E. Bogomolny, O. Bohigas, W. D. Evans, M. Freidlin, R. Grigorchuk, J. Keating, J. Kigami, M. Kotani, S. Molchanov, B. Pavlov, G. Rozenblyum, U. Smilansky, M. Solomiak, A. Terras and many others), as well as many junior researchers (one undergraduate student and dozens of graduate students and postdocs have participated in the program).

The participants represented many areas of mathematics (algebra, combinatorics, graph theory, number theory, PDEs, mathematical physics, operator theory, probability, dynamical systems), as well as physics, chemistry and computer science.

**Support** was provided by various sources: EPSRC 85,000 Euros, SCOTTISH 16,500 Euros, SPECT 14,000 Euros, SAT 10,000 Euros, JMEMBER 5,000 Euros, LMS 5,000 Euros, CPS 500 Euros and NSF $70,000.

# Structure

## **Workshops**

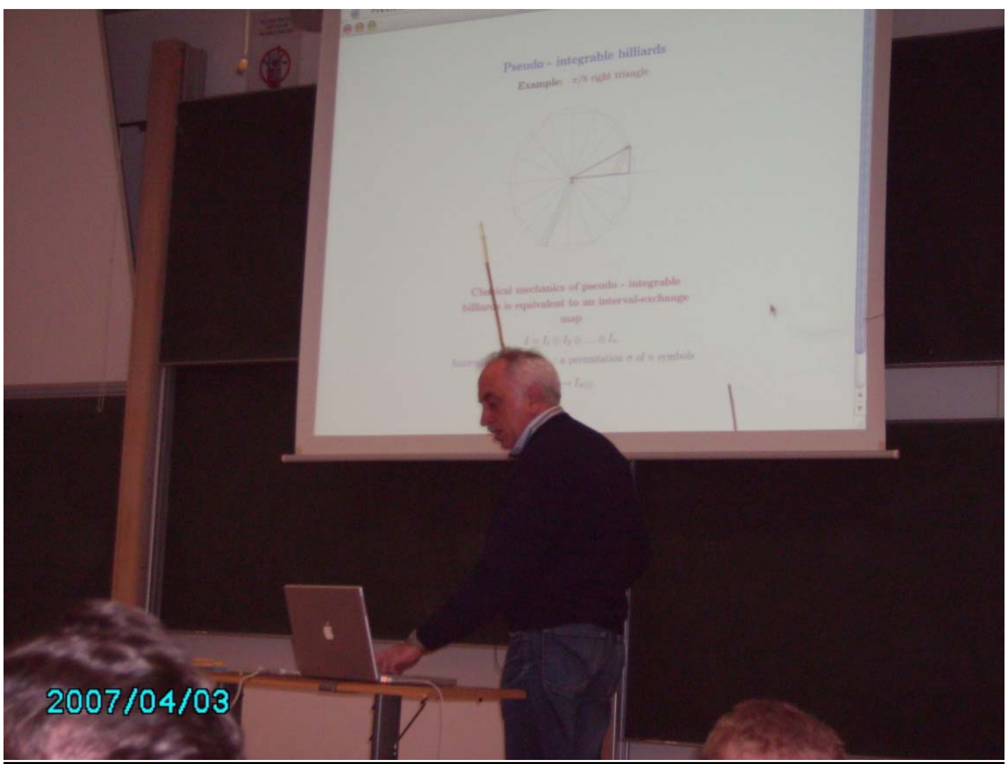
Besides the regular activities involving long and short time visitors, the program included four workshops.

**The tutorial "Analysis on graphs and other discrete structures" 10 - 15 January 2007 at Gregynog** was devoted to introducing students and junior researchers, as well as a few more senior scientists interested in learning the subject, to the main structures and method available. The tutorial involved 37 participants (mostly graduate students and postdocs, as well as one undergraduate student) from Czech Republic, Israel, Japan, Russia, Switzerland, UK and USA. The following series of 16 lectures were given: Professor T. Sunada (Meiji Univ., Japan) - *Spectral geometry of discrete Laplacians,* Professors P. Kuchment (Texas A & M Univ., USA) and P. Exner (Prague, Czech Republic) - *Quantum graphs and their applications,* Professor A. Teplyaev (Univ. of Connecticut, USA) - *Analysis on fractals.* Two guest lectures were given by Professor U. Smilansky (Weizmann Institute of Science, Israel) - *Spectral statistics* and Professor A. Valette (Institut de Mathématiques, Switzerland) - *Ramanujan Graphs*


During a lecture in Gregynog

Three very successful example sessions accompanying the lectures were conducted by postdocs J. Harrison and B. Winn (Texas A&M University).

**<u>Workshop "Quantum Graphs, their Spectra and Applications" April 2 - 5 2007 at the INI</u>** involved 79 senior and junior participants from Australia, Brazil, Czech Republic, France, Germany, Holland, Israel, Italy, Japan, New Zealand, Russia, Sweden, Switzerland, UK and USA. The workshop assembled a diverse group of mathematicians and physicists working in or interested in entering this fast developing and fascinating area. The workshop was devoted to studying the so called quantum graphs that have emerged recently. These are graphs considered as 1D  CW-complexes equipped with differential, rather than customary difference operators. This model has developed due to numerous applications arising in several areas of mathematics, sciences, and engineering, e.g. in nanotechnology, microelectronics, quantum chemistry, superconductivity, optics (photonic crystals), waveguides, etc. Such graphs, besides being in many cases useful low-dimensional models of complex systems, are also used as toy models for studying difficult issues such as Anderson localisation and quantum chaos. The methods used or expected to be useful in analysis on quantum graphs come from a very wide range of topics: algebra, combinatorics, PDEs, spectral theory, micro-local and complex analysis, to name a few. 26 lectures were delivered, covering a variety of interrelated topics, e.g. spectral theory, heat propagation, and quantum chaos on quantum graphs; Bose-Einstein condensate models; control and inverse problems for quantum networks, nodal properties of eigenfunctions of quantum graph Hamiltonians. Most talks triggered active discussion and interaction among participants.




E. Bogomolny (France) delivers a lecture at the workshop

**Workshop "Graph Models of Mesoscopic Systems, Wave-Guides and Nano-Structures" 10 - 13 April 2007 at the INI** was a logical continuation of the previous one, but with more applied tilt. It involved 52 participants (mathematicians, physicists and chemists) from Australia, Austria, Belgium, Czech Republic, France, Germany, Israel, Italy, Japan, New Zealand, Poland, Russia, Sweden, UK and USA , who delivered 23 lectures. The topics included in particular modeling newly discovered two-dimensional crystals (graphene), electronic transport in circuits of quantum wires, quantum graphs for quantum information transmission, quantum random walks, quantum chaos, graph models for the fast developing spintronics, models for slowing down light, and many others. As in the previous workshop (and even to a higher degree), active and often across the fields communication among participant was triggered.


During a lecture, workshop April 10-15

The last **workshop "Analysis on Graphs and Fractals" (A Satellite LMS Meeting at Cardiff University) took place 29 May - 2 June 2007**. About 50 participants from Czech Republic, France, Italy, Japan, Sweden, Switzerland, UK and USA delivered 24 talks. Unlike the previous two, the workshop was mostly devoted to analysis on purely discrete (combinatorial) graphs and fractals. In particular, new approaches were presented that link some classes of discrete groups (often of intermediate growth) with monodromy groups for rational mappings and spectral theory of fractals. It also included several applied talks, e.g. on Bose-Einstein condensate models and Anderson localization.


Michael Aizenman, a member of the National Academy of Sciences, USA delivers a lecture on Anderson localization at the Cardiff workshop.

**<u>Seminars</u>**

Besides the workshop lectures, 50 seminar talks on a wide range of topics related to the main theme were delivered by participants at the INI. The full list is available at **http://www.newton.cam.ac.uk/programmes/AGA/sem.html**

Most long term participants were also delivering talks at other universities across the UK.

## Outcome and achievements

The goals of the program were two-fold: first and foremost, to build an across-the-fields community of people working on problems of similar nature and to share tools developed in each area; secondly, to make progress in some outstanding problems of the field, i.e. analysis of Laplace operators on fractals, studying various zeta-functions on graphs, understanding localization phenomena on randomly perturbed graph systems, understanding spectral properties of various classes of graphs, e.g. Schreier and Caley graphs, crystal lattice graphs, etc.
One can definitely say that the community building effort was a success, since a large amount of active communication was triggered between people coming from number theory, algebra, combinatorics, mathematical physics, physics and chemistry. An example, for instance, was the intensive communication of the prominent physicist U. Smilansky (Weizmann Institute, Israel)

with the leading number theorists H. Stark and A. Terras (UC San Diego) that has led to his recent work on zeta functions on graphs.

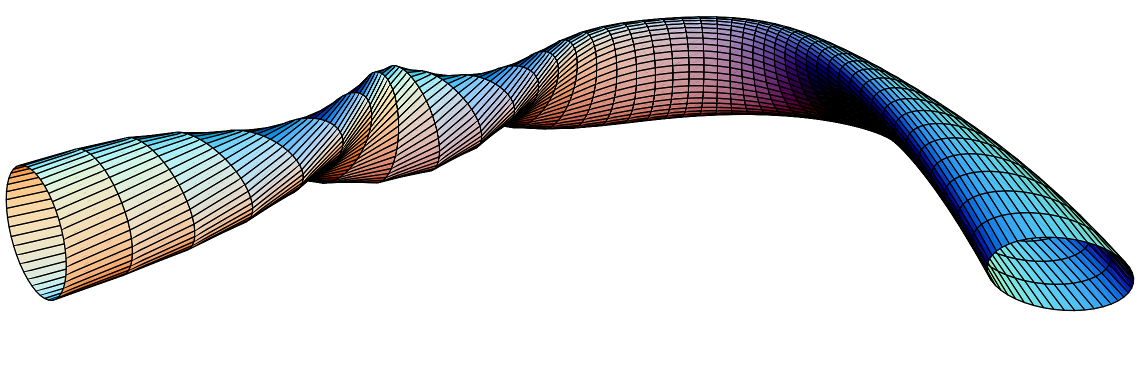
Albeit no-one could realistically expect that an outstanding problem, such as for instance existence of delocalized states in Anderson localization, would be resolved during the program, progress in this and many other directions has been made by participants, e.g. in understanding the nodal behavior of eigenfunctions of quantum (and combinatorial) graph Hamiltonians, graph zeta-functions, spectra of various classes of graphs, analysis of fractal Laplacians, etc. Significant new developments in these and other directions were announced in workshop and seminar talks, e.g. a breakthrough in long standing problem of approximating spectra of Dirichlet Laplacians in thin branching domains was reported in talks by D. Grieser and by B. Vainberg.

## Publications

The program has lead to many articles published, submitted, or in preparation, full list of which is not available to the organizers at the moment. One can only mention that besides the 25 INI preprints and 50 papers for the planned volume (see below), the organizers are aware of quite a few other publications coming out as a result of the program.

25 **preprints of the AGA program** have been posted on the INI server.

A **volume "Analysis on Graphs and its Applications"** (editors P. Exner, J. Keating, P. Kuchment, T. Sunada and A. Teplyaev) is planned to be published (either with Cambridge University Press, or American Math. Society, both of which have expressed their interest). The editors have received 50 paper submission promises from the program participants (the target line for the submission is the end of October 2007); several of the papers have already been submitted and are being peer reviewed. It is believed that the volume, when published, will become a definitive source in this emerging field.



The geometry of a curved quantum waveguide. The effects of twisting and bending are demonstrated on the left and right part of the picture, respectively. (Courtesy of D Krejcirik)