Algebraic Lie Theory (ALT)

Isaac Newton Institute, 12 January to 26 June 2009

Final Report from the Organisers: M. Geck, A. Kleshchev, G. Röhrle (July 2009)

1. Basic theme and background information

Lie theory has profound connections to many areas of pure and applied mathematics and mathematical physics. In the 1950's, the original analytic theory was extended to an algebro-geometric context so that it also makes sense over arbitrary algebraically closed fields, in particular, fields of positive characteristic: This is what we mean by "Algebraic Lie Theory". The central themes are understanding fundamental objects such as finite and infinite dimensional Lie algebras, reductive groups, quantum groups and Hecke algebras of various kinds, as well as their representation theories.

A driving force has always been the abundance of challenging, yet very basic problems, like finding explicit character formulae for representations. An indication of the complexity and the difficulty of the problems is that even the representations of the symmetric group (that is, the Weyl group of an algebraic group or a Lie algebra of type A) in positive characteristic are not understood!

The introduction of geometric methods (in the 1970s) has revolutionised the field. It led to a flow of new ideas between several disciplines, and produced spectacular advances. Outstanding problems are understanding categories of representations (especially in positive characteristic) in geometric terms. The ideas of "geometrization" and "categorification" now play a fundamental role in the development of the subject. New structures continue to arise from connections with other areas of mathematics and mathematical physics, like the emerging theory of W-algebras.

Given the wide spectrum of background motivations, the aim of the programme was to provide a forum for the discussion, the interaction and the further development of the various recent methods, advances and applications of Algebraic Lie Theory. The excellent working environment of the Institute provided stimulating surroundings for the programme.

2. Structure

The programme intentionally involved a broad variety of research areas connected by the general theme "Algebraic Lie Theory". Apart from day-to-day informal discussions, there were regular seminars ranging from two to four talks each week (organised by long-term participant P. Achar).

Due to the diversity of topics, a core feature of our programme was the organisation of workshops and special lecture series with a focus on particular aspects of current research, to be detailed below. These attracted large numbers of additional participants. Feedback from participants of these workshops was generally extremely positive.

• **Instructional Workshop** (12 - 23 January)

This initial, two-week long meeting was organised by M. Geck, A. Kleshchev and G. Röhrle. It attracted 71 participants, among which a large portion of early career researchers. The workshop consisted of 10 introductory mini-courses (3-5 lectures each) which were given by leading experts on a variety of topics which were to play a major role in the programme. There was a mixture of classical-style lectures (each morning) and informal discussions/problem sessions (each afternoon), which allowed for a close interaction between the speakers and the participants. This workshop also formed part of the activities supported by the EPSRC network grant *Representation Theory Across The Channel* (held by M. Geck and I. Gordon).

• **Algebraic Lie Structures with Origins in Physics** (23 - 27 March)

This meeting was organised by P. Etingof, A. Kleshchev, M. Nazarov and A. Premet. The 23 invited one-hour lectures were attended by over 100 participants. The idea of the workshop was to bring together mathematicians and mathematical physicists working in such (overlapping) areas as W-algebras, Yangians, vertex algebras, characteristic p Lie theory, conformal algebras, chiral algebras, quantum groups, Hecke algebras, Cherednik algebras, infinite dimensional Lie algebras, as well as related representation theory, geometry, combinatorics, and applications. The meeting has provided rare and very important opportunities, especially for young researchers. A special feature of the meeting which we did not plan in advance, but which has played out very well, was that it was held concurrently with the workshop *Quantum Discrete Integrable System* run by the parallel program in the Institute. This has increased already high "cross-pollination" opportunities provided by our meeting. As a result, several high profile talks of our conference were extremely well attended by participants of both workshops.

• **Categorification and Geometrization from Representation Theory** (14 - 17 April)

This meeting—a satellite workshop at the University of Glasgow—was organized by K. Brown, I. Gordon, U. Kraehmer, N. Reshetikhin, R. Rouquier and C. Stroppel. It attracted over a 100 participants and was made up of two separate events: a 2-day introductory workshop with 3 introductory mini-courses (3 lectures each) and a workshop with 18 invited one-hour lectures. The notion "categorification" goes back to Crane and Frenkel, motivated by mathematical physics, and in particular by the hope to construct higher dimensional topological quantum field theories. It is becoming increasingly clear that this notion is the connecting principle behind a number of new developments in both Lie theory and topology. This was the first conference on "categorification" in a broad sense and brought together people from quite different areas of mathematics (representation theorists, topologists, algebraic as well as symplectic geometers, mathematical physicists), all either working on "categorification" from their own perspective, or interested to learn about developments in this relatively new field.

• **Lecture series by Professor Laurent Lafforgue** (May) and
• **Spitalfields Day - Automorphic Forms and the Langlands Programme** (13 May)

Throughout the month of May, Professor Laurent Lafforgue, the Distinguished Rothschild Visiting Professor of our programme, gave a series of lectures (8 hours in total) explaining part of his (as yet unpublished) present research work on Langlands' functoriality principle, in which he tries to construct explicitly kernel functions for Langlands' transfer of automorphic representations. As a complement to this high-profile lecture series, with the help of the London Mathematical Society, we organised a Spitalfields Day with 4 invited one-hour lectures by internationally leading experts which centered around recent developments in the theory of automorphic forms, the Langlands Programme and related areas. A highlight was Gerard Laumon's talk about Ngo Bao-Chau's work on the so-called "Fundamental Lemma", a celebrated result in the theory of automorphic forms conjectured by Langlands, Shelstad, Waldspurger. There were about 60 participants.

• **Group Theory, Geometry and Representation Theory: Abel Prize 2008** (27 - 29 May)

This short meeting, organised by M. Geck, G. Röhrle and J. Saxl (jointly with the Department of Pure Mathematics and Mathematical Statistics at Cambridge) was designed to celebrate the award of the Abel Prize 2008, jointly to John Thompson and Jacques Tits, "for their profound achievements in algebra and in particular for shaping modern group theory". In view of the classification of finite simple groups, the theory of abstract finite groups essentially relies on the theory of reductive algebraic groups over fields of positive characteristic. The 12 invited one-hour talks covered a broad area so deeply influenced by Thompson and Tits. The large assembled audience (about 80) appreciated in particular the lectures by Serre and Thompson.

• **Representation Theory and Lie Theory** (22 - 26 June)

This was the concluding workshop of the programme, organised by M. Geck, A. Kleshchev and G. Röhrle. It attracted over a 100 participants and consisted of 23 invited one-hour talks covering a wide spectrum of topics ranging from classical Lie theory to the modern use of geometric methods in representation theory. The goal was to review the state of the art and to map out new directions. From all the feedback we received, the workshop was perceived as a major success. Especially it was commented that the average age of the speakers was remarkably low (compared to similar high-profile events) and that it was impressive to see how the subject is carried further by a new generation of top young researchers.

3. Outcome and achievements

The programme attracted well over a 100 researchers overall (visiting fellows and programme participants) with about 70% overseas participants. We were particularly pleased about the large numbers of early career researchers from all over the world participating in the various workshops. Participants gave nearly 60 invited talks at universities throughout the UK.

A major achievement of the programme was to provide a forum for presenting and discussing new ideas and developments, by bringing together as many researchers as possible from as wide a variety of areas as possible. We believe that a highly significant part of the outcome of the programme will be in the long-run, as a result of newly established collaborations and

the focalisation on new research directions. Here are some of the major themes which are bound to play a dominant role:

• Rouquier's vast programme for developing a "higher" representation theory of Kac–Moody Lie algebras. A special case of it was essential in the proof of Broué'e abelian defect group conjecture for symmetric and general linear groups (Chuang, Rouquier). The general version leads to categorifications of fundamental objects like canonical bases of quantum groups. It also gives rise to a new class of associative algebras defined independently by Khovanov and Lauda. Rouquier himself gave a series of lectures at the instructional conferences and a talk at the final workshop.

• The study, initiated and currently being developed by a generation of young mathematicians (Fiebig, Juteau, Mautner, Williamson, ...), of perverse sheaves with coefficients in positive characteristic. This is leading to a new attack on classical and long-standing open problems in the modular representation theory of finite and algebraic groups including, for example, the representations of the symmetric group in positive characteristic (Juteau). One of the major issues is to understand the failure of the "Decomposition Theorem" (due to Bernstein, Beilinson, Deligne) in the theory of perverse sheaves in positive characteristic. As a possible way around it, Juteau, Mautner and Williamson initiated the study of so-called "Parity Sheaves".

• One of the declared aims of the programme was to achieve progress on the open problem of understanding the relations between representations in characteristic zero and in positive characteristic, where major examples are given by Lusztig's character formula for representations of algebraic groups in positive characteristic (which is known to hold for "large" characteristic, with no known explicit bound) or James' conjectures on modular representations of Hecke algebras. Up to now, and despite the combined efforts of many mathematicians, these problems have resisted any definite solution, but we can now see that the work of Fiebig, Juteau, Mautner, Williamson and others may lead to a better understanding of the difficulties involved and—hopefully—to new ways of looking at them.

• The study of a new class of associative algebras, defined independently by Khovanov and Lauda (by diagrammatical methods) and Rouquier (see above), the original motivation being the "categorification" of quantized enveloping algebras. This is a primordial example of the importance of interaction between seemingly only loosely related areas. Among other and exciting new applications, these algebras provide natural gradings on various classical objects (Hecke algebras, Specht modules, ...) and currently form the subject of extensive study (Ariki, Brundan–Kleshchev, Brundan–Stroppel, Rouquier, Varagnolo–Vasserot, ...).

• The theory of finite W-algebras, which had their origin in mathematical physics. It is related to a number of classical subjects about algebraic groups and Lie algebras, like the geometry of the nilpotent cone, Slodowy slices, and so on. Connections with the related concept of generalised Gelfand–Graev representations of finite reductive groups remain to be investigated. Various problems concerning the representation theory of finite W-algebras (highest weight theory, existence of finite-dimensional or even 1-dimensional representations) have recently formed the subject of intensive study (Brundan–Kleshchev–Goodwin, Losev, Premet, Goodwin–Röhrle–Ubly).

Feedback on the programme from participants was overwhelmingly positive; the stimulting atmosphere of the Institute was particularly appreciated. Given the intensity and vibrance of new research activities (see above), we are enthusiastic about pushing the momemtum further and arranging a short follow-up event in a couple of years.

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