Arithmetic Geometry

(5^{th} January 1998 - 3^{rd} June 1998)

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Introduction

One of the origins of Arithmetic Geometry is the study of diophantine equations, where the goal is to solve polynomial equations in integers or rational numbers. The simplest example (going back at least to the Babylonians) involves rational solutions of $x^2 + y^2 = 1$. They correspond to right angled triangles in which all sides have integral length (such as (3,4,5) or (5,12,13)). In this case the set of all solutions can be described explicitly; either in terms of geometry of the circle, or by a simple formula involving complex numbers.

The interplay between algebra, number theory and geometry which is characteristic of the subject was anticipated already in the middle of the 19th century. Kronecker's vision of a unification of number theory and algebraic geometry had never been completely forgotten, but it came fully into life only after Grothendieck's revolution in algebraic geometry in the 1960's. One of the most spectacular developments has been the introduction to the subject of topological invariants that combine very subtle arithmetic and geometric information. Another important feature has been the growing importance of analytic methods, such as techniques from transcendence theory or hermitian geometry.

Arithmetic geometry is a vast subject. The programme concentrated on the following areas: arithmetic of algebraic cycles, motives and motivic cohomology, diophantine geometry, Arakelov geometry, values of L-functions. Other areas, such as modular forms and p-adic methods, were given lesser prominence, because of recent programmes at the IAS in Princeton and Centre Emile Borel in Paris that were devoted to them.

Seminars

There were regular seminars on Tuesdays and Wednesdays, which gave participants an opportunity to speak about their work. Some of the Wednesday seminars were more specialized. In January and February there was a series of talks devoted to the work of Voevodsky and Suslin. In May and June there were seminars on the Gross-Zagier formula and its generalizations, with four lectures by S. Kudla providing the main focus.

Meetings

Arithmetic of Algebraic Cycles and Motivic Cohomology, 23-27 February, 1998

This workshop was organized by J.-L. Colliot-Thélène and U. Jannsen. It was attended by about 60 participants from the following countries: Canada, France, Germany, India, Italy, Japan, Netherlands, Poland, Russia, Spain, Sweden, Switzerland, UK and USA.

Special Lectures on Applications of Model Theory to Diophantine Geometry, 17-19 March, 1998

This series of lectures was organized by A. Pillay. The speakers were A. Macintyre, A. Pillay and T. Scanlon. The main goal was to make a contribution to the growing interaction of model-theorists and algebraic/arithmetic geometers. The talks by Pillay and Scanlon gave an exposition of Hrushovski's work (and of their own contributions) on the Mordell-Lang conjecture for function fields in positive characteristic. Macintyre's talk was concerned with the model theory of the nonstandard Frobenius.

EU Summer School: Arithmetic Geometry

This two week Summer School was supported by the European Union, the Leverhulme Foundation and the Isaac Newton Institute.

Part I: Instructional Conference, Current Trends in Arithmetical Algebraic Geometry, 23-28 March, 1998

The goal of this instructional conference was to bring the audience of predominantly young researchers (postdocs and PhD students) in direct contact with some of the most important current research. There were four lecturers, A. Goncharov (*Polylogarithms, regulators, values of L-functions*), M. Nakamaye (*Diophantine approximation on algebraic varieties*), P. Salberger (*Manin's conjecture on points of bounded height*) and V. Voevodsky (*Motivic homotopy theory*), each giving 4-5 one-hour lectures. While these talks presented very advanced and recent material, the speakers succeeded very well in presenting it with enough details and preliminaries, so that the large and uneven audience would catch most of what was said. The organisers encouraged all participants to ask questions, and, indeed, there were many, both during the lectures and at the end of them. Notes had been written by the speakers prior to the meeting, and they got widely distributed. There were about 80 participants, from Austria, Bulgaria, Czech Republic, France, Germany, India, Italy, Netherlands, Russia, Spain, Sweden, UK and USA.

Part II: Rational Points, 29 March - 3 April, 1998

This was a combination of a research and a survey conference, organized by J.-L. Colliot-Thélène and P. Swinnerton-Dyer. The general aim of the conference was to give the participants a global view of what is known and what is to be expected of rational points, depending on the place of varieties in the geometric classification. There were about 100 participants, from Austria, Belarus, Bulgaria, Croatia, Czech Republic, France, Germany, Hungary, India, Israel, Italy, Netherlands, Russia, Spain, Sweden, Switzerland, UK and USA.

Computational Results in Arithmetic Geometry, 14-16 April, 1998

This workshop was organized by N. Smart and N. Stephens. It was attended by 54 people from the following countries: France, Germany, Hungary, Japan, Spain, UK, USA . There was a large contingent from the UK including a number of PhD students.

Arakelov Theory, Values of *L*-Functions, 29 June - 3 July, 1998

This workshop was organized by J. Nekovár and C. Soulé. The goal was to describe new developments in Arakelov theory and to discuss recent advances relating this field to values of L-functions. There were about 80 participants, from France, Germany, India, Israel, Italy, Japan, Poland, Romania, Russia, Spain, Sweden, Switzerland, UK and USA.

The programme and its achievements

The rest of the report describes the topics covered by the programme and some of the concrete results obtained by the participants during the programme or shortly afterwards. One of the main goals of the programme was to reflect the diversity of the subject (with limitations alluded to in the Introduction) and at the same time concentrate on areas of high research activity. Roughly speaking, the programme consisted of three periods of two months length. Each period had a different focus, even though there were some overlaps between them.

The First Period

The first two months of the programme and the first workshop were devoted to arithmetic of **algebraic cycles, motivic cohomology** and related topics.

A fundamental problem of algebraic geometry is a description of all subvarieties of a given algebraic variety, e.g. of all algebraic curves on a given surface. Over complex numbers, this appears to be a purely geometric problem. Over the field of rational numbers **Q** the question also involves arithmetic. In its full generality the problem is completely intractable. However, one can introduce a linearized version of the problem, which is hopefully more accessible. The simplest example is that of points on curves. An *elliptic curve* E is given in affine coordinates by an equation

$$E: y^2 = x^3+ax^2+bx+c.$$

The set of solutions (x,y), together with the unique point 0 at infinity, has a natural structure of an abelian group (three intersection points of E with any line add up to 0). The set of solutions in the complex field forms a torus. When a,b,c are in **Q**, the solutions with coordinates $x,y \in \mathbf{Q}$ form a *finitely generated* abelian group: this is a famous theorem of Mordell.

One can no longer add single points on curves of higher degree. For example, on the curve

$$y^2 = x^5 + ax^4 + bx^3 + cx^2 + dx + e$$

one can add only (equivalence classes of) pairs of points. For a general curve one considers formal linear combinations of points with integral coefficients, called divisors. Two divisors are equivalent if there is a suitable "deformation" between them. The equivalence classes of divisors again form an abelian group. Mordell's theorem was generalized to equivalence classes of divisors by A. Weil. For an algebraic variety X of arbitrary dimension d one can consider linear combinations of subvarieties of X of fixed dimension i, called cycles (for $0 \leq i \leq d-1$). Their equivalence classes (again defined in terms of certain deformations of cycles) form the *Chow group* $CH_i(X)$. This definition is reminiscent of that of homology groups or bordism groups in algebraic topology. Chow groups are very subtle invariants, difficult to compute. Only the case $i = d-1$ is well understood, when the theory is classical and very close to that of divisors on curves. For other values of i the situation is much more complicated. An analogue of the Mordell-Weil finiteness result is one of the main open questions.

In the late 1970s and early 1980s, Bloch and Beilinson formulated far-reaching conjectures linking Chow groups to higher algebraic K-theory and to the structure of the mysterious category of mixed motives. These conjectures suggested that the structure of Chow groups even for varieties over the complex numbers is of arithmetic nature. Consideration of special varieties as a test for the conjectures has played an important rôle. Motives, invented in the mid-1960s by Grothendieck, are elusive building blocks of varieties. Usually they cannot be observed directly, only through their various manifestations (the analogy to quarks in particle physics seems to be quite apt). Beilinson extended Grothendieck's ideas and conjectured the existence of a general cohomological formalism of motivic sheaves. In the mid-1980s, Bloch defined motivic cohomology for non-singular varieties in terms of his "higher Chow groups". In the 1990s, Voevodsky (partly in collaboration with Suslin) started to develop the theory of motivic sheaves (more precisely, a triangulated version of the theory). One of its most spectacular applications was Voevodsky's proof of Milnor's conjecture linking Milnor *K*-theory modulo 2 to Galois cohomology (and to the Witt ring) of an arbitrary field.

These recent developments were reflected in the programme. There was a series of special seminars on the work of Suslin and Voevodsky in January and February. Voevodsky gave a series of four lectures on his theory at the Instructional Conference in March, as well as a lecture at the first workshop. Inputs from motivic cohomology and Voevodsky's results were used by several participants, e.g. by T. Szamuely, who wrote part of his PhD thesis on higher class field theory during his stay at the Newton Institute. The thesis was defended at Orsay in July 1998. Motivic complexes were applied by E. Peyre to the study of negligible classes in Galois cohomology, as introduced by Serre. This approach yields results on unramified cohomology of quotients in higher degrees. Suslin's homology groups were used by A. Schmidt and M. Spiess, who studied tamely ramified class field theory of surfaces over finite fields. Other concrete applications of the new foundational work on motivic cohomology appeared in the work of Kahn (A spectral sequence for geometrically cellular varieties), Geisser-Levine (*K*-theory of fields in characteristic p) and Totaro (Chow rings of classifying spaces) presented at the February meeting.

In the related area of higher class field theory, U. Jannsen worked on his paper proving Kato's conjecture on a higher dimensional version of Hasse's exact sequence for the Brauer group of a number field. Further progress on their long-term project in higher class field theory was made by U. Jannsen and S. Saito. Their results include computations of part of the cohomology of the Kato complex for a variety over a p-adic field in terms of its special fibre and some injectivity results for the reciprocity map modulo $n : SK_1(X)/n \rightarrow \pi_1^{ab}(X)/n$ (as in Szamuely's work, assuming the Bloch-Kato conjecture for Milnor's K_3 modulo n). Earlier work of K. Kato in higher class field theory was used by R. Parimala and V. Suresh to prove new results on quadratic forms over function fields of curves over p-adic fields : for p odd any such form in more than 10 variables is isotropic. Higher dimensional class field theory originated in the work of A. Parshin more than 20 years ago. While at the Newton Institute, A. Parshin pursued his study of higher-dimensional local fields, in particular recently discovered connections to the algebraic theory of integrable systems and infinite Grassmannians. He had valuable discussions on this topic with G. Wilson and G. Segal.

O. Gabber gave lectures on his purity theorem for étale cohomology on arbitrary regular schemes, as well as on his recent result on the independence of l for intersection cohomology. Gabber's expertise was of great benefit to many participants. B. Poonen, who attended the whole programme as a Rosenbaum Fellow, obtained new results in several, quite different, directions. He completed a joint work with Stoll on the Cassels-Tate pairing for principally polarized abelian varieties. He had a joint work with J.-L. Colliot-Thélène on the Hasse principle (described below). He was one of the main speakers at the meeting "Computational results in Arithmetic Geometry". And he also found during his visit the proof of a result in diophantine geometry which extends both the Mumford-Manin and the Bogomolov conjectures.

Another topic was the study of cycle class maps from Chow groups to étale cohomology and of Chow groups of special classes of varieties. A general local-global conjecture on the image of the Chow groups in étale cohomology has been put forward by J.-L. Colliot-Thélène. It encompasses the classical Cassels-Tate dual exact sequence and the conjecture that for zero-cycles of degree one, the Brauer-Manin obstruction to the Hasse principle is the only one. Both results are known for curves if finiteness of the Tate-Safarevic group is assumed. Under the same assumption, Colliot-Thélène proved much of the general conjecture for ruled surfaces.

More elaborate varieties were considered at the February meeting. C. Schoen presented his striking example of a three-dimensional variety over the complex field for which the Chow group of 1-cycles modulo some positive integer *n* is not finite. A number of talks were devoted to the construction of so-called indecomposable elements in some of Bloch's higher Chow groups. The existence of such specific elements has played an important rôle in obtaining finitess statements for the torsion of Chow groups of certain classes of surfaces. Talks on this topic were given at the meeting by Sreekantan, Otsubo, Spiess (In the same direction, A. Langer gave a seminar talk on an axiomatic approach towards proving finiteness of *p*-primary torsion for CH_0 of surfaces). The search for indecomposable elements over the complex ground field is also an active topic (talk by Collino). At the same meeting, talks on various aspects of motivic cohomology for arbitrary varieties were presented by by Voevodsky (Triangulated category of motives over Spec(**Z**)), Hanamura (Intersection motivic cohomology, joint work with A. Corti), H. Gillet (Comparison of filtrations on algebraic *K*-theory, joint work with C. Soulé). A new Riemann-Roch theorem for flat bundles was presented by Hélène Esnault (joint work with S. Bloch). Various ärithmetic" filtrations on the Chow groups of complex varieties have been proposed. S. Saito (joint work with M. Asakura) described recent results on higher Abel-Jacobi maps which enable one to detect elements in some steps of the filtration. W. Raskind (joint work with X. Xarles) defined *p*-adic intermediate Jacobians.

The Second Period

The middle period of the programme (March and April) was devoted to **diophantine geometry**, i.e. to the study of rational points on algebraic varieties defined over number fields or function fields. It is by now well understood that the essential behaviour of rational points on a given variety reflects very closely the geometry of complex points of the variety. For curves, there is a fundamental trichotomy: curves of genus zero, for which the Hasse principle (in Legendre's form (*) below) holds; curves of genus one, for which Mordell's theorem is the main result (leading to asymptotic estimates on the number of points of a given height, as first noticed by Néron many years ago) but where many important questions still await an answer; curves of genus at least two, for which Faltings in 1983 proved Mordell's conjecture (the number of rational points is finite). Another proof of Mordell's conjecture, in the spirit of classical diophantine approximation, was later given by Vojta (futher developments in this direction were covered by Nakamaye in his series of lectures at the Instructional Conference). Neither Mordell's theorem nor Faltings' theorem have yet been made effective (except in special cases). For higher dimensional varieties, there is a well-established feeling that the geometric classification, which in its modern developments is part of the Mori programme, should influence the rational points in a similar fashion. A quantitative version is provided by Vojta's conjectures, motivated by Nevanlinna theory in complex geometry.

One of the most surprising recent developments in this area is due to E. Hrushovski, who proved deep results in diophantine geometry over function fields using methods of model theory. As part of the programme, A. Pillay and T. Scanlon gave a series of lectures on Hrushovski's work and its generalizations. A similar but slightly different gathering of logicians and number theorists occurred a few months earlier at the MSRI in Berkeley. There appears to be a genuine interest on both sides to pursue this mutually beneficial interaction of the two fields.

A topic much discussed during the second part of the programme was the Hasse principle, together with the related problem of weak approximation.

Obstructions to the Hasse principle

Is there an elementary criterion for solvability of (certain classes of) diophantine equations? More than 200 years ago, Legendre showed that the diophantine equation

$$ax^2+by^2+cz^2 = 0 \ (*)$$

(where a,b,c are non-zero integers) has a solution in integers x,y,z not all zero whenever the obvious necessary conditions are satisfied. The obvious conditions are that a,b,c be not all of the same sign (so that there is a nontrivial solution in the real numbers) and that all the congruences

$$ax^2+by^2+cz^2 \equiv 0 \pmod{p^n}$$

(p a prime number, $n \geq 1$) have primitive solutions (i.e. with at least one of x,y,z prime to p). Following Hensel, we can rephrase these necessary conditions in a more concise fashion: the equation (*), which defines a non-singular conic over the field of rational numbers $\mathbf{Q}$, should have non-trivial solutions in all completions of $\mathbf{Q}$, namely in the real field and in the p-adicfields $\mathbf{Q}_p$. Legendre's result was generalized to quadratic equations in any number of variables over $\mathbf{Q}$ and over arbitrary number fields by Minkowski and Hasse, respectively.

Since then, similar local-global results were proved for many algebraic varieties (= systems of polynomial equations), most of them homogeneous spaces of linear algebraic groups. However, it was recognized already in the 1930's that such a local-global principle ("Hasse principle") for the existence of rational points did not apply to all algebraic varieties. Among various counterexamples given over the years, perhaps the simplest one is due to Selmer: the equation

$$3x^3+4y^3+5z^3 = 0$$

has non-trivial solutions in all completions of $\mathbf{Q}$, but not in $\mathbf{Q}$ itself.

In 1970, Manin proposed a general algebraico-geometric interpretation: most of these counterexamples to the Hasse principle could be accounted for by the existence of non-trivial elements in the Grothendieck-Brauer group of varieties in question, together with the famous reciprocity law of class field theory. Is this obstruction of Manin to the Hasse principle for the existence of rational points the only one for all nonsingular algebraic varieties? It was shown in the 1980's that this was indeed the case for certain special classes of varieties, most of them unirational. More recently, conditional results assuming various well-known but daring conjectures were obtained for interesting classes of surfaces, including some surfaces given by equations

$$a_0 x_0^4 + a_1 x_1^4 + a_2 x_2^4 + a_3 x_3^4 = 0 \ (**)$$

For more general varieties one expected counterexamples, but it was only in 1997 that Skorobogatov found one that did not appeal to any conjectures. This is a (hyperelliptic) surface over the rationals, given by the system of affine equations

$$(x^2+1)y^2 = (x^2+2)z^2 = 3(t^4-54t^2-117t-243) \ (***)$$

The search for a general theory behind this example is one of the directions where some progress was achieved during the programme.

Descent for open varieties was used by J.-L. Colliot-Thélène and A. Skorobogatov who investigated varieties fibred over a projective line in the case when the principle is known for its fibres and the number of "bad" fibres is small. The Hasse principle can fail even in families of varieties; J.-L. Colliot-Thélène and B. Poonen constructed one-parameter families of principal homogeneous spaces under abelian varieties such that each fibre over a rational point of the base is a counterexample to the Hasse principle. R. Heath-Brown completed his work on the solubility of diagonal cubic diophantine equations $\sum p_i x_i^3 = 0$ (where p_i, $i = 1,\dots,4$ are prime numbers congruent to 2 modulo 3). He showed that there is always a non-trivial rational solution, assuming Selmer's parity conjecture (itself a consequence of the finiteness of the Tate-Safarevic group of elliptic curves). He proved similar conditional results for equations in five variables. Conditional results on the Hasse principle for surfaces of the form (**) were obtained by R. Pinch and P. Swinnerton-Dyer, who assumed the finiteness of the Tate-Safarevic group and Schinzel's hypothesis. The method uses explicit 4-descent and has numerical applications. In the first three works mentioned, the Brauer-Manin obstruction is the only obstruction for the existence of rational points. Whether or not this is the case in the fourth work has not yet been clarified but it is quite likely. A new proof of the Hasse principle for quasi-split algebraic groups over number fields was given by P. Gille. During his stay at the Newton Institute, A. Skorobogatov completed his work related to the example (***). This work was taken up by D. Harari, who devised a rather systematic way of producing varieties with rational points for which weak approximation fails, but for which the Brauer-Manin obstruction is not enough to prove this. The varieties have non-abelian fundamental groups; this is the key point, and it also seems to be the key to Skorobogatov's example.

Another topic involved counting points of bounded height. The height of a rational point is, roughly speaking, equal to the maximum number of digits in the numerators or denominators of its coordinates. At the end of the 1980's, a new research programme was started by Manin and his collaborators. Partly on the basis of numerical evidence, partly out of a belief in the power of geometry, they conjectured that if a Fano variety possesses rational points, then the number of rational points of height at most H and located away from a certain exceptional subvariety behaves like $CH\,(\log H)^{t-1}$ when H grows. Here $C > 0$ is some constant and $t \geq 1$ is the rank of the Picard group of the variety. (A classical example is that of projective space, other examples include more general homogeneous spaces.) The constant C was made precise by Peyre (using a Tamagawa measure on adelic points), a factor was added later by Batyrev and Tschinkel, who proved the conjecture for toric varieties and disproved it for general Fano varieties. For important classes of varieties, such as cubic surfaces, the conjecture is still open. One important new aspect in the theory is the introduction of universal torsors by Peyre and by Salberger, independently.

At the Instructional Conference in March, P. Salberger gave a series of lectures on Manin's conjectures. While at the Institute, he completed a long paper describing his point of view on the use of universal torsors, in particular for (split) toric varieties, where he can prove the conjectures in a manner completely different from that of Batyrev/Tschinkel. E. Peyre reformulated the classical circle method from analytic number theory from the point of view of universal torsors - this provides a good control of what should be the main term in the estimates. E. Peyre and Yu. Tschinkel gave a description of Peyre's constant (multiplied by the factor found by Batyrev and Tschinkel) in a manner suitable for explicit computations. This enabled them to produce more numerical evidence on the asymptotic growth of the number of rational points on diagonal cubic surfaces.

The Rational Points Conference (second part of the EU Conference) was one of the highlights of this Period. According to the geometric point of view taken on rational points, it started with a survey talk (by A. Corti) on the Mori programme of classification of varieties over algebraically closed fields. It then went on to consider the properties of points of varieties according to their place in the classification.

A) *Rational points on varieties of general type*: There were talks on the connection between the Lang conjecture and uniform upper bounds in the Mordell conjecture (Abramovich), an Arakelov version of the Masser-Wüstholz proof of Faltings' isogeny theorem (Bost), the *abc* conjecture for affine open sets of an abelian variety (Buium), a survey on the product theorem and related topics (Faltings), Nevanlinna theory and Vojta's conjectures (McQuillan), effective determination of rational points on curves of genus at least two (Poonen) and a strong form of the *abc* conjecture (Vojta).

B) *Curves of genus one and families of curves of genus one*: Along with talks by Heath-Brown, Skorobogatov and Swinnerton-Dyer on topics mentioned earlier on, there were talks on bounds for torsion on elliptic curves (Merel), elliptic curves with high ranks (Mestre) and the average rank of a one parameter family of elliptic curves (Silverman).

C) *Counting points of bounded height on Fano and similar varieties*: There were talks on various aspects of Manin's conjectures (Peyre, Strauch, Tschinkel).

D) *Qualitative aspects*: Various aspects of the Brauer-Manin obstruction appeared in several talks (Harari, Skorobogatov, Swinnerton-Dyer); other topics included integral points on quotients of affine spaces by finite groups (Beukers) and rational points on certain moduli spaces (Shepherd-Barron).

The meeting on *Computational results in Arithmetic Geometry* was focused on explicit algorithms. There were talks on a variety of topics ranging from theoretical work on modular curves through to applications in cryptography. Of particular benefit was the timely coming together of the main researchers in the field of algorithms to find rational points on curves of genus greater than one. There has been much progress in recent years on this topic and a number of lectures discussed the various approaches to the problem.

In this more explicit direction, F. Beukers and C. Smyth worked on effective computation of torsion points lying on subvarieties of tori.

The Third Period

The last two months of the programme and the final workshop were devoted to **Arakelov theory and values of *L*-functions**. What is Arakelov theory about? In the analogy between number theory and geometry, the ring of integers $\mathbf{Z} = \{ 0, \pm 1, \pm 2, \ldots\}$ is very close to the polynomial ring $\mathbf{C}[X]$ in one variable over complex numbers. Geometrically, $\mathbf{C}[X]$ is the ring of functions on the affine line $\mathbf{A}^1_{\mathbf{C}}$ over $\mathbf{C}$ and $\mathbf{Z}$ is the ring of functions on another one-dimensional object called Spec($\mathbf{Z}$). The projective line $\mathbf{P}^1_{\mathbf{C}}$ over $\mathbf{C}$ is obtained from $\mathbf{A}^1_{\mathbf{C}}$ by adding one point at infinity. Existence of an analogous point at infinity for Spec($\mathbf{Z}$) was well understood already in classical algebraic number theory. Around 1970, Arakelov considered the projective line $\mathbf{P}^1_{\mathbf{Z}}$ over $\mathbf{Z}$. This is an archetype of an *arithmetic surface* - a two-dimensional object with one dimension of arithmetic nature (as in Spec($\mathbf{Z}$)), the other one geometric. Arakelov discovered that by adding to $\mathbf{P}^1_{\mathbf{Z}}$ a copy of $\mathbf{P}^1_{\mathbf{C}}$ at infinity one gets an arithmetic object very close to the projective plane $\mathbf{P}^2_{\mathbf{C}}$ over $\mathbf{C}$. For example, two curves of degrees m resp. n in the projective plane intersect at exactly mn points (counting multiplicities). Arakelov developed similar intersection theory on any compactified arithmetic surface; the contributions from infinity were computed using Green's functions, i.e. fundamental solutions of the Laplace equation. In the 1980's, Arakelov intersection theory was generalized to arbitrary dimensions by Gillet-Soulé and others. A general arithmetic Riemann-Roch theorem was proved. Subsequently this geometry was used in the proof of important diophantine results, for example in the proofs by Faltings and Vojta of finiteness theorems for rational points on subvarieties of abelian varieties. A couple of years ago, another diophantine result was proved by Ullmo and Zhang: the Bogomolov conjecture, according to which algebraic points of a curve of genus greater than one are discrete in the Jacobian equipped with the Néron-Tate topology.

Apart from an opportunity for newcomers to the field to meet for the first time, the goal was to investigate a new direction for Arakelov theory, namely its link to modular forms and values of *L*-functions. It has been noticed for some time that the Mahler height of certain polynomials can be expressed as a value of an *L*-function (Smyth, Deninger). Also, the celebrated theorem of Wiles suggests to study Arakelov invariants of modular curves in order to understand arithmetic of elliptic curves. The self-intersection of the relative dualizing sheaves of modular curves has been computed by Abbès, Ullmo and Michel. Last year, Bost, Kühn and Kramer computed a similar quantity for the universal family of elliptic curves over a modular curve. These last computations appear to be related both to the arithmetic Riemann-Roch theorem (with singularities) and to the famous *Gross-Zagier formula*, which expresses the height of Heegner points on a modular curve as a value of (the derivative of) its *L*-function. So there is hope that the two topics can be related. A special seminar was devoted to the Gross-Zagier formula. After introductory talks, a series of lectures by S. Kudla gave an overview of "Kudla's programme", the aim of which is to provide a conceptual approach to the Gross-Zagier formula and its higher dimensional generalizations. S. Zhang presented his work on the higher weight version of the formula. R. Borcherds gave a lecture on his new approach to a theorem of Gross-Kohnen-Zagier about the relative position of Heegner points in the Jacobian of a modular curve and its generalization to higher dimensions.

In the subject of *values of L-functions* the goal is to ëxplain" certain naturally occurring numbers, such as
$1 + 2 + 3 + \dots = -1/12$ or $1 + 1/2^4 + 1/3^4 + \dots = \pi^4/90$. Deep conjectures of Beilinson and Bloch-Kato predict that these values are closely related to the structure of the category of mixed motives and to motivic cohomology, with the link provided by regulator maps or height pairings. The Gross-Zagier formula is a typical example of this phenomenon. There has been a growing interest in explicit constructions of elements in motivic cohomology and of regulator maps. A whole new industry of "polylogarithms" is involved in this enterprise. The name comes from close links to Euler's polylogarithm functions $\mathrm{L}i_m(z) = \sum_{k \geq 1} z^k/k^m$. An introduction to the whole circle of ideas around polylogarithmic complexes, their conjectural relation to motivic cohomology and values of *L*-functions was given by A. Goncharov in his lectures at the Instructional Conference. He also presented an explicit construction of regulator maps on the level of Bloch's complexes representing higher Chow groups. While at the Institute, A. Goncharov continued his work on ``higher cyclotomy'' - the values of multiple polylogarithms at roots of unity and their mysterious relations to modular forms. J. Tate formulated a generalization of B. Gross's conjecture on refined special values of Artin's *L*-functions and their derivatives. A few weeks later, D. Burns showed that Tate's conjecture is a consequence of equivariant Bloch-Kato conjectures. A surprising link between epsilon constants (long studied by specialists in the Galois module structure) and Arakelov-type invariants was discovered by M. Taylor, B. Erez and their collaborators. They had discussions with C. Soulé on the subject. A Riemann-Roch Theorem in the context of non-archimedean Arakelov theory was proved by H. Gillet and C. Soulé. They also found a better proof of a comparison between two spectral sequences in algebraic *K*-theory. The non-archimedean Arakelov theory was also used by K. Künnemann to define local heights for cycles on abelian varieties. Furthermore, he started a collaboration which H. Tamvakis in order to prove the so called ärithmetic standard conjectures" in the case of Grassmann varieties; even for these simple varieties, these conjectures in Arakelov theory happen to require a very new and difficult combinatorics. Other aspects of the theory, such as p-adic methods, were also represented. Theory of *p*-adic integration and *p*-adic regulators was being developed by A. Besser and, from a slighly different perspective, by P. Colmez. A first complete account of K. Kato's results on Euler systems for modular forms of weight 2 was written down by A. Scholl, who put together scattered fragments of Kato's lectures and filled in all technical details. J. Nekovár generalized his earlier results on syntomic regulators to the case of semistable reduction. He also continued to develop theory of Selmer complexes,

including duality theory, *p*-adic heights and Iwasawa theory for certain families of Galois representations.

The interaction between Arakelov theory and values of *L*-functions was at the heart of the final meeting, together with new developments in the first (equivariant Arakelov theory, singular metrics) and the second topic (order of vanishing of L-series at the center of symmetry). There were three lectures on different aspects of a generalized Gross-Zagier formula (Rapoport, Tipp, Zhang). Other links between these two theories appeared in several lectures, e.g. in a new definition of Borcherds'new modular forms on the moduli space of K3-surfaces in terms of Quillen metrics instead of infinite products (Yoshikawa), or in the interpretation of the epsilon constants in terms of these metrics (Chinburg, Taylor et al.). Similar links appeared elsewhere in the programme: Goncharov's work on regulators lead to the definition of higher Arakelov K-theory and Colmez's results on the so-called Chowla-Selberg formula (proved by Lerch half a century before them) suggested a relation to the arithmetic Riemann-Roch formula. Altogether, the general impression was that much more is to be discovered about the connection between the two fields.

Conclusion

Arithmetic Geometry is well and alive. It involves both beautiful general theories and a flurry of new methods, questions, and results. The programme reflected this diversity. It was also a unique occasion for many participants to start new and unexpected collaborations that will, hopefully, bear fruit in the future.

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