Mathematics and Applications of Fractals

(January to April 1999)

Report from the Organisers: RC Ball (Warwick); KJ Falconer (St Andrews)

Scientific Background and Objectives

Throughout this century mathematicians have studied the sets that are now known as 'Fractals'. An enormous body of theory, 'Geometric measure theory', was developed to analyse geometric properties of sets and measures of a very general form, though at the time these were thought of as mathematical curiosities in their own right, rather than with a view to modelling natural phenomena. In the 1970s, partly as a result of the advent of fast computation, it was realised that such sets could provide realistic representations of physical phenomena and that this body of mathematical theory ought to have useful applications. Moreover scientists came to appreciate how some very rich structures (and subsequently processes upon them) could be described semi-quantitatively in terms of statistical scale invariance. This suggested self-organisation in the mechanisms which created the structures, an idea familiar from the theory of critical phenomena, which thereby became applicable to a very wide range of problems, notably in systems far from equilibrium. In the process fractal ideas not only provided the basis for understanding 'new' problems like aggregation and growth, but have also led to 'old' phenomena such as Brownian motion and intermittency being reinterpreted.

This led to an explosion of activity in developing the mathematics of fractals and in applying them across science and beyond. Nevertheless, on the mathematical side, activity was concentrated where rigorous work was tractable, in particular on 'static' problems such as the calculation of dimensions in idealised instances. On the scientific side much of the work was merely 'fractal-spotting' which did not deserve to impress the mathematical community, proceeding too rapidly for contributions concerned with mathematical rigour to have much impact.

About 10 years ago a further concept came to the fore, namely that of multifractal measures, which gave a model for different moments scaling with non-trivially different powers. The development of multifractals has also been marked by a gulf between the mathematics and the physical applications.

In the last few years developments have ranged across the whole spectrum of mathematics and science. For example, mathematicians have started to address problems of a more 'dynamic' nature relating to physical problems, (albeit in idealised models), such as diffusions, PDEs and harmonic measures on fractals and on domains with fractal boundary. New mathematical techniques have recently been introduced into the study of fractals and multifractals, for example tangent measures and renewal theory methods. Multifractal theory has started to be placed on a formal mathematical (*ie* measure theoretic) basis, and progress is being made towards a rigorous explanation of observed phenomena such as ‘negative dimension’. Measures of fractality other than dimension are being introduced, such as new generalisations of lacunarity. Scaling terms with complex exponents have been observed in applications such as stress analysis for scale invariant structures; such exponents have also been

noticed in some mathematical descriptions. Quite rich but certainly non-rigorous arguments have been developed by physicists for the self-consistency of multifractal growth, and the apparent variation in dimension across aggregation clusters is being addressed. The distribution of galaxies in space and its time development has progressed from a confrontation between fractal speculation and uniform density assumptions to a more serious discussion of cross-over length scales in increasingly rich data sets; this is helped by recent data which reaches much further out in distance.

A major aim of the Programme was to bring together mathematicians and physicists, so as to increase awareness by mathematicians of applications requiring mathematical development and by scientists of what mathematics is available. A number of areas were identified as particularly appropriate and timely for development.

These included the mathematical development of useful ‘dynamic’ fractal and multifractal models. In particular the general theory of differential equations on fractal domains or domains with fractal boundaries, aggregation models, *etc* needed to be advance with an eye to the requirements of applications. The definition of differential operators such as the Laplacian on fractals should be extended to a more all-embracing definition that might improve our appreciation of relations between exponents such as the walk dimension, spectral dimension and the resistivity-distance exponent.

The characterisation and measurement of fractal structures (and their interpretation) needed to be developed. Dimension alone is inadequate for quantifying fractals, and generalisations of lacunarity, and classification of fractals under Lipeomorphism are needed. In particular, descriptions that might relate to the mechanism by which the fractal object or measure was generated are highly desirable.

Multifractal analysis has recently developed in many directions, with diverse applications and a bulk of mathematical theory available. The various strands of the subject should be brought together and awareness of the different facets of the subject increased.

Geometric measure theory has provided the mathematical foundation of the subject for many years, and the techniques available should be reviewed.

Mathematically based protocols are needed to identify that a structure is fractal, as is an analysis of the extent to which the geometry of fractals defined ‘in the limit’ carry over to a finite range of scales. Similarly, the information that can be usefully measured from a single realisation of a statistical fractal should be addressed.

The inheritance from critical phenomena and the Renormalisation Group could be ripe for pursuing. Although a major part of the historical background to fractals in physics, they have not been much developed mathematically in the fractal context.

The distribution of galaxies in space and how it developed over time has, with the rapid increase of available data, reached the stage of facing reconciliation or developing into extraordinary controversy.

The theory of growth processes and in particular DLA (Diffusion Limited Aggregation) has not advanced significantly for some years. It may be time to overcome some of the sticking points in this area.

Organisation

The overall planning was undertaken by Robin Ball and Kenneth Falconer, with planning help from Martin Barlow and assistance with seminar organisation from Jacques Levy-Vehel and Ben Hambly.

Apart from the main workshops, the Programme generally revolved around two formal seminars each Tuesday afternoon, and an informal discussion, generally led by two participants, on Thursday mornings. In addition, a variety of talks were contributed at other times, and open working discussions on particular topics were arranged on an ad hoc basis.

Two seminars in the Newton Institute Seminar series were presented by programme members: B Duplantier, *Conformal invariance and multifractality* and M Barlow, *Random walks and some fractal graphs*, and B Mandelbrot and K Falconer gave public lectures to packed houses during the national Science, Engineering and Technology Week.

The staff of the Isaac Newton Institute provided tremendous help to organisers and participants in many different ways, and were praised by many participants for their professional but friendly support.

Participation

Altogether the Programme involved about 40 long-term participants, 70 short-term participants, 8 affiliated members with many others taking part in the workshops. Several talks attracted mathematicians from elsewhere in Cambridge and Britain and from the parallel *Turbulence* Programme. Conversely, many Programme members visited other Cambridge departments and British universities to give talks and take part in discussions. Feedback suggests that the Programme succeeded in stimulating interaction and collaborations between participants, including between those with widely-differing scientific backgrounds.

Workshops

EC Summer School: Multifractals - Mathematics and Applications

Organisers: KJ Falconer (St Andrews), RC Ball (Warwick), L Olsen (St Andrews)

This School was held at the beginning of the Programme, from 4-8 January 1999. The 70 participants had backgrounds from pure and applied mathematics, physics, engineering and computer science and came mainly from EU countries.

The School brought together mathematicians and physicists interested in multifractals and their applications and stimulated exchanges of ideas between participants with differing backgrounds. The week began with some introductory presentations on multifractal theory and then the main lectures typically consisted of a general exposition of a topic leading to ideas at the frontiers of research. Topics covered included geometric theory; negative dimensions; calculation and interpretation of multifractal spectra in specific cases such as conformal systems, self-affine sets and Brownian-type processes; multifractal properties of signal processes and wavelet applications; multifractal time; growth models, especially DLA (diffusion limited aggregation); pattern forming systems; multifractal properties of harmonic measures on percolation clusters, Brownian boundaries, *etc*; conformal methods.

A book of survey papers on multifractals, based on some of the presentations, is in preparation. The main lecturers were: R Ball (Warwick), B Duplantier (Saclay/ Inst H Poincare), K Falconer (St Andrews), TC Halsey (Exxon), M Hastings (MIT), S Jaffard (Paris), J Levy-Vehel (INRIA), B Mandelbrot (Yale), R Mauldin (North Texas), L Olsen (St

Andrews), N Patzschke (Jena), I Procaccia (Weizmann Inst), R Riedi (Rice), SJ Taylor (Sussex), C Tricot (Clermont II).

EU Fractals Network Meeting

Organisers: R Ball (Warwick), L Pietronero (Rome)

This two-day meeting, held on 12-13 February 1999, was organised as part of the EU Network on Fractals that is running from October 1997 to September 2000, with the lectures open to all. The meeting consisted of short talks by members of the 12 Network teams from institutions across Europe on work in progress and open problems.

The emphasis of the meeting was on physical aspects of fractals and their applications. Topics covered included lacunarity; galaxy distributions; fractal growth; self organisation; fracture; earthquakes and landscapes; granular; dynamics; economics and time series.

The meeting attracted about 45 participants.

Spitalfields Day on Geometric Measure Theory

Organiser: K Falconer

This one-day meeting on 5 March 1999 was organised as a ‘Spitalfields Day’ supported by the London Mathematical Society. The 50 participants included many research students, particularly from UK.

The four lectures were given by G David (Orsay), J Harrison (Berkeley), P Mattila (Jyvaskyla) and D Preiss (University College London) and covered topics including rectifiability, Lipschitz maps, singular integrals, quasi-minimal sets and bounded currents.

Workshop on Differential Equations and Physics on Fractals

Organisers: M Barlow (Vancouver), R Ball (Warwick), H Herrmann (Paris)

This week-long meeting was held on 22-26 March 1999 and attracted about 70 participants from all parts of the world.

The main lecturers were: A Aharony (Oslo), M Barlow (Vancouver), R Ball (Warwick), R Bass (Connecticut), M van den Berg (Bristol), C Vassilicos (Cambridge), B Derrida (ENS Paris), K Falconer (St Andrews), J Harrison (Berkeley), H Herrmann (ESPCI Paris), R Hilfer (Stuttgart), J Kigami (Kyoto), S Kusuoka (Tokyo), B Mandelbrot (Yale), U Mosco (Rome), L Sander (Michigan), B Sapoval (EP Paris), R Stinchcombe (Oxford), R Strichartz (Cornell), D Vassiliev (Sussex), T Vicsek (Eotvos).

Topics covered included: Laplacian and heat equation on domains with fractal boundary; definition and properties of the Laplacian and other operators on fractal domains; diffusions, heat and wave equations, eigenvalue problems and nonlinear PDEs on fractals; physical processes on fractals; DLA and conformal maps; variational fractals.

Achievements

The Programme undoubtedly achieved its aim of bringing together scientists from a wide variety of backgrounds and has stimulated many new collaborations. It was satisfying to see a great deal of interaction between those with very different approaches to fractals and their applications. From the reports of participants a great deal of research was done at the Institute, much of which is ongoing. Many papers are already being written, but many more are likely to result from ideas developed during the Programme. Whilst the benefits cannot be judged so soon after the end of the Programme, the following paragraphs outline some areas of activity, of course such a selection is inevitably highly selective.

Multifractal theory

The Programme started with an EC Summer School on Multifractals and this stimulated discussion and research in the following weeks. A continuing aim is to calculate the multifractal spectrum of classes of measures and functions and look for new features. In this direction, J Barral, B Mandelbrot, J Peyriere and R Riedi analysed ‘multifractal pulses’ and other measures constructed using products which do not depend on a fixed scaling ratio. In particular B Mandelbrot presented recent results on products of harmonics of periodic functions. A Manning developed a neat approach to give the existence of the generalised dimensions in a way that has computational applications.

On the stochastic side, SJ Taylor, N-R Shieh and Y Peres obtained new results on the multifractal structure of Galton-Watson trees, and SJ Taylor and X Hu gave a multifractal analysis of a general subordinator, that is a monotone increasing process with stationary independent increments. SJ Taylor and N-R Shieh showed that the multifractal formalism breaks down for occupation measures associated with Brownian motion and super-Brownian motion. L Olsen suggested a functional version of the multifractal formalism to restore this breakdown.

Multifractal theory of functions was also advanced. S Jaffard developed a two-parameter approach to detect ‘chirps’ (ie intense local oscillations) as well as Holder exponents; this approach is closely related to wavelet theory. He also progressed his work on lacunary wavelet series. J Levy-Vehel worked on multifractal analysis of time-series, and in particular introduced a definition of dimension of a curve that is computationally tractable.

Considerable interest was shown in multifractal time, which has been used, for example, in modelling fractal time series that occur in finance. A mathematical theory of this was presented by B Mandelbrot and R Riedi.

Multifractal aspects of harmonic measure on percolation clusters and Brownian perimeters were also discussed intensively, see below.

Geometric measure theory

Many look to geometric theory to provide the rigorous foundation of fractal geometry in a general context, going back to the work of A Besicovitch on density, regularity, rectifiability and projections. Thus it was natural for a core of Programme members to concentrate on such topics, particularly around the Spitalfields Day on Geometric Measure theory. J Harrison presented a new approach to geometric aspects of currents which attracted considerable interest. This allows the classical theorems of Stokes, Green, etc to be formulated in a consistent way on fractal objects. This work was developed and extended at the INI and some progress is being made towards unifying this approach with the definitions of the Laplacian on fractals in common use. J Harrison also showed how these techniques could be applied to problems in vision.

M Csornyei revived interest in problems on projections of sets from points by obtaining elegant new results in an area that had been dormant since the 1970s.

D Preiss presented new results, obtained with B Kircheim and M Chlebik, that answered several long open problems on the existence of Lipschitz functions with finitely many gradients.

The problem of quantifying lacunarity was addressed in various ways. R Ball presented a definition involving 3-point correlations. E Jarvenpaa and M Jarvenpaa considered the notion of porosity of sets and extended the definition to measures.

Further progress was made on understanding the geometry of packing measures, which is markedly less well-behaved than that of Hausdorff measures, the covering analogue. K Falconer, M Jarvenpaa and P Mattila constructed examples demonstrating the lack of stability of the packing measure of sections of sets.

K Falconer and C Tricot made progress on the structure of packing measures defined by convolution kernels which occur in certain geometric questions.

M Lapidus stimulated interest in complex dimensions of fractals, embarking on studies of complex dimensions of random fractal strings and Brownian excursions with B Hambly, and on complex dimensions of multifractals with S Jaffard.

K Falconer continued his investigation of the 'horizon problem' - that is the relationship between the fractality of a (random) surface and its horizon. He obtained a complete solution for the case of an isotropic Brownian surface, and the solution for fractional Brownian surfaces assuming a natural property of the maximum of the corresponding (1-dimensional) fractional Brownian function.

Partial Differential Equations on Fractals

A workshop on PDEs on Fractals was held at the end of March, and this brought together world experts in the field, many of who stayed on as Programme participants. This area is still young and there are many outstanding problems, but progress was made in several directions at the INI.

M Barlow and T Kumagai studied the short time asymptotic behaviour of the heat equation for a multifractal measure, obtaining surprising results for the spectral behaviour. With B Hambly they realised that their pointwise results fit into the context of multifractal analysis.

M Barlow and R Bass considered divergence operators on fractal domains with large scale structure resembling the Sierpinski carpet, obtaining versions of Moser's elliptic Harnack inequality, which provided an interesting contrast to the parabolic Harnack inequalities.

Random recursive fractals are a source of challenging problems. B Hambly and T Kumagai were able to construct a Laplacian and locally regular Dirichlet forms for a large class of such fractals using Schauder's fixed point theorem. B Hambly, Y Peres, J Taylor and Q Liu used ideas from branching processes to obtain a type of multifractal spectrum for the local spectral dimension of such random fractals.

J Kigami and M Lapidus wrote a paper on the volume measure associated with self-similar sets, leading to an expression for the spectral volume of a self-similar set with respect to a trace of some power of the Laplacian.

In discussion with others, J Kigami made considerable progress on properties of heat kernels on self-similar sets, including on the Nash inequality, the continuity and positivity of the kernel and the continuity of the solution.

A Gangal and K Kolwankar introduced a new kind of differential equation on fractals involving fractional derivatives, generalising the Fokker-Planck equation and diffusion equations.

Harmonic measure on fractals

B Duplantier announced some remarkable results on the multifractal spectrum of harmonic measure on planar sets such as percolation clusters, polymers and the external boundary of Brownian paths. Using techniques involving conformal invariance and quantum gravity he obtained an exact formula for the multifractal spectra of such harmonic measures, thus answering many long-standing problems. Whilst at the INI he extended his ideas to Potts clusters and other models. With R Ball and T Halsey he uncovered a symmetry of multifractal spectra corresponding to an invariance under the exchange of interior and exterior domains with respect to certain random fractal frontiers. Numerical work by A Aharony and B Mandelbrot has corroborated these predictions.

Growth models and DLA

DLA (Diffusion Limited Aggregation) is a growth model for pattern-forming systems such as viscous fingering, colloidal aggregation and electrodeposition. The Programme revived interest in DLA theory which had developed little in the previous 15 years. B Davidovitch announced numerical results showing that the fluctuations in the radius of a DLA cluster are anomalously small compared to the typical fluctuations of fractal and multifractal properties of DLA. During the programme R Ball, B Davidovitch, L Sander and T Halsey developed theoretical arguments to explain this phenomenon. R Ball, M Hastings and T Halsey worked on theoretical approaches based on a fugacity expansion on tip-splitting events which will allow better computation of certain DLA properties.

C Bandt presented recent work on the Bak-Sneppen model of self-organised criticality, and his discussions at INI led to rigorous derivations of geometrical properties of the model.

R Stinchcombe made progress on scaling collective particle systems with stochastic dynamics via their quantum spin representation.

Other physical models

J Feder, with input from A Aharony and R Stinchcombe, made steady progress on fractal models for friction, an area which is still not fully understood.

M Filoche and B Sapoval continued work on the question of linear transfer across irregular or fractal interfaces when they are random or deterministic, a question that is important to the understanding of porous electrodes in batteries.

Galaxies

R Ball, G Caldarelli, B Mandelbrot and L Pietronero discussed the question of galaxy distribution in the light of recently available data. In particular progress was made in the relationship of lacunarity to galaxy distribution.

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