Isaac Newton Institute for Mathematical Sciences

Discrete Integrable Systems

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Final Report

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Theme of the Programme

Many processes in physics are mathematically described by differential equations, and this description reflects the smoothness of natural processes as we often experience in macroscopic phenomena. However, there are also many physical processes that are of an inherently discrete nature, and are better described by *difference equations* rather than differential equations. In particular, in physics at the subatomic scale, the assumption of a space-time continuum is no longer always adequate due to the fact that quantum mechanics takes over, and it has been speculated that a coherent theory of quantum gravity requires an inherently discrete description of the fundamental interactions in physics. To describe these discrete phenomena we need to develop the mathematical theory of difference equations much farther than has been done so far.

Difference equations are mathematical equations involving functions whose arguments are shifted by integer or other finite steps. If one would take the limit that the step size becomes infinitesimally small one often recovers a corresponding differential equation, but prior to the limit the equations are essentially nonlocal. This is one reason why the theory of difference equations, in spite of the numerous applications e.g. in numerical analysis, control theory, mathematical biology and economics, has lagged behind the analogous theory of differential equations, as this nonlocality renders such systems both richer as well as more difficult to treat. Furthermore, the fact that several distinct difference equations may reduce to one and the same differential equation after a continuum limit, adds to the difficulty of singling out a preferred difference equation as a model for a specific physical phenomenon. It is here that the notion of *complete integrability* can play a crucial role, since it implies in many cases not only the exact solvability of the equations, but also the presence of a rich algebraic and geometric structure. One expects that it is exactly these properties that are responsible for the fact that integrable equations, in spite of their rareness, appear time and again in applications and physical situations.

Although there is no accepted general definition of a (discrete) integrable system, the many examples of such systems, comprising both ordinary and partial difference equations, discrete-time many-body systems, nonlinear dynamical mappings and models in quantum mechanics, share a core of distinguishing features such as the existence of complete (possibly infinite) sets of conservation laws, exact solutions (such as multi-soliton and finite-gap solutions), new analytical methods (such as inverse scattering and algebro-geometric approaches) and the emergence of novel mathematical phenomena which are hallmarks of integrability, such as multidimensional consistency, the phenomenon of singularity confinement and the connections with discretizations of classical differential geometry (*difference geometry*).

In recent years a number of intriguing connections has emerged between the field of discrete integrable systems and various areas of mathematics such as algebraic geometry, number theory, differential geometry and representation theory, in particular:

- Algebraic geometry of rational surfaces and birational maps
- Difference Galois theory and isomonodromic deformation theory
- Quantum and non-commutative discrete integrable systems
- Representation theory, special functions and quantum and cluster algebras
- Difference geometry, and symmetries and conservation laws of discrete systems
- Diophantine problems in number theory and Nevanlinna theory in complex analysis

From these connections a host of new insights has been gained and new mathematical objects have been defined. The coming together of these various strands of research made the DIS programme very timely, and by focusing strongly on these new trends and developments within the subject, bringing together the most prominent researchers active on the interface of the various disciplines, it has endeavoured to open the possibility of fostering new new breakthroughs in this field of research.

Some of the major open problems the programme has addressed, through the various workshops and the interaction between the participants were the following:

1. The construction and analysis of explicit solutions of the lattice equations that have emerged from recent classification problems;
2. the broadening of the classifications to wider classes of integrable discrete equations;
3. the deepening of the understanding of the algebraic geometry behind certain classes of integrable difference equations and associated dynamical systems;

4. the understanding of the parallels between analytic approaches to difference equations and problems in number theory;
5. the development and understanding of canonical structures in integrable multidimensional systems and associated quantum theories;
6. the definition and analysis of special functions associated with integrable discrete systems defined through analytic difference equations;
7. the development of the algebraic theory of difference equations, their associated Galois theory and their formulation in terms of model theory.

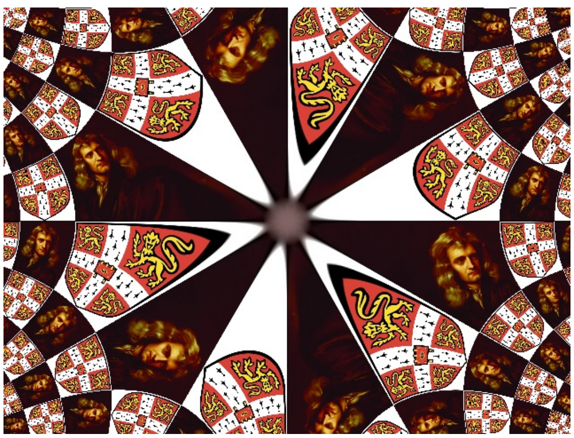

FIGURE 1: *z^3-discrete conformal map of Sir Isaac Newton's portrait and the Cambridge University logo (created by Dr C. Mercat and reproduced with his permission)*

Structure of the Programme

The programme ran over five and a half months and involved the participation of a large number of visiting fellows, ordinary participants and workshop participants, with a healthy mixture of high-profile experts in the field as well as promising early-stage researchers. There were 67 visiting fellows and 28 programme participants. To date, 39 preprints have appeared on the INI preprint list as part of the DIS programme. As well as regular talks given as part of the programme, participants also gave invited seminar talks at Cambridge, Glasgow, Imperial, Kent, Leeds, Loughborough, Open University, Oxford, Queen Mary and UCL.

There was a schedule of two formal seminars each week, interlaced with informal seminars that were organised on an ad-hoc basis, often at the initiative of the participants themselves or on the basis of "popular demand". Among these were several small sequences of lecture series, such as the series of three lectures given by the Rothschild Professor, Prof. J.J. Duistermaat, on *QRT Maps and Elliptic Surfaces*, and the series of three lectures given by Prof. M. Noumi on *Point Configurations and the Elliptic Painlevé Equation*. These series of lectures had a significant influence on the interaction between the two groups of researchers working on integrable mappings and discrete Painlevé equations. A number of so-called Monday Seminars, intended for a wider audience, were given, e.g. by Profs. S.P. Novikov, J.J. Duistermaat and N. Joshi (the latter taking place during the event organised for Isaac Newton Institute correspondents). A weekly communal dinner at one of the Cambridge restaurants constituted an informal way of bonding between the participants and with their accompanying partners. A day-excursion to Bletchley Park in June formed an interesting interlude for participants helping to break the usual pattern of interaction.

It should be mentioned that it was fortunate that the DIS programme coincided with the ALT programme on *Algebraic Lie Theory*, and some nontrivial overlap in interest occurred between the two programmes, with at least two occasions where a participant gave a talk in both programmes, whilst many seminars were attended by participants from both groups.

Apart from these ongoing activities, the programme included four workshops, two of which took place at the Isaac Newton Institute, and two satellite workshops at the University of Glasgow and the University of Leeds respectively. Each of these has a member of the DIS organising team among their organisors, but were otherwise led by a team of invited workshop organisers. A brief description of these workshops follows.

Quantum Integrable Discrete Systems, 23-27 March, 2009

This workshop was organised by Prof. O. Ragnisco in collaboration with Dr J. Avan (Cergy-Pontoise), Dr A. Hone (Univ. of Kent) and Prof. R. Quispel (as DIS representative). It brought together a mixture of mathematicians and physicists working on discrete models of space-time, and discrete integrable systems in the quantum setting, either from a theoretical point of view, or in application to quantum field theory and statistical mechanics. The workshop concluded with a special one-day meeting on discrete aspects of space and time, including talks by the Nobel physicist G. 't Hooft, and the Fields medallist S. Novikov.

The wide range of talks in the workshop revealed that while physical processes are traditionally formulated as taking place in a continuum, there is an enormous potential for further dialogue between mathematics and physics in the realm of discrete descriptions using lattices and iterated maps. In particular, using a lattice description of space-time may be the only way to avoid the problems due to divergences that usually arise in attempts to create a quantum theory of gravity.

Indeed, 't Hooft's talk invited us to consider building up four-dimensional space-time from a discrete basis of flat pieces. On the integrable side, S. Sergeev and V. Bazhanov presented new results on integrable three-dimensional quantum field theory, which revealed beautiful connections with the geometrical structure of classical discrete integrability, as explained in the talk by A. Bobenko.

The talk by C. Korff showed that the ultradiscrete version of XXZ spin chains (crystal limit) leads to the operator structure of conformal field theories, while L. Amico described how integrable spin chains are applicable to problems in condensed matter physics.

S. Novikov argued that a triangular lattice (rather than a square one) is the most natural discrete model of the plane, at least if one wishes to discretize complex analysis, and encouraged us to think about what properties should be preserved when we move from a continuum model to a lattice description.

Overall there was an interesting interaction between different people concerned with discrete structures in mathematics and physics, and the breadth of the various talks meant that nearly all the participants should have been stimulated by being exposed to some ideas outside the areas they were previously familiar with.

Geometric Aspects of Discrete and Ultra-discrete Integrable Systems, 30 March-3 April, 2009

This workshop was held at the University of Glasgow and was organised by Drs C. Gilson, C. Korff and J. Nimmo (local organisers) in collaboration with Prof. O. Ragnisco (as DIS programme representative). The workshop focused strongly among other issues, on connections with tropical geometry on the one hand, and with discrete Painlevé equations and the corresponding algebraic geometry on the other hand. Since the workshop covered a topic in which there is a strong involvement of Japanese scientists, a substantial representation from that country was present, which was made possible through additional funding from various Japanese and British-Japanese collaborative sources as well as local funding (e.g. a PMI2 Japanese/British Research Cooperation Award, the Sasakawa Fund, the Daiwa Foundation, and the Glasgow Mathematical Journal trust).

The workshop has led towards a comprehensive understanding of the present status of discrete and ultra-discrete integrable systems. It covered a wide range of subjects, including lattice equations and Yang-Baxter maps, geometric crystals and combinatorics of crystal bases, box-ball systems and tropical geometry. These intimately interlaced subjects were well presented, with special focus on their tropical geometric aspects.

Recent results on Yang-Baxter maps were presented by A. Veselov and S. Kakei. F. Nijhoff described elliptic solutions of the ABS lattice equations. T. Takenawa explored integrable cellular automata using a tropical version of Fay's identity and Y. Yamada presented a Lax formalism for Sakai's elliptic difference Painlevé equation.

Algebraic Theory of Difference Equations, 11-15 May, 2009

This was a satellite workshop, held at the University of Leeds and organised by Profs A. Pillay (Chair), A. Fordy, A. Mikhailov (local organisers) in collaboration with Prof. J.-P. Ramis (Toulouse) and Prof. F.W. Nijhoff (as DIS representative). The meeting was supported by the INI and an additional funding from the LMS. It involved researchers from at least three usually separate communities: from integrable systems, from differential and difference Galois theory, and key representatives from the model theory community in mathematical logic. Natural links between these subjects have emerged in recent years and that came strongly across from the lectures of key note speakers as well as from the interactions between participants. There were well over 50 participants, and among them there were prominent experts from all three fields, such as G. Casale, M. Gekhtman, C. Hardouin, S.P. Novikov, J.-P. Ramis, T. Scanlon, M. Singer, M. van der Put, H. Umemura and L. di Vizio. Apart from research talks a number of more elementary "tutorial" type talks were given by some of the main experts (Halburd, Scanlon, Singer and Ramis), in order to build bridges between the various disciplines and to arrive at a common language. These tutorials were highly appreciated by the participants as was evident from the feedback. In terms of coverage the meeting included the following subjects: *i)* the Galois theory of difference and q-difference equations, *ii)* isomonodromic deformation problems and Painlevé difference equations, *iii)* integrability of analytic difference equations, *iv)* difference algebraic geometry, *v)* dynamics of rational maps, *vi)* connections with the model theory of difference fields. Highlights of the meeting included the strong overlap of results in the talk by G. Casale, which depended upon difference Galois theory in the sense of Malgrange, and results in Scanlon's talk which depended on some basic dichotomies in the model theory of difference fields. Another highlight was the application of model theoretic ideas to the Galois theory of q-difference equations as explained by M. van der Put. Furthermore, the connection between Galois theory and isomonodromic deformation theory became apparent in the contributions from J. Sauloy and J.-P. Ramis.

Discrete Systems and Special Functions, 29 June - 3 July 2009

The final workshop of the programme took place at the Isaac Newton Institute, and was organised by Prof. P. Clarkson and Dr A. Olde Daalhuis (Edinburgh University) in collaboration with Prof. M. Noumi and Dr R.

Halburd (as DIS representatives). The workshop attracted 66 participants and continued a recent fruitful trend of bringing together researchers from integrable systems, special functions and orthogonal polynomials. There are deep connections between these fields. For certain choices of parameters, the discrete Painlevé equations admit special solutions given in terms of solutions of second-order discrete linear equations. These equations are natural and important in the theory of special functions. In turn, the first occurrence of a discrete Painlevé equation in the literature (although it was not recognised at the time) appears to have been as the three-point recurrence relation for a system of orthogonal polynomials.

Further connections between the discrete Painlevé equations and orthogonal polynomials were presented in talks by W. van Assche and A. Its. K. Kajiwara clarified how the apparent differences between the hypergeometric solutions of the symmetric and asymmetric q-Painlevé equations could be understood by using birational representations of Weyl groups. E.M. Rains gave a monodromy interpretation of Sakai's elliptic Painlevé equation in which the surfaces arising in Sakai's construction are moduli spaces of second-order linear elliptic difference equations (and some related objects). S. Ruijsenaars described the use of special Hilbert-Schmidt operators to construct orthonormal joint eigenvector bases of the commuting Hamiltonians for the elliptic Calogero-Moser quantum N-particle systems of non-relativistic and relativistic type. The talks by W. Hereman, E. Hubert and E. Mansfield described advances in symbolic computation for discrete systems.

Outcomes and Achievements

The overall success of the programme was that it drew together experts from different disciplines within the mathematical sciences, including experts as well as early-stage researchers working in areas like complex analysis, algebraic geometry, representation theory, Galois theory, spectral analysis, the theory of special functions, graph theory, difference and differential geometry, mathematical physics and naturally integrable systems. The main theme being the crossroads of these and other fields, the programme allowed for new synergies to emerge and new collaborations to sprout.

We list here a number of highlights that deserve special mention:

- The development of a novel variational formalism for integrable lattice systems based on Lagrangian multiforms, in which the geometry of the space of independent variables forms an essential part of the least-action principle. This is the first example of a canonical structure that incorporates the key property of multidimensional consistency. (Lobb, Nijhoff, Quispel).
- The establishment of the full soliton solutions to the celebrated ABS lattice equations and the first construction of elliptic solutions of the latter equations. These are the first steps finding solutions some of the newly established equations from the celebrated ABS classification. (Atkinson, Hietarinta, Nijhoff)
- Full classification of octahedral lattice equations in three dimensions, leading to an exhaustive list of such equations in the scalar case. This work was reported during two of the workshops of the programme. (Adler, Bobenko, Suris)
- Development of a generalized symmetry method for discrete equations, which allows for a classification of difference equations beyond the cases considered so far (i.e. affine linear scalar cases). Furthermore, one can relax the multidimensional consistency criterion of integrability, leading to new cases. (Levi, Yamilov, Hydon, Viallet).
- Development of the theory of singular finite-gap operators, which applies in particular to the case of real but singular potentials which violate self-adjointness. These can be properly defined using indefinite inner product structure, and spcific applications to the continuous and discrete Fourier transforms have been worked out. (Novikov, Grinevich).
- Nevanlinna theory and difference equations. The extension of results for the difference operator to Cartan's theory of value distribution for meromorphic functions from $\mathbb{C}$ into $\mathbb{C}P^n$. Analytical and numerical explorations of new analogies in Diophantine approximation of key ideas from Nevanlinna theory. (Halburd, Korhonen).
- The construction of a systematic approach to periodic initial value problems on the two-dimensional lattice. The discovery of elegant closed-form expressions for all integrals of the resulting maps in the integrable case has opened novel avenues of proving the Liouville-Arnold integrability of these maps. (van der Kamp, Tran, Quispel)
- New uses of Okamoto's space of initial conditions for Painlevé equations. Duistermaat showed how the explicit construction of Okamoto's space of initial conditions for the first Painlevé equation leads to a new and elegant proof that this equation has the Painlevé property. He and Joshi explored combining the methods of Okamoto and Boutroux to obtain asymptotic information.
(Duistermaat, Joshi)
- Interpretations of the discrete Painlevé equations using discrete geometry.

  Several new collaborative projects have started to find discrete geometric interpretations of discrete Painleve equations, to explore higher dimensional integrable mappings arising from the geometry of point configurations and also to clarify possible links between discrete integrable hierarchies and discrete Painleve equations. (Doliwa, Hone, Nimmo, Noumi, Viallet)