Free Boundary Problems in Industry

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Introduction

It was a bold step for one of the bastions of theoretical mathematical research - the Isaac Newton Institute, in Cambridge, England - to focus its first-ever short programme on mathematics-in-industry. The bare facts are that about 70 researchers, all of an intensely collaborative nature, came together for each of three one-week sessions to discuss free boundary problems in the food, glass, and metals industries. Free boundary problems had been selected because of their proven ubiquity in industrial research, coupled with their mathematical fascination. What emerged were some dramatic revelations about the insights that all kinds of mathematics may soon be able to offer all branches of industry, ranging from high-tech to low-tech.

New, Unstructured Study Group Format

Many frameworks have been devised to bring mathematics to bear on industrial problems as effectively as possible. Well-established clinics, study groups, and workshops are held every year around the world, all geared to moving academic expertise across the chasm into industry in order to solve specific problems. The very act of posing an industrial problem in mathematical terms has been shown, time and again, to lead to new insight and new mathematics.

The Newton programme had quite a different format - each of the week-long sessions was an unstructured coming together, catalysed by a handful of overview presentations by industrialists on Monday, followed by three days of spontaneous open discussions. The output was an astonishing list of topics considered ripe for further specific studies, compiled on Friday (and available at http://www.newton.ac.uk/programs/ old_progs/FBP/).

Suffice it to say here that intense debates were generated on topics ranging from the modelling of mastication, especially so-called bolus formation, to the 'fictive temperature' of a solidifying glass, and from the extrusion of molten chocolate to the hot tearing of mushy alloys.

There were two particularly eye-catching examples from the food industry. The first, illustrated in Figures 1-4, concerns the manufacture of crumpets. ('Crumpet' is one of those words that mean different things in different places, but it definitely does not mean 'English muffin'.) Like many breads, cakes, and biscuits, a crumpet is formed from dough that has been suitably kneaded and proved. (Why dough needs to be kneaded and proved, explained at the microstructural level, is a mathematical research topic in itself.)

In baking, most dough turns into a bubbly mixture, as in bread, with the solidified dough forming a more or less connected matrix. Crumpets, however, are heated in a special way, on a hot plate at 230°C. After a few seconds, crumpet dough takes on the 'columnar' morphology seen in Figure 4. To predict the conditions under which such fingering occurs - what a challenge for free boundary theorists! The difference in topology between a crumpet and bread is all important: Without its high genus, the crumpet would not be penetrated so easily by molten butter and jam, and it would lose all its distinctiveness.

A second, equally mouth-watering problem concerns the 'enrobing' of cake, biscuits, or ice cream with a thin layer of chocolate that is initially molten. Chocolate is a wonderful material that can solidify into at least six phases, and yet the only phase that is nice to eat is meta-stable! Fortunately, molten chocolate can be thought of as a Newtonian liquid, and we are led to the free boundary problem shown in Figure 5 for a biscuit.

We need to solve the slow flow equations $0 = -\nabla p + \mu\nabla^2 u + \rho g,\ \nabla \cdot \mathrm{u} = 0$ (1)

for the pressure p and velocity u in the flow region Ω, where m is the chocolate viscosity, ρ the density, and g gravity, with no slip conditions on B; crucially, there is no traction on the free surface $\partial\Omega$, which is also a material surface. Discussions of similar problems with circular symmetry [1] have revealed an interesting nonuniqueness in the nearly unidirectional flow that occurs towards the base of B. But the most important prediction concerns the thickness of the chocolate at the 'shoulder' of B: No one will buy your biscuits if the coating at the shoulder is too thin.

The key to the diversity and novelty of the Newton programme was the ability of the academics and industrialists to confront each other on an equal footing, with everyone free to probe and enunciate questions of philosophy and strategy among themselves. At one time, for example, we had rigorous thermodynamicists arguing with expert chocolate manufacturers; such interactions created as much astonishment as fervour. This eclectic approach had one dramatic consequence. Unlike settings in which specific problems are addressed, and in which it is mutually advantageous for industrial and academic researchers to work together, the Newton programme, lacking a common goal, allowed such frankness that wild disagreements were common. Even well-hardened study group veterans found themselves becoming intellectually overstressed!

The academics present had a rare glimpse into how industrialists select problems that they consider suitable for the attention of academics, and the industrialists were able to see the full scope and capabilities of modelling and analysis. It was these revelations to both groups that allowed the overwhelming plethora of opportunities to emerge as candidates for future targeted research.

Challenge to the Mathematics Community

One overriding implication surprised us all. If the mathematics community is ever to fulfil its potential as a source of new insights into industrial problems (rather than leaving industrial researchers to do their own mathematics), its members must learn to recognise that mathematics is everywhere, underlying activities that range from eating to flying. The task is so vast that mathematical scientists must learn to group their activities in as manageable a way as possible if they are to avoid wasteful duplication of effort and present a coherent picture to students, politicians, and others. It is a tough job for a mathematician, let alone a politician, to overview an activity that encompasses so much of science - especially when it is so unstable to small modifications to well-established models.

Take traffic flow, for example. A crude way to model dense flows is to adopt the kinematic wave approach [2], where the vehicle velocity u and flux ru are related by the conservation law $\partial\rho/\partial t + \partial/\partial x\ (\rho u) = 0$ (2)

It is common to model driver reaction with a decreasing function $u(\rho)$, say $u = 1$ - r, where $0 < \rho < 1$. At a recent Study Group in Shanghai, the traffic was bicycles (and thousand-foot traffic jams were commonplace at right turns). How do you control such traffic so as to minimise congestion? (When cars and bikes are interacting, the situation is even more of a nightmare to model.)

Colin Please and Alistair Fitt came up with the following neat idea for the case in which a fraction λ of the cyclists want to turn right. Write $u = u^*(1 - \rho)$, where the constant $u^* = 1$ on the highway and $u^* = \alpha < 1$ at the turn. The maximum flux around the turn is then $\alpha/4$ (the flow is 'choked' in gas dynamics parlance). Thence, a simple conservation-of-mass argument shows that jams will occur whenever $\lambda > \alpha/4$. There seem to be many generalizations of this kind of argument.

A few years ago, a Japanese Study Group in Kobe considered an equally fascinating transportation problem. Residents living near tunnel mouths on the Shinkahsen line experience house-rattling shock waves a few seconds before the emergence of each 200-mph train. This is another new problem for experts in hyperbolic PDEs - with the twist in this case coming from the effect of wall drag on tunnels longer than a mile or so.

One modelling approach is to modify the traditional inviscid gas dynamics equations, comprising (2) together with conservation of momentum

$\partial/\partial t\ (\rho u) + \partial/\partial x\ (p + \rho u^2) = 0$ (3)

and energy $\partial(\rho(e + \frac{1}{2} u^2))/\partial t + \partial(\rho u(e + \frac{1}{2} u^2))/\partial x + \partial(\rho u)/\partial x = 0$ (4)

by inserting a term proportional to -ru|u| on the right-hand side of (3). This term, a crude 'Fanno' approximation for the wall drag exerted on the air pushed ahead of the train, is important only over relatively long distances. The accurate incorporation of this term into asymptotic and numerical solutions continues to challenge applied mathematicians in the UK and Japan.

To get back to the Newton programme, and another class of problem altogether, the workshop revealed a beautiful inverse problem concerning the manufacture of windshields. In the 'sag' process, the wind-shield frame is positioned more or less horizontally, and an initially flat glass sheet is lowered onto the frame. Heaters are placed to make the sheet sag to a shape specified by the auto manufacturer.

In the simplest model, the windshield can be thought of as a viscous plate (or, better, as a viscoelastic shell) whose bending stiffness is a function of temperature, and hence of position, which is to be determined. Thus, we need to find the coefficients of an elliptic PDE with simply supported boundary conditions whose solution is as close as possible to a prescribed function. But this is not just a hyperbolic problem for the bending stiffness - near the corners of the frame, the windshield can easily 'lift off', thereby losing its convexity (even a 1-mm lift-off can be a disaster!). Hence, we have a novel contact problem embedded in the inverse problem. And what if the auto designer were to demand that the windshield be developable...?

With all these problems awaiting mathematical attention, how is our community to respond? In particular, how can we as mathematicians broaden our range of activities without diluting our knowledge so that we can make the contributions of which we are uniquely capable? This is a difficult question whose answer will depend on many factors. A crucial one will be the inevitable observation that mathematics-in-industry fares much better in some mathematics communities than others. The Newton meeting would not have revealed what it did had we not benefitted from a hard core of study group old hands from the UK and Europe, all of whom were used to entering into discussions on practical problems in an untrammeled way.

Of course, mathematicians all over the world engage in mathematics-in-industry on a more or less one-to-one level. Researchers involved in these time-consuming and academically low-profile activities often pay a heavy price - a lack of publications and a sense of ostracism - even though the ideas are every bit as exciting and important as those in traditional academic activity. But if these researchers remain voices in the applied mathematics wilderness, what hope is there for the wider realisation that the Newton programme revealed?

References

[1] EO Tuck, M Bertwick, J Van der Hoek, The Free Boundary Problem for Gravity-driven Unidirectional Viscous Flows, IMA J. Appl. Math. 30 (1983), 191-208
[2] GB Whitham, Linear and Nonlinear Waves, Wiley, New York, 1974

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