Foams and Minimal Surfaces

29 July to 23 August 2002

Report from the Organisers: A Kraynik (Sandia), H Stone (Harvard), E Terentjev (Cambridge), DL Weaire (Dublin)

Background

Foams are familiar gas-liquid systems of wide importance to industry, exhibiting many properties of interest to physicists, mathematicians, engineers and other applied scientists. In mathematical terms, their governing principle (at least in equilibrium) is the minimisation of surface area. The study of idealised models of foam therefore throws up a long list of significant mathematical problems, some of them associated with the name of Joseph Plateau and his classic 1873 book. In recent years computation has given fresh impetus to the subject, since it enables us to explore the consequences of idealised models for complex disordered foams by accurately simulating them. Some of the problems have direct bearing on potentially significant areas of application, such as in determining the properties of metallic foams.

Programme Outline

This programme was conceived as an opportunity to maximise the cross-fertilisation of ideas between mathematicians, physicists and allied disciplines such as chemical engineering. It was successful in attracting world-leading figures from all these communities. Particularly notable are Thomas Hales, who in recent years has produced proofs of classic minimal conjectures (the Kepler problem and the honeycomb problem mentioned below), and Ken Brakke, originator and developer of the Surface Evolver software which is widely used in problems of surface energy minimisation.

The schedule of topics moved progressively from refined minimal problems to physical experiments and their simulation. At the outset, Hales outlined his strategy for the honeycomb and other problems. That the 2D honeycomb minimises line length for cells of equal size and any shape has been recognised for centuries, but proving this has been intractable until now unless restrictions such as straight edges are imposed. The general proof of Hales is what he calls an ‘engineering’ solution, patched together out of various inequalities which rigorously entrap every possible case.

What about the 3D case, the so-called Kelvin Problem? In 1887 Kelvin conjectured that surface area was minimised by a regular stacking of his ‘tetrakaidecahedron’. His remarkable notebooks, in which some of his sudden insights into this problem are recorded, are stored in the Cambridge University Library, and participants were kindly given permission to scrutinise them during the programme itself. His conjecture was overthrown by the computational discovery of the Weaire-Phelan structure in 1994, and a proof by Sullivan and Kusner that this indeed has a lower surface area. But the proof that it is an absolute minimum remains to be found, and Hales commented (not very optimistically) on this difficult challenge.

Another challenge was thrown up by Andrew Kraynik, in the course of describing extensive simulations of large samples of disordered foam. It appeared that in the case of bubbles of equal volume, the lowest possible energy in such a bulk structure is that of a regular dodecahedron, obeying Plateau’s laws for angles and curvature. However, Ken Brakke was able to show, by conformal transformation, that the regular dodecahedron is a saddle point, and that the energy *can* be lower for asymmetric bubbles. Yet another challenge concerns an upper bound on the shear elastic modulus of a 2D disordered foam. In addition, John Sullivan adduced a long list of rather general conjectures about periodicity, pressures and other properties of equilibrium foams. In some cases the physicists protested that the answers were ‘obvious’ and indeed were able to show the value of physical insights.

Careful analysis of the statistics of the cellular patterns that represent foams in two dimensions have thrown up several intriguing correlations; these were reviewed by Schliecker, Delannay and others. The case of three dimensions has proved more difficult. For example, what is the 3D equivalent of von Neumann’s law in 2D, which relates the growth rate of a cell to its number of sides? Sascha Hilgenfeldt explained the relevance of theorems of Minkowski which concern integrals of mean curvature of a cell. These lead to a 3D growth law which plays the role of that of von Neumann, if only in an average sense.

The phenomena of drainage (the motion of liquid through a foam) and rheology (which deals with the flow of the foam) lie at the frontier of current research. Howard Stone and others debated current ideas on drainage, for which an adequate phenomenology now distinguishes between foams with effectively rigid or free surfaces. Rheology is a notoriously difficult subject to get to grips with, and the description of foam flow at finite shear rates remains far from settled. Reinhard Höhler represented the current efforts to measure rheological properties reliably. While at the Institute, Masao Doi was able to derive and present a constitutive model for viscous foams that is based on affine film deformation. Another difficult question remaining to be resolved is concerned with normal stress differences.

It is much too soon to say that a truly comprehensive practical theory of foam behaviour is available. Many of its individual ingredients, mentioned above, remain poorly defined, and their couplings (e.g., drainage and rheology, drainage and coarsening) are still to be properly understood. Often the results of experiments are rationalised afterwards, based on the influences of the different surfactants used. This made for lively discussion.

Participants were joined throughout the programme by colleagues from different departments in Cambridge. Several participants also made visits to these departments to discuss ‘foam-inspired’ topics of mutual interest. On several occasions we were joined by researchers from Schlumberger Cambridge Research. Also, Mike Ashby from the Department of Engineering made a special presentation on the mechanical properties of metal foams.

The programme was enlivened by occasional glimpses of the history of the subject in the form of biographical sketches. Some of the personalities included Kelvin, Riemann, Minkowski and Cyril Stanley Smith.

The typical schedule for a given day was for no more than four half-hour talks or focussed one-hour discussions. Throughout the meeting the discussion was lively and interactive and it was particularly valuable that participants from different backgrounds generally attended all of the talks. To allow participants to solicit feedback on problems of current interest, we also held small sets of short five-minute talks with five further minutes allotted for discussion. This proved particularly successful.

Special Events

Surface Evolver Extravaganza
The workshop started with a Hewlett-Packard day, the 'Surface Evolver Extravaganza', in which several of the world's leading authorities on the use of Brakke's Surface Evolver entertained us with various applications from triply periodic crystal structures to knots. The day finished with a presentation by Brakke on the stability of soap film junctions, after which we studied protein foams from a local brewery.

Soap Bubble Geometry Contest
The Soap Bubble Geometry Contest was an opportunity to educate the general public. Frank Morgan performed demonstrations and set questions to test the audience's knowledge of how soap films interact. The contest was well attended, and the youngest participants, less than ten years old, showed great enthusiasm.

Presentation of Sculpture
Prior to the contest, John Sullivan presented the Institute with a minimal surface sculpture (below) as a token of gratitude on behalf of the participants. The sculpture will be displayed in the Institute building.

Outcome and Achievements

The communities which were brought together interacted well, and many future collaborations should result which would not have emerged without such a stimulus. Some such collaborations were already active during the programme. Ken Brakke kindly volunteered to conduct practical sessions in the use of the Surface Evolver. These tutorials introduced several new users to the program and its applications.

It was particularly pleasing to observe the fruitful discussions between mathematical scientists from many different intellectual backgrounds. The Newton Institute is ideally suited and designed to facilitate such interactions. The general conclusion was that two-phase mixtures of academics can be as interesting, profitable and unpredictable as the mixture of gas and liquid that constitutes a foam.

Back to Top