

FINAL REPORT

INTERACTIONS BETWEEN DYNAMICS OF GROUP ACTIONS AND NUMBER THEORY

Issac Newton Institute, June 9 — July 4, 2014

Organisers: Anish Ghosh (TIFR), Alexander Gorodnik (Bristol), Barak Weiss (Tel Aviv)

Advisory Committee: Ben Green (Oxford), Dmitry Kleinbock (Brandeis), Philippe Michel (EPFL)

SCIENTIFIC BACKGROUND

In the last decade there have been a number of important breakthroughs in Number Theory and Diophantine Geometry, where progress on a longstanding problem has been achieved by utilising techniques originated from the theory of homogeneous dynamical systems. This has led to a number of new promising research directions that fuse ideas from these diverse areas and have the potential to be useful in much more general contexts. This programme explored the following topics broad range of topics which are on the forefront of current research:

- *Multiple ergodic averages* play a fundamental role in ergodic theory, and their asymptotic behaviour has been analysed in the ground-breaking works of Host, Kra and Ziegler using dynamics on nilmanifolds. The ideas developed in this work played an important role in more recent works of Green and Tao on *arithmetic progressions in primes* and on the *Hardy–Littlewood prime tuple conjecture*. These fundamental problems in number theory are related to dynamical properties of flows of nilmanifolds.

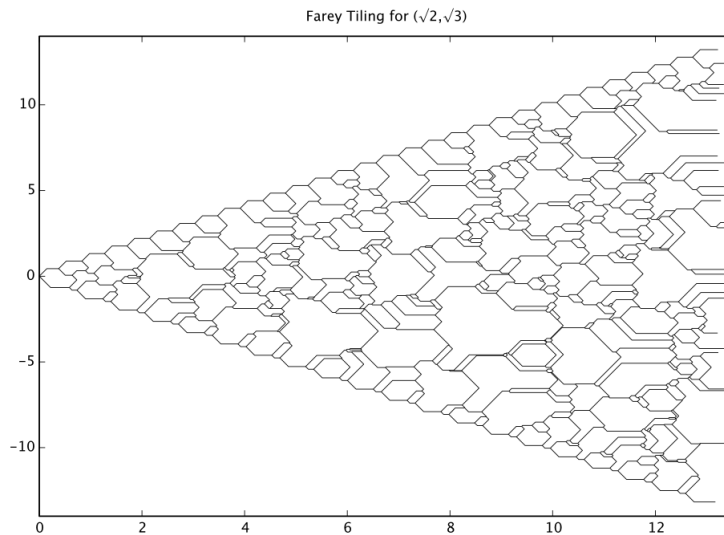


FIGURE 1. This picture, produced by Y. Cheung, represents dynamics of the Weyl chamber flow.

- The study of dynamics on nilmanifolds has found another striking application in a work of Flaminio and Forni who obtained new *estimates on exponential sums* that were not accessible by techniques from classical analytic number theory. Recently these estimates were further improved by Wooley using a very different approach.

- An important breakthrough in the *subconvexity problem for L -functions* was made by Michel and Venkatesh by combining number-theoretic and dynamical systems techniques. This problem is closely related to the distribution of orbits of compact tori on homogeneous varieties that has been explored in the recent important works of Einsiedler, Lindenstrauss, Michel, and Venkatesh.
- Remarkable progress on the problem of *quantum unique ergodicity* in the arithmetic setting has been achieved by Lindenstrauss using dynamical systems techniques and Holowinsky and Soundararajan using analytic number theory techniques. Their work is closely related to the subconvexity problem for L -functions.
- The theory of Diophantine approximation is another prime example where analytic and dynamical systems techniques go hand by hand. The best result in the direction of the *Littlewood conjecture in Diophantine approximation* was obtained by Einsiedler, Katok, and Lindenstrauss via analysing the behaviour of orbits of higher rank abelian actions, but the solution of the very much related *conjecture of W. Schmidt on Diophantine approximation with weights* was recently obtained by Badziahin, Pollington, and Velani using classical techniques.

The main aim of this programme was to bring together researchers working on the above topics, strengthening existing ties and building new connections between these rapidly changing, fertile subjects.

STRUCTURE

The programme brought together close to fifty experts and young mathematicians working in the ergodic theory of group actions on homogeneous and moduli spaces, Diophantine approximation and analytic number theory. The start of the programme was marked by an introductory Easter school “Dynamics and Analytic Number Theory” at Durham University, funded by a European Research Council grant held by A. Gorodnik, the London Mathematical Society, and the Clay Mathematics Institute. This school was intended for postgraduate students, postdoctoral scholars as well as more senior mathematicians who wanted to get a head start and learn more about the topics of the programme. This event allowed the dissemination of important ideas and techniques among a much broader audience, and many participants of the school returned to follow the more advanced part of the programme.



The first three weeks of the programme comprised of a few talks every day with the majority of the day left free for participants to discuss mathematics with each other. There were social events including pizza evenings and a tour of the city. We designed the schedule in such a way that participants were able to learn new ideas and techniques as well as make progress on their research projects and collaborations. The relaxed atmosphere of the programme and the lovely ambience of the Newton institute were very popular with the participants and resulted in a very productive time. During the last week of the programme we held a workshop that covered the latest developments in the field.

WORKSHOPS AND LECTURES

- **School “Dynamics and Analytic Number Theory”**

Durham University, 31 March — 4 April, 2014

Organisers: D. Badziahin (Durham), A. Ghosh (TIFR), A. Gorodnik (Bristol),
N. Peyerimhoff (Durham), T. Ward (Durham), B. Weiss (Tel Aviv).

This event featured a collection of lecture series given by distinctive speakers such as Austin, Bugeaud, Einsiedler, Forni, Kontorovich, Velani, and Wooley. It intended to cover some of the important research areas related to the programme in an accessible way. The school was very successful and attracted more than 60 participants working in both dynamical systems and number theory.

- **Clay Senior Lecture by Y. Benoist**

Newton Institute, June 16, 2014



This lecture covered some of the recent important developments regarding the establishing of spectral gap for actions on compact Lie groups.

- **Clay Senior Lecture by A. Eskin**

Newton Institute, June 30, 2014

This lecture discussed a new approach to the important problem of convergence of certain averages in ergodic theory, and some far-reaching applications of this method.

- **Mordell Lecture by A. Eskin**

Newton Institute, July 3, 2014

This lecture covered the recent ground-breaking results of Eskin and Mirzakhani regarding the description of orbit closures for certain group actions on moduli spaces of compact Riemannian surfaces.

- **Workshop “Groups, Numbers, and Dynamics”**

Newton Institute, 30 June — 4 July, 2014

Organisers: A. Ghosh (TIFR), A. Gorodnik (Bristol), B. Weiss (Tel Aviv).



This workshop attracted over 50 participants with very diverse backgrounds and focussed on the latest ground-breaking developments in the field. It featured prominent speakers including Y. Benoist, V. Beresnevich, V. Bergelson, S. G. Dani, M. Einsiedler, A. Eskin, G. Forni, D. Kleinbock, G. A. Margulis, S. Mozes, A. Nevo, S. Velani and T. Wooley. The talks in the workshop were of uniformly very high quality and the state of the art in metric Diophantine approximation, analytic number theory and the ergodic theory of group actions was presented. Though the workshop schedule was necessarily busier, participants still found the time during breaks to talk and collaborate. For instance, M. Einsiedler, S. Mozes, N. Shah and U. Shapira made progress on a joint project during this time.

OUTCOME AND ACHIEVEMENTS

The participants judged the programme to be an overwhelming success, and it is likely to have substantial scientific impact in the years to come. Many new collaborations were started and significant progress was made on many others. Thomas Ward made progress on a book which he is writing jointly with M. Einsiedler and E. Lindenstrauss titled “Entropy in dynamics”. J. An, V. Beresnevich, and S. Velani made progress on a joint paper on the topic of Schmidt Games, and S. G. Dani and A. Nogueira made progress on a joint research project exploring Diophantine approximation by primitive vectors. A. Ghosh, A. Gorodnik and A. Nevo finished their paper titled “Best possible rates of distribution of dense lattice orbits in homogeneous spaces”. J. Athreya and J. Chaika worked on a project on the Hausdorff dimension of non ergodic interval exchange maps, and B. Kra and B. Host made progress on their joint project exploring return times in dynamical systems.



T. Wooley obtained close to best possible estimates for Weyl sums equipped with lower order coefficients restricted to sets of measure one, motivated in part by results of Flaminio and Forni. During the programme period, there were many talks by young mathematicians discussing their research, many of whom were junior members of the institute. A major highlight during the programme was a series of lectures by Y. Benoist, Clay Senior Scholar in residence on his groundbreaking joint work with J. Quint on stationary measures. Benoist's lucid and detailed lectures were very beneficial to the participants. Most participants went away with new projects and new ideas and several grant proposals are being planned as a result of consultations during the programme. As part of the INI Outreach programme, Athreya, Ghosh and Gorodnik gave talks at the University of York, and Athreya, Eskin, and Chaika gave talks at Warwick University. The programme ended with the participants keen to return for a follow up!