Geometry and Topology of Fluid Flows

4 September to 17 December 2000
Report from the Organisers: H Aref (Urbana-Champaign), T Kambe (Tokyo), RB Pelz (Rutgers), RL Ricca (UCL)

Introduction

The realization that the mathematical disciplines of topology and geometry are extremely valuable in furthering our understanding of fluid flows has been evolving steadily for many years within the fluid mechanics community. This realization was the impetus for many workshops and conferences: a 1989 IUTAM Symposium on Topological Fluid Mechanics held in Cambridge (UK), a 1990 NATO workshop on The Global Geometry of Turbulence held in Rota (Spain), a 1991 five-month programme on Topological Fluid Mechanics at the Institute for Theoretical Physics at the University of California, Santa Barbara, and a 1992 workshop on Singularities in Fluids, Plasmas and Optics in Crete.

With these collaborative precedents between pure and applied mathematicians, physicists and engineers, and the need to enhance the dialogue between these disciplines in mind, we planned and organised this programme at the Newton Institute directed towards understanding phenomena in fluid mechanics and magnetohydrodynamics (MHD) that appear to involve substantial questions of topology and geometry.

The programme brought together experts in widely varying fields to work on a set of research problems at the crossroads of fluid mechanics, modern topology, dynamical systems, analysis, and PDEs. The Newton Institute, with its first-class facilities and optimal working environment, provided an ideal location to encourage the fruitful collaboration between pure and applied sciences.

Since Poincaré's seminal work, it has been recognised that the global geometric point of view is essential for understanding Newtonian mechanics. Modern differential geometry and topology have brought new insight into mechanical and fluid-mechanical systems having symmetry and geometrical structures. Integrals of motion or conservation laws were found to be closely related to the structure of the topological space. The existence of certain integrals of the equations of MHD and inviscid hydrodynamics, for example, can be interpreted in terms of the linking of magnetic fields and vortex lines.

The initial success in using sophisticated methods from geometry and topology to solve problems in fluid mechanics suggests that at least some of these methods belong in the "tool kit" of the applied mathematician. Researchers in a number of areas of fluid mechanics and MHD are realising that an infusion of mathematical knowledge which is non-traditional in these subjects can greatly enhance their ability to understand and explain the phenomena that they observe.

Application Areas

A few important application areas ripe for mathematical input were addressed in the programme by bringing many experts together for collaboration over an extended period, having daily seminars and expository lectures and running numerous workshops.
The topics fall roughly into four categories: the application of dynamical systems and topology to stirring and chaotic advection, the application of differential geometry to fluid mechanics, the application of topology to magnetic and classical fluid flows, and the application of geometric and topological concepts to the problem of regularity/blowup in hydrodynamics.

Stirring and Chaotic Advection

The application of topological concepts to the kinematic problem of movement of a scalar field in a given flow has provided a mathematical theory for the process, created quantitative measures for mixing, and allowed a framework for design and optimization techniques for the enhancement or suppression of mixing. Numerous experts shared recent developments and developed collaborations with other participants.

Geometry of Fluid Motion

Perhaps half of the lectures and discussions in the programme centred on the application of differential geometry to hydrodynamics and MHD. A dynamical system can be visualised as a field of vectors in a phase space, on which a solution is an integral curve. Geometrical theory then provides a means of describing the global properties of the family of solution curves, which fill up the entire phase space. The group theoretical approach to hydrodynamics considers that the fluid motion is a geodesic curve on a group of volume-preserving diffeomorphisms with right-invariant metric given by the kinetic energy.
This provides a Lagrangian description of the particle motion of an ideal fluid, allowing the Lagrangian stability to be determined by the curvature of the geodesic, and unifies ideal fluid flow with other systems like Korteweg de Vries and MHD.

Topological Ideas in Fluid Mechanics

Topological concepts have not only been applied successfully to kinematics, but have provided very physical constraints on the dynamics of ideal fluids and MHD.The knottedness of vortex and magnetic field lines is preserved in time allowing application of many concepts of knot theory to such flows. In particular, the complexity of a vortex tangle can be quantified using measures like the linking number. Energy relaxation methods based on various energy functionals have been applied to non-trivial topologies of various physical systems.

The Hydrodynamic Regularity / Blowup Problem

One year ago, this problem was elevated to one of seven 21th century, Hilbert problems. The Clay Mathematics Institute has offered $1m for proving that the equations of hydrodynamics are regular or alternatively that spontaneous blowup can occur in finite time. Many discussions and seminars during the programme centred on this question. Numerical solutions have exposed a possible asymptotic behaviour with self-similar scaling in the inner layer. Concepts from both topology and geometry were applied. Work on the curvature and structure of the manifold was initiated.

Meetings and Workshops

Four workshops were held during the programme and were specifically designed to provide either an introduction to topology and geometry and the application areas above, or a state-of-the-art assessment of recent advances in these areas.

NATO ASI: Pedagogical Workshop on *Geometry and Topology of Fluid Flows*, 11 - 22 September 2000
Organisers: RL Ricca and A Shnirelman
This workshop was held at the beginning of the programme and, as its name suggests, was designed to introduce and provide a background in the subject areas of the programme. The topics and speakers included Differential Geometry and Knot Theory (Langevin, Kauffman, Weber), Topological Fluid Mechanics and MHD (Berger, Khesin, Moffatt, Ricca, Spiegel), Topological Kinematics and Advection (Aref, Boyland), Geometry of Fluid Flows (Misiolek, Kambe, Shnirelman, Vladimirov) and Finite-time Singularities (Childress, Frisch, Hornig, Pelz). There was a poster session for all others. There were 85 researchers in attendance, many from NATO participant countries, and the proceedings will be published by Kluwer.
Spitalfields Day: *In Search of the Ideal Knot*, 12 October 2000 with a follow-on day of talks
Organiser: RL Ricca
Six world specialists presented results and latest discoveries on mathematical and physical knots.
Speakers included Stasiak (Lausanne), Niemi (Uppsala), Winfree (Arizona), Fernandez-Ranada (Complutense), Maddocks (EPFL), Sumners (Florida State) and O'Hara (Japan).
Satellite Workshop at the University of Warwick (MIR@W) on *Singularities in Classical, Quantum and Magnetic Fluids*, 20 - 23 October 2000
Organisers: X He, R Mackay, S Nazarenko and R Pelz
Speakers included Zakharov, Kuznetsov, Pelz, Okamoto, Moffatt, Gibbon, He, Cowley, Greene, Rubenchick, Shnirelman, Stuart, Kerr, Turitsin, Barenghi and Grauer. Funding was provided by LMS, the University of Warwick and INI.
The main aim of this workshop was to initiate an exchange of ideas and methods of studying singularities among researchers working in different fields such as classical Navier-Stokes and Euler fluids, superfluids, magnetohydrodynamics and fluid-like nonlinear properties of optical media.
Royal Society Discussion Meeting on *Topological Methods in the Physical Sciences*, 15 - 16 November 2000
The organisers were Arnold, Bruce, Moffatt and Pelz. Speakers included Kontsevich, Nekrasov, Hannay, Varchenko, Khesin, Marsden, Misiolek, Moffatt and Holmes. This meeting was a joint effort with the concurrent INI programme, Singularity Theory.
The first day was primarily string theory, while the second was on dynamical systems and hence more relevant to our programme.
BRIMS Day: *Differential Geometry in Fluid Dynamics and Dynamical Systems*, 20 November 2000 with a follow-on day of talks
The organiser was Tom Kambe. Speakers included Brenier (Paris), Fukumoto (Kyushu), Holm (Los Alamos), Kambe (Tokyo), Khesin (Toronto), Misiolek (Notre Dame), Ebin (SUNY) and Wocher (Schoedinger Inst).

Summary

The main goals were to inject ideas from modern differential geometry and topology into fluid mechanics and to inspire new directions in these mathematical fields from discussions on the major problems in fluid mechanics. There was a great deal of exposure to new disciplines and interaction between participants with a large range of backgrounds. Many new collaborations started during the programme. The large number of such interactions is the main measure of its success.

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