Higher Dimensional Complex Geometry

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Report from the Organisers: A Corti (Cambridge), M Gross (Warwick), M Reid (Warwick)

Programme Overview

Algebraic geometry is the geometrical study of solutions of systems of polynomial equations. This programme was centred around 3-folds, that is, solution sets of 3 dimensions. The subject of 3-folds has received much attention in the past 20 years because of significant progress in the classification programme. The aim is to classify solution sets into three broad classes of geometry: positive, zero and negative curvature. This idea goes back to the treatment of conic sections by the Greeks and non-Euclidean geometries in the 19th century, but it is only in the past 20 years that a general picture has emerged in the context of higher-dimensional algebraic geometry. The main features of this picture have been established for 3-folds but remain conjectural for higher dimensional algebraic varieties. Today the field is moving in two main directions. One is to work out explicit consequences of the general theory for special classes of 3-folds. The other direction is to establish the general features of the theory in higher dimensions.

Algebraic geometry is a central subject in mathematics today. A trend has been established over the past 50 years where key ideas of 19th century algebraic geometry, such as moduli spaces, deformations, enumerative problems and motives, have been exported to other styles of geometry (differential, symplectic, analytic, special), PDEs, and mathematical physics, where they have been the basis for the development of major new theories. Algebraic geometry is in turn cross-fertilized by development in all these fields.

Geometry, and especially algebraic geometry, is increasingly the language of theoretical physics and string theory. A recent area of interaction and enormous activity over the past 10 years has been the field of mirror symmetry which, for mathematicians, started with some enumerations by physicists of rationally parametrised curves on Calabi-Yau 3-folds (a special class of 3-folds of zero curvature), and has been studied subsequently from many different points of view. A more recent area of interaction started with the realisation by some physicists, among them Douglas, that

D-branes in type IIa string theory form a derived category, a concept first discovered by algebraic geometers in the 1960s. This trend constituted a key area of focus for the programme.

This was not the first major international event to concentrate on 3-folds: it came after symposia and meetings in Warwick (1982), Utah (1988 and 1992), Warwick again (1995) and RIMS Kyoto (1997). We can now see that these events have been instrumental in shaping the field as we know it today, each stimulating progress and anticipating some cultural change. We can hope that our programme will be recognised in the future as having been equally important.

Mathematical Themes

We describe the outcomes of the programme in more detail in the final section below. Here we sketch the main themes and how they fit in to the general advancement of the subject. We see clearly how new ideas and approaches, many significant simplifications in the theory and an increased relevance of explicit methods have arisen right across our spectrum of interests.

The programme had two central themes: the classification of all varieties, in particular the Minimal Model Programme (MMP) in 3 and 4 dimensions; and the detailed study of special classes of varieties such as Fano 3-folds, but especially Calabi-Yau 3-folds and their many relations with physics.

A famous bottleneck in the proof of MMP is the proof that flips exist. A flip is a surgery operation on varieties that occurs in codimension 2: for 3-folds this means cutting out a finite collection of curves and stitching up the variety with new curves in their place. In particular, flips are an ingredient of classification not needed in the surface case.The programme was a platform for understanding new work by Shokurov that has provided far simpler proofs in the case of 3-folds, and furnished the first proof for 4-folds. We worked through these new ideas, revisiting the foundations of the subject in this new light. In a series of lectures given in tandem by members of the programme, we worked through each part of Shokurov's manuscript. Already Corti was able to give a short course, suitable for motivated graduate students, that presented an outline of every part of a proof of 3-fold flips.

There was also a great deal of activity surrounding the study of Calabi-Yau manifolds and mirror symmetry. Calabi-Yau 3-folds occupy one slot in the classification of 3-folds, analogous to that of elliptic curves or K3 surfaces in lower dimensions. Since the late 1980s they have attracted interest because of their deep connections with string theory in physics. This programme ran in parallel with the M-Theory programme at the Newton Institute, which was of great benefit to us, and in fact crucial for this point of view. Gross led the work in this area, and gave a number of lectures from the mathematical point of view. There is an array of related fronts, most prominently work on derived categories, equivalences between them and the McKay correspondence, but also work on birational geometry and the Kähler cone, etc. Much of the work done in this area has been inspired by work done by physicists, many of whom attended for some part of the programme.

Organisation

The six month period of the programme was marked by three large meetings involving many visitors. Between these focal points, there was much activity among permanent and longer-term participants. We describe each component in detail below, but briefly they are as follows. During 4 weeks of January, a satellite workshop, or 'teach-in', was run by Reid in Warwick. This comprised introductory lecture courses in foundational material, as well as more specialised research talks. It was aimed at younger mathematicians and newcomers to the subject, and hoped to give them a leg-up into the subject and a life belt for use during the rest of the programme. Midway through the programme, the large Clay School focussed attention on activity related to physics, and brought dozens of young researchers to

Cambridge. And towards the end, in late June, a week-long Euroconference was the climactic event of the whole programme.

These set-piece events were complemented by a great deal of activity on the main areas of the programme. Throughout the programme, Corti and Reid led efforts to work through Shokurov's proof of 4-fold flips. Gross, Douglas, Szendröi and others drove work on Calabi-Yaus. A meeting of the ABC-KLM network (amazingly, the acronym stands for "algebraic geometry and boundary conformal field theory") in May also coincided with the programme, again with mutual benefit. And, of course, there were weekly seminars - usually six a week and sometimes more, including a number of mini-series of lectures by programme participants on a specific subject.

Meetings and Workshops

Warwick Teach-in on 3-folds
Satellite Workshop at the University of Warwick,
7 January - 1 February 2002

This was a month-long school, with the purpose of helping young geometers get into 3-folds, and to prepare them and others for the programme. It involved about 20 graduate students, most of whom came from overseas, and a similar number of other young geometers.

Each of the main organisers gave a series of seminars: Reid on classification and birational rigidity; Corti on the X-method, one of the central and much-feared technical tools of the subject; and Gross on mirror symmetry. These were very well attended, attracting both beginners and people familiar with parts of the theory but still needing warming up. The graduate students, perhaps especially the large Polish group, formed independent survival classes.

In addition, Kachi and Takagi, young researchers in the field, gave introductory lectures on birational geometry in general, and on the Shokurov approach to flips in particular. There were also research seminars, and a dozen or so talks by graduate students.

The meeting closed with a football match in which England beat the rest of the world 7-6, a feat not repeated in Japan six months later. Contemporary notes record that Reid was the most highly valued player, followed religiously by his team both on the field and afterwards at the bar.

Clay Mathematics Institute School on Geometry and String Theory
Workshop, 25 March - 19 April 2002

A report on this school appears elsewhere, but since in practice it was an important part of our programme for many of the participants, we say a few words about it here. This school was organised jointly with the M-Theory programme. Its subject was geometry and string theory, so it was an essential component of our Calabi-Yau stream.

The school gave intense training in many areas close to current work in string theory. The lectures were given both by mathematicians and physicists, with speakers from the two disciplines working hard to highlight the connections. For many of our participants, this school was a unique opportunity to come face-to-face with the physicists and the physics that they had known only from the mathematical point of view.

Higher Dimensional Complex Geometry
Euroconference, 24 - 28 June 2002

The EAGER-HDG conference at the Newton Institute was a major international event in algebraic geometry. It was attended by more than 100 participants. The scientific schedule centred on the main topics of the 6-month programme. While making an effort to secure a few keynote speakers well in advance, we left a large part of the schedule undecided to make room for recent progress in other areas. For instance, we heard about Voisin's recent major results on long-standing conjectures on the syzygies of canonical curves. As a result, the conference was very successful, with many interest-ing talks by young speakers on recent advances.

We had coordinated senior lectures surveying progress in 3-folds on birational geometry (Mella, Reid), rational connected varieties and classification (Campana, Kollár) and the flipping problem (Ambro, Kawamata). There were several impressive talks by leading young mathematicians. In particular we note Bridgeland's spectacular work on P-stability and derived categories on K3s, and Siebert's work with Gross on smoothings of Calabi-Yau varieties. Lectures by Fantechi and graduate talks by Terouanne and Degeratu covered the McKay correspondence. Four plenary talks (and two graduate talks) were by young Romanians, three of them working collaboratively in the USA on a variety of subjects, testifying to the emergence of an outstanding group of researchers.

There was an evening of three graduate talks organised by the students covering all the keynote subjects of the conference. The students (Budur, Degeratu, Terouanne) described their research work in an informal environment that encouraged discussion.

The key aim of the meeting was to encourage Europeans, particularly scientists at the beginning of their careers, to work on 3-folds. And we regard this to mean both fundamental developments internal to the field and outside applications to other parts of mathematics and mathematical physics.

Weekly Seminars and Mini-series

Outside conference times, we ran regular seminars, typically on Thursday and Friday afternoons, although at its height throughout the week. As well as using the Newton Institute's own advertising, we circulated details through the COW seminar series bulletins. COW is an active seminar circuit for algebraic geometry throughout the UK.

There were three mini-series, each of three lectures by a single speaker: Gross on his work with Siebert on degenerations of Calabi-Yau varieties; Ingalls on noncommutative surfaces; and Brown on the computer algebra associated with a database of polarised K3 surfaces.

Outcome and Achievements

At this stage it is too soon to judge many of the outcomes, or to determine which of the sections of the programme had the most impact. The nature of mathematical research, characterised as it is by unforeseen advances and unexpected connections, renders it futile to try to predict its course in advance.

Here we summarise some of the work that took place during the programme and was directly related to one of the main streams of the programme. Of course, there are many omissions. As usual in such meetings, many participants took the opportunity to complete outstanding work and start new collaborations, many of which we will only know about in a few years' time. Here we mention some of the papers that were completed; for a more complete list see the appendices to this Annual Report.

Shokurov's Results on pl-flips

In 1999, Shokurov announced a proof of 4-fold flips, one of the main steps in the extension of classification to 4-folds. Reid edited Shokurov's 250-page manuscript and a number of people began working on it. During the programme, this document was studied in detail. At a series of 23 lectures, and with seven different speakers, each section was dissected, and we met many new ideas. Ambro, Corti, Fujino, Kawakita, McKernan, Takagi and Uehara all gave talks on parts of the proof.

An immediate outcome has been Corti's simplified proof of 3-fold flips following Shokurov's ideas. This is available in abbreviated form from Corti's webpage. Corti gave a graduate course outlining this proof. To many of us, it was amazing to see what has been a long-standing hurdle exposed at this level.

Classification

Looking first at work on flips, Corti's paper 3-fold flips after Shokurov details a proof of 3-fold flips following Shokurov's ideas. Iskovskikh has written a first version of a paper entitled B-divisors and functional algebras after Shokurov. Both Ambro and Takagi have written papers following Shokurov's work and on the classification of Fano 3-folds, and Fujino has written three papers on the subject, one jointly with Takagi. Fujino also wrote a fourth paper Algebraic fiber spaces whose general fibers are of maximal Albanese dimension.

We are planning a book, containing our digested version of Shokurov's proof. Its precursors, the Astérisque volumes of Clemens-Kollár-Mori and Kollár et al., came out of the two Utah seminars and have been the central references of the subject for a decade. Our book will follow the style of these, having chapters written by various programme participants, with overall editorial responsibility by Corti.

Kollár, Smith and Corti (almost) finished a book on Rational and nearly rational varieties while two of the authors were together. The subject of rationality, and its close relations, has been a major component of work in classification throughout its history, and even now remains largely a mystery.

In the more explicit vein, several first drafts of papers were written around the subject of Fano 3-folds, especially from the point of view of 'unprojection'. These include a revised version of Papadakis-Reid's Kustin-Miller unprojection without complexes, Reid's draft Quasi-Gorenstein unprojection and Reid-Suzuki's Cascades of projections from log del Pezzo surfaces. In another application, Brown-Zucconi have written the first of a series of planned papers on the pliability of Fano 3-folds.

Many other people did important work at the Institute: Lazarsfeld, McKernan (who finished 4 papers), Mustata, Prokhorov, Shokurov and more.

Calabi-Yaus and Mirror Symmetry

One of the most active areas at present is the study of derived categories on Calabi-Yau manifolds. This has received a great deal of impetus from physics, especially in the work of Douglas, Moore, Aspinwall and others. During his stay Bridgeland carried out research into stability conditions on derived categories as suggested by Douglas. It appears likely that these ideas will provide some very exciting new methods in mirror symmetry and the study of derived categories. He also worked together with Thomas, working out examples of this notion in symplectic geometry.

There were many other contributors to this subject: Caldararu worked with Bridgeland and Gross on various applications of derived categories to mirror symmetry; Szendröi carried out work with Grojnowski and others on automorphisms of derived categories; Kawamata applied methods of derived categories to study birational geometry in higher dimensions; and Horja worked on automorphisms of derived categories. Nakajima also studied automorphisms of derived categories of moduli of K3 surfaces.

In the general area of mirror symmetry, Barannikov gave several lectures at the Clay School on his approach to higher dimensional mirror symmetry. Batyrev and Matverov continued work on toric aspects of mirror symmetry, in particular writing the paper Mixed toric residues and Calabi-Yau complete intersections. Gross and Siebert continued their work on a new approach to mirror symmetry using log geometry which provides an algebro-geometric version of the Strominger- Yau-Zaslow conjecture. An initial version of the resulting huge paper is expected in the next few months, and the programme included four talks on this exciting subject. Kawamata wrote a first version of his paper D-equivalence is K-equivalence (available as math.AG/0205287) while at the programme.

In a more physical vein, Wendland and van Enckevort worked on the boundary between string theory and algebraic geometry, and Donagi also participated in the Strings 2002 conference (see the report on the M-Theory programme).

There was also some activity concerning manifolds of special holonomy of a different type: Sawon and O'Grady studied hyperkähler manifolds, while Kovalev developed a new construction of G2 manifolds, resulting in a paper Coassociative K3 fibrations of compact G2 manifolds.

In a more arithmetic direction, Yui studied

L-functions of Calabi-Yau manifolds, writing four papers in total. Schoen, in three papers, studied their Brauer groups and associated Tate- Shafarevich groups. Schoen and Thomas started a collaboration based on Thomas' ideas on nodes and algebraic cycles.

Finally, with two of the authors at the programme, a new textbook Calabi-Yau manifolds and related geometries by Gross, Huybrechts and Joyce was completed.

Warwick Teach-in on 3-folds

The Warwick satellite workshop brought together many young researchers, and several learned of new problems there that may well become part of their theses. And a number of PhD students were actually able to complete their theses. Warwick students Leng and Ryder were both regular visitors, and both collaborated substantially with other visitors to complete their theses. Leng's work on the McKay correspondence may be seen as the first steps in an ambitious extension of the project. Ryder's work with Corti, Reid and Takagi sets much of Cheltsov's work on elliptic and K3 fibrations on solid footing, working out very substantial examples. Tokyo student Suzuki collaborated with many of the participants on her thesis, especially in her work on classification of Fano 3-folds of higher Fano index.

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