# Homotopy Harnessing Higher Structures

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**Paul Goerss, John Greenlees, Stefan Schwede, Ulrike Tillmann**

The sixteen years since the last Isaac Newton Institute Programme in homotopy theory in 2002 have reshaped the subject. Old problems have been solved using new methods, new methods led to new ideas, new ideas to new problems, and new problems to new theorems. The result is a rebirth: established mathematicians have reinvented their research programmes and there is an entire cohort of new researchers taking the field in new directions.

In retrospect, we can see that this fundamental change occurred as the mathematicians in the field confronted and solved an array of related problems. These included the task of making sense of Witten's insights on the elliptic genus, of finding a provable formulation of Mumford's conjecture, the related development of topological field theories, and attempts to make rigorous Morava's programme using the geometry of formal algebraic groups to organize stable homotopy theory. In each case, there was enormous progress after the introduction and the study of higher homotopical structure: for example, finding a universal target for Witten's sigma orientation led to the Hopkins-Miller theorem on topological modular forms and the parallel emergence of the new field of derived algebraic geometry. It was no longer enough to know some structure up to homotopy: we needed higher order information, such as knowledge of the homotopy type of the space of all possible solutions to some problem. In some sense this idea goes back to Kan and Quillen, but the spectacular new examples and the methods developed to understand them transformed the field.

The Homotopy Harnessing Higher (HHH) Structures programme took full advantage of this model. There were four week-long and well-attended workshops with interlocking themes, but we left the rest of the time relatively unstructured, with traditional one-hour seminars on Tuesdays and an informal Gong Show (four ten-minute talks stimulating many conversations) on Thursdays. The unstructured time between was filled with the sharing and development of ideas, often as part of active joint exploration. The Newton Institute collaboration space was ideal, and participants filled the space with activity, conversation, fueled by copious quantities of coffee.

The Newton Institute was a perfect venue: this is an international field and bringing together leading mathematicians from many countries simultaneously advanced the field and boosted UK activity. We had participants at all stages of their careers, mainly from continental Europe, the United States, and the UK, but we also had visitors from Asia and Australia. There were quite a few participants in residence for large sections of the programme, to provide continuity, but there was also a gentle shift as an exceptional summer turned to autumn, from mathematicians more directly in abstract homotopy theory to those more concerned with applications, especially to the algebraic topology of manifolds. This progression was also reflected in the workshops: the first workshop focused on the foundations of homotopy theory, the second on equivariant and motivic homotopy theory, the third on chromatic homotopy and derived algebraic geometry and the fourth and final conference was specifically on manifolds.

### Higher structures

The introductory workshop “Higher structures in homotopy theory” made for a perfect start of the program. It was a great illustration both of the rejuvenation of the subject and of the paradigm shift in how we think about the fundamentals of the field. For the entire week, the main INI lecture room was packed with an enthusiastic crowd, including many early career researchers, eager to learn how the new higher categorical language can be used to clarify, extend and interconnect conceptual statements and even concrete calculations in algebraic topology.

The two main lecture series well exemplified how the general mindset in homotopy theory has changed in the past ten or fifteen years. In her talks, Emily Riehl highlighted the common structure behind different model for infinity-categories and convinced the audience of the versatility of the invariant “model-agnostic” approach. Thomas Nikolaus recast half a century of development in algebraic K-theory into a fascinating four hour narrative starting from a homotopy invariant notion of  “group completion”, and leading up to previously invisible structures such as spectral lambda-operation on algebraic K-theory spectra.

One of the lasting lessons of the whole program is how liberating the new language of infinity-categories has been for the field, and that this new point of view will play a predominant role in the subject for years to come. For example, the work of Nikolaus, Scholze, and Antieau on topological cyclic homology, cyclotomic spectra and topological Cartier modules, or the construction of Bachmann and Hoyois of normed motivic spectra are both hard to imagine without the flexibility and power of the new higher categorical machinery. It became a running joke during the rest of the program, alluded to many times in the Gong Show talks in the weekly Thursday session, that “to play with the cool kids”, one should formulate the results in the language of infinity-categories.

**Equivariant and motivic homotopy theory**

Equivariant homotopy theory has been a fundamental component of algebraic topology since its inception but the solution of the Kervaire invariant one problem by Hill-Hopkins-Ravenel gave new depth and direction of to the field. The HHR work not only reemphasized the power of equivariant homotopy theory but it also highlighted Galois-like phenomena, complementing the traditional group actions on manifolds, and shone a bright light on the importance of understanding subtle the higher homotopical of ring spectra.

Motivic homotopy is a newer field, brought to the forefront by Voevodsky's proof of the Milnor conjecture. It has since emerged has a vibrant field bridging homotopy theory and algebraic geometry. This development has been led by established mathematicians, such as Marc Levine, but there is also emerging cadre of leaders relatively early in their careers. This includes researchers from homotopy theory, such as Marc Hoyois, and from algebraic geometry, such as Arvand Asok. from both Galois actions have made it a type of equivariant homotopy theory from the start, with various realizations taking values in equivariant topology. Both the traditional application to the study of quadratic forms and also the use of equivariant homotopy theory for calculation were well represented.

The workshop on these topics was one of the revelations of the program. It is not new to bring together equivariant and motivic topologists, and we certainly expected a certain synergy, but we were surprised and delighted by the close and tangible cross-fertilization. Some of the hardest calculations of stable homotopy groups of spheres now use the motivic and equivariant methods (represented in Behrens's talk), and in the other direction common structures are apparent in Balmer spectra, Smith theory and Adams spectral sequence methods give motivic calculations.  One of the places where the interaction between the fields is most fruitful is in derived algebraic geometry and chromatic homotopy theory as represented in the third workshop (for example in the treatment of tmf with level structure by equivariant methods, and embodied by Beaudry's talk on Picard groups in the equivariant context).

There were other common themes as well: descent, multigrading, and the slice filtration in equivariant and motivic homotopy theory. There was new impetus to more traditional problems as well: for example, to the study of quadratic forms by motivic homotopy theory and the use of equivariant homotopy theory for hard calculation both received a big play. And to start the workshop, we had a discussion of group actions on manifolds; indeed, one of the most charming moments from the whole program came in Dugger's talk on the classification of involutions on surfaces when it emerged that this question itself originated from motivic homotopy theory.

There is an important secondary phenomenon in the study of equivariant and motivic homotopy theory: many rich fundamental structures only become apparent in the more complicated context,  from Mackey and Tambara functors to multiplicative norm maps and equivariant commutativity (represented in talks by  Bachmann, Bohmann, Kedziorek, Gerhardt).  The application to trace methods in K-theory embodying the ideals of the programme was advertised above.

**Chromatic homotopy and derived algebraic geometry**

Chromatic stable homotopy theory had its origins in the early 1970s, with the realization by Quillen, Morava, Landweber, and others that there was a very strong connection between cohomology theories with a natural theory of Chern classes and smooth 1-parameter formal groups. Since then, going on fifty years, homotopy theorists have used the algebraic geometry of formal groups to organize calculations and the search for large scale phenomena. It remains a guiding paradigm for the field.

Derived algebraic geometry has its roots in many diverse areas of mathematics, from mathematical physics to algebraic *K*-theory. One of the foundational examples was the theory of topological modular forms, developed by Hopkins, Miller, and their coauthors to understand elliptic cohomology theories, a particular example of the sort of cohomology theory seen in chromatic homotopy theory. A basic observation of derived algebraic geometry is that we can do geometry not simply over commutative rings, but over commutative ring spectra.

The two fields come together with the observation that stable homotopy theory can be framed as the study of modules over the sphere spectrum. As with any ring, the sphere spectrum has a prime ideal spectrum, which was analyzed in the 1980s by Hopkins and Smith in their work on periodicity and nilpotence; in particular, the prime ideals are exactly those predicted by the geometry of the moduli stack of formal groups. These include the standard primes found in the integers, but also infinitely many more: one for each pair $(p,n)$ where $p$ is a prime and $n$ an positive integer. Localization at these primes is gives us $K(n)$-local homotopy theory, which has been a rich area of study for many years, and the new points of view from derived algebraic geometry have given new tools for assembling information from various primes. This is known as transchromatic homotopy theory, and there has been remarkable recent progress in that area.

These two themes, chromatic homotopy theory and derived algebraic geometry, were built into the programme right from the start, featuring strongly in two of the workshops, and present in talks throughout the semester. The very first Gong Show featured two talks in chromatic homotopy theory, and the very last talk to the term was on assembly maps in *K*-theory, a core tool in derived algebraic geometry. There were significant advances in both fields as well during the programme. For example, Beaudry, Goerss, Hopkins, and Stojanoska have given a partial solution to the Linearization Hypothesis, a conjecture about the equivariant homotopy type of a dualizing object in $K(n)$-local homotopy theory. A key step in this work were conversations between Vesna Stojanoska and her INI officemate, Jesper Grodal on classical results about maps out of classifying space of elementary abelian $p$-groups. Another very nice suite of results on Whitehead theory for periodic homotopy groups by Barthel, Heuts, and Meier, work that was a direct result of a collaboration started and completed at the Newton Institute.

The third workshop featured talk series from Agnés Beaudry on $K(n)$-local homotopy theory, Ben Antieau on derived algebraic geometry, the team of Tobias Barthel and Nat Stapleton on transchromatic theory, and from Niko Naumann on Lurie's approach to elliptic cohomology theories. Naumann's series clearly displayed the power of the relationship, with tools and ideas of derived algebraic geometry deployed to understand the very difficult Hopkins-Miller theorem on the structure of derived Lubin-Tate space. The interaction infused some of the individual talks as well; for example, in Lennart Meier's talk on level structures in topological modular forms.

**Manifolds**

There has been and remains a strong interplay between the study of manifolds and homotopy theory. Classically, Thom's cobordism theorem illustrates the power that stable homotopy theory can have in the classification of manifolds while Quillen's universal group law puts cobordism theory and hence manifolds at the very heart of chromatic homotopy theory. A more modern incarnation of this relation is the interplay of higher structures in form of cobordism categories and moduli spaces, factorisation homology and embedding spaces, configuration spaces and $E_n$ algebras. Much of the recent activity grew out of the study of topological quantum field theories and the embedding calculus at the end of the last century.

Our fourth workshop — simply called "Manifolds" — showcased recent developments in this fast moving field. The first of the two mini lecture series was dedicated to the groundbreaking work by Galatius, Kupers, and Randal-Williams on $E_n$ cell complexes and its application to homology stability of mapping class groups. The second series was delivered by Willwacher and Turchin on their and their collaborators' dramatic use of graph complexes for the study of embedding spaces, diffeomorphism groups and moduli spaces. Other highlights included lectures delivered by Raptis on joint work with Steimle expressing Waldhausen K-theory as the classifying space the h-cobordism category; by Hebestreit on a break-through $E_n$ collaboration identifying the classifying space of an algebraic cobordism category of Poincare complexes with a derived version of Grothendieck-Witt theory; by Wahl on the homotopy invariance of operations in string homology; and by Malkiewich on trace methods for the study of periodic points of dynamical systems.

Though this subcommunity prefers to work with more concrete geometric objects it too had to acknowledge the power of the language of infinity and derived categories.

**The Rothschild Lecture**

In the midst of program we had our Rothschild Distinguished Visiting Professor Lars Hesselholt gave a talk on higher algebra and arithmetic. His abstract for the lecture began "This talk concerns a twenty-thousand-year old mistake: The natural numbers record only the result of counting and not the process of counting." This provocative statement was prelude to the subtle hypothesis that if we did arithmetic, algebra, and algebraic geometry over commutative ring spectra, and not simply over commutative rings then we had access to ideas and techniques that could illuminate some of the hardest questions of number theory and related arithmetic geometry. This was partly hyperbole to make a point, but it also is a deep reflection of what Professor Hesselholt himself has accomplished: his work with Madsen and others connecting algebraic K-theory and algebraic number theory exactly fits this rubric.

**Conclusion**

This sketch names at least six identifiable areas in homotopy theory, and the review above shows that many participants were closely involved across multiple sub disciplines. This is a symbiotic continuum of activity facilitated by an enormously powerful and flexible set of unifying foundations. The remarkable outcomes in the various fields, both those reported in the programme and those that come to fruit later are witness to the effectiveness of these ideas and a hint of further progress to come.