Dispersive hydrodynamics: mathematics, simulation and experiments, with applications in nonlinear waves

Final report

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Brief background/historical information

A standard paradigm in quantum mechanics is concerned with the notion of wave-particle duality. Such a duality is realised in the macroscopic world by the propagation of solitons—modelled by localised, nonlinear wave solutions of partial differential equations that interact elastically like particles. Taking this as the starting point for a microscopic description of a physical system, a natural question is how to study large, ordered and disordered collections of nonlinear waves. A powerful concept in this regard is that of nonlinear wave modulations from which emerges a hydrodynamic description of waves. The corresponding Whitham modulation equations provide the macroscale description of nonlinear waves analogous to the Euler equations describing the collective motion of gas or liquid particles. The study of the macroscopic dynamics, or hydrodynamics, of large collections of nonlinear waves is known as dispersive hydrodynamics.

Since simple linear superposition does not hold for nonlinear partial differential equations (PDEs), the mathematical description of a large collection of nonlinear waves is therefore quite complex and challenging. The inherent multiple time and spatial scales of collective nonlinear wave motion lead to natural scalings and small parameters. Various approaches to the asymptotic analysis of collective nonlinear wave motion have been developed, including Whitham modulation theory, zero dispersion limits for integrable systems, and large-scale numerical computation.

The modulation theory developed by Whitham [77] leads to a system of quasilinear, first order PDEs in conservation form describing slow variations of (quasi-)periodic, nonlinear multiphase wavetrains; these equations, known as the Whitham modulation equations, have been used to describe dispersive shock waves (DSWs), the fundamental nonlinear wave structures emerging as a result of dispersive regularisation of hydrodynamic singularities. The theory of DSWs initiated in the classical paper by Gurevich and Pitaevskii [45] has grown into a significant area of applied mathematics, with numerous physical applications [32]. Closely related to Whitham modulation theory is the rigorous theory of semiclassical, zero-dispersion limits for integrable systems, which originated in the groundbreaking work of Lax and Levermore [54] and was further revolutionised by the development of powerful nonlinear steepest descent methods of Deift and Zhou for singular limits of Riemann-Hilbert problems arising from inverse scattering [23]. Modelling of complex real-world hydrodynamic phenomena often requires a suitable description of random and chaotic behaviours. The theory of integrable turbulence [82] fuses integrability and randomness into a mathematical framework to interpret extreme events such as rogue waves. Recent developments that connect randomness to dispersive hydrodynamic phenomena include the kinetic theory of soliton gases [29,33], the statistical mechanics of many-body quantum systems [26], and the use of random matrix theory to describe probabilistic solutions of integrable equations [22].

Despite these successes, there are major open problems in this budding field of research that the HYD2 programme aimed to address. At the highest level, there is a need for deepening the connections between rigorous analysis, mathematical modeling, numerical simulation, and experiments. In what follows, important open problems are identified.

One major goal in the field is to build a general description of dispersive shock waves, akin to the mathematical theory of conservation laws and classical shock waves (weak solutions, entropy conditions, etc). While modulation theory is a promising tool for such a description, it lacks mathematical rigour and has primarily been developed for one-dimensional problems. On the other hand, the mathematical structure of integrable systems enables the rigorous construction of dispersive shock waves, but the techniques only apply to a restricted class of dispersive nonlinear wave equations. The field has its sights on the dual goals of i) explaining wave phenomena in physical and numerical experiments and ii) developing a rigorous and general mathematical theory to further clarify the dynamics.

Multidimensional dispersive hydrodynamics has received much less attention than problems in one spatial dimension and time. In this vein, a major outstanding problem in the field is the theory of multidimensional modulation equations of dispersive hydrodynamics. Specifically, a modulation theory in multiple space dimensions is in the early stages of development, alongside inverse scattering theory for integrable systems. The field is poised to make advances on the complete classification of systems of integrable, dispersive PDEs and multidimensional bi-Hamiltonian structures of hydrodynamics type, as well as the development of a theory for non-diagonalisable integrable systems of hydrodynamic type. These developments are motivated by ongoing experimental research on multiscale nonlinear wave propagation and interactions in multiple dimensions.

The complexity of dispersive hydrodynamics increases significantly when randomness is present in the description of nonlinear waves. Here, the dynamics are described by stochastic processes. But, again, there are two competing aims: a general description and mathematical rigour. Open problems include the rigorous construction of a soliton gas, understanding of soliton gas thermodynamics, application of soliton gas theory to physical observations and laboratory experiments.

At its core, dispersive hydrodynamics is inherently an applied discipline so the physical observation of multiscale nonlinear wave phenomena and its mathematical description encompass a large class of open problems in the field. Each of the aforementioned open problems—the rigorous analysis of dispersive hydrodynamics in a general context, multidimensional dispersive hydrodynamics, and integrable turbulence/soliton gas—would all benefit from the development of laboratory and field measurements. Promising application areas include geophysical fluid dynamics (surface/internal water waves), nonlinear optical/matter waves, and material science.

The HYD2 programme witnessed developments in each of these problem areas, described below in the Scientific outcomes section.

Programme timeliness, scope and outline

From 21 June to 2 July 2021, prior to the beginning of the HYD2 programme, the bridging workshop *New horizons in dispersive hydrodynamics* was held. This workshop was organised as a way to remain connected to researchers in the field despite the severe disruptions due to the global pandemic. Its subject matter surveyed the field of dispersive hydrodynamics. This two week, online-based workshop consisted of 51 presentations and 47 additional participants, demonstrating significant interest in dispersive hydrodynamics and paved the way for an engaging HYD2 programme.

The six-month HYD2 programme ran 4 July–16 December, 2022 and was structured around four overarching themes, occurring sequentially in time: Modulation theory and dispersive shock waves, Analysis of dispersive hydrodynamic systems, Random phenomena and dispersive hydrodynamics, and Physical applications. Within each theme, one or more workshops and special events heralded new research developments and directions for the field. The events associated with each theme are highlighted below.

Throughout the duration of the HYD2 programme, 115 junior and senior participants presented their research in seminars while new collaborations developed, facilitated by the INI's conducive, collaborative physical environment. Junior researchers self-organised a weekly seminar with a non-traditional structure: three slides, discussion and no senior participants. Due to the large number of junior participants (31), these seminars were well-attended. More traditional one-hour seminars were regularly held in between workshops. See Table 1 for the numbers.

Modulation theory and dispersive shock waves

This theme, running for the first two months of the programme, included the 5-day workshop *Modulation theory and dispersive shock waves* and the 1-day workshop *Women in dispersive equations day.*

The kickoff, 5-day workshop highlighted research from all four programme themes with an emphasis on Whitham modulation theory. Whitham modulation theory has developed into a powerful mathematical tool to describe the intermediate and long-time behaviour of solutions to conservative or nearly conservative nonlinear dispersive equations. A number of innovative developments and applications of modulation theory were discussed.

Among the five plenary talks, 29 presentations, and follow-on discussions, several topical areas emerged during the 5-day workshop:

- Whitham modulation theory and dispersive shock waves for non-integrable model equations including those with higher order dispersion or nonconvex nonlinear flux;
- Developing and applying multidimensional modulation theory.
- Rapid transitions between two nonlinear periodic travelling waves and their connections to modulation theory;
- The rigorous connection between hyperbolicity of the modulation equations and stability of periodic travelling waves;
- Applications of modulation theory to specific physical systems in geophysics, optics, ultracold atomic gases, and materials;
- Exact solutions describing dispersive hydrodynamics in integrable systems.

These topical areas sparked significant discussion and set the tone for the remainder of the programme, with the four remaining 5-day workshops revisiting, expanding and building upon these topics in greater detail.

The 1-day workshop, scheduled the Monday following the 5-day workshop, highlighted the research of junior and senior female mathematicians who study nonlinear dispersive equations. The eight presenters covered a range of topics from the rigorous (long time asymptotics of dispersive shock waves, N-soliton solutions with $N \to \infty$, instability of Stokes waves in water), to applications (lattices, elasticity, fibre lasers, ultracold atomic gases) and numerics. The workshop concluded with an engaging panel discussion on challenges women face in the mathematical sciences. The discussion elicited significant audience participation, with concrete experiences shared by female and male researchers alike from a number of countries. By highlighting diverse regional, cultural, and gender experiences, participants were left with a broader view of the realities of an academic career in the mathematical sciences.

Analysis of dispersive hydrodynamic systems

Running from September through mid-October, this theme included two 5-day workshops: Analysis of dispersive systems and the satellite workshop Integrable systems and applications at the University of Loughborough. Following these workshops, a one-day visit to the University of Oxford's OxPDE research group was organised. The two workshops brought attention to two analysis approaches to dispersive hydrodynamics. Analysis that applies to non-integrable as well as integrable PDEs was contrasted to methods that require the mathematical structure of integrable PDEs.

The *Analysis of dispersive systems* workshop consisted of 22 hour-long presentations covering the following range of analysis problems:

- Exact solutions and the well-posedness of nonlinear boundary value problems;
- Use of inverse scattering theory to obtain special solutions and long time asymptotics of dispersive hydrodynamic phenomena;
- The analysis of random nonlinear waves;
- The construction and stability of traveling waves arising in dispersive hydrodynamics;
- Wave breaking, collapse, and blow up;
- The derivation and solutions of higher order nonlinear models in applications.

The workshop included a small poster session with several student presentations.

The satellite workshop *Integrable systems and applications* included 26 oral presentations and six poster presentations that broadly covered the following topics.

- Soliton gases;
- Rogue waves;
- Multidimensional nonlinear wave solutions, stability, modulations, and numerical simulations in the Kadomtsev-Petviashvili (KP) equation and other multidimensional integrable equations;
- Riemann problems and solitons in dispersive hydrodynamics;
- Modeling of oceanic internal waves;

• Nonlinear waves subject to dissipation in applications.

The 1-day visit to Oxford included five presentations highlighting a range of dispersive hydrodynamic phenomena and interactions with the OxPDE research group.

Random phenomena and dispersive hydrodynamics

This programme theme from mid-October to mid-November was kicked off by the 5-day workshop *Statistical mechanics, integrability and dispersive hydrodynamics* and immediately followed by the Open for Business event *From dispersive hydrodynamics to forecasting, machine learning and back.* During this period, the research focus shifted to stochastic nonlinear wave dynamics in a variety of settings.

The *Statistical mechanics, integrability and dispersive hydrodynamics* workshop included five plenary talks, 30 presentations, and an afternoon poster session on the following topics:

- Non-equilibrium physics and the theory of generalised hydrodynamics;
- Wave turbulence in nonlinear lattices, optics, and fluids;
- Integrable turbulence;
- Soliton and breather gases, theory, numerics and experiment;
- Random matrices, theory and applications;
- Integrable and non-integrable nonlinear lattices with random initial conditions.

Immediately following the workshop was an Open for Business event that attracted eight presenters on the dispersive hydrodynamics of weather/climate forecasting and the use of machine learning in nonlinear wave dynamics. The event concluded with a robust audience discussion on the appropriate uses of machine learning in weather and climate modeling.

Physical applications

From mid-November to the conclusion of the programme in mid-December, the theme of physical applications of dispersive hydrodynamics predominated. This portion of the programme included a variety of application areas represented in the *Physical applications* 5-day workshop followed by the Open for Business event *Physical applications of dispersive hydrodynamics*.

With four plenary talks and 30 presentations, the *Physical applications* workshop covered applications in:

- Fluid dynamics;
- Bose-Einstein condensates;
- Nonlinear optics;
- Surface water waves;
- Internal waves;
- Quantum computing;
- Nematic liquid crystals;
- Granular crystals;

• Elastic bars.

The variety of application areas was striking. These talks included both mathematical analysis and experiments on Riemann problems & dispersive shock waves in optical fibers, solitary wave fission and Faraday waves in fluids, atom lasers, internal waves and their collisions, breathers from solitary wave and cnoidal-like wave interactions in fluids, rogue waves in ocean measurements, an inverse energy cascade in 2D light turbulence, soliton gases in optical fibers and deep water tanks, solitons in granular crystals, an equilibrium 2D quantum vortex gas, and solitons from Google's quantum computer.

The follow-on Open for Business event highlighted applications in fibre optics, photorefractive crystal optics, optical microresonators, rogues waves in the ocean, and ocean circulation. A discussion amongst the audience and seven presenters ensued on how to characterise rogue waves from observations and relate this to theories that have been developed.

Scientific outcomes

With approximately 285 technical presentations across weekly programme seminars, a 2-week bridging workshop in summer 2021, five week-long workshops, and four 1-day events, the programme has imparted an indelible mark on the mathematics and science of dispersive hydrodynamics. It is difficult to summarise all these results. There were numerous scientific interactions, new collaborations, technical results, networking opportunities, and even new job opportunities that opened up for several junior and senior participants. The programme's scale is quantified in Table 1.

One of the major scientific accomplishments of this programme was the co-location and interaction of experts in PDE analysis, modeling, computation, and experiments. The growth in the field of dispersive hydrodynamics is driven by bringing these perspectives and approaches together in efforts to address the challenging problems of multiscale nonlinear waves. Toward the goal of achieving a general theory of dispersive shock waves and dispersive hydrodynamics, significant advances on Whitham modulation theory and dispersive shock waves for a variety of non-integrable model equations were reported. These included a class of Hamiltonian PDEs, PDEs with higher order dispersion, hyperbolic systems with periodic coefficients or initial data, discrete lattice equations, and PDEs of non-evolutionary type. These results are driven by new observations in ultracold atomic physics, nonlinear optics, material science, and fluid dynamics presenting challenges to modelling and theory.

One theme of the research programme was the study of random nonlinear waves. This is a challenging problem area that witnessed some important breakthroughs during the programme. The integrable systems community began to reckon with how to evolve probability distributions of solitons according to integrable evolutionary equations (integrable turbulence). Such random ensembles of solitons, called soliton gases, have recently attracted attention from a broad spectrum of researchers in nonlinear mathematics and physics. One approach to describe a soliton gas is to use modulation theory of exact N-phase (quasi-periodic) solutions in the limit $N \to \infty$ that results in integro-differential kinetic equations for the probability density of the state of the system. Another approach involves the $N \to \infty$ limit of N-soliton solutions associated with so-called primitive potentials. During the programme, these two approaches were reconciled for a special class of soliton gases termed soliton condensates. The theory of soliton condensates for the Korteweg-de Vries equation, formulated and constructed during the programme, revealed deep connections between soliton gases and fundamental, deterministic dispersive hydrodynamic structures such as DSWs and rarefaction waves.

| Activity | Number |
|---|---|
| Programme participants | 115 in-person, 73% senior, 27% junior ¹ |
| Technical presentations | 285 programme and workshop presentations |
| Programme seminars | 44 |
| Junior researcher-only seminars | 11 |
| New horizons in dispersive hydro- | |
| dynamics 2-week bridging virtual | 51 presenters & 98 participants (all virtual) |
| workshop in June–July 2021 | |
| Modulation theory and dispersive | 34 presenters & 67 in-person 12 virtual participants |
| shock waves workshop | 54 presenters & 67 m-person, 12 virtuar participants |
| Women in dispersive equations day | 8 presenters & 39 in-person 7 virtual participants |
| workshop | |
| Analysis of dispersive systems work- | 22 presenters & 73 in-person 8 virtual participants |
| shop | |
| Integrable systems and applications | 26 presenters & 53 in-person, 3 virtual participants |
| workshop | |
| Visit to Oxford's OXPDE group 1- | 5 presenters & 22 in-person participants |
| day event | |
| Statistical mechanics, integrability | |
| and dispersive hydrodynamics work- | 35 presenters & 78 in-person, 4 virtual participants |
| shop | |
| From dispersive hydrodynamics to | |
| forecasting, machine learning and | 8 presenters & 52 in-person, 18 virtual participants |
| back Open for Business day | |
| Physical applications workshop | 34 presenters & 78 in-person, 12 virtual participants |
| Physical applications of dispersive | |
| hydrodynamics Open for Business | 7 presenters & 24 in-person, 14 virtual participants |
| day | |
| New research collaborations ² | 34 (71% of respondents) |
| New research directions ² | 34 (71% of respondents) |
| New funding proposals ² | 17 (35% of respondents) |
| Talks at other UK institutions ² | 15 (31% of respondents) |

Table 1: Quantitative metrics of scientific impact. Data provided by INI staff and workshop organisers.

 122 junior, US-based participants received support from NSF grant DMS-1941489: 17 programme participants and 5 workshop-only participants.

²Based on 48 programme survey respondents (42% response rate).

Yet another promising cross-disciplinary connection presented in a number of talks and prominently featured in numerous discussions was the one between the kinetic theory of soliton gases and generalised hydrodynamics (GHD)—the hydrodynamic theory of integrable quantum and classical many body systems. Parallels between the theories of soliton gases and GHD sparked researchers to use the results from one field to the benefit of the other. In GHD, thermodynamics and hydrodynamics are emphasised, with the free energy, correlation functions and external force fields being the focus. In the context of soliton gases, integrable structure and wave phenomena are emphasised in settings that include modulational instability, DSWs and rogue waves. The soliton gas-GHD connections revealed during the programme have prompted the organisation of the five day workshop "Emergent Hydrodynamics of Integrable Systems and Soliton Gases", which will take place at the Centre International De Recontres Mathématiques (France) in November 2023.

The work on soliton gases reported and discussed in the programme has since inspired physical experiments in fibre optics and water waves. The optical experiment was performed at Lille University where the interaction of a test soliton with a dense optical soliton gas was investigated. The effect of nonlinear soliton refraction observed and measured in the experiment imparted the first quantitative confirmation of the kinetic theory of soliton gas for the focusing nonlinear Schrödinger equation. A further series of water wave experiments on the interaction of two spectrally monochromatic deep water soliton gases performed in a 150 m long wave flume at École Centrale de Nantes have provided additional convincing evidence for the efficacy of spectral kinetic theory.

A major challenge that lies ahead is the construction of a rigorous theory of soliton gases that would lay a firm mathematical foundation for the kinetic theory.

This area was recognised to be sufficiently fertile, that a new INI Satellite Programme at Northumbria and Newcastle Universities called *Emergent phenomena in nonlinear dispersive* waves was proposed, has been approved, and will take place in summer 2024.

Major advances developing and applying multidimensional modulation theory were presented for a class of nonlinear Schrödinger equations, the Kadomtsev-Petviashvili (KP) equation, and the two-dimensional Benjamin-Ono equation, all important nonlinear wave equations that arise in applications. Symmetric reductions of multidimensional dispersive hydrodynamics, for example to KP modulations that are time-independent or independent of one space dimension, were presented. These reductions were proven to be integrable and, despite a reduction in modulation dimension, the full PDE solutions that they describe remain dynamical and multidimensional. These and other reductions, such as to ring waves, represent a bridge between the theories of single and multidimensional dispersive hydrodynamics.

As if to prove the the need for multidimensional dispersive hydrodynamics, two-dimensional water wave tank experiments produced modulated oblique line solitons, interactions, and random wave fields. Experiments in Bose-Einstein condensates showcased multidimensional wave turbulence and the dispersive hydrodynamic propagation of an atom laser. The field is moving in the direction of multidimensional dispersive hydrodynamics in order to describe observed physical phenomena.

Nonlinear wave interactions are core features of dispersive hydrodynamics. In addition to the aforementioned N-phase and N-soliton solutions, new properties of solutions of integrable systems involving the interaction (nonlinear superposition) of solitons and (quasi-)periodic waves were produced, building upon seminal work that appeared fifty years ago [52, 53]. These solutions, referred to as soliton lattice defects or travelling breathers, were shown to play a fundamental role in the interaction of dispersive shock waves and solitons as well as soliton condensates, a special kind of soliton gas. In addition to their mathematical development, the first experiment to realise these solutions was presented at the programme. The theory and

observations motivate further study of the interaction between solitons or solitary waves and (quasi-) periodic wavetrains in both integrable and non-integrable systems.

It was fitting to have the *Physical applications* workshop at the end of the programme where all of the theoretical work that dominated much of the prior presentations got refracted through the prism of reality. The variety of dispersive hydrodynamic media with which physicists, engineers, and even applied mathematicians are actively undertaking new experiments is remarkable. It is worth highlighting this diversity:

- Classical fluid dynamics of interfacial fluids: soliton fission, analogue black holes, wave turbulence, breathers from soliton-cnoidal wave interactions
- Internal waves: solitary wave collisions and the role of sea ice, undular bores
- Quantum hydrodynamics of Bose-Einstein condensates: phase-wound superfluid mixtures, two-dimensional quantum vortex gases, wave turbulence, atom lasers
- Optical fibres: the dispersive Riemann problem, soliton gas
- Spatial optics: turbulent 2D fluids of light
- Granular crystals: solitary waves and dispersive shock waves
- Surface gravity waves: soliton gas in deep water, one- and two-dimensional soliton gas in shallow water

Bridging between observation and integrable systems were new efforts to utilise the inverse scattering transform (IST) in the analysis of ocean and fibre optic time series data as well as the evolution of non-integrable systems. Despite the lack of integrability, IST provides a convenient way to analyse real-world data and motivates further study of perturbed integrable systems in the dispersive hydrodynamic context.

Publications and programme legacy

A number of research articles were seeded, worked on, and completed during the programme. Below are 21 INI preprints and other published works that have so far been identified.

A. Abeya, G. Biondini, G. Kovacic and B. Prinari, "Inverse scattering transform for a two-level Maxwell-Bloch systems with inhomogeneous broadening and one-sided nonzero background"

M. Bertola, R. Jenkins, and A. Tovbis, "Partial degeneration of finite gap solutions to the Korteweg-de Vries equation: soliton gas and scattering on elliptic backgrounds"

G. Biondini, A. Abeya, and M. A. Hoefer, "Whitham modulation theory for the defocusing nonlinear Schrödinger equation in two and three spatial dimensions"

R. Camassa, G. Falqui, G. Ortenzi, M Pedroni, and T. T. Vu Ho, "Simple two-layer dispersive models in the Hamiltonian reduction formalism"

V. Caudrelier, A. Gkogou and B. Prinari, "Soliton interactions and Yang-Baxter maps for the complex coupled short-pulse equation"

J. Cole, G. El, M. A. Hoefer, X.-D. Luo, and M. J. Ablowitz, "Soliton-mean field interaction in Korteweg-de Vries dispersive hydrodynamics"

T. Congy, G. El, G. Roberti, A. Tovbis, S. Randoux and P. Suret, "Statistics of extreme events in integrable turbulence"

T. Congy, A. Tovbis, G. Roberti, and G. El, "Dispersive hydrodynamics of soliton condensates for the Korteweg-de Vries equation" F. Demontis, G. Ortenzi, G. Roberti, and M. Sommacal, "Rogue wave formation scenarios for the focusing nonlinear Schrödinger equation with parabolic-profile initial data on a compact support"

B. Doyon, G. Perfetto, T. Sasamoto, and T. Yoshimura "Emergence of Hydrodynamic Spatial Long-Range Correlations in Nonequilibrium Many-Body Systems"

M. A. Hoefer, A. Mucalica, and D. Pelinovsky, "KdV breathers on a cnoidal wave background" E. R. Johnson and J. S. Marshall, "The high-speed submerged hydrofoil"

A. Rybkin, "Norming constants of embedded bound states and bounded positon solutions of the Korteweg-de Vries equation"

A. Rybkin, "A continuous analog of the binary Darboux transformation for the Korteweg-de Vries equation"

P. Suret, S. Randoux, A. Gelash, D. Agafontsev, B. Doyon, and G. El, "Soliton gas: theory, numerics and epxeriments"

P. Suret, M. Dufour, G. Roberti, F. Copie, G. El and S. Randoux, "Soliton refraction through an optical soliton gas"

Y. Mao, S. Chandramouli, W. Xu, and M. A. Hoefer, "Observation and scattering of traveling breathers in a two-fluid system"

G. Orsatti, M. Bertola, and T. Grava, "Soliton shielding of the focusing nonlinear Schrödinger equation"

H. Spohn, "Hydrodynamic Scales of Integrable Many-Particle Systems"

P. Sprenger, T. Bridges and M. Searer, "Traveling Wave Solutions of the Kawahara Equation Joining Distinct Periodic Waves"

B. Prinari, "Inverse Scattering Transform for Nonlinear Schrödinger Systems on a Nontrivial Background: A Survey of Classical Results, New Developments and Future Directions"

Motivated by the interest in the programme, a group of programme participants submitted a proposal to Cambridge University Press to establish the new *Journal of Nonlinear Waves*. At the time of this writing, the proposal is in the final stages of review. It is anticipated that the first issue of the journal will be a special issue drawing on contributors from this programme.

To further build on the impacts of this programme, the already mentioned satellite programme at Northumbria and Newcastle Universities called *Emergent phenomena in nonlinear dispersive waves* was proposed and approved, and will take place in summer 2024.

Illustrations



Figure 1: Participants in Workshop 1, 11 July–15 July, 2022 on Modulation theory and dispersive shock waves.



Figure 2: Participants in Women in dispersive equations workshop, 18 July, 2022.



Figure 3: Participants in Workshop 2 on Analysis of dispersive systems, 5–9 September, 2022.



Figure 4: Participants in the visit to Oxford, 22 September, 2022.



Figure 5: Participants in Workshop 4 on *Statistical mechanics, integrability and dispersive hydrodynamics*, 17–21 October, 2022.



Figure 6: Participants in Workshop 5 on Physical applications, 5–9 December, 2022.

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