**FINAL REPORT**

**INI programme"Mathematics for the Fluid Earth", October-December 2013**

**Organisers: M. Cullen, V. Lucarini, B. Pelloni and S.Vaienti**

## Brief background

There is increasing interest in advanced mathematical tools and techniques in the field of earth sciences, and a great need for developing better communication between mathematicians and earth scientists.

The programme's aim was to bring together different mathematical and physical communities to increase communication both ways, and to foster increased interaction among those fields that can contribute to understanding and predicting the behavior of atmosphere and ocean dynamics.

This programme followed on from a very large programme at INI on Atmosphere-Ocean Dynamics, held in 1996. More recently, there have been programmes on the 'Mathematics of the climate System' in 2010 and 'Adaptive Multiscale Methods for the Atmosphere and Ocean', which focussed on statistical and numerical methods respectively.

## The topic of the programme

The programme had three main themes, chosen for the potential for significant progress on the mathematical side and in terms of the investigation of the planet Earth: *a) Dynamical Systems and Statistical Mechanics; b) Extreme Events; c) Partial Differential Equations.* Indeed, the 1996 programme brought together mathematicians working on partial differential equations arising in geophysics, and dynamical meteorologists and oceanographers. This created a lot of momentum on many topics in the area, including the appreciation of the importance of different asymptotic limits. The MFE programme continued and complemented these efforts by focussing on the statistical mechanics, the understanding of extreme events and the partial differential equations underpinning the evolution of the atmosphere and ocean.

One of the original purposes of the programme was the interaction among people working in ergodic theory and more applied researchers in non-linear physics and climatology. The study of the statistical properties of dynamical systems was the topic most emphasized as a bridge between the two communities, in particular through the study of extreme value distributions and perturbations.

The dynamical systems perspective also served as a link towards theme c) . There is by now a significant body of mathematical theory of the partial differential equations governing the atmosphere and ocean. Inevitably this has focussed on systems which are easy to analyse, such as those which are based on a decomposition of the equations into fast waves governed by a constant-coefficient wave equation and a 'slow' dynamics governed by potential vorticity and advection. The need is to push this towards more realistic sets of equations and global descriptions of the dynamics.

## The programme

The format of the programme was quite free but intense, with several talk each week, series of talks by one speaker on a specific topic ( or example by I. Gallagher and V. Lucarini) and two thematic workshops, plus two satellite workshop before and after the programme itself.

The first workshop focused on extreme events, but several experts in the theory of partial differential equations relevant to meteorology and oceanography were invited as participants, together with geophysical fluid dynamics experts. A second workshop was organised specifically on the latter theme, and was well attended by both mathematicians and dynamicists.

The first satellite workshop, in preparation for the programme, took place in Reading (Jan 2013) and focused on "Non-equilibrium Statistical Mechanics and the Theory of Extreme Events in Earth Science". The second one, titled "Mathematics for the Fluid Earth" and more wide ranging in topics, took place at the LMS in London (February 2014). It was intended as an opportunity for key scientists who had not been able to attend the main programme. Highlights were the presence at this workshop of Gallavotti, Dijkstra and Saint-Raymond.

**Scientific outcomes**

Considerable progress was reported during the programme in the analysis of multiscale systems using time averaging and showing convergence or good approximation to asymptotic limit solutions. Examples were given in the expository lectures by Gallagher (Paris 7) and in the work of Cheng (Surrey) and Wingate (Exeter).

The need to study asymptotic limits on scales larger than the deformation radius (often called the semi-geostrophic limit) was widely recognised. In this regime the unbalanced motion takes the form of inertial rather than gravity waves (Zeitlin, Vanneste). The asymptotic limit solutions do not compromise the thermodynamics, which is very important for studying large-scale circulations. However, the decomposition into slow dynamics and fast waves governed by a constant coefficient wave equation is no longer possible. The wider recognition of this issue will hopefully lead to progress in this area. There was substantial interest in the study of the slow dynamics in the semi-geostrophic limit. There were a number of useful informal discussion sessions on the applicability of the semi-geostrophic equations. While they explain the coherent structures that are observed in this regime in the atmosphere and ocean, they do not explain the breakdown into stratified turbulence or other motions that occurs on smaller scales. Other sets of equations have to be used to do this. Progress in the study of the semi-geostrophic equations has primarily been in the special case of constant rotation, which is not an appropriate assumption in this regime in the atmosphere, though it is relevant in the ocean. In that case, rigorous results have been obtained by optimal transport, and a new form of relaxed solution, widening the physical applicability of the results, was presented by Feldman and Tudorascu. Useful discussions took place involving also Rubtsov and Lopes-Filho on extending the theory to the variable rotation case and on proving uniqueness of the solutions. The latter is required to allow the semi-geostrophic solutions to be used as limits of the Euler equations.

Another approach to this topic was studied by Oliver and Vasylkevyich. They used direct approximation of Hamilton's principle to derive limit equations in the same regime, which have higher regularity than solutions of the semi-geostrophic system. While the resulting equations cannot always be written in physical space, they can still be shown to approximate the solutions of the Euler equations. It is not yet clear whether this approach can be used to generate the frontal solutions that the semi-geostrophic equations generate.

A different theme much emphasised during the programme are statistical properties of dynamical systems. This was particularly considered according to the following viewpoints:

A] To show that a deterministic dynamical systems exhibits (probabilistic) limit theorems when it has strong mixing properties and relaxes to its invariant state. During the workshop a lot of attention was brought to one of these limit theorems, namely the *extreme value distributions (EVD)*. Contributions by Ghil, Faranda, Freitas and Freitas, Holland, Lucarini, Kuna, showed the wide range of applications of extreme value theory to dynamical systems and two aspects were particularly stressed:

(i) the clustering of exceedances for observables along orbits visiting the neighborhood of periodic points (Freitas-Freiteas, Faranda). This leads to the introduction of the *Extremal Index* (EI), which is a parameter that quantifies the amount of clustering and that, together with the usual behavior around generic points, gives a complete description of the asymptotic distributions in all the regions of the phase space.

(ii) the application of the EVD to dynamical systems with some sort of weak hyperbolicity (e.g. Henon's attractors), and the use of more general observables (with respect to those usually invoked at item (i) and related to local recurrences), with the aim to explore the geometrical properties of the attractors and the relationship with other dynamical indicators like the Lyapunov exponents and the entropy (Kuna, Holland, Lucarini). This could have more concrete applications to meteorology as shown by Ghil and Faranda-Vaienti; the latters presented a very efficient tool to analyze climate time series throughout the EVT and whenever the signal is thought to be affected by observational noise (see point B below).

Another statistical property was the object of talks and discussions, namely the decay of correlations, that constitutes the first step to prove limit theorems like the central limit theorems, the weak-invariance principle, the large deviations etc. It is interesting to observe that this topic was mostly considered and investigated according to the second approach which for simplicity of expositions we could call perturbed dynamical systems and which is the object of the next item. Entropy production was also at the center of discussions due to various contributions to this field by Lucarini and Ruelle.

B] Perturbing a dynamical system would address several questions. The response of the systems to the perturbation, its parametric dependence on the latter, its stability and the (eventual) return to the original configurations were considered and exposed by V Baladi from a mathematical point of view and by Lucarini and Ruelle with a more physical perspective. The *linear response* was one of the more unifying topics of the workshop and a few works will surely arise from those discussions.

Random dynamical systems and sequential dynamical systems were also treated, in particular by Vaienti and Aimino. In the context of randomly perturbed systems, two results have been presented, the first concerning the proof of extreme value distributions under observational noise (see also above), and the second about annealed and quenched limit theorems for random expanding systems. In the context of sequential dynamical systems, Vaienti presented the proof of polynomial decay for the loss of memory in the case of Pomeau-Manneville like maps with perturbed slopes, and Aimino announced the proof of concentration inequalities for the sequential systems studied by Conze and Raugy. Other statistical approaches to dynamical systems were presented by Beck and Rondoni, in particular to discuss stochastic models of quantum liquids and their connections to the statistical mechanics of 2-dimensional turbulence.

We summarise our assessment of the results of the programme with the following quotation by D Ruelle about the impact of low-dimensional dynamical systems for the understanding of more complex systems. His last sentence is beautiful synthesis of what happens at Newton and we hope this experience will be repeated in the future.

"*The theory of smooth dynamical systems, especially ergodic aspects, ...needs reliable general conceptual tools, in spite of its limitation. (Only certain classes of dynamical systems are rigorously understood, hyperbolic systems being most important from the present viewpoint, and it is not always easy to guess what results proved for hyperbolic systems have greater applicability. Nevertheless, a combination of mathematical work and computer testing gives rather convincing results). To summarize: the present program has put together two somewhat distant scientific areas, in a way that makes perfect sense, but need not be easy to realize. This challenge has been met in a very satisfactory way by the MFE program.*"

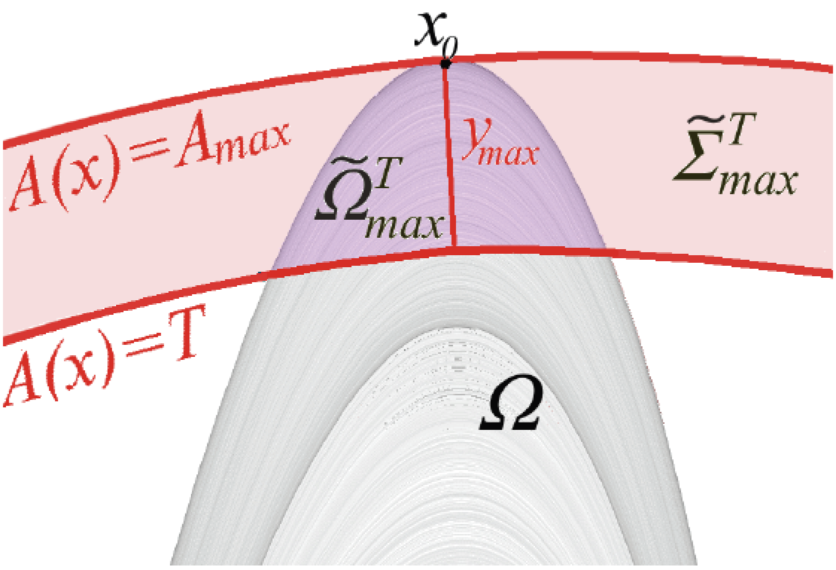


**Figure 1: The extremes of the observable A(x) for a dynamical system possessing attractor Ω can be studied by looking at the geometry of the intersection between the attractor and the level sets of A(x).**

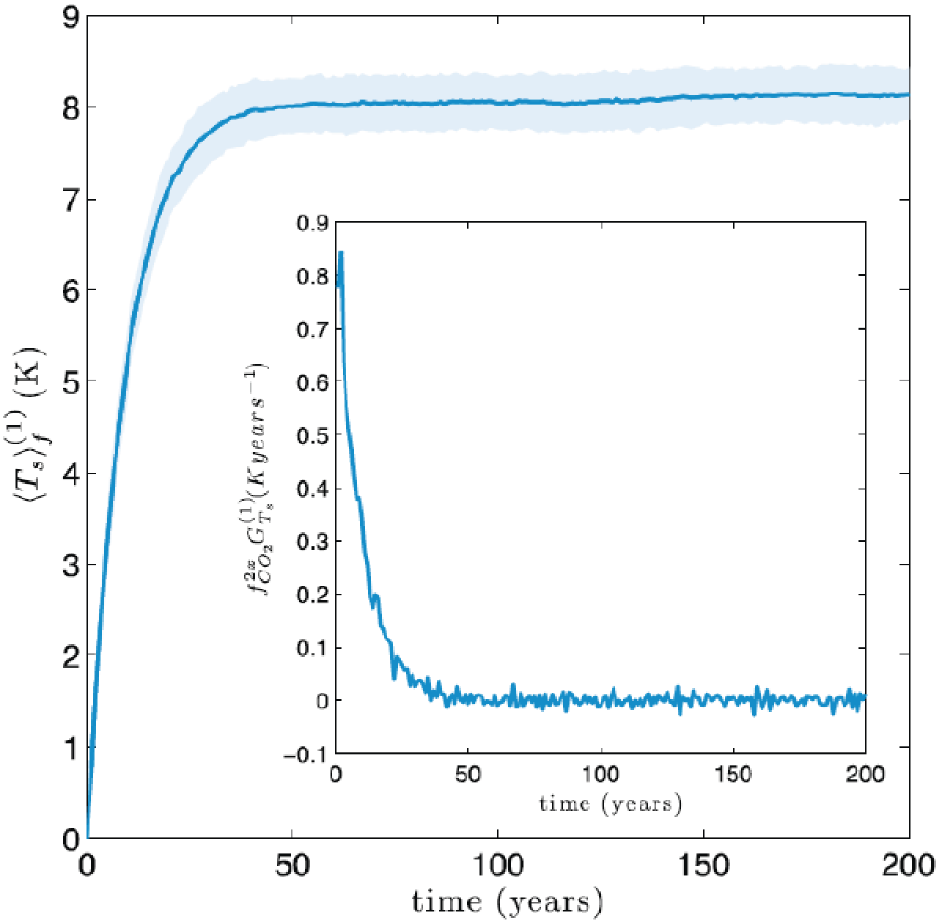


**Figure 2: Using the Ruelle Response theory, it is possible to construct the Green function describing the response of the globally averaged surface temperature to increasing $CO_2$ concentrations.**

**Highlights and resulting collaborations**

*Submitted Papers:*

C. Beck, Phys. Rev. Lett. 111, 231801 (December 2013)

Bin Cheng, "Improved Accuracy of incompressible approximations of compressible Euler equations", submitted (and on ArXiv)

Yong Li And Paul D. Williams, Analysis Of The Raw Filter In Composite-Tendency Leapfrog Integrations, submitted to *Journal Of Computational Physics*

Javier Amezcua And Paul Williams, The Composite-Tendency Raw Filter In Semi-

Implicit Integrations, submitted to *Quarterly Journal Of The Royal Meteorological Society*

Paul D. Williams And Christopher W. Kelsall, Zonal Jet Formation In Numerical Simulations Of A Large Rotating Annulus Experiment, submitted to *Journal Of The Atmospheric Sciences*

D. Ruelle, Nonequilibrium statistical mechanics of turbulence

Roytsov, Roulstone and Banos, Invariants of Monge-Ampère operators and some 3D meteorological models

F. Bouchet, J. Laurie, and O. zaboronski, Langevin dynamics, large deviations and instantons for the quasi-geostrophic model and two-dimensional Euler equations, 2014, submitted to *J. Stat. Phys*

Lucarini, Pascale, Wouters, Blender, Ragone, Herbert, Mathematical and Physical Ideas for climate science, submitted to *Rev, Geophysics*

Ragone, Lucarini, Lunkeit, A new framework for climate sensitivity and prediction, submitted to *Climate dynamics*

Bodai, Lucarini, Lunkeit, Global Instability in a Sellers-type model, submittted to *Climate Dynamics*

R.Aimino, H. Hu, M. Nicol, A. Torok, S. Vaienti, Polynomial Loss Of Memory For Maps Of The Interval With A Neutral Fixed Point, Submitted, Http://Arxiv.Org/Pdf/1402.4399.Pdf,

D. Faranda, X. Leoncini, S. Vaienti, Mixing Properties In The Advection Of Passive Tracers Via Recurrences And Extreme Value Theory,
Submitted, **Arxiv:1402.3798**

D. Faranda, S. Vaienti, Extreme Value Laws For Dynamical Systems Under Observational Noise, Submitted, Arxiv:1308.5624

Ya. Pesin, Yun Zhao, "Scaled Entropy for Dynamical Systems", Preprint, PSU, 2014 submitted to *J. of Statistical Physics*

**Book planned**

Valerio Lucarini, Davide Faranda, Jeroen Wouters (U. Hamburg, Germany); Jorge Freitas, Ana Moreira Freitas (U. Oporto, Portugal); Mark Holland (U. Exeter, UK); Matthew Nicol (U. Houston, USA); Mike Todd (U. St. Andrews, UK); S. Vaienti (U. Toulon, France)., ***Extremes and Recurrence in Dynamical Systems***

**Other outputs**

Planned Workshops/Conferences: "GEOMETRIC METHODS IN DYNAMICAL SYSTEMS", CIRM, 31 March -- 4 April 2014.

Special issue of New Journal of Physics on "Stochastic Flows and Climate Statistics" Edited by Brad Marston and Paul Williams
http://iopscience.iop.org/1367-2630/focus/Focus%20on%20Stochastic%20Flows%20and%20Climate%20Statistics

MCRN mini-grant:
A computational approach to measuring the geometrical properties and spatial distribution of tundra permafrost lakes
**Ivan Sudakov**, Luke Mander, Christopher Boulton, **Peter Cox** and Timothy Lenton

Planning (started during the programme) of a proposal for an EPSRC *programme grant* involving several of the participants (Kuna, Holm, Lucarini, Pelloni, Zimmer)

## Collaborations

Baladi, Demers, Liverani: Decay of correlations of Sinai billiard FLOW.

Baladi, Kuna, Lucarini: Generalized Response theory

Cox, Huntingford, Williams: Climate sensitivity