Moduli Spaces 4 January – 1 July 2011

Organisers: P. E. Newstead, L. Brambila-Paz, O. Garcia-Prada, R. Thomas

Scientific background

Algebraic geometry is a key area of mathematical research of international significance. It has strong connections with many other areas of mathematics (differential geometry, topology, number theory, representation theory, etc.) and also with other disciplines (in the present context, particularly theoretical physics). Moduli theory is the study of the way in which objects in algebraic geometry (or in other areas of mathematics) vary in families and is fundamental to an understanding of the objects themselves. The theory goes back at least to Riemann in the mid-nineteenth century, but moduli spaces were first rigorously constructed in the 1960s by Mumford and others. The theory has continued to develop since then, perhaps most notably with the infusion of ideas from physics after 1980.

The programme focussed on three topics of current interest: stability in derived categories, the relationship of moduli spaces to topology, Teichmüller theory and hyperbolic geometry especially in the context of Higgs bundles, and moduli of bundles and coherent systems on algebraic curves.

Programme structure

For the most part, the programme had an open structure to allow the maximum time for discussion. There were periods of concentration on stability of complexes and stability of varieties, representations of surface groups and Higgs bundles, and vector bundles and coherent systems. However at all times there was provision for those working on other aspects of the programme and for interactions between the different areas. The fixed features of the programme were an introductory school and three workshops; in addition there were a Spitalfields Day covering all aspects of the programme and a one-day meeting on vector bundles and coherent systems.

Workshops

Introductory School

5-14 January 2011

Organisers: L. Brambila-Paz, O. Garcia-Prada, P. E. Newstead, R. Thomas

The aim of this very successful school (there were 134 registered participants) was to introduce young mathematicians to the foundations of the theory of moduli spaces, some of its major developments and tools used for its study. It included lecture courses on derived categories and stability conditions, moduli of varieties, stacks, geometric invariant theory, deformation theory, Ringel-Hall algebras and applications to moduli, representations of surface groups and Higgs bundles, principal bundles and augmented bundles, and a lecture on moduli spaces and physics. There were also tutorials associated with each of the lecture courses.

Representations of surface groups and Higgs bundles

Oxford satellite, 14-18 March 2011

Organisers: O. Garcia-Prada, W. Goldman, T. Hausel, N. J. Hitchin, P. E. Newstead

The workshop concentrated on the relationship of moduli spaces to topology, Teichmüller theory and hyperbolic geometry - this relationship takes place in the study of representations of a surface group in a real Lie group and involves Higgs bundle theory, bounded cohomology, Anosov systems, cluster varieties and tropical algebraic geometry. The workshop included lectures by experts on the latest developments in this area. There were 61 registered participants.

Derived categories

11-15 April 2011

Organisers: T. Bridgeland, J. Stoppa, R. Thomas

This workshop focussed on derived categories of coherent sheaves and their interactions with moduli, stability, invariants, mirror symmetry, string theory, geometric representation theory and knot theory. It was preceded by an informal week of talks. The workshop itself attracted 117 registered participants.

Closing conference

27 June – 1 July 2011

Organisers: P. E. Newstead, L. Brambila-Paz, O. Garcia-Prada, R. Thomas

This workshop incorporated the 2011 workshop of the Research Group Vector Bundles on Algebraic Curves. The scientific programme consisted of 13 invited talks and 9 contributed talks; the latter were chosen from 24 submissions and those whose talks were not selected were invited to present posters. The talks covered various aspects of the theory of moduli. There were 116 registered participants.

Outcome and achievements
Many new results emerged from the programme. Many collaborations were consolidated and new projects were launched. The detailed comments below are divided into sections corresponding to the focus topics of the programme. Asterisks indicate preprints on the INI server.

Derived categories
One of the most currently active areas in moduli spaces is in defining invariants of 3-folds using derived categories of sheaves. The theory is clearly not in its final form, with great progress being made and reported on during the 6 month activity.

As a first step, one needs a notion of stability to extract moduli spaces from the derived category. This is the most problematic part of the programme at the moment, and has only been overcome in special cases (for instance for the abelian category of sheaves, rather than the whole derived category, or for certain types of stable pairs) using ad-hoc methods and Geometric Invariant Theory constructions. A celebrated general theory of stability in derived categories has been developed by Bridgeland, one of our workshop organisers and a participant. During the conference Bayer, Bertram, Macri and Toda** made and reported on a breakthrough in constructing such Bridgeland stability conditions on 3-folds, partially solving a 5-year old problem and making an unexpected link with Fujita's conjecture in Mori theory. Others such as Maciocia, Yoshioka and Smith gave talks on progress in this field; Smith reporting on exciting physics-inspired work with Bridgeland. Work by Maciocia and Meachan* on Bridgeland stability on principally polarised abelian varieties came to fruition during the programme.

Different ways of extracting invariants from such moduli spaces form a foundational area involving exciting new methods; the world's experts in this area (Behrend, Fantechi, Joyce, Kiem, Li, Lieblich) explained on-going work, and new perspectives. In particular Kiem and Li made substantial progress on wall-crossing formulae using Kirwan blow-ups. The state of the art looked very different at the end of the 6 months than it did in the introductory school at the beginning. More powerful motivic and categorified versions of the numerical invariants are now defined in some cases and look likely to be understood fully soon. Bryan, Davison, Nagao, Szendröi and Reineke told us about exciting calculations in different cases which show a beautiful theory is close to being discovered.

Where moduli spaces can be constructed, the emphasis is now on computing the resulting invariants and relating them to other invariants, such as the long-established Gromov-Witten invariants, and the long-conjectured BPS invariants of Gopakumar-Vafa coming from string theory. The world's leading figures in this area, such as Bryan, Diaconescu, Göttsche, Pandharipande, Shende and Toda spent extended periods at the Newton Institute working intensively in fevered discussions with Thomas. Toda completed two papers on generalised Donaldson-Thomas invariants**. By the end we had seen a complete understanding of how all 3 theories match perfectly for plane curves in Calabi-Yau 3-folds (A support theorem for Hilbert schemes of planar curves, Migliorini and Shende, arXiv:1107.2355). Göttsche also developed a new and deeper motivic version of his famous conjecture, motivated by Kool-Shende-Thomas' recent proof. In discussions with Shende he also made a remarkable discovery about its link to Welshinger invariants counting real curves. This exciting and completely unexpected link is sure to be a big topic of research for the next decade.

The interaction of moduli and physics is famous and well-documented; it seemed particularly successful here as in particular Diaconescu's physical insights into BPS invariants influences the mathematicians' approaches, and vice-versa. It was also exciting to see the next generation emerge, such as our CPS scholars Davison and Shende. Garcia-Fernandez started a collaboration with Ross on stability conditions for pairs expected to have applications in theoretical physics. There were huge numbers of graduate students interested in this area, and they commented on how much they learned and how helpful it was to their research to interact with the leaders in the field.

A related topic that received a lot of attention was derived categories of quiver representations, and their

invariants. This is both a model for the sheaf theory, and a place one can do explicit calculations, exposing rich and beautiful structure. Most of the world's experts visited and exchanged ideas, from Nakajima in mathematics to Hanany in physics. The latter announced a whole new class of hyperkähler spaces, generalising Nakajima's famous construction, causing a stir amongst the mathematicians. Cautis and Nakajima explained beautiful categorifications of Nakajima's algebra actions on cohomology of quiver moduli. Reineke completed work on quantized Donaldson-Thomas invariants for m-loop quivers.

A driving force behind this whole subject is the 20 year old mirror symmetry conjecture, and Gross, Iritani, Keel, Pantev, Soibelman and Zaslow all explained substantial breakthroughs in its understanding. It looks likely that proofs from a number of different perspectives will emerge in the next few years; as they are linked and unified we can finally expect a proper understanding of this remarkable phenomenon that has occupied so many brilliant minds for so long.

Higgs bundles and character varieties
Research was completed for joint work by Logares, Muñoz and Newstead on Hodge polynomials of character varieties and a paper has been submitted*; this involved discussions with Hausel and Thomas. Joint work of Garcia-Prada, Heinloth and Schmitt on the computation of the motive of the moduli space of rank-4 Higgs bundles confirmed conjectures of Hausel and Rodriguez-Villegas; a paper has been submitted for publication*. Some progress on the generalization to higher rank has been made.

Pantev and Simpson worked on the functoriality of the ramified non-abelian Hodge theory correspondence and settled the compatibility of pushforward for morphisms of relative dimension one.

A joint paper of Balaji with Seshadri on parahoric bundles was completed; this involved discussions with Garcia-Prada, Biquard, Mundet and Narasimhan. Substantial progress was made on the writing up of a joint paper of Biquard, Garcia-Prada and Mundet on parabolic G-Higgs bundles and representations of the fundamental group of a punctured surface. There was much interaction and discussion on parahoric bundles by Balaji, Biquard, Garcia-Prada, Heinloth and Mundet, comparing algebraic and analytic approaches.

Progress was made on the writing up of a joint paper by Garcia-Prada and Ramanan on involutions on the moduli space of Higgs bundles and real forms of a complex semisimple Lie group. Progress was also made on a joint project of Garcia-Prada and Schaffhauser on real Higgs bundles over a real curve, i.e. a complex curve equipped with an antiholomorphic involution.

Progress was made in the writing up of two papers on Higgs bundles for Lie groups of Hermitian type, one for general groups by Biquard, Garcia-Prada and Rubio and another on the group SO*(2n) by Bradlow, Garcia-Prada and Gothen.

A joint paper by Goldman and Toledo on relative character varieties and cubic surfaces was completed and submitted*.

Research on a joint project of Franco, Garcia-Prada and Newstead on Higgs bundles on elliptic curves was completed and will be written up shortly. A joint paper by Alvarez-Consul, Garcia-Fernandez and Garcia-Prada on coupled Yang-Mills equations and Kähler metrics was completed*.

There were many discussions on moduli spaces of vortices by Bradlow, Manton, Romao, Mundet, Speight and other programme participants.

Vector bundles and coherent systems
There has been much exciting work recently on rank 2 stable bundles with 4 sections on algebraic curves. The primary object of this is to determine the smallest degree for which such bundles exist and hence to settle the rank 2 case of a conjecture of Mercat. The current work was sparked by a paper of Grzegorczyk, Mercat and Newstead (drawing on earlier work of Voisin); following this Farkas and Ortega produced a series of counter-examples to Mercat's conjecture, which were extended by Lange and Newstead. During the programme, following discussions with Farkas, a postscript was added to the paper of Grzegorzcyk, Mercat and Newstead answering some questions raised in it* and Lange and Newstead extended their work further; subsequently they have obtained a best possible result and a paper has been submitted*. There is still much

work to be done, but it seems that we may be close to giving a complete answer to this question.

Osserman completed a paper on Brill-Noether loci for fixed determinant*. Teixidor had previously shown that, for determinant of large degree, these loci have a component of the expected dimension. During the programme, following discussions with Osserman and Teixidor, Grzegorczyk and Newstead showed that, in certain cases where the determinant has smaller degree, the corresponding moduli spaces of coherent systems possess a component of the expected dimension, which is sometimes unique, thus showing that in these cases Osserman's bound is sharp.

Newstead also had discussions with Tommasini on coherent systems the results of which will form part of his thesis, and with Lelli-Chiesa on Gieseker-Petri divisors on the moduli spaces of curves.

There were many other discussions on the moduli of stable bundles, involving Balaji, Bhosle, Biswas, Bradlow, Brambila-Paz, Choe, Gomez, Mata, Mistretta, Newstead, Ortega, Ramanan and Teixidor among others.

Other work
Much other work, not fitting the above categories, took place during the programme. In particular, papers by Hulek and Grushevsky on intermediate jacobians of cubic threefolds* (Hulek also reports useful discussions with Alexeev, Farkas, Lange, Shepherd-Barron and Totaro), by Lange and Ortega on the Prym map for non-cyclic triple coverings* and by Mukai on the Igusa quartic and Steiner surfaces* were completed. Voisin wrote a paper during her stay and held discussions with many people, including Mukai, Sankaran, Thomas, Brambila-Paz and Totaro. Lehn held discussions with King which opened a new perspective. Pratoussevitch developed some new ideas on higher spin bundles and Gorenstein quasi-homogeneous surface singularities, which she discussed with Bayer, Giansiracusa, Newstead and Nitsure. Nitsure continued his collaboration with Neumann and obtained new results on gerbal stratifications of moduli stacks and Lange's universal extensions. Giansiracusa spoke to Baranovsky and Doran about potential projects. Belkale had discussions with Biswas, Boysal and Narasimhan among others.

Publications
A book based on the courses given in the introductory school is in preparation. Many papers will be published, some of which are already available as INI preprints and on arXiv.

Impact on UK mathematics
We cannot do better than to quote the following (thanks to Burt Totaro): “The programme started with excellent introductory lectures to a packed audience of graduate students and postdocs, including series of five lectures each by Daniel Huybrechts on stability conditions, Valery Alexeev on moduli spaces, and Kai Behrend on stacks. The rest of the workshop included some of the world's greatest geometers, including Lothar Göttsche, Nigel Hitchin, Terry Tao, Carlos Simpson, Jun Li, Tom Bridgeland, Gavril Farkas, Richard Thomas, Max Lieblich, Hiraku Nakajima, Rahul Pandharipande, Shigeru Mukai, Alexander Kuznetsov, Eric Zaslow, Dominic Joyce, Claire Voisin, Nick Shepherd-Barron, Sean Keel, Mark Gross, Simon Donaldson, Prakash Belkale and William Goldman. That list includes two Fields Medalists, and most of those listed have been invited speakers at the International Congress of Mathematicians. Many of them are from the U.S. or Japan, and so they don't come too often to the U.K. The Newton Institute programme on Moduli Spaces has been a great contribution to mathematical research in the U.K.”

Another feature of the programme was the large number of visits made by participants to universities in the UK. This has undoubtedly benefitted UK research in the area covered by the programme and has counteracted the fact that many UK researchers who were invited to participate in the programme were either unable to do so or were able to come for very limited periods due to other commitments.