Mathematics and Physics of Anderson Localization: 50 years after

Final Report

Organizers: Professor Y. V. Fyodorov (Nottingham), Professor I. Goldsheid (Queen Mary, London), Professor T. Spencer (IAS, Princeton), Professor M. R. Zirnbauer (Cologne)

Basic theme and background information

In his seminal paper “Absence of diffusion in certain random lattices” (1958) Philip W. Anderson discovered one of the most striking quantum interference phenomena: particle localization due to disorder. Cited in 1977 for the Nobel prize in physics, that paper was fundamental for many subsequent developments in condensed matter theory. In particular, in the last 25 years the phenomenon of localization proved to be crucial for the understanding of the Quantum Hall Effect, mesoscopic fluctuations in small conductors, as well as some aspects of quantum chaotic behaviour.

Random Schrödinger operators are an area of very active research in mathematical physics and mathematics. Here the main effort is to clarify the nature of the time dynamics and the underlying spectrum. In particular, it has been proved that in dimension one all states are localized, and that in any dimension the random Schrödinger operator has dense point spectrum for large enough disorder. Some open mathematical questions of major importance include the long-time evolution of a quantum particle in a weakly disordered medium and the existence of absolutely continuous spectrum in three dimensions. The expected transition from localized (point spectrum) to extended eigenstates (absolutely continuous spectrum) has also to be addressed.

The goal of the programme was to bring together the world leaders in the spectral theory of random Schrödinger operators and theoretical physicists successfully working on the problem of Anderson localization, and, in this way, to try to bridge the existing language gap between the communities and create an environment conducive to fruitful collaboration between physicists and mathematicians on problems of common interest.

The programme ran from 14 July to 19 December 2008.

Structure

The programme activities were organized around four workshops (three at the Newton Institute, and a satellite meeting at Gregynog Hall, University of Wales). The programme was attended by around 100 long and short-term participants. Professors P. Anderson, J. Froehlich (Rothschild Visiting Professor), D. Thouless, and F. Wegner (Microsoft Research Fellow) were among our most distiguished participants. Between the workshops, a seminar schedule of typically two to three seminars per week was run, with the majority of participants presenting their research in that framework. Many participants of the programme used the opportunity to travel round the UK and gave talks at quite a few British institutions including Birmingham, Bristol, Brunel, Edinburgh, Imperial, Lancaster, Manchester, Nottingham, Oxford, UCL, and Warwick, as well as in various departments in Cambridge itself. At the encouragement from the Director of the Institute we organized a few cross-programme seminars with the participants attending the programme on turbulence. Those meetings were very useful in exposing non-specialists to the present state of research and to outstanding issues in both fields, and for identifying both similarities and differences in questions asked and methods used.

Workshops

- 14 - 25 July 2008: *Anderson Localization Transition: Introductory Training Course*
  Organiser: M.R. Zirnbauer (Cologne)
  Participants: 61

  This training course was mainly directed at researchers in early stages of their careers (typically at postdoctoral level), but a few more mature colleagues, interested to learn more on topics related to Anderson localization, were also present. The audience represented a rather balanced mixture of mathematicians, mathematical and theoretical physicists. The main goal was to provide the participants with an introduction to the subject, by exposing them to ideas, terminology and analytical techniques of the rigorous as well as the heuristic kind. Methods used in the study of Anderson localization by mathematicians and by theoretical physicists were reviewed by experts from both communities.

  The list of topics covered included: phenomenology of Anderson localization (T. Spencer); introduction to the spectral theory of random Schrödinger operators (L. Pastur); introduction to supermatrix techniques and the nonlinear σ-model (Y.V. Fyodorov); rigorous techniques for 1D and quasi 1D systems (I. Goldsheid); rigorous methods in the statistical mechanics of phase transitions (D. Brydges); critical phenomena in two-dimensional disordered systems (A. Ludwig).

  In addition to the scheduled lectures, some of the younger participants had a chance to present their recent research.

- 18 - 22 August 2008: *Anderson Localization and Related Phenomena*
  Organisers: Y.V. Fyodorov (Nottingham), I. Goldsheid (Queen Mary, London), T. Spencer (Princeton) and M.R. Zirnbauer (Cologne)
  Participants: 60

  This one-week workshop brought together mathematicians and theoretical physicists considered to be leaders in the study of various mathematical and physical aspects of the theory of random Schrödinger operators, Anderson localization phenomena, and related topics. Presentations were selected with the intent to review the current state of the art of the field in the theoretical and mathematical physics communities.

  Among the topics that were addressed during the workshop were: the nature of critical phenomena associated with localization-delocalization transitions; the existence and statistical properties of extended states for $D > 2$ and the behaviour in the critical dimension $D = 2$; supersymmetric methods and nonlinear σ-model techniques; the localization-delocalization phenomena associated with the Integer Quantum Hall Effect, including new materials like graphene; localization in the presence of a random magnetic field; asymptotic behaviour of products of random matrices and associated Lyapunov exponents; spectra, localization and delocalization in disordered systems described by non-selfadjoint operators; localization in systems with nonlinearities; and localization-delocalization phenomena in disordered systems of interacting quantum particles.

- 2 - 6 November 2008: *Delocalization Transitions and Multifractality*
  Organizers: F. Evers (Karlsruhe) and Y.V. Fyodorov (Nottingham)
  Participants: 39

  As is well known, by changing the disorder potential in a single-particle quantum Hamiltonian, one may induce a transition from localized to extended eigenstates (the Anderson delocalization transition). Precisely at the point of the transition one expects a very non-trivial intensity pattern reflecting intriguing "multifractal" properties of the critical eigenstates. (See Figure 1.) This picture extends to a broader class of quantum systems, including, most prominently, those exhibiting Quantum Hall-type transitions. It also has interesting counterparts beyond quantum mechanics in other disorder-induced critical phenomena.

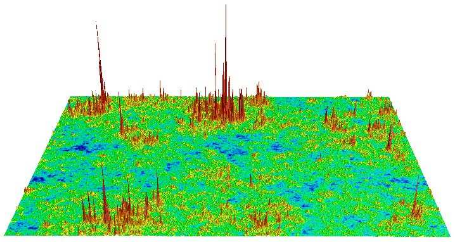

Figure 1: Intensity profile of a multifractal wave function at the critical point of Quantum Hall-type transition (courtesy of F Evers, A Mildenberger, and A Mirlin). The intensity in green areas is typical, in blue areas is anomalously low and in brown areas is anomalously high.

In recent years considerable effort has been invested in performing large-scale numerical simulations of systems of that kind. This has allowed both the testing of highly non-trivial analytical predictions for various critical statistics, and has provided additional insights and the background for further theoretical understanding. The four-day workshop co-organised with the Karlsruhe Institute of Technology, Germany, brought together the researchers successfully working on the problem of Anderson delocalization transition, with emphasis on multifractality and its implications. The talks reviewed the state of the art in both numerical and analytical communities, discussed conceptual as well as technical difficulties, and mapped new directions of research in this fascinating area of theoretical physics. In addition, a few, more mathematical talks related to various instances of multifractal measures and other probabilistic questions in systems with disorder.

- 15 - 19 December 2008: *Classical and Quantum Transport in the Presence of Disorder*
  Organisers: Y. V. Fyodorov (Nottingham), I. Goldsheid (Queen Mary, London), T. Spencer (Princeton) and M. R. Zirnbauer (Cologne)
  Participants: 54

  This was the concluding conference of the programme. In addition to describing recent results in quantum localization, the conference explored some emerging connections between classical and quantum transport. The topics included quantum localization and diffusion, random walks in a random environment, quantum chaos and resonances, and the effects of nonlinearity on wave propagation.

Outcomes, achievements, and publications arising

The scope of the problems addressed during the programme not only covered diverse aspects of Anderson localization, but frequently touched on a much broader range of topics in the theory of disordered systems and related fields (e.g. the theory of random matrices). Among the many contributions, we would like to highlight, in particular, the following two:

- The work by M. Disertori, T. Spencer, and M.R. Zirnbauer, who rigorously analysed a supersymmetric hyperbolic σ-model which is believed to contain most of the essential features of the Anderson localization phenomenon. In particular, in that framework they proved the existence of a kind of quasi-diffusion in $D = 3$ spatial dimensions. This model is equivalent to a random walk in a highly correlated random environment. These results will be presented at an invited talk at the International Congress of Mathematical Physics (Prague, August 2009).
- The work by V. Chulaevsky and Yu. Suhov on multi-particle localization theory, which presents the first essential steps towards extending the standard mathematical methods to the case of a few interacting particles. Remarkably, the cumbersome original proofs were

considerably simplified and generalised by incorporating ideas and heuristic insights provided by the fellow physicists present on the programme (mainly, B. Shapiro).

Of the many other interesting results obtained by long-term participants we would like to mention the works on level repulsion of Wigner matrices (L. Erdos, B. Schlein, and H.-T. Yau); on fractal superconductivity close to the delocalization transition (M. Feigelman and V. Kravtsov); on pre-freezing transition in disorder-induced multifractal exponents (Y. Fyodorov); on the effects of nonlinearity in 1D random systems (S. Fishman); on a new approach to constructive control of Lyapunov exponents (I. Goldsheid), on ballistic transport in disordered graphene (A. Mirlin and P. Ostrovsky); on localization in quantum graphs (S. Molchanov); on quantum transport in chaotic cavities (D. Savin and B. Khoruzhenko); on eigenvector localization for random band matrices with power law band width (J. Schenker); on the capacitance of quasi one-dimensional wires (M. Skvortsov and M. Zirnbauer); on colour-flavour transformations and characteristic polynomials of real random matrices (Y. Wei and B. Khoruzhenko); and on the mean density of complex eigenvalues for an ensemble of random matrices with prescribed singular values (Y. Wei and Y. V. Fyodorov).

All of the above mentioned work will certainly give rise to a number of publications, and quite a few of these are already available in the form of electronic preprints at uk.arxiv.org. Beyond that, V. Chulaevsky and Yu. Suhov are working at present on the book “Multi-Particle Multiscale Analysis” to be published by Birkhäuser in 2009 or 2010. L. Pastur and M. Shcherbina will publish the book “Eigenvalue Distribution of Random Matrices” (American Mathematical Society), considerable parts of which were written during their participation in the programme.

In conclusion, we would like to mention that the main challenges to the coherence of the programme were posed by the diversity of the participants’ backgrounds. Judging by responses from programme participants, the activity indeed succeeded in engaging physicists and mathematicians in a constructive dialogue, with mathematicians particularly appreciative of the opportunities to learn from the physical ideas and heuristic arguments of their physicist colleagues.