Noncommutative Geometry
24 July – 22 December 2006

Report from the Organisers:
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Scientific Background

It is very natural to think that space and time are primary notions and that physics should be based on them. It would, however, be more reasonable to say that our notions of geometry emerged in the 19th century from and together with the classical physics of the day, a process that led ultimately to Einstein's formulation of gravity as curvature. Geometry now also plays a role all branches of mathematics, indeed wherever one finds a continuum space of interest.

This continuum assumption of classical mechanics was, however, already shattered in the 1920's with the discovery of the quantum nature of the phase space of the microscopic mechanical system describing an atom. Such a system manifests itself through discrete spectral lines and its basic laws, such as the Ritz–Rydberg law of spectroscopy, are in direct contradiction with a continuum picture of the phase space. Heisenberg was the first to understand that for a microscopic mechanical system the coordinates, namely real numbers such as the positions and momenta $x, p, \ldots$, that one would like to use to parameterize points of the phase space, actually do not commute. More recent developments such as string theory similarly imply that our classical geometrical framework is too narrow to describe in a faithful manner the physical spaces of great interest when one deals with microscopic systems: one needs some form of 'noncommutative' geometry in which the "algebra of coordinates" is no longer commutative.

Although the first examples of noncommutative spaces came from quantum mechanics, there turn out to be a great many others of interest, such as the leaf spaces of foliations, the space of Penrose tilings, the noncommutative torus $\mathbb{T}^2_\theta$ which plays a role in the quantum Hall effect and in M-theory compactification and the space of $\mathbb{Q}$-lattices related to a spectral interpretation of the zeros of the L-functions of number theory. The new theory of 'noncommutative geometry' that has developed in the last three decades adapts all classical geometric concepts and tools to this new class of geometric spaces, leading to far reaching generalizations such as cyclic cohomology and K-homology and extending, for example, the celebrated Atiyah-Singer index theory to such operators.

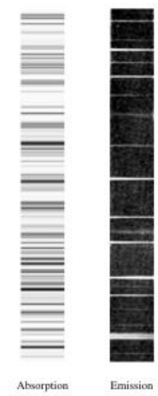

Figure 1: Emission and absorption spectra

Also emerging in the 1980s were objects called 'quantum groups', initially as generalised or 'quantum' symmetries in certain physical models but soon connected to knot theory, the theory of 3-manifolds and to the representation theory of algebraic groups. Lie groups are key geometrical objects and similarly quantum groups have contributed in part to what could be called an 'algebraic stream' to noncommutative geometry. They also have applications for example in the bookkeeping of divergences in quantum field theories and in new models for noncommutative spacetime. One of the goals of the programme was to help reconcile the algebraic and the cyclic cohomology sides of noncommutative geometry. Another goal was to bring out the full range of its applications.

Structure of the Programme

It should be clear that scientists working in noncommutative geometry come from diverse backgrounds spanning the entire gamut from theoretical particle physics to the most pure mathematics. One of our goals in structuring the programme was to create a genuine melting pot bringing together different expertise. At the same time we provided a certain amount of focus in the form of 3 workshops (described below). Not counting the workshops, there were 91 programme participants of which 35 were 'Visiting fellows', 56 were 'Programme participants' and 2 were affiliate participants. Of the participants some 19 stayed two months or more (in several cases almost the entire period) providing a central core during the different programme phases.

In terms of emphasis the operator algebras and cyclic cohomology side was uniformly represented particularly by leading figures such as Connes, Nest, Plymen and Wodzicki across the programme, Moscovici, Tsygan and Rieffel in early months and Wassermann in later months. The algebra side was represented particularly by van den Bergh and Smith in the later months. The physics was particularly represented by Schwarz in the first half of the programme. Quantum groups were represented particularly by Woronowicz and Majid across the programme. The other many participants had interests spread across the programme. Among them, we were particularly happy that some junior postdocs and PhD students played an active and extended part.

Seminars were given by participants throughout the programme. Not counting the workshops, 58 seminars were given with topics covering the whole range of the field.

In addition, a very special feature of the programme was a ten-hour lecture course by Connes on his new approach to the standard model of particle physics based on noncommutative geometry.

Another special feature of the programme was a public panel discussion featuring Connes, Sir Roger Penrose, J Polkinghorne and others speculating about the true nature of space and time and its broader context. This was linked to the second workshop below and was very much appreciated both by participants and members of the public.

Workshops

Noncommutative Geometry and Cyclic Cohomology, 31 July - 4 August 2006

Organisers: A Connes, H Moscovici, R Nest, G Yu

This first workshop focussed on the pure mathematics of noncommutative differential geometry, centered on cyclic cohomology. Some particular themes were strong progress in noncommutative Chern-Weil theory and the cyclic cohomology of Hopf algebras, the Baum-Connes conjecture and related developments in algebraic K-theory, and quantum group methods.

There were 60 participants, many of them from North America. Thanks to an external grant from the NSF obtained by J Roe and N Higson at Penn State, we were able to fund many participants including students, while many senior participants used their own grants to cover much of their expenses. The new Junior Member scheme was also very helpful in funding more junior UK postdocs and students and allowing an excellent level of UK participation. These remarks also apply to the other workshops below.

The pure mathematical level of the 23 lectures was extremely high. The talks of Baum and Plymen linking noncommutative geometry, the Baum-Connes conjecture and the Langlands programme were considered particularly exciting. Talks of Moscovici and Nest related noncommutative geometry to Hopf algebras and quantum groups and there were several more talks on Hopf-cyclic cohomology. High-level talks by Connes, Tsygan, Wodzicki and others covered other aspects of the field. There were also several interesting talks by junior participants as well as a good selection of posters.

Noncommutative Geometry and Physics: Fundamental Structure of Space and Time, 4 - 8 September 2006

Organisers: A Connes, C Hull, S Majid and A Schwarz

This workshop, generously supported by the Templeton Foundation, aimed to bring together mathematicians and physicists to consider the fundamental nature of space and time. Are space and time a continuum, discrete or something different from both of these at the tiniest scales? Is time intrinsically generated? What is the picture of spacetime arising from string theory and quantum gravity? The central role of noncommutative geometry in answering these questions was explored.

The workshop was very well attended with over 90 participants from all over the world. Key among 28 talks, Connes spoke about a new way of thinking of the different kinds of elementary particles observed in Nature as extra 'noncommutative directions' added to our usual four-dimensional space time. Several experts spoke about the renormalisation of infinities in quantum field theory using noncommutative geometry. Moreover, we were able to complement the noncommutative side with a day of interdisciplinary talks in which Sir Roger Penrose spoke about spin networks and a new vision of cosmology, A Taylor at the Royal Observatory in Edinburgh gave a survey of what is known and not known about dark matter and dark energy from the point of view of astronomical evidence, and C Laemmerzahl gave a survey of the modern search for quantum gravity effects from the point of view of an experimental physicist involved in several such projects. The connections between these talks and noncommutative geometry were considered to be a most stimulating aspect of the workshop. There were also talks on the nature of time and on string theory in connection with noncommutative and generalised geometries as well as talks on other approaches to quantum gravity. One of the surprising lessons from the operator theory side of noncommutative geometry is that noncommutative spaces in some sense generate their own time evolution in contrast to ordinary geometries which are in this deep sense ‘static’. Majid showed how the same feature can arise in the algebraic approach as an anomaly or obstruction to differential calculus while preserving a classical or quantum group symmetry. In addition, there were several poster sessions. Our gaol was to approach the topic broad-mindedly and we believe this resulted in a somewhat unique and stimulating workshop.

The Nature of Space and Time: An Evening of Speculation, 7th September 2006, Emmanuel College. Organisers: J Butterfield and S Majid

This was a separate evening event within the workshop. J Butterfield, a noted philosopher, chaired a public panel discussion in which five panelists: *Connes, M Heller, Majid, R Penrose and J Polkinghorne* presented and then discussed their deepest and more speculative thoughts on the nature of space and time, led by questions from the floor. We were happy to see a very wide spectrum in the over 200-strong audience ranging from undergraduate students and members of the public through to world authorities such as S Hawking. The event has led to a contract with CUP for a book of essays by the panelists along with the cosmologist A Taylor.

Trends in Noncommutative Geometry, 18 - 22 December 2006

Organisers: A Connes, MA Rieffel, SP Smith

This workshop aimed to expose some of the most exciting new developments relating noncommutative geometry to other topics in pure and applied mathematics. Topics included aperiodic patterns, the theory of foliations, fractal and infinite-dimensional geometry, number theory, algebraic geometry, among others. Emphasis was on the algebra and number theory sides as particular areas of fruitful interaction.

There were 103 participants, many from the UK and mainland Europe. The 29 talks included key lectures by Connes and P Cartier, B Zilber, J Hunton and T Stafford. Talks of Connes and Consani led the number theory section with work on motives, L-functions and a spectral interpretation of the Riemann-Weil formulae in terms of traces.

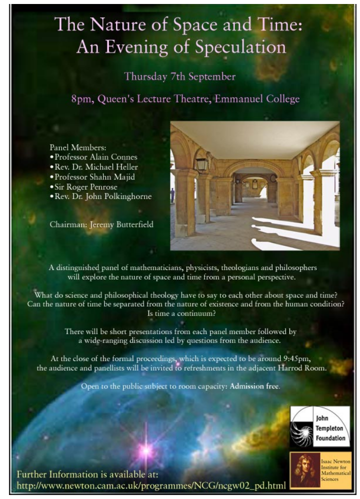

Figure 2: Publicity poster for the ‘Evening of speculation’ on space and time, produced *in house* by staff-member Josie Camus

Connes outlined an approach to the Riemann hypothesis viewed as a result about the ‘prime at infinity’ and a remarkable analogy with quantum gravity.

A very special feature of the workshop organisation was to give ‘prime time’ exposure to the work of particularly promising young researchers, and this was particularly evident in the first three days of the workshop. This did result in a certain number of 1/2-hour talks but we were also able to offer several full length and plenary slots. Other talks on the algebraic side, on aperiodic tilings and on quantum groups allowed us to cover what we felt were some of the most exciting trends for the future. In addition, many participants expressed the view that they had found the workshop ‘eye-openning’ on each being exposed to a side of noncommutative geometry that they had not previously appreciated. The workshop was considered particularly successful and a positive close to the programme as a whole.

Outcome and Achievements

The programme achieved most of its stated goals. One senior north-American participant described it as the ‘most stimulating period of his academic life’ as a result of which he could now see how his work (in algebra) could be made relevant to a much wider community. Several of our core participants expressed similar sentiments about a considerable broadening of their understanding of noncommutative geometry. This had been a primary goal of the organisers and can be expected to be very positive for the development of the field in years to come.

Another goal of the programme was to provide international exposure and contacts to UK researchers in noncommutative geometry at all levels. The UK already has a strong international reputation in different strands of the field, but this was the first event to bring researchers together to interact with each other and with international figures for an extended period. Several collaborations have emerged here and some of them are listed below. Another side of the UK interaction was the junior member scheme as well as the Scottish and LMS funds which together allowed many students and younger postdocs (and some senior researchers) from the UK to take part, thus helping a new generation to find their footing in the field. In addition, international participants to the programme gave 30 seminars elsewhere in the UK.

At the time of writing, some 47 works were reported by participants as publications, preprints or preprints in preparation completed or arising during the period of the programme. Several of them have been archived on the internet rather than as NI preprints. Among books, Connes and M Marcolli completed a book connecting noncommutative geometry, motives and quantum field theory. Beggs and Majid began and completed three chapters of a book on noncommutative Riemannian geometry in an algebraic approach. Lykova worked on a research monograph on the homology of topological algebras. Connes, Heller, Majid, Penrose, Polkinghorne and Taylor initiated a book of speculative essays on the nature of space and time, which should come to fruition in 2007.

Turning to specific results and collaborations, some highlights were as follows. Clearly standing out was the completion of a breakthrough paper by Connes, Marcolli and A Chamsedine, which also formed the basis of Connes' lecture course. Theoretical physicists often consider the possibility of extra dimensions in spacetime beyond the 4 directly observed, but these usual Kaluza-Klein ideas imply unobserved physics due to coordinate transformations in the extra dimensions. If instead one adds noncommutative directions expressed in a simple finite-dimensional algebra one obtains a much better conceptual fit with what is observed in the standard model, namely gravity and Yang-Mills theory with nothing extra. In the recent work Connes overcame some technical problems in this approach to find for the first time exactly the right finite-dimensional algebra of 'extra dimensions' along with an essentially unique representation of it that then exactly encodes the structure of elementary particle physics as it appears today. A Dirac operator in the noncommative extra dimensions encodes the many parameters in the standard model but with 3 further constraints that could be viewed as actual predictions for particle physics. These include hitherto undreamt of relations between particle masses. One of the technical problems overcome was also overcome independently by J Barrett from the UK. Connes, Barrett and others subsequently enjoyed many discussions on this topic, leading to several ideas for further progress. Several junior and senior participants reported extremely helpful conversations with Connes for example after giving their seminar or over dinner or a beer. Connes also produced other papers during the period.

Schwarz completed a key project with Movshev using noncommutative supergeometry and homological algebra to classify supersymmetric deformations of 10D super-Yang Mills theory, of interest in string theory. With Vologodsky he also began a study of the integrality of Gopakumar-Vafa invariants and also completed a paper with X Tang. S Barannikov, crediting particularly discussions with Schwarz, completed a key work on noncommutative Batalin-Vilkovisky geometry.

E Beggs and Majid worked on a project that could ultimately explain the role of the stress-energy tensor as emerging from a semiclassical obstruction to noncommutative geometry (this could provide insight into the problem of 'dark energy'). They meanwhile completed two unrelated papers, the most important of which provided a new categorical framework for 'complex conjugation' in noncommutative geometry adequate for the first time to include quantum groups at roots of unity. S Brain and Majid developed a systematic 'noncommutative twistor theory' benefitting from discussions particularly with G Landi and SP Smith. Majid and Smith collaborated on understanding what could be called *-algebraic geometry, with some first results concerning the classical Moebius bundle.

Smith spoke extensively with several other participants and potentially initiated several collaborations. Also very accessible and very visible was R Nest who completed several papers including key ones with A Gorokhovsky and with B Tsygan on noncommutative geometry and formal deformations. More than a few participants reported very helpful discussions with Nest.

Nest also brought several postdocs or students with him and his work on the Baum-Connes conjecture for quantum groups led to many interesting discussions with those coming from the algebraic theory of quantum group principal bundles. Moscovici, Ranjipour and Kahlkahli led much active discussion on Hopf cyclic cohomology. These works were part of research that took place on the interface between quantum group methods and those from cyclic cohomology.

Other very visible participants included M Rieffel who made a breakthrough in one of his previously stalled key projects, which he credits to

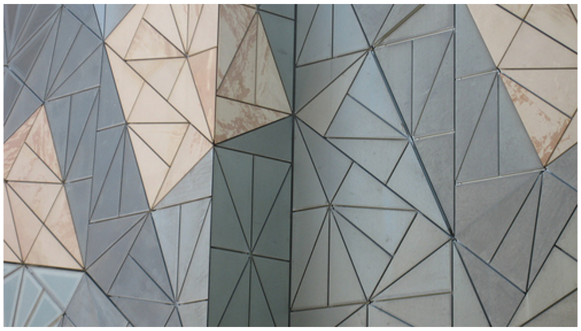

Figure 3: Aperiodic pinwheel pattern on a building in Federation Square, Melbourne. Tiled on the plane, the triangles appear in an infinite number of rotations. Photo by M Subritzky

the excellent environment. He also progressed his work on reformulating the Dirac operator on spheres and homogeneous spaces benefitting from conversations with Landi and Majid. Landi for his part completed a key work on isospectral deformation of the Dirac operator on the quantum 4-sphere. Similarly, R Plymen from the UK made significant progress on his work with Baum on the interface between noncommutative geometry and noncommutative number theory. He was also able to complete a work on K-theory and elliptic representations of SL(N). J Hunton from the UK made a breakthrough in his work with J Kellendonk on torsion in the cohomology and K-theory of aperiodic patterns. He also extended previous results that cover the K-theory of examples such as the pinwheel pattern illustrated.

On the algebraic side T Stafford completed three projects on noncommutative projective surfaces, on noncommutative blowups and, along with M van den Bergh, on noncommutative crepant resolutions and rational singularities. Van den Bergh also worked with A Bondal on the moduli space of the stability conditions on K3-surfaces. Another core participant G Wilson, from the UK, largely completed a key manuscript on Calagero-Moser spaces and significantly progressed a second on the adelic Grassmannian of an algebraic curves.

This is by no means a full account of the research done during the programme but gives an indication of some of the directions in which progress was made, with emphasis on those present for longer periods. Almost all participants expressed praise for the staff and an excellent environment at the Newton Institute.