Nonlinear Hyperbolic Waves in Phase Dynamics and Astrophysics

27 January to 11 July 2003

Report from the Organisers: CM Dafermos (Brown), PG LeFloch (CNRS), EF Toro (Trento)

Scientific Background

Quasilinear hyperbolic systems in divergence form, commonly called “hyperbolic conservation laws”, govern a broad spectrum of physical phenomena in compressible fluid dynamics, nonlinear material science, general relativity, etc. Such equations admit solutions that may exhibit various kinds of shocks and other linear and nonlinear waves (propagating phase boundaries, fluid interfaces, gravitational waves, etc.) which play a dominant role in multiple areas of physics: astrophysics, cosmology, dynamics of (solid-solid) material interfaces, multiphase (liquid-vapour) flows, combustion theory, etc. In recent years, major progress has been made in both the theoretical and numerical aspects of the field, while the number of applications has skyrocketed. Hyperbolic models arising in applications often face serious mathematical difficulties related to the occurrence of discontinuities, coordinate singularities, resonance between two or more wave speeds, elliptic regions in phase space, etc. The challenge for mathematicians is to comprehend the properties of nonlinear waves and their relationships with the dynamics of many physical phenomena. The activities during the programme were organised around five main themes described below.

Structure of the Programme

Nonclassical Shock Waves and Propagating Phase Boundaries

The first period of concentration (January– February) was centred around the concept of a kinetic relation, which determines the propagation of undercompressive waves such as those arising in phase transition dynamics. This idea was put forward by material scientists and applied mathematicians in recent years. The mathematical modeling of such problems and the theoretical and numerical aspects of the kinetic functions were both discussed. Several series of lectures were given by long-term participants, on continuum mechanical models of evolutionary structural transformations (Knowles) and on diffusive- dispersive singular limits for systems of conservation laws when both viscosity and capillarity effects play a role (Bedjaoui).

Mathematical Aspects of the Dynamics of Phase Transitions
Spitalfields Day, 10 February 2003
Organisers: CM Dafermos and PG LeFloch

This event was supported by the London Mathematical Society, and was an opportunity to review the state-of-the-art in the field and to initiate several collaborations between long-term participants. The meeting covered a broad range of topics, including the mathematical analysis of the kinetic relation (Asakura, Corli, Sablé-Tougeron), the dynamics of phase boundaries in solids (Berezovski, Knowles, Pence, Thanh) and the discretisation of continuous models (Friesecke, Shearer, Truskinovsky).

Well-Posedness Theory of Systems of Conservation Laws

Recent developments on the general theory of entropy solutions to one-dimensional hyperbolic systems of conservation laws were covered during this second concentration period. Over the two-week period 10–21 March, a total of six series of lectures was offered on:

• Fundamental concepts of the theory of shock waves and their connection with thermodynamics and continuum mechanics (Dafermos)

• Conservation laws with diffusion or relax-ation (Marcati, Serre)

• Vanishing viscosity solutions (Bressan)

• Well-posedness theory for general hyper-bolic systems (LeFloch)

• Boltzmann, Euler and Navier-Stokes equa-tions of gas dynamics (Liu)

Mathematical Theory of Hyperbolic Systems of Conservation Laws
Workshop, 24–28 March 2003
Organisers: CM Dafermos and PG LeFloch

This workshop was very successful. It attracted a large and enthusiastic participation from many of the worldwide experts in the field and provided a special opportunity to investigate current trends on the mathematical theory of shock waves and conservation laws. All contributions consisted of one-hour presentations, allowing plenty of time for questions and discussions during the breaks. Two poster sessions were also organised. The main themes treated during the week were:

• General properties of hyperbolic systems arising in continuum physics (Godunov, Brenier, Ferapontov)

• Kinetic models from mathematical biology and discrete velocity Boltzmann equations (Perthame, Tzavaras)

• Well-posedness theory of systems of cons-ervation laws, especially existence theory for general flux-functions and classical and nonclassical entropy solutions (Iguchi, LeFloch, Trivisa, Liu)

• Models of diffusive relaxation approx-imations (Marcati, Serre)

• Vanishing viscosity approximations (Bres-san)

• Euler and Navier-Stokes equations includ-ing results on critical thresholds in Eulerian dynamics (Tadmor)

• Asymptotic behaviour (Nishibata)

Multidimensional Hyperbolic Problems

In April, the programme concentrated on multi-dimensional aspects of hyperbolic conservation laws, including computational methods with applications to multiphase flows. Several series of three lectures each were given on multidimensional Riemann problems and related issues (Chen, Keyfitz) and on mixed equations of steady transonic flow dynamics (Morawetz). The equations of transonic flow are particularly challenging: change of type of the equations, degeneracy at the sonic line and the appearance of free boundary problems which describe shock interfaces. Building on her earlier results on mixed equations for transonic flow, CS Morawetz presented new existence theorems for viscous flows, including an application of Noether's theorem to the Tricomi equation. PD Lax's inspiring lectures presented an illuminating overview on the zero-dispersion limit for the Korteweg-de Vries equation.

Multiphase Fluid Flows and Multi- Dimensional Hyperbolic Problems
Workshop, 31 March–4 April 2003
Organisers: J Ballmann, PG LeFloch, R LeVeque and EF Toro

This workshop attracted a large international audience of about 80 participants from all over the world. The talks fell within three categories: first, multidimensional hyperbolic problems including mixed models for transonic flows (Morawetz, Chen), self-similar solutions to the Riemann problem in two space dimensions (Keyfitz) and the asymptotic stability of non-planar Riemann solutions (Frid); second, numerical methods, especially the evolution Galerkin schemes (Noelle) and the positive schemes for systems of conservation laws (Lax); third, a large variety of applications, especially the simulation of cavitation processes (Ballmann), the propagation of interfaces (Nikiforakis, Greenberg, Marquina), wave structure for elastic-plastic flow (Menikoff) and various problems from fluid dynamics (Toro, Drikakis, Falle, Peregrine).

Computational Methods for Complex Fluid Flows

During May, there were two main concentrated periods of activity. A short course (LeVeque, Shu, Toro) was organised over four days on numerical methods for hyperbolic conservation laws. The course was aimed primarily at young engineers and scientists within various disciplines in which modern numerical methods for hyperbolic systems are used. Support for young UK scientists was provided via the Institute's Junior Membership scheme. There were about 45 participants, 30 of whom were junior scientists and the remainder senior participants. The contents of the course included both basic theoretical aspects of conservation laws and notions on numerical methods for hyperbolic equations (Riemann solvers, high-order schemes). To illustrate the potential applicability of the methods three lectures were offered on traffic flow modelling (Shu), environmental fluid dynamics (Toro) and astrophysics (LeVeque).

Very High-Order Numerical Methods for Hyperbolic Conservation Laws
Workshop, 27–30 May 2003
Organisers: N Nikiforakis and EF Toro

This four-day workshop was about current research on approaches for constructing numerical methods of very high-order accuracy for solving hyperbolic conservation laws. The emphasis was on algorithms that allow high accuracy (third order and above) in both space and time, for one- and multi-dimensional problems and for problems including additional algebraic source terms (balance laws) or higher order derivatives (viscosity, capillarity). The talks in the workshop fell within three categories: algorithm design, numerical analysis and applications. A central objective of the workshop was to include all major current approaches for hyperbolic equations, aiming at bringing together various schools of thought. Topics studied included: finite volume methods, total variation diminishing, essentially non-oscillatory schemes and variants, adaptive finite-element methods, discontinuous Galerkin methods, multi-dimensional upwinding, adaptive meshes and applications. The workshop enjoyed the participation of about 40 scientists from several countries. All contributions consisted of one-hour informal presentations, allowing plenty of time for questions and lively discussions.

Hyperbolic Models in General Relativity
The last period of the programme (June–July) was devoted to hyperbolic aspects of Einstein's field equations of General Relativity. Several major aspects of the mathematical theory were covered. This period started with short advanced courses on the Cauchy problem for the Einstein vacuum equations (Klainerman), shock wave solutions of the Einstein equations (Temple), linear hyperbolic equations in a black hole geometry (Finster) and the Dirac sea and the principle of the fermionic projector (Finster again). S Klainerman and I Rodianski presented a new conjecture on the optimal regularity of the solutions to the Einstein equations. The programme included also some activity on the interaction of gravity with other force fields, described by Einstein's equations coupled to the Yang-Mills, Maxwell, and Dirac equations (Smoller, Finster). These coupled, highly nonlinear equations have been shown to admit solutions with surprising properties.

Recent Activity on Numerical General Relativity
Hewlett-Packard Day, 2 June 2003
Organisers: PG LeFloch, and JM Stewart

The speakers discussed recent numerical work in General Relativity, addressing central issues related to cosmology, the collision of two black holes, the generation of gravitational waves and shock waves, etc. The themes included the most recent progress in numerical relativity concerning the binary black hole inspiral and merger (Hawke, Seidel), recent advances in numerical relativistic magnetohydrodynamics motivated by applications in astrophysics (Marti) and the Geroch reduction and the issue of axisymmetry in numerical relativity (Stewart).

Hyperbolic Models in Astrophysics and Cosmology
Euroconference, 23–27 June 2003
Organisers: CM Dafermos, PG LeFloch, JA Smoller and JM Stewart

This meeting was devoted to the Einstein field equations of General Relativity and covered both the general mathematical theory and various applications in astrophysics and

cosmology. It brought to Cambridge the front-line specialists in mathematical general relativity. A large number of young researchers from the European Community participated in this event and the conference fostered exchanges between several research centres of excellence working on general relativity, including Cambridge, Golm, Oxford, Princeton, Southampton and Trieste. There were about 20 lectures of one hour each plus two poster sessions, leaving plenty of time for lively discussions between participants.
A presentation of the most recent research on existence theory for the Einstein equations was provided by Friedrich, Klainerman, Reula and Rodianski. The other themes were:
• Late-time behaviour of solutions (Rendall)
• Gravitational waves and shock waves (Griffiths, Stewart, Temple)
• Interaction of gravity with other force fields (Smoller, Linden)
• Recent progress on numerical methods in general and special relativity (Bona, Font)
• Geometric aspects of Lorentzian manifolds (Zeghib)
• The interior of charged black holes and the problem of uniqueness in general relativity (Dafermos)
• Many other topics such as black hole spacetimes and scattering problems (Komis-sarov, Lindblad, MacCallum, Mason, Melnick)

Outcome and Achievements

The programme proved very popular and brought to Cambridge a large number of short- and long-term participants, including both young and senior researchers. All participants were very favourably impressed by the working conditions offered by the Newton Institute and many would have liked to come for a longer period.
Weekly seminars, short courses, workshops and conferences provided valuable opportunities for reviewing the state of the art for physical models, techniques of mathematical analysis and numerical analysis. Many challenging problems in the field were discussed. Significant work was carried out by the participants and new bridges arose between the most recent developments in the general mathematical theory of shock waves and the areas of applications that are currently most active.
For instance, the programme helped to increase the interest of the applied mathematics community in relativistic fluid models. The lectures by H Friedrichs and JM Stewart were very helpful to clarify the most important issues in General Relativity and initiate exchanges between the long-term participants.
Progress was also made on nonclassical travelling waves with viscosity and capillarity terms and the concept of nucleation and its connection with the stability or instability of interfaces in phase transition dynamics.

One of the most unexpected and lasting outcomes of the programme was the launching of a new mathematical research journal, the *Journal of Hyperbolic Differential Equations*, devoted precisely to the topics treated during this programme. Almost all of the members of the editorial board were long-term participants, and many participants have already submitted papers written at the Institute for publication in the *Journal*.
