Nonlinear Partial Differential Equations

8 January to 6 July 2001
Report from the Organisers: H Brezis (Paris), EN Dancer (Sydney), JF Toland (Bath), NS Trudinger (Aust. Nat. Univ.)

Scientific Background

Nonlinear partial differential equations lie at the frontier of contemporary mathematics with deep theoretical challenges linked to diverse applications. This programme emphasised equations of elliptic and parabolic type, which traditionally model steady states and evolving processes. The programme was divided into four interrelated themes:

- Geometric evolution equations;
- Fully nonlinear equations;
- Variational problems with singularities;
- Reaction diffusion equations.

The first two themes were pursued mainly during the first three months of the programme and the last two themes during the last three months. The activities under each theme culminated in a workshop. The first two were integrated in a two- week workshop supported by the EC, as a Euroworkshop entitled Geometric Evolutions and Nonlinear Elliptic Equations, organised by B Andrews and NS Trudinger, from 26 March to 6 April 2001. A workshop on Variational Problems with Singularities, organised by H Brezis and F Bethuel, was held from 25 to 29 June 2001, while the final activity of the programme, covering the last theme above, was a Euroconference entitled Nonlinear Elliptic Equations and Transition Phenomena, organised by EN Dancer and H Brezis, from 2 to 6 July 2001.

Many of the leading international researchers in these areas participated, typically as long-term visitors or just for the workshops which, together with the weeks immediately preceding, were the most exciting and stimulating periods of the programme. Moreover the blend and standard of mathematics and ensuing synergies generated during those times must rank as the best ever in these areas.

Apart from the main activities a shorter workshop on Multiscaling, organised by G Friesecke (Oxford), was held from 9 to 11 April 2001; and a Spitalfields Day, entitled PDEs Today, consisting of survey talks on a range of topics in partial differential equations, was held in London on 11 May. Outside of the workshop periods, there were regular seminars, normally two or three each week. There were a total of 156 participants in the programme (48 long stay, 118 short stay), drawn from 22 countries and including 33 from the United Kingdom.

The following is a selection of particular research activities embraced by the programme classified according to theme, which should also give some indication of the huge scope of scientific terrain impacted upon by the programme research.

Outcome and Achievements

Geometric Evolution Equations and Fully Nonlinear Equations

As already indicated the focus of the first half of the programme was fully nonlinear elliptic equations and geometric evolution equations. These are areas which have seen spectacular advances in recent years, including the resolution of long-standing open problems in general relativity, Riemannian geometry, affine differential geometry and the calculus of variations, along with striking applications to areas such as optimal transportation, meteorology, image processing and crystal growth. Many of the leading researchers in these areas were assembled together for the first time. As well as hearing about the great breakthroughs in theory and applications by those who made them - such as Huisken (Tubingen) and Ilmanen (Zurich), who solved the Penrose conjecture using geometric evolutions, and Brenier (Nice), Caffarelli (Texas) and Cullen (Reading) who separately pioneered the application of the Monge-Ampère equation to transport problems and meteorology - the participants engaged in various joint research projects. Andrews combined with Guan (McMaster) and Ma (Shanghai) to apply geometric flow methods to the Christoffel-Minkowski problem. Trudinger and Wang (Canberra) completed their work on the solvability of the Plateau problem for Gauss curvature and answered another long-standing problem in affine geometry, namely that affine completeness of strongly convex hypersurfaces, of dimension larger than one, implies Euclidean completeness. Kuo (Taiwan) and Trudinger established Schauder estimates for fully nonlinear difference equations. There were also interactions with researchers in complex flows, Cao (Texas A&M) and Zhu (Princeton), and complex Monge-Ampère equations, Cegrell (Umea). One of the pioneers of the theory of fully nonlinear equations determined by elementary symmetric functions of curvatures, Ivochkina (St Petersburg), even liaised with the companion programme on Macdonald polynomials, presenting a seminar explaining the underlying arithmetic of symmetric polynomials used in nonlinear PDEs. All of the geometric flow researchers appreciated the opportunity to interact with Galaktionov (Bath), a leading researcher in (non-geometric) blow up for parabolic equations.

Variational Problems with Singularities

The theory of variational problems with singularities has become a highly important area of nonlinear mathematics in recent years, which links deep theoretical studies with applications to areas such as material science and superconductivity. The programme activities in this theme were largely concentrated in the June workshop, although there were extended visits by Sigal (Toronto) and Sternberg (Indiana). Of particular value were discussions between the Paris group, led by Brezis, and the North American group working on Ginzburg-Landau equations, and discussions between these two groups and those participants whose main interest was sharp transitions in reaction diffusion equations. The workshop itself was highly successful. Among many outstanding lectures were those of Ambrosio (Pisa) on optimal maps in mass transport problems, Sternberg and others on Ginzburg-Landau equations, including those by Sandier (Paris) and Serfaty (Cachan) on the distribution of vortices, as well as remarkable presentations by Soner, Shafrir, Jerrard, Alama, Bronsard, Sigal, Almog and Alberti. Various lectures described phenomena arising in physics, not related to type 2 superconductivity, but which now can be handled by the new machinery developed for the study of Ginzburg-Landau vortices (Aftalion and Mallick). Various aspects of minimal surfaces were covered in lectures by Chang (Beijing), Simon (Stanford) and Kirchheim (Leipzig).

Reaction Diffusion Equations

The emphasis in this theme was on sharp transition layers for semilinear elliptic equations, with activity largely concentrated in the last month of the program. There were long-term visits by Gui (Vancouver), Sigal (Toronto), Sternberg (Indiana), Wei (Hong Kong) and Yan

(Sydney), as well as numerous short-term visitors and conference attendees in the last week. A great deal of collaborative work occurred especially on sharp transitions and critical exponent problems, likely to result in many joint works over a longer timescale. In particular, there was much discussion on the use of the recent work on the DeGiorgi Conjecture in low dimensions (due to Gui-Ghousoub, Ambrosio, Cabre and others) and a better understanding of the profile of solutions obtained by Gamma convergence (due to Modica and Kohn-Sternberg among others). As already indicated, there was a great deal of fruitful discussion on the relationship between work done on the Ginzburg-Landau equations (by Brezis, Sigal, Sternberg and many others) and that on sharp transition layers for scalar equations (by Dancer, Gui, Wei, Yan and others), leading to much better understandings between the two groups.
The final Euroconference on Nonlinear Elliptic Equations and Transition Phenomema drew about 50 participants, with outstanding lectures on recent developments by Gui on the DeGiorgi problem, Lin and Chen (Taiwan) on critical exponent problems, Wei on applications to biological systems and Berestycki (Paris) on travelling waves. The presence of Nirenberg (New York) during this part of the program was greatly appreciated by all participants.

Other activity

The participation of Toland (as an organiser) and Plotnikov (Novosibirsk) broadened the scope of the programme somewhat, by introducing concern for specific rigorous PDE problems arising in mechanics. Toland interacted substantially with another organiser, Dancer, and with Buffoni (Lausanne) with whom he was preparing a monograph on the application of analytic varieties to differential equations. Plotnikov, who was a long-term participant, spoke about his work on fully nonlinear equations and contributed to all aspects of the programme.

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