Emergent Phenomena in Nonlinear Dispersive Waves

Final report

Isaac Newton Institute Satellite Meeting PDW

Northumbria University and Newcastle University, Newcastle, UK

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Background

Emergence is a powerful concept that plays a fundamental role in many areas of modern mathematics and physics. In a system composed of a very large number of elementary constituents, e.g. classical or quantum particles, even with simple laws of interaction, the observed, emergent macroscopic behaviour can be highly nontrivial. This coarse-graining from microscopic to macroscopic scales is at the heart of hydrodynamic theories for inhomogeneous, dynamic many-body systems. These theories, which include classical fluid mechanics, describe various, sometimes exotic, systems and phenomena such as the non-equilibrium dynamics of Bose-Einstein condensates and electron flows in graphene. A very different kind of hydrodynamics arises when the microscopic system's constituents are waves. The subject of dispersive hydrodynamics—the theory of multiscale phenomena in nonlinear dispersive waves—has attracted much attention recently and was the focus of the 6-month HYD2 programme "Dispersive hydrodynamics: mathematics, simulation and experiments, with applications in nonlinear waves" at the Isaac Newton Institute in July-December 2022. The 4-week satellite PDW programme "Emergent phenomena in nonlinear dispersive waves" aimed to capitalise on the newfound collaborations and scientific problems that emerged over the course of the HYD2 programme.

A well-understood, classical paradigm in quantum mechanics is wave-particle duality, which is spectacularly realised in the propagation of solitons—localised, nonlinear wave solutions of partial differential equations (PDEs) that interact elastically and pair-wise like particles [52]. Taking this as the starting point for a microscopic description of a physical system, a natural question is how to study large, ordered and disordered collections of nonlinear waves. A powerful concept in this regard is that of nonlinear wave modulations from which emerges a hydrodynamic description of waves. The corresponding Whitham modulation equations [51] provide the macroscale description of nonlinear waves analogous to the Euler equations describing the collective motion of gas or liquid particles. Succinctly stated, this is the study of the *emergent macroscopic dynamics, or hydrodynamics, of large collections of nonlinear waves* also known as *dispersive hydrodynamics* [9].

Dispersive hydrodynamics has enjoyed a resurgence of activity since around 2005, with physical experiments—notably observations of dispersive shock waves (DSWs) in superfluids [35] and nonlinear optics [30, 50]—and mathematical breakthroughs that include the DSW fitting method [22], the universality of dispersive wave breaking [19], strong asymptotics of semiclassical limits for the focusing nonlinear Schrödinger (NLS) equation [8,48], a general description of the Whitham modulation equations for Hamiltonian systems [5], multidimensional dispersive and dispersionless integrability [25,37], and the rigorous connection between the stability of periodic traveling waves and the hyperbolicity of the Whitham modulation equations [6]. Stemming from the INI 6-month HYD2 programme *Dispersive hydrodynamics* are new lines of research on generalised Riemann problems [27], Whitham shocks [45, 46], multidimensional modulation theory [1,10], and generalised breathers [34,39]. A closely related topic prominently featured in the HYD2 programme was the dynamics of soliton interactions with dispersive hydrodynamic mean flows [2, 26, 42, 43]. These research directions inspired some of the major themes of the one-month PDW satellite programme.

As noted above, the broad and fundamental topic of the stability of nonlinear dispersive waves is connected to the hyperbolicity of the Whitham modulation equations. This connection originates from the classical work of Whitham [51], Lighthill [38], and Benjamin and Feir [4] that has since been refined and applied in numerous contexts. It is now known as the theory of modulational instability [55]. One of the fundamental and practical outcomes of wave evolution in modulationally unstable systems is the generation of rogue waves—localised coherent structures of distinctly large amplitude emerging from an otherwise regular oscillatory background. Additional, modern mathematical approaches to nonlinear wave stability analysis feature the inverse scattering transform (IST) based techniques, finite gap theory, Evans functions, and Floquet-Lyapunov theory that have proven increasingly successful in the study of wave instabilities [13,33,36,40]. In connection with integrability, these techniques allow one to derive explicit stability criteria [15, 49], and provide selection criteria for the emergence of rogue waves.

The HYD2 programme witnessed the intersection of two previously disparate lines of research: soliton gases and generalised hydrodynamics (GHD). The study of random soliton ensembles in integrable PDEs, initiated by V. Zakharov in 1971 [53], has gained renewed interest, becoming a rapidly growing field of research in its own right [20, 21, 24, 31]. Independently, the theory of emergent hydrodynamics for integrable many-body systems such as the Lieb-Liniger and Heisenberg spin chain models, dubbed GHD, was pioneered in 2016 [7, 14] and is now an extremely active field of research, with many theoretical results and experimental verifications, including an accurate description and understanding of the quantum Newton cradle experiment on oscillatory behaviours of trapped cold atomic gases [44]. Key contributors to soliton gas and GHD theory and experiment were present at HYD2 programme, where intriguing mathematical and physical intersections between the two theories began to be explored, paving the way for promising cross-disciplinary exchange [11]. Major experimental developments in water waves [41, 47] and superfluids [12] provide a strong physical motivation for the corresponding, important theoretical developments related to nonlinear modulational instability, the emergence of rogue waves [28, 29], and various extensions of GHD [3, 16].

The above developments formed the scientific background of the one-month PDW satellite programme at Northumbria University, Newcastle.

Programme scope and outline

The PDW satellite programme was structured around three overarching themes: (i) Emergent hydrodynamics of nonlinear dispersive waves in integrable and nonintegrable systems, numerical simulation, and physical applications; (ii) Stability of nonlinear dispersive waves and emergent phenomena in unstable systems; (iii) Soliton gases, generalised hydrodynamics and statistical mechanics of integrable systems. The PDW programme included a five-day workshop on Frontiers of Dispersive Hydrodynamics held at Newcastle University and a one-day Women in Dispersive Hydrodynamics workshop. Two public lectures on the programme's topics were given at the Newcastle Lit&Phil Library and there were nine research seminar talks. These events are highlighted below.

Frontiers of Dispersive Hydrodynamics Workshop

Organisers: Magda Carr (Newcastle University), Vincent Caudrelier (University of Leeds), Thibault Congy (Northumbria University), Ryan Doran (Newcastle University), Mark Hoefer (University of Colorado Boulder), Karima Khusnutdinova (Loughborough University)

The 5-day workshop (29 July - 02 August) highlighted research from all three programme themes. It hosted about 80 participants and also featured a wide-ranging poster session with over 20 posters by junior and senior mathematicians. The workshop also included a fluids lab tour at Newcastle University, guided by Magda Carr.

Among the five plenary talks, 32 invited presentations, and follow-on discussions, several topical areas emerged during the 5-day workshop:

- Recent advances in modulation theory and dispersive shock waves, particularly in non-convex and discrete systems.
- Interaction of solitary waves with periodic waves in integrable systems
- Soliton gas theory (deterministic and random aspects) and its connection with generalised hydrodynamics
- IST and asymptotic Riemann-Hilbert problem techniques
- Numerical approaches to multiscale nonlinear wave phenomena
- Applications (including physical experiments and field observations) in geophysical fluid dynamics, nonlinear optics and condensed matter physics

Women in Dispersive Equations Workshop

Organisers: Antonio Moro (Northumbria University), Barbara Prinari (University at Buffalo), Matteo Sommacal (Northumbria University)

The 1-day workshop, held on 12 August at Northumbria University, highlighted the research of junior and senior female mathematicians who study nonlinear dispersive equations. The eight presenters covered a range of topics from the rigorous (fractal properties in linear PDEs with discontinuous initial data, asymptotics of random solitons, spectral theory of self-adjoint Dirac operators with periodic potentials) to applications in shallow and deep water waves. The workshop concluded with an engaging panel discussion on challenges women face in the mathematical sciences. The discussion elicited significant audience participation, with concrete experiences shared by female and male researchers alike from different countries. By highlighting diverse regional, cultural, and gender experiences, participants were left with a broader view of the realities of academic careers in the mathematical sciences.

Public talks

The PDW programme featured two public talks delivered in the historic Newcastle Lit&Phil library:

1. The Rulpidon: a fascinating sculpture for a mathematician by Prof. Sylvie Benzoni-Gavage
2. Solitons and the Ocean by Prof. Henrik Kalisch

These talks were very well attended by both the general public and programme participants.

Programme demographic statistics

The PDW programme hosted nearly 90 participants (including workshop participants) of which about 25% were female participants and 35% early career researchers. The participation in this four-week programme was global, with about 25% of the participants from the UK and the rest attending from the USA, Canada, Germany, Italy, France, Norway, Saudi Arabia, Australia, China, Brazil and Japan.

The significant participation of early career researchers was aided by NSF conference grant DMS-2339212. This grant provided travel support for 13 US-based junior participants (6 graduate students, 5 postdocs, and 2 junior faculty) to participate in workshops and/or the programme. These 13 participants account for about 1/3 of the 31 total junior participants.

Scientific highlights and outcomes

The programme provided a significant boost to research efforts, new collaborations, and the mentoring of junior researchers in the field of dispersive hydrodynamics. One of the scientific accomplishments of this programme was the co-location and interaction of experts in PDE analysis, modelling, computation, and experiments. The growth in the field of dispersive hydrodynamics is driven by bringing these perspectives and approaches together in efforts to address the challenging problems of multiscale nonlinear waves.

During the PDW programme significant advances on Whitham modulation theory and dispersive shock waves for a variety of non-integrable model equations were reported, particularly for discrete lattice systems and PDEs with higher order dispersion. The related topics of generalised Riemann problems and soliton-mean field interactions in dispersive hydrodynamics systems that emerged during the HYD2 programme have become the subject of intense discussions in the PDW programme with several related workshop talks and a collaborative effort aimed at a better understanding of the relation between the lack of integrability of a dispersive hydrodynamic system and non-convexity of the associated system of modulation equations.

A major theme of the PDW programme was the study of random nonlinear waves with the particular emphasis on the emergence aspects in integrable multiscale systems. This is a challenging problem area that witnessed some important recent breakthroughs. One of the notable outcomes of the parent HYD2 programme that has had a profound effect on the satellite PDW programme was bringing together two previously disjoint communities of researchers involved in the theory of soliton gases and the GHD. The PDW programme hosted some of the leading experts from both areas, and the complementary soliton gas and GHD perspectives on the statistical mechanics of random waves in integrable systems have become a source of many fruitful discussions. The topical discussions concerned multidimensional soliton gases (particularly for the Kadomtsev-Petviashvili equation), soliton gases in multi-component systems (such as the Manakov system), the interaction of soliton gases with dispersive hydrodynamic mean fields and mathematically rigorous approaches to incorporating randomness into the recently developed Riemann-Hilbert theory of deterministic, or regular, soliton gases. A seminar talk by S. Randoux on the physical realisation of soliton gases in different experimental platforms, including deep water waves, optical fibres and electrical transmission lines, sparked a particular interest by highlighting a number of pertinent mathematical questions with practical implications related to the breakdown of integrability on the soliton gas dynamics. This talk also highlighted the utility of the IST in the analysis of experimental time series data as well as the evolution of non-integrable systems. Despite the lack of integrability, IST provides a convenient way to analyse real-world data and motivates further study of perturbed integrable systems in the dispersive hydrodynamic context. Another significant topic of discussions at the interface of integrability and randomness was related to the emergence of dispersive hydrodynamics in random matrix models.

The overarching theme of physical applications of dispersive hydrodynamics permeated the entirety of the programme. Recent theoretical and experimental findings in water waves, nonlinear optics and condensed matter physics were the focus of several workshop presentations, and a major topic of discussions during the programme. The concrete applications considered included gravity surface and internal waves in geophysical flows, light propagation in optical fibres, matter waves in Bose-Einstein condensates and nonlinear waves in granular flows. Overall, the interdisciplinary aspects of dispersive hydrodynamics were among the main highlights of the PDW programme, where rigorous mathematical considerations were often inspired by experimental findings.

The programme witnessed many collaborations, both well-established and newly formed. Particular examples include the focused discussion/collaboration groups on the Riemann-Hilbert approach to the description of soliton gases (Bertola, Girotti, Grava, Jenkins, McLaughlin, Tovbis), semi-classical limits of integrable systems (Miller, Bilman, Lu), soliton-mean field interactions (Congy, El, Hoefer, Gavrilyuk), soliton gas-GHD connections (Bastianello, Biondini, Bonnemain, Doyon, El, Roberti, Suret), DSWs in nonconvex and discrete systems (Chandramouli, Hoefer, Kevrekidis, Sprenger), inverse scattering transform and applications (Biondini, Caudrelier, Prinari, Gkogkou, Zhang), dispersive hydrodynamics in random matrix models and multidimensional integrable systems (Benassi, Clarkson, Del'Atti, Kodama, Moro), and others.

One of the most significant outcomes of the HYD2 programme and prominently highlighted during the PDW programme was the establishment of the new Journal of Nonlinear Waves (JNW) by Cambridge University Press that is envisioned as the home for the field of nonlinear wave phenomena, broadly defined. A significant part of the JNW Editorial Board consists of HYD2 and PDW programme participants. A Special Issue on Dispersive Hydrodynamics and Applications is planned for 2025, where a collection of selected contributions from participants in both INI programmes will be published. This Special Issue will be a lasting monument to the HYD2 and PDW programmes, and a touchstone for future developments in this active field of physical applied mathematical research.

Publications and Grant Applications

A number of research articles were seeded, worked on, and completed during the programme. The preprints resulted from the PDW research collaborations so far (as of November 2024) are listed below.

1. T. Bonnemain, B. Doyon, G. Biondini, G.Roberti, and G.A. El, Two-dimensional soliton gas, arXiv:2408.05548
2. G. Biondini, G. A. El, X.-D. Luo, J. Oregero, and A. Tovbis, Breather gas fission from elliptic potentials in self-focusing media, arXiv:2407.15758
3. S. Gavrilyuk and M. Ricchiuto, A geometrical Green-Naghdi type system for dispersive-like waves in prismatic channels, arXiv:2408.08625

4. R. Jenkins and A. Tovbis, Approximation of the thermodynamic limit of finite-gap solutions of the focusing NLS hierarchy by multisoliton solutions, arXiv:2408.13700

5. M. Bertola, T. Grava and G. Orsatti, $\bar{\partial}$-problem for focusing nonlinear Schrödinger equation and soliton shielding, arXiv:2409.14825

6. D. S. Agafontsev, T. Congy, G. A. El, S. Randoux, G. Roberti, and P. Suret, Spontaneous modulational instability of elliptic periodic waves: the soliton condensate model, arXiv:2411.06922

Several grant applications arising from activity during the programme have been submitted, with a number of pending applications to be submitted in 2025. The funding bodies involved include the EPSRC, the Royal Society, ANR (France), NSF (USA) and the Simons Foundation (USA).

Illustrations

Figure 1: Participants in the Workshop "Frontiers of Dispersive Hydrodynamics", 29 July - 11 August 2024.

Figure 2: Participants of the "Women in Dispersive Equations" workshop, 12 August 2024.

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