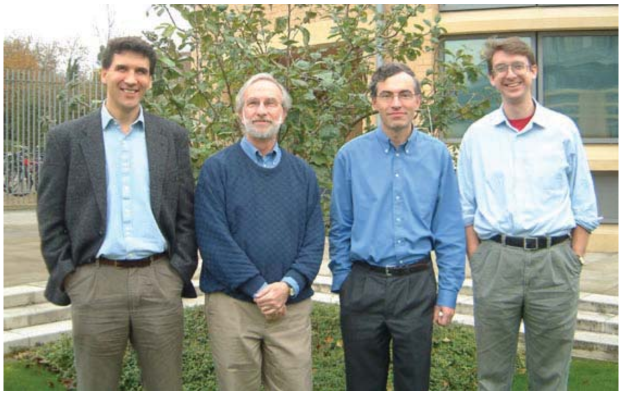
Pattern Formation in Large Domains

1 August to 23 December 2005

Report from the Organisers:
JHP Dawes (Cambridge), M Golubitsky (Houston), PC Matthews (Nottingham) and AM Rucklidge (Leeds)

S Wilkinson



AM Rucklidge, M Golubitsky, PC Matthews and JHP Dawes

Scientific Background

Pattern formation is the study of the spontaneous appearance of structure in nature and in the laboratory. Natural examples include sand ripples, geological structures such as the Giant's Causeway, cloud formations, animal coat markings, the synchronous flashing of fireflies and animal gaits. Laboratory examples span a diverse range of disciplines including fluid mechanics, granular media, chemistry (both at macroscopic and nanometre scales) and nonlinear optics.

This broad range of motivating examples is mirrored in the similarly broad range of techniques that have been brought to bear on their analysis: dynamical systems theory, group representations, asymptotic analysis for differential equations and computational methods. The field is thus well suited to the kind of multidisciplinary interaction that an Isaac Newton Institute programme encourages.

Observations of similar features in many different natural and experimental systems point to a kind of universality that should be manifest in the underlying mathematics, and to the possibility of scientific progress.

Although regular patterns with a single length-scale such as stripes and hexagons are well understood, many systems naturally generate more complex structures when given the freedom to fill an experimental or computational domain that is large compared to the pattern length-scale. Examples of these more complex structures include rotating spiral waves, spatially quasiperiodic patterns and 'spatiotemporal chaos' (of many kinds). These present challenges to our current theoretical understanding. A key aim of the programme was to bring experts in these different mathematical areas together with experimental scientists who work directly in the areas of application.

Structure of the Programme

The programme was structured around a series of five one-week workshops (four at the Newton Institute and a satellite meeting at the University of Surrey) which introduced and developed the central themes of the programme.

An additional two-day workshop on *Pattern Formation in Growing Domains* coincided with the visits of various participants concerned with pattern formation problems where system parameters naturally vary over time. Such problems occur naturally in many pattern-forming systems motivated by mathematical biology.

A one-day workshop in memory of Prof L Kramer of the University of Bayreuth was held on 11 December 2005, organised by A Newell and A Buka. Friends and family gathered to discuss Lorenz Kramer's life and work and his seminal contributions to pattern formation across many disciplines.

M Golubitsky gave a stimulating and very well attended Rothschild Visiting Professor lecture with the title *Pattern Formation in Coupled Systems*.

The organisers are grateful to the Leverhulme Trust for funding the visit of M Cross for the entire

programme, through a Leverhulme Visiting Professorship.

A total of 42 long-stay participants provided continuity in the programme. Together with the 38 short-stay participants, a purposefully lower level of activity between workshops enabled time for reflection and digestion of the workshop topics, and for new ideas and collaborations to emerge. A regular timetable of two seminars and two pub visits per week struck a balance between formal discussions and informal collaborative work.

Programme participants travelled extensively through the UK, giving seminars and forging links with many researchers in related disciplines.

Workshops

Pattern Formation

Marie Curie Training Course, 1–5 August 2005

Organiser: JHP Dawes

The programme began with a training course designed for research students and post-doctoral researchers from many scientific backgrounds to appreciate and absorb the standard mathematical techniques in the field, and to point the way to current open questions and challenges. The 71 participants at the workshop included 38 research students and 11 post-doctoral researchers, from 10 countries. Much of the support for the workshop was provided by the EC.

The variety of techniques used by theorists encompasses pure mathematical areas, such as representation theory; traditional 'applied mathematics' areas, for example multiple scales asymptotics; and, increasingly, numerical simulation. With this in mind, the course was structured around three sets of four lectures each, given by JHP Dawes (symmetric bifurcation theory), M Cross (spatially extended pattern formation) and jointly by L Tuckerman and D Barkley (numerical methods). The morning lectures were complemented by afternoons spent in less formal problem-solving sessions and computing labs using interactive software written by M Cross and D Barkley. Three late afternoon lectures covered the experimentalist's viewpoint (G Ahlers and R Ecke), and theoretical developments in the dynamics of coupled cell systems (M Golubitsky).

As well as providing a succinct summary of the state of the art, the course provoked lively interactions between longer-term participants and younger scientists, particularly during the poster sessions. The broad mixture of subject areas and international diversity of the participants, both students and longer-term participants resident during the week, made for a thoroughly enjoyable and extremely successful start to the programme.

Developments in Experimental Pattern Formation

Marie Curie Conference, 8–12 August 2005

Organisers: R Ecke, AM Rucklidge and H Swinney

The aim of this workshop, which was supported by the EC, was to bring together experimentalists working on disparate systems in physical science in order to focus on the common issues of pattern formation and provide theoretical challenges for the remainder of the programme. The workshop attracted 68 researchers from 12 countries, of whom 23 were from the UK. The workshop consisted of a series of 50-minute invited lectures from the world leaders in the field. These talks covered a wide range of problems, from the classical ones of thermal convection and the Faraday wave experiment, to situations where insights from a pattern formation perspective are only just being applied: granular flows, turbulence, electric discharges in plasma, nonlinear optics, foams, columnar joints, slime-mold aggregation and patterns in the arrangement of leaves and seeds in plants. In addition there were two poster sessions and, courtesy of S Morris, two experimental demonstrations. Many of the talks and posters provoked animated discussion, and the workshop closed with an overall summary and discussion session led by P Hohenberg; this highlighted some of the common features and theoretical challenges, including chaos in spatially extended systems, localised states and the influence of noise.

Theoretical Aspects of Pattern Formation

Satellite Meeting at the University of Surrey, 19–23 September 2005

Organisers: I Melbourne, AM Rucklidge and B Sandstede

This workshop was supported jointly by the London Mathematical Society, EPSRC and the

S Wilkinson

Participants at the workshop on 'Theory and Applications of Coupled Cell Networks'

Institute of Advanced Studies (IAS) at Surrey. It consisted of 19 lectures and 7 posters, covering a broad range of topics. A total of 51 participants, including 5 research students, attended the workshop, with 21 of them participating in the parent programme at the Isaac Newton Institute either before or afterwards.

Among the presenters were experimentalists, theoretical physicists and chemists, and applied and pure mathematicians. The five lectures on 20 September were sponsored by the IAS and were aimed at a broader audience, with the goal of surveying the state of the art in the field and outlining promising future directions.

The workshop focussed mainly on theoretical aspects of pattern formation and discussed problems arising from, amongst other topics, the formation of bacterial colonies, excitable media and reaction-diffusion systems, bifurcations of spiral waves, flows in thin liquid films, temporal intermittency and spatiotemporal chaos, coarsening and the dynamics of defects, and the formation of patterns in turbulent flows.

Theory and Applications of Coupled Cell Networks

Workshop, 26–30 September 2005

Organisers: P Ashwin, S Coombes, JHP Dawes and M Golubitsky

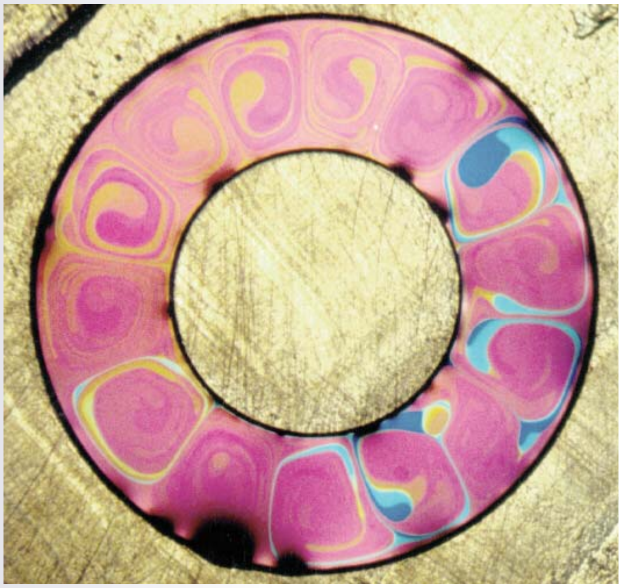
The central idea motivating this workshop was that of a dynamical system distributed over the nodes (or 'cells') of a network. A total of 86 participants from 16 countries gathered for a rather hectic workshop schedule. Thirty-three participants were younger researchers, ensuring a lively poster session.

Comparing the local dynamics of individual cells in a network leads to notions of symmetry and synchrony in the overall dynamics, linking this workshop directly to the general themes of the programme. But the network structure also brings new mathematical questions and challenges. With the first presentation of the workshop, M Golubitsky gave an overview of recent extensions of equivariant local bifurcation theory to coupled cell systems where the idea of a symmetry group is replaced by that of a symmetry groupoid; this remained a theme for the week, with later speakers presenting sometimes surprising mathematical results developed in this new framework. Closely related to this were talks discussing the interplay between network architecture and dynamics. Other developing lines of mathematical attack include the measurement of topological characteristics of pattern formation and a systematic treatment of global bifurcation theory.

Interwoven with the mathematical questions, presentations by (mathematical) biologists discussed a host of specific applications: to neuroscience, to systems biology, and to cell physiology and development. Many participants wrote at the end of the week of the enormously stimulating variety of the talks and the sense of adventure and excitement about the challenges that lie ahead, both in the mathematics and in the potential for understanding, prediction and control in the applications.

The workshop was supported by the US Office of Naval Research Global, the European detachment of the US Air Force Office of Scientific Research, and EPSRC.

S Morris



Electroconvection pattern formed in an annular smectic liquid crystal film. Colour variations correspond to local variations in film thickness

Pattern Formation in Fluid Mechanics

Workshop, 12–16 December 2005

Organisers: E Knobloch, PC Matthews and MRE Proctor

Many of the key motivating examples for the study of pattern formation arise from fluid mechanics, in particular thermal convection in a horizontal fluid layer (the Rayleigh–Bénard problem), the Taylor–Couette experiment of flow between coaxial rotating cylinders, and Faraday waves at the free surface of a vertically vibrated horizontal fluid layer. This workshop addressed unsolved questions in these classic systems, such as the imperfect bifurcation at the onset of Taylor–Couette flow and pattern selection in the Faraday experiment. Talks on the Rayleigh–Bénard problem included novel effects such as non-uniform heating, the addition of time-dependent vertical acceleration, or closely related systems such as electroconvection in liquid crystals.

Talks and poster sessions covered many other fluid mechanical phenomena, including avalanches, magnetic fluids and de-wetting. Theoretical aspects discussed included mechanisms for localised patterns, behaviour of spiral waves, defect motion in patterns and the derivation and use of amplitude equations for travelling waves.

There were 48 participants: 13 from the UK and the remainder predominantly from France, Germany, Spain, Russia and the USA. Financial support from the ICIAM99 fund is gratefully acknowledged.

Outcome and Achievements

It was clear throughout the programme that participants had purposeful and productive visits. In the short summaries below we select some particular highlights, and must apologise to participants for not being able to provide a comprehensive account of the programme's activities in the space available.

Our broad overall aim was realised, in that theoreticians and experimentalists interacted in unexpected and mutually beneficial ways that has provoked renewed interest both in understanding particular physical phenomena, and in extending the available theoretical techniques. A major outcome of the programme is thus the establishment of many new international and interdisciplinary collaborations.

Rayleigh–Bénard Convection

The problem of thermal convection in a thin horizontal fluid layer continues to motivate the field. During the programme two particular issues were addressed, and the combinations of theorists and experimentalists involved in both provide good examples of the kinds of new interactions the programme generated. Intermittent 'bursting' dynamics in convection in an inclined layer has been investigated experimentally by K Daniels. She collaborated with E Knobloch and M Golubitsky in a project to explain this theoretically as the result of a mode interaction between stripe patterns in two different orientations to the inclination. For horizontal layers in medium-size domains, much remains to be done examining the onset of time-dependent dynamics with increasing thermal forcing: G Ahlers plans further experiments, following useful discussions with E Knobloch, M Golubitsky and AM Rucklidge.

Taylor–Couette Flow

It is illustrative of the complexity of nonlinear phenomena that there are still unresolved issues in this classic fluid mechanical problem, and it continues to be the focus of both experimental and theoretical work. Ongoing discussions between AM Rucklidge, A Champneys, M Cross, T Mullin and A Cliffe synthesised a range of recent new theoretical ideas with experimental results, leading to new insights on the onset of the pattern-forming instability and the influence of the end walls in the apparatus.

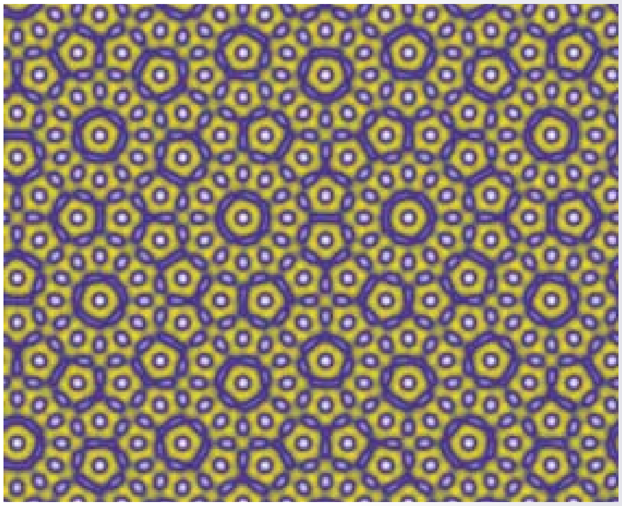
De-wetting Patterns

L Pismen and E Knobloch worked on the dynamics of liquid droplets on melting or freezing solid surfaces. Along with discussions between U Thiele and E Knobloch on drop motion in homogeneous inclined surfaces, and the strong likelihood of experimental interaction through R Richter using ferrofluids, this points to a promising new area of application for pattern formation methods. As observed above, such motivation for experimental investigation, and collaboration with theorists, would have been much harder without the programme.

Localised States

Many pattern-forming systems are observed to generate localised spots of activity. Experimental observations of this kind were discussed by several speakers at the second and fifth workshops, including S Residori and W Firth (in nonlinear optics), H-G Purwins (in dielectric gas discharge experiments) and R Richter (in a ferrofluid, near the Rosensweig instability). Numerical results showing localised patterns were presented by E Knobloch and D Lloyd.

Many authors, including participants A Champneys, G Lord and E Knobloch, have contributed to theoretical analysis of the existence of localised patterns near a subcritical bifurcation, and during discussions it was realised that there should be close connections with a rather different mechanism for localisation examined by PC Matthews, S Cox, MRE Proctor and JHP Dawes. Such a synthesis of ideas helps explain the origin of the localised states: for example combining the Ginzburg–Landau approach with the description of the set of steady solutions in terms of a spatial dynamical system. E Knobloch and JHP Dawes separately completed papers examining specific model problems in which localised states appear. New collaborations are under way involving several of the above participants and B Sandstede.



AM Rucklidge

Approximate quasipattern solution of a set of model equations designed to encourage mode interactions between waves oriented at 30° from each other. The resulting pattern has 12-fold rotational symmetry but no translational symmetry

Pattern Dynamics

One outstanding issue in pattern formation is the motion of grain boundaries and its influence on large-scale coarsening dynamics of the pattern. This topic was discussed by M Cross, A Newell, J Vinals, N Ercolani, M Paul, P Hohenberg, PC Matthews and B Sandstede, and new computational and analytical research was initiated.

More strongly driven pattern forming systems often exhibit spatiotemporal chaos in various forms. K Daniels, M Cross and M Schatz discussed the use of spatiotemporal chaos as a mechanism for passive scalar transport, and it is envisaged that further collaboration will combine experimental and large-scale numerical simulations of such dynamics.

More theoretical treatment of ideas of generic instabilities of patterns in large domains led to fruitful discussions between PC Matthews, AM Rucklidge, M Cross and I Melbourne. There are delicate theoretical questions concerning long-wavelength instabilities and the existence of spatially quasiperiodic solutions, and progress was made in identifying the difficulties in applying standard theoretical tools.

Theoretical Issues in Model Equations

The importance of simple, clearly defined, model problems in understanding and disentangling issues in nonlinear systems cannot be overstated. Two particular examples of this came to light. Firstly, the 'Nikolaevsky equation' for so-called 'soft-mode' turbulence may or may not prove to be the generic equation of its kind. This issue was addressed by PC Matthews, M Tribelsky, R Wittenberg, S Cox, MRE Proctor and JHP Dawes, and collaborative work is continuing. In a similar vein, the typical dynamics of systems with stochastic forcing is of central importance in identifying those features of experimental results that are due to uncontrollable external noise. G Lord, M Cross and AM Rucklidge collaborated on this problem.

Faraday Waves and Quasipatterns

Understanding of the complex mode interactions involved in the Faraday experiment has improved as a result of a synthesis of experimental, theoretical and numerical work by J Fineberg, J Porter, A Skeldon, M Silber, PC Matthews and AM Rucklidge. In addition to the close comparison of experimental results and theoretical predictions, there are difficult technical issues surrounding the existence of spatially quasiperiodic structures. Discussions at the satellite workshop between AM Rucklidge, I Melbourne and E Wayne opened up new possibilities for establishing the existence of spatially quasiperiodic patterns using ideas related to KAM theory.

Heteroclinic Dynamics

Heteroclinic orbits are objects in phase space that account for the appearance of intermittent temporal dynamics. They appear naturally in symmetric systems and as such have been well studied, at least in low-dimensional problems. For higher-dimensional dynamical systems many questions are unanswered. The programme generated significant new interactions that should lead to deeper understanding of these issues. H Kori and P Ashwin collaborated on heteroclinic cycles in coupled oscillator networks. M Cross and JHP Dawes worked on a model for the 'domain chaos' state in rotating Rayleigh–Bénard convection. JHP Dawes and A Pikovsky discussed general issues in a new class of PDE problems that exhibit travelling wave dynamics with interesting similarities to heteroclinic dynamics in ODEs. These new directions should provide methods and techniques that enable greater general progress to be made in the study of higher-dimensional examples.

Symmetric Dynamics and Bifurcation Theory

One of the central pillars of the theory is the exploration of dynamical systems in the presence of symmetry. A range of new bifurcation phenomena appear and there are a number of subtle issues. D Chillingworth and R Lauterbach renewed their collaborative efforts on forced symmetry-breaking and the identification of heteroclinic orbits in such bifurcation problems. A seminar by R Lauterbach on group actions where bifurcation to steady states cannot be guaranteed led to further work with PC Matthews. In addition, collaborations between A Dias, F Antoneli and PC Matthews have resulted in a paper on new generally applicable methods for the computation of the number of invariant and equivariant polynomials for symmetric Hopf bifurcation. A Dias and PC Matthews collaborated with A Rodrigues on the particular case of S_n symmetry, relevant to the coupled cell theme below.

Coupled Cell Networks and Motifs

Together with I Stewart, T Elmhirst and many other collaborators, M Golubitsky continues to develop a general theory for the dynamics of coupled cell networks, extending the idea of global system symmetries to account for local 'symmetries' in a network of similar units. For the first time, experimental verification of his theoretical results for these systems was carried out by T Mullin and N McCullen using coupled electronic oscillators. This collaboration would not have come about without the Newton Institute programme, and provides clear impetus for further theoretical study of coupled cell systems. The 'feed-forward' network that was analysed seems to have a practical application, as a narrow bandwidth filter–amplifier.

Much additional motivation for the study of coupled cell systems comes from biology: in particular genetic regulatory and protein signalling networks. Discussions between M Golubitsky, J Tyson and U Alon enabled an extremely fruitful exchange of ideas in what could turn out to be a very important area for future research. E Crooks and M Golubitsky worked on the idea of 'objective structures' and hope to relate this to the groupoid approach investigated by M Golubitsky, I Stewart and co-workers. It is clear that these conversations, in particular, represent the beginnings of exciting new directions for the field.