STUDYING THE EVOLUTION OF OPEN QUANTUM SYSTEMS VIA CONDITIONAL WIENER INTEGRALS

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Abstract

Representation of the propagator for open quantum systems in the form of an integral with respect to conditional Wiener measure is obtained. Numerical methods of computation of conditional Wiener integrals can now be applied to calculate characteristics of open quantum systems. Example of calculation of the propagator in the case of harmonic oscillator is presented. Comparison with another approach to description of time evolution of open quantum systems is discussed.

1 Introduction

Within the framework of the path integral approach Feynman and Vernon have offered [1] the model in which a quantum system is considered in interaction with its environment. The time evolution of such an open quantum system can be described with the help of a density operator $\hat{\rho}(t)$. Matrix elements of this operator in coordinate representation can be written in the form [1]:

$$\langle x|\hat{\rho}(t)|x'\rangle = \int dx_0 \int dx J(x,x',t;x_0,x'_0,t)\langle x_0|\hat{\rho}(0)|x'_0\rangle,$$

where the propagator J is written as a double path integral:

$$J = \int\limits_{(0,x_0)}^{(t,x)} Dx(\tau) \int\limits_{(0,x'_0)}^{(t,x')} Dx'(\tau) \exp\left\{\frac{i}{\hbar}(S[x(\tau)] - S[x'(\tau)])\right\} F_{in}[x(\tau), x'(\tau)]. \tag{1}$$

Here

$$S[x(\tau)] = \int\limits_0^t \left(\frac{m}{2}\dot{x}^2(\tau) - V[x(\tau)]\right) d\tau$$

is a classical action for the system under consideration in an external field with potential $V(x)$; $F_{in}[x(\tau), x'(\tau)]$ is a functional describing interaction of the system with its environment.

This model envelops vast variety of phenomena. In particular, the processes with dissipation of energy can be described in its framework. From a methodological point of view the problem of description of time evolution of open quantum systems has the most general statement because all real systems are open. Within the restrictions of the Lagrangian formulation of quantum mechanics we can also consider Feynman and Vernon's approach as the most general one.

The main problem of the approach consists of determination of the influence functional F_{in} and of calculation of the propagator (1). It is possible to find an explicit expression for the propagator only in some special cases. Generally one has to use some sort of approximations applying either the perturbation theory [1] or numerical methods.

Besides that, there is a problem of mathematical substantiation of the Feynman path integral theory. The main difficulty here is that Feynman integrals contain integrations of the type $\int \exp\{iS(q)\}dq$. The value $|\exp\{iS(q)\}|$ is always equal to the unity so that the integrand represents undamped oscillations and this integral is not determined [2]. It generates the problem of determination of a countably-additive measure in a trajectory space. There are various approaches to construction of the rigorous theory of Feynman integrals. Neither of them is universal and conventional [3].

In this paper we suggest an approach to definition of the double Feynman path integral representing propagator for an open quantum system. The purpose of this work is to obtain a formula suitable for application of the numerical methods of approximate calculation of functional integrals with respect to conditional Wiener measure. For this purpose we have used the results obtained earlier by R.Cameron.

2 Cameron's approach to definition of Feynman integral

Cameron has used the known resemblance between Feynman integrals and mathematically well defined Wiener integrals.

We shall consider the set C of all continuous on the section $[0,t]$ functions (trajectories) $x(\tau)$ satisfying the condition $x(0) = 0$. Let the section $[0,t]$ be divided on n parts $\Delta t = t/n$ which for simplicity are chosen equal. Let's introduce designations:

$$dW_\sigma^n(\vec{x},t) = \prod_{j=1}^{n}\left[\frac{1}{\sqrt{2\pi\sigma\bigtriangleup t}}\exp\left\{-\frac{(x_j - x_{j-1})^2}{2\sigma\bigtriangleup t}\right\}\right]dx_1...dx_n, \tag{2}$$

where $x_j = x(j\bigtriangleup t), j = 0,1,...,n; x_0 = 0, \vec{x} = (x_1,...,x_n)$.

Then the integral (an average value) of a functional $F[x(\tau)]$ on Wiener measure with a real positive parameter σ is defined as follows [4],[5]:

$$\int_C F[x(\tau)]dW_\sigma(x) = \lim_{n\to\infty}\int_{R^n} F(\vec{x})dW_\sigma^n(\vec{x},t). \tag{3}$$

Here $F(\vec{x}) = F[x_n(\tau)]$, $x_n(\tau)$ is a rectangular function such that $x_n(j\bigtriangleup t) = x(j\bigtriangleup t)$, $j = 0,1,...,n$. [1]

Functional integrals obtained by Feynman can be written in the form (3) if one replaces σ by $i\sigma$ in this equality. Cameron [4] used the equality (3) with an imaginary parameter σ as a definition of a Feynman integral. In order to give a meaning to such a definition it is necessary to prove existence of the limit in r.h.s. (3).

[1] In his original paper [4] Cameron has used somewhat different expressions.

Using the following property of the expression (2):

$$dW^n_{p\sigma}(\vec{x},t) = dW^n_\sigma(\vec{x}/\sqrt{p},t),$$

where p is a positive number, Cameron has written the integral (3) as follows: [2]

$$\int\limits_C F[x(\tau)]dW_\sigma(x) = \int\limits_C F[\sqrt{\sigma}x(\tau)]dW(x). \tag{4}$$

The integrals in this equality can be considered as a definition of an analytical (in a necessary area) function of a complex parameter σ, which for real positive values of σ coincides with the Wiener integral (3). When σ tends to the imaginary unity i the limit of this function gives the equality

$$\int\limits_C F[x(\tau)]dW_i(x) = \int\limits_C F[\sqrt{i}x(\tau)]dW(x) \tag{5}$$

which was considered by Cameron as a basic computation formula for a Feynman integral. This equality expresses a Feynman integral (in the Cameron's definition) through a Wiener integral.

Cameron has proven the existence of the appropriate limits in the equality (5). He has also shown that the existence of any side of the equality ensures the existence of the other one. Thus the equality (5) can be used for a definition of a Feynman integral independently from the original Cameron's definition. Namely, we shall consider the expression in r.h.s. of (5) as a definition of Feynman integral of a functional F over the set C .

This approach gives the opportunity to evaluate Feynman integrals, for a certain class of functionals, with the help of methods of calculation of functional integrals with the well defined Wiener measure.

3 Representation of the propagator in the form of an integral with conditional Wiener measure

One can put a question about the extension of Cameron's method to the case when it would be possible to define the double Feynman integral (1) by means of a double Wiener integral.

Let's consider a closed quantum system. In this case the influence functional F_{in} is equal to the unity. Then the double integral (1) can be written as a product of a Feynman integral and its complex conjugate one:

$$\int\limits_{(0,x_0)}^{(t,x)} Dx(\tau) \int\limits_{(0,x'_0)}^{(t,x')} Dx'(\tau) \exp\left\{\frac{i}{\hbar}(S[x(\tau)] - S[x'(\tau)])\right\} = \tag{6}$$

$$\int\limits_{(0,x_0)}^{(t,x)} Dx(\tau) \exp\left\{\frac{i}{\hbar}S[x(\tau)]\right\} \int\limits_{(0,x'_0)}^{(t,x')} Dx'(\tau) \exp\left\{-\frac{i}{\hbar}S[x'(\tau)]\right\}.$$

[2]The symbol σ will be omitted whenever $\sigma = 1$.

Feynman has obtained the following equality for a path integral [6]:

$$\int\limits_{(0,x_0)}^{(t,x)} Dx(\tau)\exp\left\{\frac{i}{\hbar}S[x(\tau)]\right\} = \tag{7}$$

$$\lim_{n\to\infty}\int\ldots\int\prod_{j=1}^{n}\left[\exp\frac{i}{\hbar}\left\{\frac{m}{2}\frac{(x_j-x_{j-1})^2}{\Delta t}-V(x_j)\Delta t\right\}\right]\left(\frac{m}{2\pi i\hbar\Delta t}\right)^{\frac{n}{2}}dx_1\ldots dx_{n-1}$$

Here the integration is carried out over the set $C_{[0,x_0,t,x]}$ of all continuous functions $x(\tau)$ satisfying the conditions $x(0)=x_0$, $x(t)=x$. This Feynman integral can be defined through an integral with respect to conditional Wiener measure, which differs from the considered above Wiener integral by absence of the integration on x_n in r.h.s. (3). Accordingly, instead of (2) it is necessary to use slightly different expression:

$$d\tilde{W}_\sigma^n(\vec{x},t)=\prod_{j=1}^{n}\left[\frac{1}{\sqrt{2\pi\sigma\,\Delta\,t}}\exp\left\{-\frac{(x_j-x_{j-1})^2}{2\sigma\,\Delta\,t}\right\}\right]dx_1\ldots dx_{n-1}, \tag{8}$$

where $x_j=x(j\,\Delta\,t), j=0,1,\ldots,n; x_n=x, \vec{x}=(x_0,\ldots,x_n)$. Thus the Wiener integral over the set $C_{[0,x_0,t,x]}$ is defined by the equality:

$$\int\limits_{C_{[0,x_0,t,x]}} F[x(\tau)]d\tilde{W}_\sigma(x)=\lim_{n\to\infty}\int\limits_{R^{n-1}} F(\vec{x})d\tilde{W}_\sigma^n(\vec{x},t). \tag{9}$$

It readily follows from (8) that

$$d\tilde{W}_{p\sigma}^n(\vec{x},t)=\frac{1}{\sqrt{p}}d\tilde{W}_\sigma^n(\vec{x}/\sqrt{p},t).$$

Taking into account this property, we can write the integral (9) in the form similar to (4):

$$\int\limits_{C_{[0,x_0,t,x]}} F[x(\tau)]d\tilde{W}_\sigma(x)=\frac{1}{\sqrt{\sigma}}\int\limits_{C_{\left[0,\frac{x_0}{\sqrt{\sigma}},t,\frac{x}{\sqrt{\sigma}}\right]}} F[\sqrt{\sigma}x(\tau)]d\tilde{W}(x). \tag{10}$$

The integration in the right-hand side of this equality is carried out over the set $C_{[0,\frac{x_0}{\sqrt{\sigma}},t,\frac{x}{\sqrt{\sigma}}]}$ of continuous functions $x(\tau)$ satisfying the conditions $x(0)=x_0/\sqrt{\sigma}$, $x(t)=x/\sqrt{\sigma}$. In order to exclude the parameter σ from the limits of the integration we shall make a parallel shift in the space $C_{[0,\frac{x_0}{\sqrt{\sigma}},t,\frac{x}{\sqrt{\sigma}}]}$: $x(\tau)\to y(\tau)=x(\tau)+\bar{x}(\tau)$. Here $\bar{x}(\tau)$ is a fixed function such that $\bar{x}(0)=x_0/\sqrt{\sigma}$, $\bar{x}(t)=x/\sqrt{\sigma}$. Now $x(\tau)$ belongs to the set $C_{[0,0,t,0]}$ of functions satisfying the conditions $x(0)=x(t)=0$. Under this transformation the functional integral in r.h.s. (10) is transformed with the help of the formula [5]:

$$\int\limits_{C_{\left[0,\frac{x_0}{\sqrt{\sigma}},t,\frac{x}{\sqrt{\sigma}}\right]}} F[x(\tau)]d\tilde{W}(x)= \tag{11}$$

$$\exp\left\{-\frac{1}{2}\int\limits_0^t\left(\frac{d\bar{x}}{d\tau}\right)^2d\tau\right\}\int\limits_{C_{[0,0,t,0]}} F[x(\tau)+\bar{x}(\tau)]\exp\left\{-\int\limits_0^t\frac{d\bar{x}}{d\tau}dx(\tau)\right\}d\tilde{W}(x).$$

If we take the fixed function in the form $\bar{x}(\tau) = [(x-x_0)\tau/t + x_0]/\sqrt{\sigma}$ then from (10) and (11) we obtain the formula for an integral with respect to conditional Wiener measure with a real positive parameter σ:

$$\int\limits_{C_{[0,x_0,t,x]}} F[x(\tau)]d\tilde{W}_\sigma(x) = \tag{12}$$

$$\frac{1}{\sqrt{\sigma}}\exp\left\{-\frac{(x-x_0)^2}{2\sigma t}\right\}\int\limits_{C_{[0,0,t,0]}} F[\sqrt{\sigma}x(\tau) + (x-x_0)\tau/t + x_0]d\tilde{W}(x).$$

For the purpose of application of numerical methods we shall also write this integral for the normalized conditional Wiener measure with the scale of time $[0,1]$.

One can scale the time interval using the following property of the expression (8):

$$d\tilde{W}^n_{p\sigma}(\vec{x},t) = d\tilde{W}^n_\sigma(\vec{x},pt) = \frac{1}{\sqrt{p}}d\tilde{W}^n_\sigma(\vec{x}/\sqrt{p},t). \tag{13}$$

Under this transformation it is also necessary to change the fixed function:
$\bar{x}(\tau) = [(x-x_0)\tau + x_0]/\sqrt{\sigma}$.

The normalized conditional Wiener measure satisfies

$$\int\limits_{C_{[0,0,1,0]}} d\tilde{W}^*(x) = 1 \tag{14}$$

and it is connected with the non-normalized one by the equality [5]:

$$\int\limits_{C_{[0,0,1,0]}} d\tilde{W}(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{C_{[0,0,1,0]}} d\tilde{W}^*(x). \tag{15}$$

Thus it follows from (12)-(15):

$$\int\limits_{C_{[0,x_0,t,x]}} F[x(\tau)]d\tilde{W}_\sigma(x) = \tag{16}$$

$$\frac{1}{\sqrt{2\pi\sigma t}}\exp\left\{-\frac{(x-x_0)^2}{2\sigma t}\right\}\int\limits_{C_{[0,0,1,0]}} F[\sqrt{\sigma t}x(\tau) + (x-x_0)\tau + x_0]d\tilde{W}^*(x).$$

The Feynman integral (7) corresponds to the Wiener integral in r.h.s. (16) with the parameter $\sigma = i\hbar/m$ and with the functional

$$F[x(\tau)] = \exp\left\{-\frac{it}{\hbar}\int_0^1 V[x(\tau)]d\tau\right\}. \tag{17}$$

In particular, for the potential $V(x) = 0$ from (16),(17) and (14) the known expression for a free particle follows [6]:

$$\int\limits_{C_{[0,x_0,t,x]}} Dx(\tau)\exp\left\{\frac{i}{\hbar}S[x(\tau)]\right\} = \sqrt{\frac{m}{2\pi i\hbar t}}\exp\left\{i\frac{m(x-x_0)^2}{2\hbar t}\right\}.$$

If under some conditions the equality (12) remains valid for a pure imaginary σ then it can be used for a definition of a Feynman integral on the space $C_{[0,x_0,t,x]}$ similar to the case considered above. Negative value of σ will then correspond to a definition of a complex conjugated Feynman integral. Thus let's assume by definition:

$$\int\limits_{C_{[0,x_0,t,x]}} Dx(\tau)\exp\{iS[x(\tau)]\} = \tag{18}$$

$$\frac{1}{\sqrt{i}}\exp\left\{i\frac{(x-x_0)^2}{2t}\right\}\int\limits_{C_{[0,0,t,0]}}\exp\left\{-i\int\limits_0^t V[\sqrt{i}x(\tau)+(x-x_0)\tau/t+x_0]\right\}d\tilde{W}(x)$$

and

$$\int\limits_{C_{[0,x_0,t,x]}} Dx(\tau)\exp\{-iS[x(\tau)]\} = \tag{19}$$

$$\frac{1}{\sqrt{-i}}\exp\left\{-i\frac{(x-x_0)^2}{2t}\right\}\int\limits_{C_{[0,0,t,0]}}\exp\left\{i\int\limits_0^t V[\sqrt{-i}x(\tau)+(x-x_0)\tau/t+x_0]\right\}d\tilde{W}(x).$$

The problem of validity bounds of the definitions (18) and (19) remains open.

Taking into account (18) and (19), we can write the double integral (6) in the form:

$$\int\limits_{(0,x_0)}^{(t,x)} Dx(\tau)\int\limits_{(0,x_0')}^{(t,x')} Dx'(\tau)\exp\{i(S[x(\tau)]-S[x'(\tau)])\} =$$

$$\exp\left\{i\frac{(x-x_0)^2}{2t}\right\}\exp\left\{-i\frac{(x'-x_0')^2}{2t}\right\}\times$$

$$\int\limits_{C^2_{[0,0,t,0]}}\exp\left\{-i\int\limits_0^t\Big(V[\sqrt{i}x(\tau)+(x-x_0)\tau/t+x_0\right.$$

$$\left.-V[\sqrt{-i}x'(\tau)+(x'-x_0')\tau/t+x_0']\Big)d\tau\right\}d\tilde{W}(x)d\tilde{W}(x').$$

Here $m=\hbar=1$.

In general case of an open quantum system we can define the integral (1) by the equality:

$$\int\limits_{(0,x_0)}^{(t,x)} Dx(\tau)\int\limits_{(0,x_0')}^{(t,x')} Dx'(\tau)\exp\{i(S[x(\tau)]-S[x'(\tau)])\}F_{in}[x(\tau),x'(\tau)] = \tag{20}$$

$$\exp\left\{i\frac{(x-x_0)^2}{2t}\right\}\exp\left\{-i\frac{(x'-x_0')^2}{2t}\right\}\times$$

$$\int\limits_{C^2_{[0,0,t,0]}}\Phi[\sqrt{i}x(\tau)+(x-x_0)\tau/t+x_0,\sqrt{-i}x'(\tau)+(x'-x_0')\tau/t+x_0']d\tilde{W}(x)d\tilde{W}(x'),$$

where

$$\Phi[x(\tau), x'(\tau)] = \exp\left\{-i\int\limits_0^t (V[x(\tau)] - V[x'(\tau)])\,d\tau\right\} F_{in}[x(\tau), x'(\tau)].$$

Again, the question about conditions under which the definition is valid remains open.

For practical reasons it is convenient to use integrals with normalized conditional Wiener measure. In this case from (16) one can derive the following formula:

$$\int\limits_{(0,x_0)}^{(t,x)} Dx(\tau) \int\limits_{(0,x_0')}^{(t,x')} Dx'(\tau) \exp\left\{\frac{i}{\hbar}(S[x(\tau)] - S[x'(\tau)])\right\} F_{in}[x(\tau), x'(\tau)] = \tag{21}$$

$$\frac{m}{2\pi\hbar t}\exp\left\{i\frac{m(x-x_0)^2}{2\hbar t}\right\}\exp\left\{-i\frac{m(x'-x_0')^2}{2\hbar t}\right\}\times$$

$$\int\limits_{C^2_{[0,0,1,0]}} \Phi[\sqrt{\frac{i\hbar t}{m}}x(\tau) + (x-x_0)\tau + x_0, \sqrt{-\frac{i\hbar t}{m}}x'(\tau) + (x'-x_0')\tau + x_0']d\tilde{W}^*(x)d\tilde{W}^*(x').$$

Here

$$\Phi[x(\tau), x'(\tau)] = \exp\left\{-\frac{it}{\hbar}\int\limits_0^1 (V[x(\tau)] - V[x'(\tau)])\,d\tau\right\} F_{in}[x(\tau), x'(\tau)].$$

We shall consider the equality (21) as a basic formula for calculation of the propagator for an open quantum system.

4 Example of calculation of the propagator

In order to test the formula (21) we can use the case when the propagator (1) can be found explicitly. In such a case we can compare the result of a numerical calculation of the integral in r.h.s. (21) with an exact value.

Strunz [7] has derived the propagator (1) from the equation for density operator $\hat{\rho}(t)$:

$$\frac{\partial\hat{\rho}}{\partial t} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] + \frac{1}{2\hbar}\sum_\mu\left([\hat{L}_\mu\hat{\rho},\hat{L}^\dagger_\mu] + [\hat{L}_\mu,\hat{\rho}\hat{L}^\dagger_\mu]\right). \tag{22}$$

Here $\hat{H}$ is the Hamiltonian of a quantum system, the operators $\hat{L}_\mu$ model the interaction of the system with an environment.

As it has been shown by Lindblad [8], the equation (22) is the most general form of a master-equation for Markovian open quantum systems which relate to many interesting physical problems.

Strunz has considered the environment operators of the form

$$\hat{L}_\mu = \beta_\mu\hat{x} + \gamma_\mu\hat{p}, \tag{23}$$

where β_μ and γ_μ are complex numbers. The influence functional F_{in} obtained by Strunz, in particular case $\hat{L}_\mu = \beta_\mu\hat{x}$, is of the following form:

$$F_{in}[x(\tau), x'(\tau)] = \exp\left\{-\frac{1}{2\hbar}|\beta|^2\int\limits_0^t d\tau(x(\tau) - x'(\tau))^2\right\}, \tag{24}$$

where $|\beta|^2 = \sum_\mu |\beta_\mu|^2$. For a harmonic oscillator $\left(V(x) = \frac{1}{2}\omega^2 x^2\right)$ Strunz has obtained an explicit expression for the propagator (1). In the particular case $\hat{L}_\mu = \beta_\mu \hat{x}$ his result is the following:

$$J(x, x', t; x_0, x_0', 0) = \frac{m\omega}{2\pi\hbar|\sin\omega t|} \exp\left\{\frac{i}{\hbar}S_R\right\} \exp\left\{-\frac{1}{\hbar}S_I\right\}. \tag{25}$$

Here

$$S_R = \frac{m\omega}{2\sin\omega t}\left[(x_0^2 - (x_0')^2 + x^2 - (x')^2)\cos\omega t - 2(x_0 x - x_0' x')\right],$$

$$S_I = \frac{m|\beta|^2}{8\omega\sin^2\omega t} \times$$
$$\left[((x - x')^2 + (x_0 - x_0')^2)(2\omega t - \sin 2\omega t) - 4(x - x')(x_0 - x_0')(\omega t\cos\omega t - \sin\omega t)\right].$$

For numerical evaluation of the integral in r.h.s. (21) with the influence functional from (24) one can use the formula for approximate calculation of a multiple integral with normalized conditional Wiener measure obtained in [9]:

$$\int_{C^l_{[0,0,1,0]}} F[\mathbf{x}(\tau)]d\tilde{W}^*(\mathbf{x}) \approx \frac{1}{2l}(2\pi)^{-N/2}\int_{R^N} \exp\left\{-\frac{1}{2}\sum_{k=1}^{l}\sum_{j=1}^{n_k}(u_j^{(k)})^2\right\} \times$$
$$\sum_{j=1}^{l}\int_{-1}^{1} F[U_{n_1}(\mathbf{u}^{(1)}), ..., \Sigma_j(\rho(v,t)), ..., U_{n_l}(\mathbf{u}^{(l)})]d\mathbf{u}dv. \tag{26}$$

Here l is a multiplicity of the functional integral, $\mathbf{x}(\tau) = (x_1(\tau), ..., x_l(\tau)), \mathbf{u} = (\mathbf{u}^{(1)}, ..., \mathbf{u}^{(l)})$, $\mathbf{u}^{(k)} = (u_1^{(k)}, ..., u_{n_k}^{(k)}), N = \sum_{k=1}^{l} n_k$,

$$U_{n_k}(\mathbf{u}^{(k)}) = \sqrt{2}\sum_{j=1}^{n_k} u_j^{(k)}\frac{1}{j\pi}\sin(j\pi t),$$
$$\Sigma_j(\rho(v,t)) = \sqrt{l}(\rho(v,t) - S_{n_k}(v,t)) + U_{n_k}(\mathbf{u}^{(k)}),$$
$$S_{n_k}(v,t) = 2\sum_{j=1}^{n_k}\frac{1}{j\pi}\sin(j\pi t)\cos(j\pi v)sign(v),$$
$$\rho(v,t) = \begin{cases} -t\, sign(v), & t \leqslant |v| \\ (1-t)sign(v), & t > |v| \end{cases}$$

Thus numerical evaluation of a double functional integral is reduced to calculation of the usual (Riemann) integral of multiplicity $N+1$. The accuracy of the formula (26) improves as N increases.

In Table 1 the outcomes of the numerical calculation of the real J_r and imaginary J_{im} parts of the propagator for the harmonic oscillator are given. One can compare these outcomes obtained by means of the formula (26) with the corresponding exact values $J_{ex.r}$ and $J_{ex.im}$ calculated with the help of the formula (25). The calculations have been carried out with $n_1 = n_2 = 1; \omega = |\beta|^2 = 1; x_0 = 0.3, x = 1.4, x_0' = 0.7, x' = -1; \hbar = m = 1$ for various values of time t.

Table 1.

t	J_r	$J_{ex.r}$	J_{im}	$J_{ex.im}$
0.1	-0.7962	-0.7970	-1.2312	-1.2323
0.5	-0.5415×10^{-1}	-0.5425×10^{-1}	-0.2097	-0.2100
1.0	0.3018×10^{-1}	0.3023×10^{-1}	-0.6754×10^{-1}	-0.6776×10^{-1}
1.5	0.1308×10^{-1}	0.1320×10^{-1}	-0.2587×10^{-1}	-0.2612×10^{-1}
2.0	0.216×10^{-2}	0.1426×10^{-2}	-0.598×10^{-2}	-0.6661×10^{-2}

The good agreement of the results of the numerical calculations with the exact ones even for the formula (26) with the lowest accuracy ($N = 2$) indicates the possibility of successful application of the offered approach to numerical study of time evolution of some open quantum systems. As it is seen from Table 1, the accuracy of the numerical calculations decreases as t increases. The accuracy also depends on the values of $|\beta|^2$ and ω. Such a dependence on the physical parameters stems from the features of the chosen approximation formula (26). The opportunities of employment of the offered approach can be extended owing to development of numerical methods of calculation of Wiener integrals and particularly by feasible improvement of the formula (26).

5 Comparison of the integral and differential approaches to description of time evolution of Markovian open quantum systems

The dynamics of an open quantum system can be described using Wigner disribution function instead of density operator $\hat{\rho}(t)$. A Wigner function $W(x,p,t)$ is determined by the Weil transform of a density operator [10]:

$$W(x,p,t) = \frac{1}{\pi\hbar}\int dy \langle x-y|\hat{\rho}(t)|x+y\rangle e^{2ipy/\hbar}.$$

If the Hamiltonian $\hat{H}$ in (22) is of the form $\hat{H} = \hat{H}_0 + \frac{\nu}{2}(\hat{p}\hat{x}+\hat{x}\hat{p})$ where $\hat{H}_0 = \hat{p}^2/2m + V(\hat{x})$, ν is a real number, and the environment operators are given by (23), then the corresponding equation for the Wigner function is the following [10]:

$$\frac{\partial W}{\partial t} = -\frac{p}{m}\frac{\partial W}{\partial x} + \frac{\partial V(x)}{\partial x}\frac{\partial W}{\partial p} + D_{xx}\frac{\partial^2 W}{\partial x^2} + D_{pp}\frac{\partial^2 W}{\partial p^2} + 2D_{px}\frac{\partial^2 W}{\partial p \partial x} + \quad (27)$$
$$(\lambda-\nu)\frac{\partial}{\partial x}(xW) + (\lambda+\nu)\frac{\partial}{\partial p}(pW) + \sum_{n=1}^{\infty}(-1)^n \frac{\hbar^{2n}}{2^{2n}(2n+1)!}\frac{\partial^{2n+1}V(x)}{\partial x^{2n+1}}\frac{\partial^{2n+1}W}{\partial p^{2n+1}},$$

where $D_{xx} = \frac{\hbar}{2}\sum_\mu |\gamma_\mu|^2$, $D_{pp} = \frac{\hbar}{2}\sum_\mu |\beta_\mu|^2$,and $\lambda = -\operatorname{Im}\sum_\mu \gamma_\mu^* \beta_\mu$.

This infinite order differential equation consists of a classical Fokker-Planck equation and of a quantum correction to it in the form of an infinite power series containing the small parameter $\hbar$. To resolve the equation (27) it is necessary to make a simplifying assumption that higher terms of the series are vanishing, except some special cases [10],[11]. The problem with the use of functional integrals is considered in a general form for a more broad class of potentials.

From the point of view of application of numerical methods there is a problem of stability of approximations, which is intrinsic to numerical solutions of partial differential equations but it is not actual for an integral statement of a problem. In many-dimensional case the advantage of an integral statement must be more significant since in this case the difficulties related to numerical solution of differential equations considerably increase.

6 Conclusion

Our purpose was to make accessible application of the numerical methods which development seems to be perspective to description of dynamics of open quantum systems. The approach offered in this paper requires the rigorous mathematical substantiation. We hope that such a substantiation will be obtained later and that the results of this work will be useful in practical applications.

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