On ergodic $\mathbb{Z}^d$-actions on Lie groups by automorphisms*

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1 Introduction

In response to a question raised by Halmos in his book on ergodic theory ([10], page 29) it was proved that a locally compact group admits a (bicontinuous) group automorphism acting ergodically (with respect to the Haar measure as a quasi-invariant measure) only if it is compact (see [9] for historical details and a generalisation to affine transformations; see [5] for the case of Lie groups). A substantial part of ergodic theory is now being extended to actions of $\mathbb{Z}^d$ (the group of integer d-tuples) and it is natural in this context to ask which locally compact groups admit ergodic $\mathbb{Z}^d$-actions by automorphisms. In this note we address the question for connected Lie groups, (and more generally for almost connected locally compact groups). Unlike in the case of $d = 1$, even for $\mathbb{Z}^2$ a connected Lie group admitting such an action need not be compact; e.g. the group $\mathbb{R}$ of real numbers, the automorphism group consists of multiplications by non-zero real numbers and it has subgroups isomorphic to $\mathbb{Z}^2$ which are dense, and hence act ergodically on $\mathbb{R}$. The example readily generalises to actions of higher rank abelian groups, on higher dimensional vector spaces; more generally the connected abelian Lie group $\mathbb{R}^n \times \mathbb{T}^m$ (where $\mathbb{T}^m$ denotes the m-dimensional torus) admits ergodic $\mathbb{Z}^d$-actions by automorphisms for sufficiently large d, for any $n \geq 0$ and $m \neq 1$ (see § 6 for precise results in this respect). We show here, in particular, that the general class of connected Lie groups with such actions is not much larger; we assume only existence of a dense orbit for the action, a condition which is satisfied if the action is ergodic; the condition however turns out to be equivalent to ergodicity in the present instance (see Theorem 1.1). However the abelian groups do not exhaust the class, and in fact there exist nonabelian Lie groups with ergodic $\mathbb{Z}^2$-actions on them by automorphisms (see § 6.4).

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The original question of Halmos was limited to automorphisms preserving the Haar measure. Since for $d \geq 2$ there exist noncompact groups with ergodic $\mathbb{Z}^d$-actions by automorphisms, one may ask whether some of them are measure-preserving. The answer turns out to be in the negative in the case of connected noncompact groups (see Corollary 1.3).

By a vector group we mean a Lie group which is (topologically) isomorphic to $\mathbb{R}^n$ for some n. We say that an automorphism τ of a Lie group is unimodular on a τ-invariant connected Lie subgroup S if the restriction of the derivative $\mathrm{d}\tau$ to the Lie subalgebra corresponding to S is unimodular as a linear transformation (namely, has determinant 1).

We recall that two measures are said to be equivalent if they have the same class of sets as sets of measure 0. A measure μ is said to be quasi-invariant under an action by a group H if every element of H transforms μ to a measure equivalent to μ. The Haar measure of any locally compact group is quasi-invariant under the action of the group of all bicontinuous automorphisms. An action is said to be ergodic with respect to a quasi-invariant measure if every invariant Borel set is either of measure 0 or the complement of a set of measure 0. When a continuous action on a locally compact second countable space X is ergodic with respect to a measure with full support, the orbits of almost all points are dense in X (see e.g. [10], for instance).

Theorem 1.1. *Let G be a connected Lie group. Suppose that there exists an abelian group H of continuous automorphisms of G such that the H-action on G has a dense orbit. Then the following conditions are satisfied:*

i) there exists a compact subgroup C contained in the center of G such that G/C is a vector group (in particular $[G,G]$ is contained in C);

ii) either G is abelian or $[G,G]$ is not closed, and in the latter case there exists $\tau \in H$ such that the restriction of τ to $[G,G]$ is not unimodular;

iii) the H-action preserves a σ-finite measure equivalent to Haar measure on G;

iv) the action of H is ergodic with respect to the Haar measure on G;

v) if the H-action preserves either the Haar measure on G or a finite measure equivalent to it, then G is compact.

Remark 1.2. Conclusion (i) signifies in particular that G is a two-step nilpotent Lie group and further that if it is simply connected then it is a vector group. Condition (ii) rules out more groups from admitting an action as in the Theorem. In particular, the nonabelian quotients of the Heisenberg group are excluded.

A locally compact group G is said to be almost connected if G/G^0 is compact, where G^0 is the connected component of the identity in G. Using a theorem of

Montgomery and Zippin on the structure of almost connected locally compact groups together with Theorem 1.1 we deduce the following.

Corollary 1.3. *Let G be an almost connected locally compact group and suppose that there exists an abelian group H of bicontinuous automorphisms of G whose action on G has a dense orbit. Then there exists a compact normal subgroup C of G such that the following holds:*

i) G/C is a vector group;

ii) if G is connected and finite-dimensional then C is contained in the center of G;

iii) if the action of H on G preserves either the Haar measure on G or a finite measure equivalent to it, then $C = G$ (namely G is compact).

Remark 1.4. We note that the conclusion as in assertion (ii) of the above corollary may not hold if G is not assumed to be connected and finite-dimensional; e.g. if K is a compact group then $G = K^{\mathbb{Z}^d}$ has the shift action by $\mathbb{Z}^d$, which is ergodic; choosing K to be a finite nonabelian group this shows the necessity of the connectedness condition while choosing it to be a connected nonabelian group shows the need for the finite-dimensionality condition.

Specialising to $\mathbb{Z}^d$-actions on connected Lie groups we prove the following.

Theorem 1.5. *Let G be the Lie group $\mathbb{R}^n \times \mathbb{T}^m$, where n and m are nonnegative integers, and let $d \geq 1$. Then G admits an ergodic $\mathbb{Z}^d$-action by automorphisms if and only if $m \neq 1$ and $d \geq n - [n/2] + 1$, where $[n/2]$ denotes the largest integer not exceeding $n/2$.*

In [8] it was shown that if the action of the group of all automorphisms of a connected Lie group G has a dense orbit on G then G is a nilpotent. Here we deduce Theorem 1.1 from this, via a closer study of the automorphism groups of these Lie groups. Theorem 1.1 will be proved in § 4, after various preparatory results. Corollary 1.3 is proved in § 5. Therorem 1.5 is proved in § 6 where we give also an example of a nonabelian Lie group on which there exists an ergodic $\mathbb{Z}^2$-action by automorphisms.

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2 Orbits on vector spaces

In this section we let V be a vector space over $\mathbb{R}$, of finite positive dimension; the dimension will be denoted by n. By $GL(V)$ we denote the general linear group of V. If S is a subspace of V or a Lie subgroup of $GL(V)$ then we denote by $\dim S$ the dimension of S.

Proposition 2.1. *Let H be an abelian Lie subgroup of $GL(V)$. Suppose that there exists $v \in V$ such that the H-orbit of v spans V. Then $\dim H \leq n$. Moreover, if H-orbit of V is open then $\dim H = n$, and the map $h \mapsto h(v)$, for all $h \in H$ is a homeomorphism of H onto the orbit of v.*

Proof: Let S be the stability subgroup of v under the H-action and let $h_0 \in S$. Since H is abelian h_0 then fixes $h(v)$ for all $h \in H$ and since the H-orbit of v spans V this implies that h_0 is the identity element. Thus S is the trivial subgroup. As H is a Lie subgroup this implies that $\dim H \leq \dim V = n$. Now suppose that the orbit is open. Then we also have $\dim H \geq \dim V$ and hence $\dim H = n$. Furthermore, as H is second countable, for any open orbit the map of H onto the orbit is a quotient map, and since S is the trivial subgroup in the present instance it is a homeomorphism. This proves the proposition.

For a linear transformation τ we shall denote by $\det \tau$ the determinant of τ.

Corollary 2.2. *Let H be an abelian Lie subgroup of $GL(V)$ such that the H-action on V has an open orbit. Then H contains all positive scalar transformations. The subgroup $H' = \{\tau \in H \mid |\det \tau| = 1\}$ is of dimension $n-1$ (and in particular has no open orbit on V).*

Proof: Let S be the subgroup of $GL(V)$ consisting of all positive scalar transformations. Then SH is also an abelian Lie subgroup whose action on V has an open orbit. Therefore by Proposition 2.1 we have $\dim SH = \dim V = \dim H$. Since S is a connected Lie subgroup this implies that S is contained in H. Moreover SH' is an open subgroup of H and since the latter has an open orbit in V it follows that SH' also has an open orbit in V. Hence $\dim SH' = \dim V$ and since $S \cap H'$ is trivial this implies that $\dim H' = \dim V - 1$. This proves the corollary.

Proposition 2.3. *Let H be a Lie subgroup of $GL(V)$ with an open orbit on V. Let E denote the set of points of V whose H-orbits are not open in V. Then E has 0 Lebesgue measure.*

Proof: Let $V_{\mathbb{C}} = V \otimes \mathbb{C}$, the complexification of V, and $\mathbf{GL}(V_{\mathbb{C}})$ be the group of $\mathbb{C}$-linear automorphisms of V_C. Let $\widetilde{H}$ be the Zariski-closure of H in $\mathbf{GL}(V_{\mathbb{C}})$. We realise V and $GL(V)$ as subsets of $V_{\mathbb{C}}$ and $\mathbf{GL}(V_{\mathbb{C}})$ respectively, in the usual way. Since by hypothesis H has an open orbit on V it follows that $\widetilde{H}$ has a Zariski-open orbit on $V_{\mathbb{C}}$, say Ω. Since $V_{\mathbb{C}}$ is an irreducible algebraic variety it follows that the complement of Ω in $V_{\mathbb{C}}$ is an algebraic variety of dimension at most $n-1$. Hence in particular $V - \Omega$ is a set of 0 Lebesgue measure. On the other hand for any $v \in V \cap \Omega$ the H-orbit of v in V is of the same dimension as the $\widetilde{H}$-orbit of v in $V_{\mathbb{C}}$ (see the argument in the proof of Lemma 1.22 in [15]), and hence it is open in V. Thus E is contained in $V - \Omega$ and hence has 0 Lebesgue measure.

We note also the following.

Corollary 2.4. *Let H be an abelian subgroup of $GL(V)$ such that the H-action on V has a dense orbit, and let $\widetilde{H}$ be the Zariski-closure of H in $GL(V)$. Then H is dense $\widetilde{H}$ (in the usual topology).*

Proof: Let $v \in V$ be a point whose H-orbit is dense in V. As $\widetilde{H}$ is an algebraic subgroup of $GL(V)$ all its orbits are locally closed (see [3], for instance). Since the $\widetilde{H}$-orbit of v is dense, this implies that it is also open. Hence by Proposition 2.1 the map $h \mapsto h(v)$ is a homeomorphism of $\widetilde{H}$ onto the $\widetilde{H}$-robit of v. Since the H-orbit of v is dense in V, and in particular in the $\widetilde{H}$ orbit of v, this implies that H is dense in $\widetilde{H}$.

3 Orbits on nilpotent Lie groups

Given a connected Lie group G we shall henceforth denote by Aut (G) the group of all continuous automorphisms of G (we recall that all such automorphisms are differentiable and even real analytic). We realise Aut (G) also as a group of linear transformations of the Lie algebra of G, by identifying each automorphism τ with its derivative. We note that if G is simply connected then Aut (G) is an algebraic subgroup of the group of linear transformations of the Lie algebra (see [7] for more general conditions under which this holds). A subgroup of Aut (G) will be said to be algebraic if it is an algebraic subgroup of the linear group. A subgroup is said to be almost algebraic if it is an open subgroup of an algebraic subgroup.

Proposition 3.1. *Let G be a simply connected nilpotent Lie group. Suppose that there exists an abelian Lie subgroup H of Aut (G) whose action on G has an open orbit. Then G is a vector group.*

Proof: We shall suppose that there exist a nonabelian simply connected nilpotent Lie group G and an abelian Lie subgroup H of Aut (G) having an open orbit on G, and arrive at a contradiction. Replacing H by the connected component of the identity in its Zariski closure (which is also an abelian subgroup of Aut (G)) we may assume that H is a connected almost algebraic subgroup.

Let $\mathcal{G}$ be the Lie algebra of G. Since H is an abelian connected almost algebraic subgroup of Aut (G), it can be expressed as DW, where D is a subgroup consisting of automorphisms whose action on $\mathcal{G}$ is diagonalisable over the reals (with positive eigenvalues), and W consists of automorphisms for which all eigenvalues of the action on $\mathcal{G}$ are of absolute value 1; we first decompose H as SU where S consists of semisimple elements and U consists of unipotent elements, then decompose S as DM, where D consists of elements diagonalisable over $I\!R$ and M is a compact subgroup, and put $W = MU$. We can now decompose $\mathcal{G}$ as $\oplus_{\lambda\in\Lambda}\mathcal{G}_\lambda$, where Λ is a set of characters (multiplicative homomorphisms) $\lambda : D \to I\!R^+$ (into positive reals) and $\mathcal{G}_\lambda = \{\xi \in \mathcal{G} \mid \delta(\xi) = \lambda(\delta)\xi \text{ for all } \delta \in D\}$; we assume Λ to be such that each $\mathcal{G}_\lambda$ is nonzero for all $\lambda \in \Lambda$. Then each $\mathcal{G}_\lambda$ is invariant under the H-action on $\mathcal{G}$. For each $\lambda \in \Lambda$ let W_λ denote the subgroup consisting of factors of elements of W on $\mathcal{G}_\lambda$ (when the latter is viewed as $\mathcal{G}/(\Sigma_{\mu\neq\lambda}\mathcal{G}_\mu)$). Since H has an open orbit on $\mathcal{G}$ the factor action on $\mathcal{G}_\lambda$ has an open orbit. Since $H = DW$ and the factor action of D is by scalars, by Corollary 2.2 this implies that $\dim W_\lambda = \dim \mathcal{G}_\lambda - 1$, for all $\lambda \in \Lambda$. Hence $\dim W \leq \Sigma_{\lambda\in\Lambda} \dim W_\lambda = \dim \mathcal{G} - |\Lambda|$, where $|\Lambda|$ denotes the cardinality of Λ.

Since $\mathcal{G} = \Sigma_{\lambda\in\Lambda}\mathcal{G}_\lambda$ and $\mathcal{G}$ is nonabelian, it follows that there exist $\lambda, \mu, \nu \in \Lambda$ with $\xi \in \mathcal{G}_\lambda$, $\eta \in \mathcal{G}_\mu$, $[\xi, \eta] \in \mathcal{G}_\nu$ and $[\xi, \eta] \neq 0$. Then for any $\delta \in D$ we have $\nu(\delta)[\xi, \eta] = \delta([\xi, \eta]) = [\delta(\xi), \delta(\eta)] = \lambda(\delta)\mu(\delta)[\xi, \eta]$. As $[\xi, \eta] \neq 0$ this implies that $\nu(\delta) = \lambda(\delta)\mu(\delta)$ for all $\delta \in D$. This means that any element δ of D is determined by $\{\lambda(\delta)\}_{\lambda\neq\nu}$ and hence $\dim D \leq |\Lambda| - 1$. Therefore $\dim H \leq \dim D + \dim W \leq (|\Lambda| - 1) + (\dim \mathcal{G} - |\Lambda|) = \dim \mathcal{G} - 1$. But this is a contradiction since H has an open orbit on $\mathcal{G}$. This shows that G must be abelian.

The next proposition concerns certain restrictions that are applicable to groups which are not simply connected.

Proposition 3.2. *Let G be a nonabelian connected Lie group. Suppose that G has a compact subgroup C contained in the center, such that G/C is a vector group. Suppose further that there is an abelian group H of continuous automorphisms of G whose action on G has a dense orbit. Then there exists $\tau \in H$ such that the restriction of τ to $[G, G]$ is not unimodular.*

Proof: Let $\mathcal{G}$ be the Lie algebra of G and $\widetilde{H}$ be the the connected component of the identity in the Zariski-closure of H in $GL(\mathcal{G})$ containing H, Aut (G) being realised as a group of Lie automorphisms of $\mathcal{G}$, as before. Since H is abelian $\widetilde{H}$ is also abelian. As in the proof of Proposition 3.1 we decompose $\widetilde{H}$ as DW where D is a subgroup of $GL(\mathcal{G})$ which is diagonalisable over the reals (with positive eigenvalues) and W consists of elements all whose eigenvalues on $\mathcal{G}$ are of absolute value 1. Now suppose that for all $\tau \in H$ the restriction of τ to $[G, G]$ is unimodular. Then the restriction of τ to $[\mathcal{G}, \mathcal{G}]$ is unimodular for all $\tau \in \widetilde{H}$ and in particular it holds for all $\delta \in D$. Since the action of D is diagonalisable over the reals there exists a D-invariant subspace $\mathcal{V}$ of $\mathcal{G}$ such that $\mathcal{G} = \mathcal{V} \oplus \mathcal{C}$ where $\mathcal{C}$ is the Lie ideal corresponding to C. Furthermore, we can decompose $\mathcal{V}$ as $\mathcal{V} = \oplus_{\lambda\in\Lambda} \mathcal{V}_\lambda$, where Λ is a set of characters on D with values in $I\!R^+$ and for each $\lambda \in \Lambda$, $\mathcal{V}_\lambda = \{v \in \mathcal{V} \mid \delta(v) = \lambda(\delta)v \text{ for all } \delta \in D\}$; we assume Λ to be such that $\mathcal{G}_\lambda$ is nonzero for all $\lambda \in \Lambda$.

Consider the factor action of $\widetilde{H}$ on $\mathcal{G}/\mathcal{C}$. Since the H-action on G has a dense orbit so does the H-action on G/C, and since G/C is simply connected this implies that the H-action on $\mathcal{G}/\mathcal{C}$ has a dense orbit. Hence the action of the Zariski-closure of H has a dense orbit on $\mathcal{G}/\mathcal{C}$ and as seen earlier such an orbit is open. It follows therefore that the $\widetilde{H}$-action on $\mathcal{G}/\mathcal{C}$ has an open orbit. It is easy to see that for each $\lambda \in \Lambda$, $(\mathcal{V}_\lambda + \mathcal{C})/\mathcal{C}$ is a $\widetilde{H}$-invariant subspace of $\mathcal{G}/\mathcal{C}$ and the latter is a direct sum of those subspaces. Hence the $\widetilde{H}$-action on each $(\mathcal{V}_\lambda + \mathcal{C})/\mathcal{C}$ has an open orbit. For each λ let m_λ denote the dimension of $(\mathcal{V}_\lambda + \mathcal{C})/\mathcal{C}$. Then by Corollary 2.2 the preceding conclusion implies that the subgroup of $GL((\mathcal{V}_\lambda + \mathcal{C})/\mathcal{C})$ consisting of factors of elements of W is of dimension $m_\lambda - 1$. On the other hand the subgroup of $GL(\mathcal{G}/\mathcal{C})$ consisting of factors of elements of $\widetilde{H}$ has dimension equal to the dimension of $\mathcal{G}/\mathcal{C}$, which is the same as $\Sigma_{\lambda\in\Lambda} m_\lambda$. Since $\widetilde{H} = DW$ this implies that the dimension of the group consisting of factors of D on $\mathcal{G}/\mathcal{C}$ equals the cardinality of Λ. Therefore for any $\{\alpha_\lambda\}_{\lambda\in\Lambda}$ there exists $\delta \in D$ such that $\delta(v) = \alpha_\lambda v$ for all $\lambda \in \Lambda$ and $v \in \mathcal{V}_\lambda$.

Now let $\xi_1, \ldots, \xi_n$ be a basis of $\mathcal{V}$ such that each ξ_i is contained in some $\mathcal{V}_\lambda$. Then the set $\{[\xi_i, \xi_j] \mid i, j = 1, \ldots, n\}$ spans $[\mathcal{G}, \mathcal{G}]$. In particular it contains a basis of $[\mathcal{G}, \mathcal{G}]$. This implies that there exist nonnegative integers n_λ, $\lambda \in \Lambda$, not all zero, such that the determinant of the restriction of any $\delta \in D$ to $[\mathcal{G}, \mathcal{G}]$ is given by $\prod_{\lambda\in\Lambda} \lambda(\delta)^{n_\lambda}$. Since for all $\{\alpha_\lambda\}_{\lambda\in\Lambda}$ there exists $\delta \in D$ such that $\delta(v) = \alpha_\lambda v$ for all $\lambda \in \Lambda$ and $v \in \mathcal{V}_\lambda$, this shows that the restriction of δ to $[\mathcal{G}, \mathcal{G}]$ can not be unimodular for all $\delta \in D$. This contradicts our earlier observation, and hence it follows that there exists $\tau \in H$ such that the restriction of τ to $[G, G]$ is not unimodular. This proves the proposition.

4 Proof of Theorem 1.1

In this section we complete the proof of Theorem 1.1, following the notation as in the hypothesis.

Proof of Theorem 1.1: i) It was shown in [8] that if the automorphism group of a connected Lie group G has a dense orbit on G, then G is a nilpotent Lie group. In particular the group G as in the hypothesis is nilpotent. Hence it has a unique maximal compact subgroup C and the latter is contained in the center of G. Furthermore, G/C is a simply connected nilpotent Lie group. The action of H on G factors to G/C and the factor action has a dense orbit. Recall that as G/C

is simply connected its automorphism group is an algebraic subgroup of the linear group of its Lie algebra. Let H' be the Zariski closure in $\mathrm{Aut}\,(G/C)$ of the group of automorphisms arising as factors of elements of H on G/C. Then the H'-action on G/C has a dense orbit. Since H' is an algebraic subgroup, for its action on the Lie algebra all orbits are open in their closure (see [3]) and in particular the dense orbit is open. This shows that the H'-action on G/C has an open orbit. Since H is abelian so is H' and hence by Proposition 3.1 the preceding conclusion implies that G/C is a vector space. This shows that assertion (i) holds.

ii) Suppose that G is nonabelian. Then assertion (i) as above and Proposition 3.2 imply that there exists $\tau \in H$ such that the restriction of τ to $[G, G]$ is not unimodular. Now if $[G, G]$ is closed then, being contained in the compact subgroup C, it would be compact; this is however not possible since any continuous automorphisms of a compact abelian Lie group is unimodular. This proves (ii).

iii) Now let $\mathcal{G}$ be the Lie algebra of G and $\mathcal{C}$ be the Lie subalgebra corresponding to C. Let $\Delta = \{\xi \in \mathcal{G} \mid \exp \xi = e\}$, where exp denotes the exponential map and e is the identity element in G. Then Δ is a discrete subgroup of $\mathcal{C}$ and $\mathcal{C}/\Delta$ is compact. We realise H as a subgroup of $GL(\mathcal{G})$, as earlier. Clearly Δ is invariant under the H-action on $\mathcal{G}$. We now form the semidirect product of H and Δ with respect to the action of H on Δ and denote it by Γ. The latter can be realised as a group of affine automorphisms of $\mathcal{G}$, the elements of Δ being identified with the corresponding translations. Let A be the Zariski-closure of Γ in the group of affine automorphisms of $\mathcal{G}$ (which is a real algebraic group). Clearly Δ is Zariski-dense in $\mathcal{C}$ and hence the orbits of A on $\mathcal{G}$ are precisely the inverse images of orbits of the factor action on $\mathcal{G}/\mathcal{C}$. The factor action has a dense orbit, say $\mathcal{O}'$, and since A is an algebraic subgroup the orbit is also open. Furthermore, by Proposition 2.3 the complement of $\mathcal{O}'$ in $\mathcal{G}/\mathcal{C}$ has 0 Lebesgue measure. Together with the earlier observation this implies that the A-action on $\mathcal{G}$ has an open orbit, say $\mathcal{O}$, whose complement has 0 Lebesgue measure. Therefore to prove assertion (iii) it is enough to show that the action of A on $\mathcal{O}$ admits an invariant measure equivalent to the restriction of the Lebesgue measure on $\mathcal{G}$. Let S be the stability subgroup of a point, say p, in $\mathcal{O}$ and consider the homogeneous space A/S equipped with a (locally finite) measure μ which is quasi-invariant under the action of A (on the left); such a measure exists and is unique up to equivalence of measures (see [12]). Then the map $\alpha S \mapsto \alpha(p)$ is a Borel isomorphism such that the image of μ under the map is equivalent to the restriction of the Lebesgue measure on $\mathcal{O}$. Therefore it is enough to show that A/S admits an A-invariant measure equivalent to μ. Let $\widetilde{H}$ denote the Zariski-closure of H. Then A is a semidirect product of $\widetilde{H}$ with $\mathcal{C}$. Since the action of H on $\mathcal{C}$ leaves invariant a lattice Δ it follows that the action is unimodular and therefore the action of $\widetilde{H}$ on $\mathcal{C}$ is also unimodular. Since A is the semidirect product of $\widetilde{H}$ and $\mathcal{C}$ under the action, it follows that A is a unimodular group. We note also that S does not contain any nontrivial translations. Since the quotient of A modulo the subgroup consisting of translations is abelian, this implies that S is abelian. In particular S is unimodular. As A is also unimodular this implies that A/S admits an A-invariant measure (see [14], Ch. III). This proves (iii).

iv) We follow the notation as in the proof of (iii) above. To prove assertion (iv) it suffices to prove that the Γ-action on $\mathcal{O}$ is ergodic, with respect to the restriction of the Lebesgue measure on $\mathcal{G}$. Realising $\mathcal{O}$ as A/S via the map as above, which is a homeomorphism, we see that the Γ-action on A/S has a dense orbit. By duality this implies that the S-action on A/Γ has a dense orbit. It is well-known that for flows on homogeneous spaces with finite invariant measure the action of a subgroup has a dense orbit if and only if it is ergodic (see [4], Theorem 6.1). Therefore the S-action on A/Γ is ergodic and hence by duality the Γ-action on $\mathcal{O}$ is ergodic. This proves (iv).

v) In view of (i) it is enough to prove assertion (v) only for vector groups. Now let $G = V$ be a vector group and $\mathrm{Aut}\,(G)$ be realised as $GL(V)$. Let $\widetilde{H}$ be the Zariski-closure of H in $GL(V)$. Since H has a dense orbit on V, as seen before it follows that $\widetilde{H}$ has an open orbit on V. If the action of H preserves the Lebesgue measure then each element of H has determinant ± 1. By Corollary 2.2 these two observations imply that V can not be of positive dimension. Hence V is trivial and thus G is compact.

Next suppose that the H-action preserves a finite measure equivalent to the Lebesgue measure. For a finite measure μ on V such that the support of μ spans V, the subgroup of $GL(V)$ consisting of transformations preserving μ is compact; see [6]. Therefore H must be contained in a compact subgroup of $GL(V)$. Since by hypothesis the H-action has a dense orbit this implies that V is trivial. This completes the proof of the theorem.

5 Extensions

We next use the theorems of Montgomery and Zippin on the structure of almost connected locally compact groups and deduce Corollary 1.3. A subgroup of a locally compact group is called a characteristic subgroup if it is invariant under all bicontinuous automorphisms of G; such a subgroup is necessarily normal. By a Lie group factor of a locally compact group G we mean a factor group of the form G/S which is a Lie group, S being a closed normal subgroup of G; if furthermore S is a characteristic subgroup we shall say that G/S is a Lie group factor by a characteristic subgroup. We first note the following fact which may be considered 'standard'; a proof is included for the sake of completeness.

Lemma 5.1. *Let G be an almost connected locally compact group. Then G has a unique maximal compact normal subgroup C, and G/C is a Lie group.*

Proof: By Theorem 4.6 of [13] there exists a compact normal subgroup F of G such that G/F is a Lie group. Therefore it enough to prove the lemma for the case of Lie groups (with finitely many connected components). As before let G^0 denote the connected component of the identity in G. Dimension considerations, together with the fact that the product of two compact normal subgroups is a compact normal subgroup, show that G^0 has a unique maximal compact connected normal subgroup, say M. Because of the uniqueness property M has to be normal in G. Now clearly it is enough to prove the lemma for G/M in the place of G and hence we may assume that M is trivial. The center of G^0 is a compactly generated abelian subgroup (cf. [11], Theorem 1.2, Ch. XVI) and hence has a unique maximal compact subgroup, say C. Then C is also a normal subgroup of G and passing to quotient we may assume C also to be trivial. Now let F be any compact normal subgroup of G. Since M is assumed to be trivial F is a finite subgroup. Hence $F \cap G^0$ is a finite normal subgroup of G^0 and hence it is contained in the center of G^0. Since C is assumed to be trivial this implies that $F \cap G^0$ is trivial. Therefore the order of F is bounded by the order of G/G^0. Cardinality considerations therefore show that G has a unique maximal compact normal subgroup.

Lemma 5.2. *Let G be a connected finite dimensional locally compact group. Let Q be the smallest closed subgroup of G containing every compact totally disconnected normal subgroup of G. Then Q is a compact subgroup contained in the center of G and G/Q is a Lie group.*

Proof: Since G has a unique maximal compact normal subgroup (see Lemma 5.1) it follows that Q as in the hypothesis is compact. By Theorem 4.6 of [13] every neighbourhood of the identity in G contains a compact normal subgroup F such that G/F is a Lie group, and since G is assumed to be finite-dimensional, for all sufficiently small neighbourhoods such a subgroup is totally disconnected. This implies that G/Q is a Lie group. It remains to show that Q is contained in the center of G. Now let F be any compact totally disconnected normal subgroup of G such that G/F is a Lie group. Since G/F is a Lie group all its compact totally disconnected subgroups are finite and, as G/F is connected, they are contained in the center of G/F. Thus Q/F is contained in the center of G/F. Hence for any $g \in G$ and $q \in Q$ the commutator $gqg^{-1}q^{-1}$ is contained F. Since every neighbourhood of the identity contains such a subgroup F it follows that Q is contained in the center of G. This proves the lemma.

Proof of Corollary 1.3: Let the notation be as in the hypothesis. Let C be the unique maximal compact normal subgroup of G, as obtained in Lemma 5.1. Then the H-action on G factors to G/C and the factor action has a dense orbit. Clearly G/C has no nontrivial compact normal subgroup. Therefore by the preceding observation Theorem 1.1 implies that G/C is a vector group. This proves Assertion (i) in the Corollary. Now suppose that G is connected and finite-dimensional. Let Q be the subgroup of G as in Lemma 5.2. Since Q is a compact normal subgroup, it is contained in C, as above. Then Q is invariant under all bicontinuous automorphisms of G and hence the H-action on G factors to G/Q. The factor action also has a dense orbit and, since G/Q is a Lie group, by Theorem 1.1 C/Q is contained in the center of G/Q and G/C is a vector group. Now let F be a compact totally disconnected normal subgroup of G such that G/F is a Lie group. We note that $F \subseteq Q \subseteq C$. Now, C/F is a compact nilpotent Lie group (with finitely many connected components) and hence its automorphism group is countable. Since C/F is a normal subgroup on G/F and the latter is connected this implies that C/F is contained in the center of G/F. Since every neighbourhood of the identity in G contains a subgroup F as above, by an argument as in the proof of Lemma 5.2 this implies that C is contained in the center of G. This proves Assertion (ii). Assertion (iii) follows immediately from the corresponding assertion in Theorem 1.1, applied to G/C. This completes the proof of the corollary.

6 Ergodic $\mathbb{Z}^d$-actions

In this section we prove Theorem 1.5, characterising the class of connected abelian Lie groups admitting ergodic $\mathbb{Z}^d$-actions by automorphisms and also give an example of a nonabelian Lie group with an ergodic $\mathbb{Z}^2$-action on it by automorphisms.

Proposition 6.1. *The Lie group $\mathbb{R}^n \times \mathbb{T}^m$, where m and n are nonnegative integers and $m \neq 1$, admits an ergodic $\mathbb{Z}^d$-action for $d \geq n - [n/2] + 1$, where $[n/2]$ denotes the largest integer not exceeding $n/2$.*

Proof: Let $k = [n/2]$ and $l = n - 2k$, namely 0 or 1 according to whether n is even or odd respectively. Let $p = k + l$; we note that $n - [n/2] = k + l = p$. Let H be the group $(\mathbb{C}^*)^k \times (\mathbb{R}^*)^l$. We write $\mathbb{R}^n$ as $\mathbb{C}^k \times \mathbb{R}^l$. Then on $\mathbb{R}^n$ we get an essentially transitive action of H, by scalar multiplication by the entries in the respective coordinates. We note that H is topologically isomorphic to $\mathbb{R}^p \times C$, where C is a compact subgroup with two connected components, and hence it has dense cyclic subgroup. It follows that H has a dense subgroup isomorphic to $\mathbb{Z}^{p+1}$. Thus we get an ergodic $\mathbb{Z}^{p+1}$-action on $\mathbb{R}^n$. This proves the proposition in the case when $m = 0$. Now let $m \geq 2$. Let $d_0, d_1, \ldots, d_p \in H$ be $p + 1$ elements generating a dense subgroup and $A_0, A_1, \ldots, A_p$ be the automorphisms of $\mathbb{R}^n = \mathbb{C}^k \times \mathbb{R}^l$ corresponding to them (as above). Let T be a mixing automorphism of $\mathbb{T}^m$. Let S_0 be the automorphism $A_0 \times T$ of $\mathbb{R}^n \times \mathbb{T}^m$ (the cartesian product of

A_0 and T), and for $i = 1, \dots, p$ let S_i be the automorphism $A_i \times I$, where I is the identity automorphism of $\mathbb{T}^m$. We show that the $\mathbb{Z}^{p+1}$-action generated by the commuting automorphisms $S_0, S_1, \dots, S_p$ is ergodic. We note that the H-action on $\mathbb{R}^n$ is essentially transitive and hence it is enough to show that the $\mathbb{Z}^{p+1}$ action on $H \times \mathbb{T}^m$ generated by $T_0 \times T$ and $T_i \times I$, $i = 1, \dots, p$, where T_i, $i = 0, \dots, p$, denote the translations of H by d_i, is ergodic. Since $\{d_0, d_1, \dots, d_p\}$ generate a dense subgroup of H, which is isomorphic to $\mathbb{R}^p \times C$ as above, the subgroup Λ generated by $\{d_1, \dots, d_p\}$ is discrete and cocompact in H. Therefore the space of orbits under the $\mathbb{Z}^p$-action on $H \times \mathbb{T}^m$ generated by $T_1 \times I, \dots, T_p \times I$ can be realised canonically as $(H/\Lambda) \times \mathbb{T}^m$. To show that the $\mathbb{Z}^{p+1}$-action on $H \times \mathbb{T}^m$ is ergodic it enough to show that the factor action of $T_0 \times T$ on this quotient is ergodic. Since $\{d_0, \dots, d_p\}$ generates a dense subgroup of H it follows that the translation T_0 of H/Λ by d_0 is ergodic. Since T is mixing it follows that the cartesian product $T_0 \times T$ as above is ergodic.

Proposition 6.2. *Suppose there exists a group H of continuous automorphisms of $V = \mathbb{R}^n$ such that H is isomorphic to $\mathbb{Z}^d$ and the H-action has a dense orbit on V. Then $d \geq n - [n/2] + 1$.*

Proof: Let $v \in V$ be such that the H-orbit of v is dense in V. Let $\widetilde{H}$ be the Zariski-closure of H in $GL(V)$. Then as seen earlier the $\widetilde{H}$-orbit of v is open on V. Hence by Proposition 2.1 $\widetilde{H}$ is n-dimensional. Therefore the stability subgroup of v, say S, is discrete and, being algebraic, it is a finite subgroup. Since the $\widetilde{H}$-orbit of v is open in V (and hence locally compact) and the H-orbit is dense in it, it follows that HS is dense in $\widetilde{H}$. Since $\widetilde{H}$ is an abelian real algebraic group it has a unique maximal compact subgroup, say C. Furthermore, as $\widetilde{H}$ has an open orbit on V it follows that $\dim C \leq [n/2]$; this may be seen from a decomposition as in the proof of Proposition 2.1. Let W denote the connected component of the identity in $\widetilde{H}/C$. Then W is a vector group, and $\dim W = \dim \widetilde{H} - \dim C \geq n - [n/2]$. We note that S is contained in C and hence HC is dense in $\widetilde{H}$. As H is isomorphic to $\mathbb{Z}^d$ this implies that W contains a dense subgroup isomorphic to $\mathbb{Z}^d$. Therefore $d \geq \dim W + 1 \geq n - [n/2] + 1$.

Proposition 6.3. *Let $G = \mathbb{R}^n \times \mathbb{T}^1$ for some $n \geq 0$ and let H be an abelian group of continuous automorphisms of G. Then the H-action on G has no dense orbit.*

Proof: Suppose that the H-action has a dense orbit. Let C be the maximal compact subgroup of G, namely $\mathbb{T}^1$ as in the expression for G (it is determined uniquely by the expression unlike the other factor). Let $\mathcal{G}$ be the Lie algebra of G and let $\widetilde{H}$ be the Zariski-closure of H in $GL(\mathcal{G})$ (automorphisms of G being identified with automorphisms of $\mathcal{G}$ as before). Let $\mathcal{C}$ be the Lie subalgebra of $\mathcal{G}$ corresponding to C. The group $\mathbb{T}^1$ has only one nontrivial automorphism, namely $\theta \mapsto \theta^{-1}$ for all $\theta \in \mathbb{T}^1$. It follows that for $\tau \in H$, realised as an automorphism of $\mathcal{G}$, the restriction to $\mathcal{C}$ is $\pm I$, where I is the identity automorphism of $\mathcal{C}$. Since $\widetilde{H}$ is the Zariski-closure of H it follows that $\tau(\xi) = \pm\xi$ for all $\xi \in \mathcal{C}$ and $\tau \in \widetilde{H}$. Since the H-action on G has a dense orbit and G/C is a vector group, the factor action of H on $\mathcal{G}/\mathcal{C}$ has a dense orbit. Hence the factor action of $\widetilde{H}$ on $\mathcal{G}/\mathcal{C}$ has a dense orbit. Since $\widetilde{H}$ is an algebraic group, as seen earlier, the dense orbit is open in $\mathcal{G}/\mathcal{C}$. Hence by Corollary 2.2 the factors of $\widetilde{H}$ on $\mathcal{G}/\mathcal{C}$ include all positive scalar tranformations. Since $\widetilde{H}$ is an abelian group of transformations whose restrictions to $\mathcal{C}$ are $\pm I$, this implies that there exists a $\widetilde{H}$-invariant subspace, say $\mathcal{W}$, of $\mathcal{G}$ such that $\mathcal{G} = \mathcal{C} \oplus \mathcal{W}$. Since $\mathcal{G}$ is abelian $\mathcal{W}$ is in fact a Lie subalgebra and hence there exists a unique connected Lie subgroup W with $\mathcal{W}$ as the corresponding Lie subalgebra. Then we have $G = CW$ and $C \cap W$ is discrete. Since G/C is simply connected this further implies that $C \cap W$ is trivial. Now if $\{w_i\}$ is a sequence in W such that $w_i \to g \in G$ and $g = cw$ with $c \in C$ and $w \in W$, then $w_i \to w$, which shows that W is closed. But then for any $c \in C$, $Wc \cup Wc^{-1}$ is a closed set invariant under all $\tau \in H$, which contradicts the assumption that there is a dense orbit. Hence the H-action has no dense orbit.

Theorem 1.5 is immediate from Propositions 6.1, 6.2 and 6.3. We conclude with an example of a nonabelian connected Lie group with an ergodic $\mathbb{Z}^2$-action on it by automorphisms.

Example 6.4. Let $m \geq 3$ and consider an irreducible $\mathbb{Z}^2$-action on $\mathbb{T}^m$ such that the corresponding linear action on $\mathbb{R}^m$ is diagonalisable over $\mathbb{R}$ and all the eigenvalues are positive; we recall that an action on $\mathbb{T}^m$ by automorphisms is said to be irreducible if there is no proper closed subgroup of positive dimension invariant under the action; actions as above may, for instance, be found for $m = 3$ by application of Theorem 2.8 in [16]. Let $V = \mathbb{R}^m$, $\Lambda = \mathbb{Z}^m$ and Δ be the subgroup of $GL(V)$ isomorphic to $\mathbb{Z}^2$ and leaving Λ invariant, defining the action as above. Let $\{v_1, \dots, v_m\}$ be a basis of V such that each v_j is an eigenvector of all $\delta \in \Delta$. Now let $P = \mathbb{C}$, the complex plane, and let $\mathcal{G} = P \oplus V$. We equip $\mathcal{G}$ with the Lie algebra structure (over $\mathbb{R}$) defined by $[x + iy + \Sigma_{j=1}^m x_j v_j, x' + iy' + \Sigma_{j=1}^m x'_j v_j] = (xy' - x'y)v_1$, for all $x, y, x_1, \dots x_m$ and $x'_1, \dots, x'_m$ in $\mathbb{R}$. Then $\mathcal{G}$ is a 2-step nilpotent Lie algebra; $[\mathcal{G}, \mathcal{G}]$ is the one-dimensional subspace spanned by v_1 and V is the center of $\mathcal{G}$. Let $\widetilde{G}$ be the corresponding simply connected Lie group. We shall realise V also as a subgroup of $\widetilde{G}$, via the exponential map (this is essentially for notational convenience). Now let $G = \widetilde{G}/\Lambda$, where Λ is the subgroup of V as above. We shall construct an ergodic $\mathbb{Z}^2$-action on G by automorphisms of G. This would give an example of a nonabelian two-step nilpotent Lie group admitting an ergodic $\mathbb{Z}^2$-action by automorphisms.

For any $\alpha \in \mathbb{C}^*$ and $a_1, \dots, a_m \in \mathbb{R}^+$ (positive reals) let $D(\alpha, a_1, \dots, a_m)$ be the linear transformation of $\mathcal{G}$ defined by $z + \Sigma_{j=1}^m x_j v_j \mapsto \alpha z + \Sigma_{j=1}^m a_j x_j v_j$ for all $z \in \mathbb{C}^*$ and $x_1 \dots, x_m \in \mathbb{R}$; we note that $D(\alpha, a_1, \dots, a_m)$ is a Lie automorphism of $\mathcal{G}$ if and only if $|\alpha|^2 = a_1$. Now let δ and δ' be a pair of generators of Δ and let $a_1, \ldots, a_m$ and $b_1, \ldots, b_m$ be the eigenvalues of δ and δ' respectively, corresponding to the eigenvectors $v_1, \ldots, v_m$ respectively. We note that a_1 and b_1 are not commensurable with each other, since otherwise the subset $\{tv_1 + \Lambda \mid t \in \mathbb{R}\}$ of V/Λ would be pointwise fixed by a nontrivial element δ of Δ, while in fact in view of the irreducibility of the Δ-action on V/Λ for any nontrivial δ in Δ the set of fixed points is discrete. Hence the subgroup generated by a_1 and b_1 is dense in $\mathbb{R}^+$. Therefore we can choose $\alpha, \beta \in \mathbb{C}^*$ such that $|\alpha|^2 = a_1$ and $|\beta|^2 = b_1$ and the subgroup generated by α and β is dense in $\mathbb{C}^*$.

Let Φ be the subgroup generated by $D(\alpha, a_1, \ldots, a_m)$ and $D(\beta, b_1, \ldots, b_m)$. It is isomorphic to $\mathbb{Z}^2$ and consists of Lie automorphisms of $\mathcal{G}$. Thus we get a $\mathbb{Z}^2$-action on $\widetilde{G}$, and since Λ is invariant under the action, we get a factor action on $G = \widetilde{G}/\Lambda$, by automorphisms. We shall show that this action is ergodic. To that end it is enough to prove that the action on $\widetilde{G}$ by the group of affine automorphisms generated by automorphisms from the Φ-action and translations by elements of Λ (as Λ is contained in the center of $\widetilde{G}$, the left and right translations are the same). We note that the exponential map is a Borel isomorphism such that the image of the Lebesgue measure on $\mathcal{G}$ is equivalent to the Haar measure on G and it commutes with the $\mathbb{Z}^2$-actions on the spaces. Therefore it is enough to show that the action on $\mathcal{G}$ by the subgroup generated by Φ and translations by elements of Λ is ergodic with respect to the Lebesgue measure on $\mathcal{G}$.

Since the Δ-action on V is diagonalisable, with positive eigenvalues, there exists a unique vector subgroup of $GL(V)$, say W, such that Δ is a lattice in W. Let A be the subgroup of $GL(\mathcal{G})$ consisting of all Lie automorphisms of the form $D(z, x_1, \ldots, x_m)$, with $z \in \mathbb{C}^*$ and $x_1, \ldots, x_m \in \mathbb{R}^+$, whose restriction to V is an element of W. Then Φ is contained in A and, since the absolute values of the $\mathbb{C}^*$-components of elements of A are determined by the restriction to V, it follows that Φ is a lattice in A.

Now let H be the subgroup AV of affine automorphisms of $\mathcal{G}$, V being viewed as a group of translations by elements of V, as before. Then $\Gamma = \Phi\Lambda$ is a lattice in H. We note that the complement of V in $\mathcal{G}$ is a single H-orbit, since the factors of H on $\mathcal{G}/V$ include scalar multiplications by all nonzero complex numbers. Let $p \in P \subset \mathcal{G}$, $p \neq 0$, and let $S = \{h \in H \mid hp = p\}$ be the stability subgroup of p; we note that S contained in A. We consider H/S equipped with a measure quasi-invariant with respect to the H-action (such measures exist and any two of them are equivalent; see [12]). Then the map $hS \mapsto hp$, for all $h \in H$ is a Borel isomorphism of H/S onto the orbit of p in $\mathcal{G}$, such that sets of measure 0 correspond to sets of measure 0. Since the complement of the orbit has zero measure it is now enough to show that the Γ action on H/S is ergodic. By a well-known duality principle this holds provided the S-action on H/Γ is ergodic.

Recall that H is a semidirect product of the abelian subgroups A and V. In view of the irreducibility of the Δ-action on V/Λ no nontrivial δ has a fixed point in V, and this implies that every element of V is of the form $\delta w \delta^{-1} w^{-1}$ (in multiplicative notation) and in particular we have $V \subseteq [H, H]$; since H/V is abelian we get $[H, H] = V$. Also, it can be verified that H is a direct product of the circle group with a simply connected solvable Lie group for which eigenvalues of the adjoint actions are all real (and positive). Therefore by Green's criterion (see [2], Ch. VII; see also [4] for more general results) the S-action on H/Γ is ergodic if the S-action on $H/V\Gamma$ is ergodic. Since S contained in A this would hold if the S-action on A/Φ is ergodic. Now consider the A-action on P, which is an invariant subspace. The A-orbit of p is $P - \{0\}$. Since α and β generate a dense subgroup of $\mathbb{C}^*$ it follows that the Φ-orbit of p is dense in $P - \{0\}$. Since S is also the stability subgroup of p under the A-action on P, this implies that ΦS is dense in A. Since A is an abelian group and Φ is a lattice in A, this shows that the S-action on A/Φ is ergodic. This completes the proof of the assertion that the action of Φ on G is ergodic.

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