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# Cartan-decomposition subgroups of SU(2, n)

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**Abstract.** We give explicit, practical conditions that determine whether or not a closed, connected subgroup $H$ of $G = \mathrm{SU}(2, n)$ has the property that there exists a compact subset $C$ of $G$ with $CHC = G$. To do this, we fix a Cartan decomposition $G = KA^+K$ of $G$, and then carry out an approximate calculation of $(KHK) \cap A^+$ for each closed, connected subgroup $H$ of $G$. This generalizes the work of H. Oh and D. Witte for $G = \mathrm{SO}(2, n)$.

## 1 Introduction

**Definition 1.1.** [14, Defn. 1.2] Let $H$ be a closed subgroup of a connected, simple, linear, real Lie group $G$. We say that $H$ is a *Cartan-decomposition subgroup* of $G$ if

- $H$ is connected, and
- there is a compact subset $C$ of $G$, such that $CHC = G$.

(Note that $C$ is only assumed to be a sub*set* of $G$; it need not be a sub*group*.)

**Example 1.2.** The Cartan decomposition $G = KAK$ shows that the maximal split torus $A$ is a Cartan-decomposition subgroup of $G$.

It is known that $G = KNK$ [9, Thm. 5.1], so the maximal unipotent subgroup $N$ is also a Cartan-decomposition subgroup.

If $\mathbb{R}\text{-rank}\, G = 0$ (that is, if $G$ is compact), then every (closed, connected) subgroup of $G$ is a Cartan-decomposition subgroup.

If $\mathbb{R}\text{-rank}\, G = 1$, then it not difficult to see that every (closed, connected) noncompact subgroup of $G$ is a Cartan-decomposition subgroup (cf. [5, Lem. 3.2]).

It is more difficult to characterize the Cartan-decomposition subgroups when $\mathbb{R}\text{-rank}\, G = 2$, but H. Oh and D. Witte [14] studied two examples in detail. Namely, they described all the Cartan-decomposition subgroups of $\mathrm{SL}(3, \mathbb{R})$ and of $\mathrm{SO}(2, n)$, and they also explicitly described the closed, connected subgroups that are *not* Cartan-decomposition subgroups. Here, we obtain similar results for $\mathrm{SU}(2, n)$. Unfortunately, the results are rather complicated to state.

**Notation 1.3.** Let $G = \mathrm{SU}(2, n)$ and fix an Iwasawa decomposition $G = KAN$ and a corresponding Cartan decomposition $G = KA^+K$, where $A^+$ is the (closed) positive Weyl chamber of $A$ in which the roots occurring in the Lie algebra of $N$ are positive. Thus, $K$ is a maximal compact subgroup, $A$ is the identity component of a maximal split torus, and $N$ is a maximal unipotent subgroup.

To simplify, let us restrict our attention here to subgroups of $N$.

**Theorem 1.4.** (cf. 3.4) *Let $G = \mathrm{SU}(2,n)$ and let $H$ be a closed, connected subgroup of $N$. Then $H$ is a Cartan-decomposition subgroup of $G$ if and only if*

1. *$H$ satisfies at least one of the eight conditions in Proposition* 4.1*; and*
2. *$H$ satisfies at least one of the five conditions in Proposition* 5.1.

**Theorem 1.5.** *Let $G = \mathrm{SU}(2,n)$ and let $H$ be a closed, connected, nontrivial subgroup of $N$. Then $H$ is* **not** *a Cartan-decomposition subgroup of $G$ if and only if $H$ belongs to one of the eleven types of subgroups explicitly described in Theorem* 6.1.

For subgroups $H$ that are not contained in $N$, there is no loss of generality in assuming that $H \subset AN$ (see 7.1), and that $H$ satisfies the additional technical condition of being compatible with $A$ (see 7.3). Under these assumptions, Theorem 7.4, Proposition 7.6, and Lemma 7.8, taken together, list the possibilities for $H$ and, in each case, determine whether $H$ is a Cartan-decomposition subgroup or not.

Our results require an effective method to determine whether a subgroup is a Cartan-decomposition subgroup or not. This is provided by the Cartan projection.

**Definition 1.6.** (Cartan projection) For each element $g$ of $G$, the Cartan decomposition $G = KA^+K$ implies that there is an element $a$ of $A^+$ with $g \in KaK$. In fact, the element $a$ is unique, so there is a well-defined function

$$\mu\colon G \to A^+ \text{ given by } g \in K\,\mu(g)\,K.$$

The function $\mu$ is continuous and proper (that is, the inverse image of any compact set is compact). Some properties of the Cartan projection are discussed in [1] and [7].

We have $\mu(H) = A^+$ if and only if $KHK = G$. This immediately implies that if $\mu(H) = A^+$, then $H$ is a Cartan-decomposition subgroup. Y. Benoist and T. Kobayashi proved the deeper statement that, in the general case, $H$ is a Cartan-decomposition subgroup if and only if $\mu(H)$ comes within a bounded distance of every point in $A^+$.

**Notation 1.7.** For subsets $U$ and $V$ of $A^+$, we write $U \approx V$ if there is a compact subset $C$ of $A$, such that $U \subset VC$ and $V \subset UC$. This is an equivalence relation.

**Theorem 1.8.** (Benoist [1, Prop. 5.1], Kobayashi [8, Thm. 1.1]) *A closed, connected subgroup $H$ of $G$ is a Cartan-decomposition subgroup if and only if $\mu(H) \approx A^+$.*

**Remark 1.9.** We may consider $\mathrm{SO}(2,n)$ to be the subgroup of $\mathrm{SU}(2,n)$ consisting of the real matrices. Then, because $A \subset \mathrm{SO}(2,n)$, we see that $\mathrm{SO}(2,n)$ is a Cartan-decomposition subgroup of $\mathrm{SU}(2,n)$. More generally, a subgroup of $\mathrm{SO}(2,n)$ is a Cartan-decomposition subgroup of $\mathrm{SO}(2,n)$ if and only if it is a Cartan-decomposition subgroup of $\mathrm{SU}(2,n)$. (For example, this follows from the fact that the Cartan projection for $\mathrm{SO}(2,n)$ is the restriction of the Cartan projection for $\mathrm{SU}(2,n)$.) Thus, our results generalize those theorems of H. Oh and D. Witte [14] that are directed toward $\mathrm{SO}(2,n)$.

**Remark 1.10.** One may define a partial order $\ll$ on the set of closed, connected subgroups of $G$ by

$$H_1 \prec H_2 \text{ if there is a compact subset } C \text{ of } G, \text{ such that } H_1 \subset CH_2C.$$

(So $H$ is a Cartan-decomposition subgroup of $G$ if and only if $G \prec H$.) We see from [1, Prop. 5.1] that $H_1 \prec H_2$ if and only if there is a compact subset $C$ of $A$, such that $\mu(H_1) \subset \mu(H_2)C$. Thus, it is of interest to calculate $\mu(H)$, for each subgroup $H$ of $G$. Our results solve this problem: for each (closed, connected) subgroup $H$, we give an explicit subset $U$ of $A^+$, such that $\mu(H) \approx U$. For the cases where $\mu(H) \not\approx A^+$, these results are summarized in Tables 1, 2, and 3 of Section 8, and the subset $U$ is given in a standard form that makes it easy to determine whether $H_1 \prec H_2$. Thus, we determine the order structure of the relation $\prec$, and also determine precisely where each subgroup lies in this partial order.

The interest in Cartan-decomposition subgroups is largely due to the following basic observation that, to construct nicely behaved actions on homogeneous spaces, one must find subgroups that are *not* Cartan-decomposition subgroups. (See [7, §3] for some historical background on this result.)

**Proposition 1.11.** (Calabi-Markus phenomenon, cf. [10, pf. of Thm. A.1.2]) *If $H$ is a Cartan-decomposition subgroup of $G$, then no closed, noncompact subgroup of $G$ acts properly on $G/H$.*

H. Oh and D. Witte [15, 16] used this proposition as a starting point to study the existence of tessellations. (A homogeneous space $G/H$ is said to have a tessellation if there is a discrete subgroup $\Gamma$ of $G$, such that $\Gamma$ acts properly on $G/H$, and $\Gamma\backslash G/H$ is compact.) In particular, when $n$ is even, they determined exactly which homogeneous spaces $\mathrm{SO}(2,n)/H$ have a tessellation (under the assumption that $H$ is connected). These results depend not only on the characterization of Cartan-decomposition subgroups, but also on the calculation of $\mu(H)$ for each subgroup $H$, and on the maximum possible dimension of subgroups with a given image under the Cartan projection. In [4] we use some of the results of the current paper to study tessellations of homogeneous spaces of $\mathrm{SU}(2,n)$.

Here is an outline of the paper. Section 2 describes the notation we use to specify elements of $\mathrm{SU}(2,n)$. Section 3 recalls some general results on Cartan-decomposition subgroups, and defines a representation $\rho$. Section 4 determines whether $H$ contains large elements with $\|\rho(h)\|$ approximately equal to $\|h\|^2$. Similarly, Section 5 determines whether $H$ contains large elements with $\|\rho(h)\|$ approximately equal to $\|h\|$. By combining the calculations of the preceding two sections, Section 6 determines which subgroups of $N$ are Cartan-decomposition subgroups. Then Section 7 determines which other subgroups of $G$ are Cartan-decomposition subgroups. Section 8 determines the maximum possible dimension of a subgroup of $H$ with any given image under the Cartan projection.

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## 2 Explicit coordinates in $\mathrm{SU}(2,n)$

**Notation 2.1.** We realize $\mathrm{SU}(2,n)$ as isometries of the indefinite Hermitian form

$$\langle v \mid w \rangle = v_1\overline{w_{n+2}} + v_2\overline{w_{n+1}} + \sum_{i=3}^{n} v_i\overline{w_i} + v_{n+1}\overline{w_2} + v_{n+2}\overline{w_1}$$

on $\mathbb{C}^{n+2}$. The virtue of this particular realization is that we may choose $A$ to consist of the diagonal matrices in $\mathrm{SU}(2,n)$ that have nonnegative real entries, and $N$ to consist of the upper-triangular matrices in $\mathrm{SU}(2,n)$ with only 1's on the diagonal. Thus, the Lie algebra of $AN$ is

$$\mathfrak{a}+\mathfrak{n} = \left\{ \begin{pmatrix} t_1 & \phi & x & \eta & i\mathsf{x} \\ 0 & t_2 & y & i\mathsf{y} & -\overline{\eta} \\ 0 & 0 & 0 & -y^\dagger & -x^\dagger \\ 0 & 0 & 0 & -t_2 & -\overline{\phi} \\ 0 & 0 & 0 & 0 & -t_1 \end{pmatrix} \;\middle|\; \begin{array}{c} t_1, t_2 \in \mathbb{R}, \\ \phi, \eta \in \mathbb{C}, \\ x, y \in \mathbb{C}^{n-2}, \\ \mathsf{x}, \mathsf{y} \in \mathbb{R} \end{array} \right\}, \tag{2.1}$$

where $\overline{\phi}$ or $\overline{\eta}$ denotes the conjugate of a complex number $\phi$ or $\eta$, and $x^\dagger$ or $y^\dagger$ denotes the conjugate-transpose of a row vector $x$ or $y$. Note that the first two rows of any element of $\mathfrak{a}+\mathfrak{n}$ are sufficient to determine the entire matrix.

**Notation 2.2.** Because the exponential map is a diffeomorphism from $\mathfrak{n}$ to $N$, each element of $N$ has a unique representation in the form $\exp u$ with $u \in \mathfrak{n}$. Thus, each element $h$ of $N$ determines corresponding values of $\phi$, $x$, $y$, $\eta$, $\mathsf{x}$ and $\mathsf{y}$ (with $t_1 = t_2 = 0$). We write

$$\phi_h, x_h, y_h, \eta_h, \mathsf{x}_h, \mathsf{y}_h$$

for these values.

**Notation 2.3.** We let $\alpha$ and $\beta$ be the simple real roots of $\mathrm{SU}(2,n)$, defined by $\alpha(a) = a_1/a_2$ and $\beta(a) = a_2$, for an element $a$ of $A$ of the form

$$a = \mathrm{diag}(a_1, a_2, 1, 1, \ldots, 1, 1, a_2^{-1}, a_1^{-1}).$$

Thus,

- the root space $\mathfrak{u}_\alpha$ is the $\phi$-subspace in $\mathfrak{n}$,
- the root space $\mathfrak{u}_\beta$ is the $y$-subspace in $\mathfrak{n}$,
- the root space $\mathfrak{u}_{\alpha+\beta}$ is the $x$-subspace in $\mathfrak{n}$,
- the root space $\mathfrak{u}_{\alpha+2\beta}$ is the $\eta$-subspace in $\mathfrak{n}$,
- the root space $\mathfrak{u}_{2\beta}$ is the $\mathsf{y}$-subspace in $\mathfrak{n}$, and
- the root space $\mathfrak{u}_{2\alpha+2\beta}$ is the $\mathsf{x}$-subspace in $\mathfrak{n}$.

**Notation 2.4.** For a given Lie algebra $\mathfrak{h} \subset \mathfrak{n}$, we use $\mathfrak{z}$ to denote $\mathfrak{h} \cap (\mathfrak{u}_{\alpha+2\beta} + \mathfrak{u}_{2\alpha+2\beta} + \mathfrak{u}_{2\beta})$. In other words,

$$\mathfrak{z} = \{ u \in \mathfrak{h} \mid \phi_u = 0 \text{ and } x_u = y_u = 0 \}.$$

(We remark that if $\phi_u = 0$ for every $u \in \mathfrak{h}$, then $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{z}$ and $\mathfrak{z}$ is contained in the center of $\mathfrak{h}$.)

**Notation 2.5.** For $h \in \mathrm{SU}(2,n)$, define

$$\Delta(h) = \det \begin{pmatrix} h_{1,n+1} & h_{1,n+2} \\ h_{2,n+1} & h_{2,n+2} \end{pmatrix}.$$

The following results collect some straightforward calculations that will be used repeatedly throughout the paper.

**Remark 2.6.** For

$$u = \begin{pmatrix} 0 & \phi & x & \eta & i\mathsf{x} \\ 0 & 0 & y & i\mathsf{y} & -\overline{\eta} \\ 0 & 0 & 0 & -y^\dagger & -x^\dagger \\ 0 & 0 & 0 & 0 & -\overline{\phi} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{n} \qquad \text{and} \qquad h = \exp u \in N,$$

we have

$$\exp(u) = \begin{pmatrix} 1 & \phi & x + \frac{1}{2}\phi y & \eta - \frac{1}{2}xy^\dagger & -\frac{1}{2}|x|^2 - \mathrm{Re}(\phi\overline{\eta}) + \frac{1}{24}|\phi|^2|y|^2 \\ & & & + \frac{1}{2}i\phi\mathsf{y} - \frac{1}{6}\phi|y|^2 & + i\left(\mathsf{x} - \frac{1}{6}|\phi|^2\mathsf{y} + \frac{1}{3}\,\mathrm{Im}(\overline{\phi}xy^\dagger)\right) \\ 0 & 1 & y & i\mathsf{y} - \frac{1}{2}|y|^2 & -\overline{\eta} - \frac{1}{2}yx^\dagger - \frac{1}{2}i\overline{\phi}\mathsf{y} + \frac{1}{6}\overline{\phi}|y|^2 \\ 0 & 0 & \mathrm{Id} & -y^\dagger & -x^\dagger + \frac{1}{2}\overline{\phi}y^\dagger \\ 0 & 0 & 0 & 1 & -\overline{\phi} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\Delta(h) = \begin{array}{c} -|\eta|^2 + \mathsf{x}\mathsf{y} - \frac{1}{4}|x|^2|y|^2 + \frac{1}{4}|xy^\dagger|^2 - \frac{1}{6}|y|^2\,\mathrm{Re}(\eta\overline{\phi}) \\ - \frac{1}{6}\mathsf{y}\,\mathrm{Im}(xy^\dagger\overline{\phi}) + \frac{1}{12}\mathsf{y}^2|\phi|^2 - \frac{1}{144}|y|^4|\phi|^2 \\ + i\left(\frac{1}{24}\mathsf{y}|\phi|^2|y|^2 + \mathrm{Im}(xy^\dagger\overline{\eta}) + \frac{1}{2}\mathsf{x}|y|^2 + \frac{1}{2}\mathsf{y}|x|^2\right). \end{array}$$

When $\phi = 0$, these simplify to:

$$\exp(u) = \begin{pmatrix} 1 & 0 & x & \eta - \frac{1}{2}xy^\dagger & i\mathsf{x} - \frac{1}{2}|x|^2 \\ 0 & 1 & y & i\mathsf{y} - \frac{1}{2}|y|^2 & -\overline{\eta} - \frac{1}{2}yx^\dagger \\ 0 & 0 & \mathrm{Id} & -y^\dagger & -x^\dagger \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\Delta(h) = \begin{array}{l} -|\eta|^2 + \mathsf{xy} - \frac{1}{4}|x|^2|y|^2 + \frac{1}{4}|xy^\dagger|^2 \\ +\, i\left(\mathrm{Im}(xy^\dagger\overline{\eta}) + \frac{1}{2}\mathsf{x}|y|^2 + \frac{1}{2}\mathsf{y}|x|^2\right). \end{array}$$

Similarly, when $y = 0$, we have

$$\exp(u) = \begin{pmatrix} 1 & \phi & x & \eta + \frac{1}{2}i\phi\mathsf{y} & \begin{array}{c} -\frac{1}{2}|x|^2 - \mathrm{Re}(\phi\overline{\eta}) \\ +\, i\left(\mathsf{x} - \frac{1}{6}|\phi|^2\mathsf{y}\right) \end{array} \\ 0 & 1 & 0 & i\mathsf{y} & -\overline{\eta} - \frac{1}{2}i\overline{\phi}\mathsf{y} \\ 0 & 0 & \mathrm{Id} & 0 & -x^\dagger \\ 0 & 0 & 0 & 1 & -\overline{\phi} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$\Delta(h) = \left(\mathsf{xy} + \tfrac{1}{12}|\phi|^2\mathsf{y}^2 - |\eta|^2\right) + i\left(\tfrac{1}{2}|x|^2\mathsf{y}\right). \tag{2.2}$$

**Remark 2.7.** For

$$u = \begin{pmatrix} 0 & \phi & x & \eta & i\mathsf{x} \\ & 0 & y & i\mathsf{y} & -\overline{\eta} \\ & & \cdots & & \end{pmatrix} \quad \text{and} \quad \tilde{u} = \begin{pmatrix} 0 & \tilde{\phi} & \tilde{x} & \tilde{\eta} & i\tilde{\mathsf{x}} \\ & 0 & \tilde{y} & i\tilde{\mathsf{y}} & -\overline{\tilde{\eta}} \\ & & \cdots & & \end{pmatrix}, \tag{2.3}$$

we have

$$[u, \tilde{u}] = \begin{pmatrix} 0 & 0 & \phi\tilde{\mathsf{y}} - \tilde{\phi}\mathsf{y} & -x\tilde{y}^\dagger + \tilde{x}y^\dagger + i\phi\tilde{\mathsf{y}} - i\tilde{\phi}\mathsf{y} & -2i\,\mathrm{Im}(x\tilde{x}^\dagger + \phi\overline{\tilde{\eta}} - \tilde{\phi}\overline{\eta}) \\ & 0 & 0 & -2i\,\mathrm{Im}(y\tilde{y}^\dagger) & \tilde{y}x^\dagger - y\tilde{x}^\dagger + i\overline{\phi}\tilde{\mathsf{y}} - i\overline{\tilde{\phi}}\mathsf{y} \\ & & & \cdots & \end{pmatrix},$$

and

$$\left[[u, \tilde{u}], \hat{u}\right] = \begin{pmatrix} 0 & 0 & 0 & -(\phi\tilde{\mathsf{y}} - \tilde{\phi}\mathsf{y})\hat{y}^\dagger + 2i\hat{\phi}\,\mathrm{Im}(y\tilde{y}^\dagger) & * \\ & 0 & 0 & 0 & * \\ & & & \cdots & \end{pmatrix}. \tag{2.4}$$

## 3 Preliminaries on Cartan-decomposition subgroups

**Notation 3.1.** We employ the usual Big Oh and little oh notation: for functions $f_1, f_2$ on $H$, and a subset $Z$ of $H$, we say $f_1 = O(f_2)$ *for* $z \in Z$ if there is a constant $C$, such that, for all large $z \in Z$, we have $\|f_1(z)\| \leq C\|f_2(z)\|$. (The values of each $f_i$ are assumed to belong to some finite-dimensional normed vector space, typically either $\mathbb{C}$ or a space of complex matrices. Which particular norm is used does not matter, because all norms are equivalent up to a bounded factor.) We say $f_1 = o(f_2)$ *for* $z \in Z$ if $\|f_1(z)\|/\|f_2(z)\| \to 0$ as $z \to \infty$. Also, we write $f_1 \asymp f_2$ if $f_1 = O(f_2)$ and $f_2 = O(f_1)$.

**Definition 3.2.** Define $\rho\colon \mathrm{SU}(2,n) \to \mathrm{GL}(\mathbb{C}^{n+2} \wedge \mathbb{C}^{n+2})$ by $\rho(h) = h \wedge h$, so $\rho$ is the second exterior power of the standard representation of $\mathrm{SU}(2,n)$. Thus, we may define $\|\rho(h)\|$ to be the maximum absolute value among the determinants of all the $2 \times 2$ submatrices of the matrix $h$.

We now introduce convenient notation for describing the image of a subgroup under the Cartan projection $\mu$.

**Notation 3.3.** For functions $f_1, f_2\colon \mathbb{R}^+ \to \mathbb{R}^+$, and a subgroup $H$ of $\mathrm{SU}(2,n)$, we write $\mu(H) \approx \big[f_1(\|h\|), f_2(\|h\|)\big]$ if, for every sufficiently large $C > 1$, we have

$$\mu(H) \approx \big\{\, a \in A^+ \;\big|\; C^{-1} f_1\big(\|a\|\big) \le \|\rho(a)\| \le C f_2\big(\|a\|\big) \,\big\}.$$

(If $f_1$ and $f_2$ are monomials, or other very tame functions, then it does not matter which particular norm is used.)

We have $A^+ = \{\, a \in A \mid a_{1,1} \ge a_{2,2} \ge 1 \,\}$, so, for $a \in A^+$, we have

$$\|a\| = a_{1,1} \le a_{1,1}\, a_{2,2} = \|\rho(a)\| \le a_{1,1}^2 = \|a\|^2.$$

Thus $A^+ \approx \big[\|h\|, \|h\|^2\big]$, so, from Theorem 1.8, we see that $H$ is a Cartan-decomposition subgroup of $G$ if and only if $\mu(H) \approx \big[\|h\|, \|h\|^2\big]$. This observation, which is essentially due to Y. Benoist (in a much more general context, cf. [1, Lem. 2.4]), leads to the following result.

**Proposition 3.4.** (cf. [14, Prop. 3.24]) *A closed, connected subgroup $H$ of $\mathrm{SU}(2,n)$ is a Cartan-decomposition subgroup if and only if*

1. *there is a sequence $\{h_m\}$ in $H$, such that $h_m \to \infty$ as $n \to \infty$, and $\rho(h_m) \asymp \|h_m\|^2$; and*

2. *there is a sequence $\{h_m\}$ in $H$, such that $h_m \to \infty$ as $n \to \infty$, and $\rho(h_m) \asymp h_m$.*

The following result allows us to replace $H$ by a conjugate subgroup whenever it is convenient.

**Lemma 3.5.** (cf. [1, Prop. 1.5], [8, Cor. 3.5]) *Let $H$ be any closed, connected subgroup of $\mathrm{SU}(2,n)$. For every $g \in G$, we have $\mu(g^{-1}Hg) \approx \mu(H)$.*

*In particular, $H$ is a Cartan-decomposition subgroup if and only if $g^{-1}Hg$ is a Cartan-decomposition subgroup.*

## 4 When is the size of $\rho(h)$ quadratic?

In this section, Proposition 4.1 is a list of subgroups that contain a sequence $\{h_m\}$ with $\rho(h_m) \asymp \|h_m\|^2$, and Proposition 4.3 is a list of subgroups that do not contain such a sequence. Then Proposition 4.4 shows that both lists are complete.

**Proposition 4.1.** *Assume that $G = \mathrm{SU}(2,n)$. Let $H$ be a closed, connected subgroup of $N$. There is a sequence $h_m \to \infty$ in $H$ with $\rho(h_m) \asymp \|h_m\|^2$ if either*

1. *there is an element $u$ of $\mathfrak{h}$ with $\phi_u = 0$, such that the vectors $x_u$ and $y_u$ are linearly independent over $\mathbb{C}$; or*

2. *there is an element $z$ of $\mathfrak{z}$, such that $|\eta_z|^2 \neq \mathsf{x}_z\mathsf{y}_z$; or*

3. *there are elements $u$ of $\mathfrak{h}$ and $z$ of $\mathfrak{z}$, such that $\phi_u = 0$, and $\mathsf{x}_z|y_u|^2 + \mathsf{y}_z|x_u|^2 + 2\,\mathrm{Im}(x_u y_u^\dagger \overline{\eta_z}) \neq 0$; or*

4. *there is an element $u$ of $\mathfrak{h}$, such that $\phi_u \neq 0$, $y_u = 0$, $\mathsf{y}_u = 0$, and $|x_u|^2 + 2\,\mathrm{Re}(\phi_u \overline{\eta_u}) = 0$; or*

5. *$\mathfrak{u}_{2\alpha+2\beta} \subset \mathfrak{h}$ and there is an element $u$ of $\mathfrak{h}$, such that $\phi_u \neq 0$, $\mathsf{y}_u \neq 0$, and $y_u = 0$; or*

6. *there are elements $u$ and $v$ of $\mathfrak{h}$, such that $\phi_u \neq 0$, $y_u \neq 0$, $\phi_v = 0$, $y_v = 0$, $x_v \neq 0$, $\mathsf{y}_v = 0$, and $x_v y_u^\dagger = 0$; or*

7. *$\mathfrak{u}_{2\alpha+2\beta} \subset \mathfrak{z}$, and there are nonzero elements $u$ and $v$ of $\mathfrak{h}$, satisfying $\phi_u \neq 0$, $y_u \neq 0$, $\phi_v = 0$, $y_v = 0$, $\mathsf{y}_v \neq 0$, and $x_v y_u^\dagger = -i\phi_u \mathsf{y}_v$; or*

8. *$\dim \mathfrak{h} = 3$, $\mathfrak{z} = \mathfrak{u}_{2\alpha+2\beta}$, there exist $u, v \in \mathfrak{h} \setminus \mathfrak{z}$, such that $y_u \neq 0$, $y_v = 0$, $\mathsf{y}_v = 0$, $|x_v|^2 + 2\,\mathrm{Re}(\phi_v \overline{\eta_v}) > 0$, and we have $\phi_h \neq 0$ for every $h \in \mathfrak{h} \setminus \mathfrak{z}$.*

**Remark 4.2.** In Conclusions (6) and (7), the restriction on $x_v y_u^\dagger$ is not necessary; it was included to avoid overlap with Conclusion (2). Namely, if $x_v y_u^\dagger \neq -i\phi_u \mathsf{y}_v$, then $[u,v]$ satisfies $\mathsf{y} = 0$ and $\eta \neq 0$, so Conclusion (2) holds. Also, it is not necessary to assume $\mathsf{y}_v \neq 0$ in Conclusion (7), because Conclusion (6) holds if $\mathsf{y}_v = 0$ (and $x_v \neq 0$). Thus, (6) and (7) may be replaced with the following:

(6*) there are elements $u$ and $v$ of $\mathfrak{h}$, such that $\phi_u \neq 0$, $y_u \neq 0$, $\phi_v = 0$, $y_v = 0$, $x_v \neq 0$, and $\mathsf{y}_v = 0$; or

(7*) $\mathfrak{u}_{2\alpha+2\beta} \subset \mathfrak{z}$, and there are nonzero elements $u$ and $v$ of $\mathfrak{h}$, satisfying $\phi_u \neq 0$, $y_u \neq 0$, $\phi_v = 0$, $y_v = 0$, and $x_v \neq 0$.

**Proof.** We separately consider each of the eight cases in the statement of the proposition.

(1) Let $h^t = \exp(tu)$. Replacing $H$ by a conjugate under $U_\alpha$, we may assume that $x_u$ is orthogonal to $y_u$; that is, $x_u y_u^\dagger = 0$. Then it is clear that $\rho(h^t) \asymp \Delta(h^t) \asymp t^4 \asymp \|h^t\|^2$.

(2) Let $h^t = \exp(tz)$. We have $h^t \asymp t$ and

$$\Delta(h^t) = \mathsf{x}_{tz}\mathsf{y}_{tz} - |\eta_{tz}|^2 = t^2(\mathsf{x}_z\mathsf{y}_z - |\eta_z|^2) \asymp t^2.$$

Therefore $\rho(h^t) \asymp \Delta(h^t) \asymp t^2 \asymp \|h^t\|^2$.

(3) For any large $t$, let $h = \exp(tu + t^2 z)$. Clearly, we have $|x_h| + |y_h| = O(t)$ and $|\mathsf{x}_h| + |\mathsf{y}_h| + |\eta_h| = O(t^2)$, so $h = O(t^2)$.

We have

$$\operatorname{Im}\Delta(h^t) = t^4\left[\frac{1}{2}\left(2\operatorname{Im}(x_u y_u^\dagger \overline{\eta_z} + \mathsf{x}_z|y_u|^2 + \mathsf{y}_z|x_u|^2\right)\right] + O(t^3) \asymp t^4.$$

Therefore, $\rho(h^t) \asymp t^4 \asymp \|h^t\|^2$.

(4) For any large $t$, let $h = \exp(tu)$. Then $h_{1,n+2} = it\mathsf{x}_u$, so it is easy to see that $h \asymp t$. We have $\rho(h) \asymp t^2 \asymp \|h\|^2$.

(5) Replacing $H$ by a conjugate (under a diagonal matrix), we may assume that $\phi_u = \mathsf{y}_u$. Then, by renormalizing, we may assume that $\phi_u = \mathsf{y}_u = 1$. Let $z$ be the element of $\mathfrak{u}_{2\alpha+2\beta}$ with $\mathsf{x}_z = 1$. By subtracting a multiple of $z$ from $u$, we may assume $\mathsf{x}_u = 0$. For any large $t$, let $h = \exp(6tu + 36t^3 z)$, so $h_{1,n+2}$ is real. We have

$$\operatorname{Re}\Delta(h) = (36t^3)(6t) + \frac{1}{12}(6t)^2(6t)^2 + O(t^2) \asymp t^4,$$

so $\rho(h) \asymp t^4 \asymp \|h\|^2$.

(6) For each large $t$, let $h$ be an element of $\exp(tu + \mathbb{R}v)$, such that $h_{1,n+2}$ is pure imaginary. (This exists because the sign of $-\frac{1}{2}|x|^2$ is opposite that of $\frac{1}{24}|\phi|^2|y|^2$.) We note that $x_h \asymp t^2$ and $|\eta_h| + |\mathsf{x}_h| = O(t^2)$, but $\phi_h \asymp y_h \asymp t$ and $|\mathsf{y}_h| + |x_h y_h^\dagger| = O(t)$. Thus $h = O(t^3)$ and

$$\rho(h) \asymp \operatorname{Re}\Delta(h) = -\frac{1}{4}|x_h|^2|y_h|^2 - \frac{1}{144}|y_h|^4|\phi_h|^2 + O(t^5) \asymp t^6 \asymp \|h\|^2.$$

(7) Because $x_v y_u^\dagger = -i\phi_u \mathsf{y}_v$, we have $x_v \neq 0$, so, for any large $t$, we may choose $h \in \exp(tu + \mathbb{R}v + \mathfrak{u}_{2\alpha+2\beta})$, such that $h_{1,n+2} = 0$. Thus $\phi_h \asymp y_h \asymp t$, but $x_h \asymp \mathsf{y}_h \asymp t^2$ and $|\eta_h| + |\mathsf{x}_h| = O(t^2)$. Then (because $h_{1,n+2} = 0$) it is easy to verify that $h = O(t^3)$. However

$$\operatorname{Im}\Delta(h) = \frac{1}{24}\mathsf{y}_h|\phi_h|^2|y_h|^2 + \frac{1}{2}\mathsf{y}_h|x_h|^2 + O(t^5) \asymp t^6.$$

So $\rho(h) \asymp \|h\|^2$.

(8) For any large $t$, choose $s = O(1)$, such that $\operatorname{Re}\big(\exp(su + tv)_{1,n+2}\big) = 0$. (This is possible, because $-\frac{1}{2}|x_v|^2 - \operatorname{Re}(\phi_v\overline{\eta_v}) < 0$.) Then we may choose $h \in \exp(su + tv + \mathfrak{z})$, such that $h_{1,n+2} = 0$. Then $\phi_h \asymp t$, $|x_h| + |\eta_h| = O(t)$, and $|y_h| + |\mathsf{y}_h| = O(1)$, so we have $\rho(h) \asymp t^2 \asymp \|h\|^2$. ∎

**Proposition 4.3.** *Assume that $G = \mathrm{SU}(2,n)$. Let $H$ be a closed, connected, nontrivial subgroup of $N$.*

1. *If $\dim\mathfrak{h} = 1$, $\mathfrak{h} = \mathfrak{z}$, and we have $|\eta_h|^2 = \mathsf{x}_h\mathsf{y}_h$ for every $h \in H$, then $\rho(h) \asymp h$ for every $h \in H$.*

2. *If $\phi_h = 0$ and $y_h = 0$ for every $h \in \mathfrak{h}$, $\mathfrak{z} \subset \mathfrak{u}_{2\alpha+2\beta}$, and there is some $u \in \mathfrak{h}$, such that $\mathsf{y}_u \neq 0$, then $\mu(H) \approx \big[\|h\|, \|h\|^{3/2}\big]$, unless $\dim H = 1$, in which case $\rho(h) \asymp \|h\|^{3/2}$ for every $h \in H$.*

3. *Suppose $\phi_h = 0$ for every $h \in \mathfrak{h}$, and there is some $\lambda \in \mathbb{C}$, such that $x_h = \lambda y_h$ for every $h \in H$, and we have $\eta_z = i\lambda\mathsf{y}_z$ and $\mathsf{x}_z = |\lambda|^2\mathsf{y}_z$ for every $z \in \mathfrak{z}$.*

(a) *If there is some* $u \in \mathfrak{h}$, *such that* $\mathsf{x}_u + |\lambda|^2 \mathsf{y}_u + 2\operatorname{Im}(\lambda\overline{\eta_u}) \neq 0$, *then* $\mu(H) \approx \big[\|h\|, \|h\|^{3/2}\big]$, *unless* $\dim H = 1$, *in which case* $\rho(h) \asymp \|h\|^{3/2}$ *for every* $h \in H$.

(b) *Otherwise,* $\rho(h) \asymp h$ *for every* $h \in H$.

4. *If* $y_h = 0$, $\mathsf{y}_h = 0$, *and* $|x_h|^2 + 2\operatorname{Re}(\phi_h\overline{\eta_h}) \neq 0$ *for every* $h \in \mathfrak{h} \setminus \mathfrak{u}_{2\alpha+2\beta}$ (*so* $\mathfrak{z} \subset \mathfrak{u}_{2\alpha+2\beta}$), *then* $\rho(h) \asymp h$ *for every* $h \in H$.

5. *If* $\mathfrak{z} = 0$, *there is some* $u \in \mathfrak{h}$ *and some nonzero* $\phi_0 \in \mathbb{C}$, *such that* $\phi_u \neq 0$, *and we have* $\phi_h = \phi_0 \mathsf{y}_h$ *and* $y_h = 0$, *for every* $h \in \mathfrak{h}$, *then* $\mu(H) \approx \big[\|h\|, \|h\|^{4/3}\big]$, *unless* $\dim H = 1$, *in which case,* $\rho(h) \asymp \|h\|^{4/3}$ *for every* $h \in H$.

6. *If* $\dim \mathfrak{h} \leq 3$, $\mathfrak{z} = 0$, *we have* $\phi_v \asymp y_v$ *and* $v = O\big(|\phi_v| + |\mathsf{y}_v|\big)$ *for every* $v \in \mathfrak{h}$, *and there exists* $u \in \mathfrak{h}$, *such that* $\phi_u \neq 0$, *then* $\rho(h) \asymp \|h\|^{3/2}$ *for every* $h \in H$.

7. *If* $\dim \mathfrak{h} = 2$, $\mathfrak{z} = \mathfrak{u}_{2\alpha+2\beta}$, *and* $\phi_h \neq 0$ *and* $y_h \neq 0$ *for every* $h \in \mathfrak{h} \setminus \mathfrak{z}$, *then* $\mu(H) \approx \big[\|h\|, \|h\|^{3/2}\big]$.

**Proof.** We separately consider each of the seven cases in the statement of the proposition.

(1) Because $\Delta(h) = 0$ for every $h \in H$, it is clear that $\rho(h) \asymp h$ for every $h \in H$.

(2) We have $|\eta_h| + |\mathsf{y}_h| = O(x_h)$, so $h_{1,n+2} \asymp |x_h|^2 + |\mathsf{x}_h|$ and $h_{i,j} = O(x_h) = O\big(|h_{1,n+2}|^{1/2}\big)$ whenever $(i,j) \neq (1, n+2)$. Thus, $\rho(h) = O\big(\|h\|^{3/2}\big)$.

We have $\rho\big(\exp(tu)\big) \asymp \operatorname{Im}\Delta\big(\exp(tu)\big) \asymp t^3 \asymp \|\exp(tu)\|^{3/2}$. If $\dim H > 1$, then there is some nonzero $v \in \mathfrak{h}$, such that $\mathsf{y}_v = 0$. Then, for $h \in \exp(\mathbb{R}v)$, we have $\rho(h) \asymp |x_h|^2 + |\mathsf{x}_h| \asymp h$.

(3) Replacing $H$ by a conjugate under $U_\alpha$, we may assume that $\lambda = 0$, so $x_h = 0$ for every $h \in H$, and $\eta_z = \mathsf{x}_z = 0$ for every $z \in \mathfrak{z}$ (which means $\mathfrak{z} \subset \mathfrak{u}_{2\beta}$). Therefore, the Weyl reflection corresponding to the root $\alpha$ conjugates $\mathfrak{h}$ to a subalgebra either of type (2) or of type (4), depending on whether or not there is some $u \in \mathfrak{h}$, such that $\mathsf{x}_u + |\lambda|^2 \mathsf{y}_u + 2\operatorname{Im}(\lambda\overline{\eta_u}) \neq 0$.

(4) By assumption, the quadratic form $|x|^2 + 2\operatorname{Re}(\phi\overline{\eta})$ is definite on $\mathfrak{h}/\mathfrak{z}$, so $|x|^2 + |\phi|^2 + |\eta|^2 = O\big(|x|^2 + 2\operatorname{Re}(\phi\overline{\eta})\big)$. Therefore, $h_{i,j} = O\big(|h_{1,n+2}|^{1/2}\big)$ whenever $(i,j) \neq (1, n+2)$. Furthermore, $h_{i,j} = O(1)$ whenever $i \neq 1$ and $j \neq n+2$. Thus, $\rho(h) \asymp h$.

(5) For any sequence $\{h_m\} \to \infty$ in $H$, we write $\phi_m, x_m, y_m, \mathsf{y}_m, \eta_m, \mathsf{x}_m$ for $\phi_{h_m}$, etc.

We have $\phi_m \asymp \mathsf{y}_m$. If $x_m = O(|\mathsf{y}_m|^{3/2})$, then $\rho(h_m) \asymp \operatorname{Re}\Delta(h_m) \asymp \mathsf{y}_m^4 \asymp \|h_m\|^{4/3}$. (This completes the proof if $\dim H = 1$.) If $|\mathsf{y}_m|^{3/2} = o(x_m)$, then $h_m \asymp h_{1,n+2} \asymp |x_m|^2$, but $h_{i,j} = O\big(|x_m| + \mathsf{y}_m^2\big) = O\big(|x_m|^{4/3}\big)$ whenever $(i,j) \neq (1, n+2)$, and $h_{i,j} = O(\mathsf{y}_m) = O\big(|x_m|^{2/3}\big)$ whenever $i \neq 1$ and $j \neq n+2$. Therefore

$$\rho(h_m) = O\big(|x_m|^2 |x_m|^{2/3} + |x_m|^{4/3} |x_m|^{4/3}\big) = O\big(|x_m|^{8/3}\big) = O\big(\|h_m\|^{4/3}\big).$$

If $\dim H > 1$, then there is some (large) $h \in H$ with $\mathsf{y}_h = 0$ (and hence $\phi_h = 0$). Thus $\rho(h) \asymp |x_h|^2 \asymp h$.

(6) For any sequence $\{h_m\} \to \infty$ in $H$, we show that $\rho(h_m) \asymp \Delta(h_m) \asymp \|h_m\|^{3/2}$. We write $\phi_m, x_m, y_m, \mathsf{y}_m, \eta_m, \mathsf{x}_m$ for $\phi_{h_m}$, etc.

If $\mathsf{y}_m = o(\phi_m^2)$, then $h_{1,n+2} \asymp \phi_m^4$, but $h_{i,j} = O(\phi_m^3)$ whenever $(i,j) \neq (1, n+2)$, and $h_{i,j} = O(\phi_m^2)$ whenever $i \neq 1$ and $j \neq n+2$. Thus, $\rho(h_m) \asymp \operatorname{Re}\Delta(h_m) \asymp \phi_m^6 \asymp \|h_m\|^{3/2}$.

We may now assume that $\phi_m^2 = O(\mathsf{y}_m)$. Thus, there is some $v \in \mathfrak{h}$, such that $\phi_v = 0$ and $\mathsf{y}_v = 1$. (Note that, because $y_v \asymp \phi_v$, we have $y_v = 0$.) Because $[u,v] \in \mathfrak{z} = 0$, we must have $\eta_{[u,v]} = 0$, so $x_v y_u^\dagger = -i\phi_u \mathsf{y}_v \neq 0$. In particular, $x_v \neq 0$, so $x_m \asymp \mathsf{y}_m$.

We have $h_{1,n+2} = O\big(|x_m|^2\big) = O(\mathsf{y}_m^2)$, but $h_{i,j} = O\big(|\phi_m \mathsf{y}_m| + |\mathsf{y}_m|\big) = O\big(|\mathsf{y}_m|^{3/2}\big)$ whenever $(i,j) \neq (1, n+2)$, and $h_{i,j} = O(\mathsf{y}_m)$ whenever $i \neq 1$ and $j \neq n+2$. Thus, $h_m = O(\mathsf{y}_m^2)$ and $\rho(h_m) = O(\mathsf{y}_m^3)$.

Furthermore, we have

$$\operatorname{Im}\Delta(h_m) = \frac{1}{24}\mathsf{y}_m|\phi_m|^2|y_m|^2 + \frac{1}{2}\mathsf{y}_m|x_m|^2 + O(\mathsf{y}_m^2\phi_m) \asymp \mathsf{y}_m^3,$$

because $\mathsf{y}_m|x_m|^2 \asymp \mathsf{y}_m^3$, and the terms $\frac{1}{24}\mathsf{y}_m|\phi_m|^2|y_m|^2$ and $\frac{1}{2}\mathsf{y}_m|x_m|^2$ cannot cancel (since they both have the same sign as $\mathsf{y}_m$). We conclude that $\rho(h_m) \asymp \Delta(h_m) \asymp \mathsf{y}_m^3$.

All that remains is to show $\mathsf{y}_m^2 = O(h_m)$. If $\phi_m^2 = o(\mathsf{y}_m)$, then

$$\operatorname{Re} h_{1,n+2} = -\frac{1}{2}|x|^2 + O(\phi_m^2 \mathsf{y}) \asymp \mathsf{y}_m^2,$$

as desired. If $\mathsf{y}_m = o(\phi_m^2)$, then

$$\operatorname{Re} h_{1,n+2} \asymp o(\phi_m^4) + o(\phi_m^3) + |\phi_m^4| \asymp \phi_m^4,$$

so $\mathsf{y}_m = o(\phi_m^2) = o(\phi_m^4) = o(h_m)$, as desired. Thus, we may assume that $\mathsf{y}_m \asymp \phi_m^2$. Because $x_m = \mathsf{y}_m x_v + O(\phi_m)$ and $x_v y_m^\dagger = -i\phi_m \mathsf{y}_v = -i\phi_m$, we have

$$\begin{aligned}\operatorname{Im}(h_{1,n+2}) &= O(\mathsf{y}_m) - \frac{1}{6}|\phi_m|^2\mathsf{y}_m + \left[\frac{1}{3}\operatorname{Im}\big(\overline{\phi_m}(\mathsf{y}_m x_v)y_m^\dagger\big) + O(\phi_m^3)\right] \\ &= -\frac{1}{6}|\phi_m|^2\mathsf{y}_m - \frac{1}{3}|\phi_m|^2\mathsf{y}_m + O(\phi_m^3) \asymp \mathsf{y}_m^2,\end{aligned}$$

as desired.

(7) For $z \in \mathfrak{z}$, we have $\rho(z) \asymp z$. For $u \in \mathfrak{h} \setminus \mathfrak{z}$ with $y_u \neq 0$, we have $\rho\big(\exp(tu)\big) \asymp t^6 \asymp \|\exp(tu)\|^{3/2}$. All that remains is to show $\rho(h) = O\big(\|h\|^{3/2}\big)$ for every $h \in H$.

Note that $\phi_h \asymp y_h$, and $|x_h| + |\eta_h| + |\mathsf{y}_h| = O(\phi_h)$. If $\phi_h = O(1)$, then it is obvious that $\rho(h) \asymp h$. Thus, we may assume $|\phi_h| \to \infty$. Then, because $\operatorname{Re} h_{1,n+2} \asymp |\phi_h|^2|y_h|^2 \asymp \phi_h^4$, but $h_{i,j} = O\big(\phi_h|y_h|^2\big) = O(\phi_h^3)$ whenever $(i,j) \neq (1, n+2)$, and $h_{i,j} = O(\phi_h^2)$ whenever $i \neq 1$ and $j \neq n+2$, we have

$$\rho(h) = O\big[|\phi_h|^4|\phi_h|^2 + \big(|\phi_h|^3\big)^2\big] = O\big(|\phi_h|^6\big) = O\big(|h_{1,n+2}|^{3/2}\big) = O\big(\|h\|^{3/2}\big). \quad \blacksquare$$

**Proposition 4.4.** *Assume that $G = \mathrm{SU}(2,n)$. Let $H$ be a closed, connected, nontrivial subgroup of $N$.*

1. *There is a sequence $h_m \to \infty$ in $H$ with $\rho(h_m) \asymp \|h_m\|^2$ if and only if $H$ is one of the subgroups described in Proposition* 4.1.

2. *There is* **not** *a sequence $h_m \to \infty$ in $H$ with $\rho(h_m) \asymp \|h_m\|^2$ if and only if $H$ is one of the subgroups described in Proposition* 4.3.

**Proof.** It suffices to show that $H$ is described in either Proposition 4.1 or Proposition 4.3.

We may assume

$$|\eta_z|^2 = \mathsf{x}_z\mathsf{y}_z \text{ for every } z \in \mathfrak{z} \tag{4.1}$$

(otherwise, 4.1(2) holds). Because $|\eta|^2 - \mathsf{xy}$ is a quadratic form of signature $(3,1)$ on $\mathfrak{u}_{2\beta} + \mathfrak{u}_{\alpha+2\beta} + \mathfrak{u}_{2\alpha+2\beta}$, then we must have $\dim \mathfrak{z} \leq 1$. Thus, we may assume $\mathfrak{h} \neq \mathfrak{z}$ (otherwise 4.3(1) holds).

*Case 1. Assume $\phi_h = 0$ and $y_h = 0$ for every $h \in H$ (and $\mathfrak{h} \neq \mathfrak{z}$).* We may assume $\mathsf{y}_z = 0$ for every $z \in \mathfrak{z}$, for, otherwise, 4.1(3) holds. Then, from Eq. (4.1), we have $\eta_z = 0$ for every $z \in \mathfrak{z}$. Thus, $\mathfrak{z} \subset \mathfrak{u}_{2\alpha+2\beta}$. We may assume $\mathsf{y}_h = 0$ for every $h \in H$, for otherwise Conclusion 4.3(2) holds. We conclude that 4.3(4) holds.

*Case 2. Assume $\phi_h = 0$ for every $h \in H$, and there is some $u \in \mathfrak{h}$ with $y_u \neq 0$.* We may assume that $x_h$ and $y_h$ are linearly dependent over $\mathbb{C}$ for every $h \in H$ (otherwise 4.1(1) holds). In particular, there exists $\lambda \in \mathbb{C}$, such that $x_u = \lambda y_u$.

*Subcase 2.1. Assume $\mathfrak{z} = 0$.*

*Subsubcase 2.1.1. Assume there exists $v \in \mathfrak{h}$, such that either $x_v \notin \mathbb{C}y_u$ or $y_v \notin \mathbb{C}y_u$.* We may assume there exists $w \in \mathfrak{h}$, such that $x_w \neq \lambda y_w$ (otherwise 4.3(3) holds). Furthermore, by adding a small linear combination of $u$ and $v$ to $w$, we may assume that $y_w \neq 0$ and that either $x_w \notin \mathbb{C}y_u$ or $y_w \notin \mathbb{C}y_u$. Because $x_w$ and $y_w$ are linearly dependent, there exists $\lambda_1$ $(\neq \lambda)$ such that $x_w = \lambda_1 y_w$. (Then note that we must have $y_w \notin \mathbb{C}y_u$.) Then

$$x_{u+w} = x_u + x_w = \lambda y_u + \lambda_1 y_w \notin \mathbb{C}(y_u + y_w) = \mathbb{C}y_{u+w}$$

(because $\lambda \neq \lambda_1$ and $\{y_u, y_w\}$ is linearly independent over $\mathbb{C}$). This contradicts the fact that $x_{u+w}$ and $y_{u+w}$ are linearly dependent over $\mathbb{C}$.

*Subsubcase 2.1.2. Assume $x_h, y_h \in \mathbb{C}y_u$, for every $h \in \mathfrak{h}$.* For each $h \in \mathfrak{h}$, there exist $\lambda_x, \lambda_y \in \mathbb{C}$, such that $x_h = \lambda_x y_u$ and $y_h = \lambda_y y_u$. Because $\mathfrak{z} = 0$, we must have $\mathsf{y}_{[h,u]} = 0$, so $\mathrm{Im}(y_h y_u^\dagger) = 0$, which means that $\lambda_y$ is real. We must also have $\eta_{[h,u]} = 0$, so

$$0 = -x_h y_u^\dagger + x_u y_h^\dagger = (-\lambda_x + \lambda\overline{\lambda_y})|y_u|^2 = (-\lambda_x + \lambda\lambda_y)|y_u|^2.$$

Thus $\lambda_x = \lambda\lambda_y$, so

$$x_h = \lambda_x y_u = \lambda\lambda_y y_u = \lambda y_h.$$

Therefore 4.3(3) holds.

*Subcase 2.2. Assume $\mathfrak{z} \neq 0$.* We show that either 4.1(2), 4.1(3) or 4.3(3) holds. Straightforward calculations show that conditions 4.1(2), 4.1(3) and 4.3(3) are invariant under conjugation by $U_\alpha$, so we may assume that $\lambda = 0$; that is, $x_u = 0$. Thus, we may assume $\mathsf{x}_z = 0$ for every $z \in \mathfrak{z}$, for, otherwise, 4.1(3) holds. Then we may assume $\eta_z = 0$ for every $z \in \mathfrak{z}$, for, otherwise, 4.1(2) holds; therefore $\mathfrak{z} = U_{2\beta}$. We may now assume $x_h = 0$ for every $h \in \mathfrak{h}$, for, otherwise, 4.1(3) holds. Thus, 4.3(3) holds (with $\lambda = 0$).

*Case 3. Assume there exists $u \in \mathfrak{h}$ with $\phi_u \neq 0$.* We claim that $\mathfrak{z} \subset \mathfrak{u}_{2\alpha+2\beta}$. If not, then there is some $z \in \mathfrak{z}$, such that either $\eta_z \neq 0$ or $\mathsf{y}_z \neq 0$. If $\mathsf{y}_z = 0$, then $|\eta_z|^2 \neq 0 = \mathsf{x}_z\mathsf{y}_z$, so 4.1(2) holds. On the other hand, if $\mathsf{y}_z \neq 0$, then, letting $z' = [u, z]$, we have $\mathsf{y}_{z'} = 0$ and $\eta_{z'} \neq 0$, so 4.1(2) holds once again.

*Subcase 3.1. Assume $y_h = 0$ for every $h \in \mathfrak{h}$.* We may assume that there is some $v \in \mathfrak{h}$, such that $\mathsf{y}_v \neq 0$ (otherwise, either 4.1(4) or 4.3(4) holds). Then we may assume $\mathfrak{z} = 0$ (otherwise, 4.1(5) holds).

We claim that 4.3(5) holds. If not, then there is some $w \in \mathfrak{h}$, such that $\phi_w \neq 0$ and $\mathsf{y}_w = 0$. Then $\eta_{[v,w]} \neq 0$, which contradicts the assumption that $\mathfrak{z} = 0$.

*Subcase 3.2. Assume there is some $v \in \mathfrak{h}$, such that $y_v \neq 0$.*

*Subsubcase 3.2.1. Assume $\mathfrak{z} = \mathfrak{u}_{2\alpha+2\beta}$.* Suppose, for the moment, that there exists $w \in \mathfrak{h} \setminus \mathfrak{z}$ with $\phi_w = 0$. We may assume that $y_w = 0$ (otherwise, 4.1(3) holds). Therefore $x_w \neq 0$, so 4.1(7*) holds.

We may now assume that $\phi_w \neq 0$ for every $w \in \mathfrak{h} \setminus \mathfrak{z}$. This implies that $x$, $y$, $\eta$, and $\mathsf{y}$ are functions of $\phi$; in particular, $\dim \mathfrak{h} \leq 3$. Also, because $\mathfrak{z} \neq 0$ and $u, v \notin \mathfrak{z}$, we must have $\dim \mathfrak{h} \geq 2$.

We claim $\dim \mathfrak{h} = 2$ (so 4.3(7) holds). If not, then $\dim \mathfrak{h} = 3$, so there exist $u, w \in \mathfrak{h}$, such that $\phi_u = 1$ and $\phi_w = i$. Because $\phi_{[u,w]} = 0$, we must have $[u, w] \in \mathfrak{u}_{2\alpha+2\beta}$. Therefore $0 = x_{[u,w]} = y_w - iy_u$, so $y_w = iy_u$. Furthermore,

$$0 = \mathsf{y}_{[u,w]} = -2i \operatorname{Im}(y_u y_w^\dagger) = -2i \operatorname{Im}(-i|y_u|^2) = -2i|y_u|^2,$$

so $y_u = 0$. Then $y_w = iy_u$ is also 0. This implies $y_h = 0$ for every $h \in H$. This contradicts the fact that $y_v \neq 0$.

*Subsubcase 3.2.2. Assume $\mathfrak{z} = 0$.* Lemma 4.5 below implies that either 4.3(6) or 4.1(6*) holds. ∎

**Lemma 4.5.** *Let $H$ be a closed, connected subgroup of $N$, such that $\mathfrak{z} = 0$, and assume there exist $u, v \in \mathfrak{h}$, such that $\phi_u \neq 0$ and $y_v \neq 0$. Then either $H$ is described in* 4.3(6) (*and in* 5.2(4), *which is the same*), *or $H$ is a a Cartan-decomposition subgroup* (*and is described in* 4.1(6*) *and* 5.1(2)).

**Proof.** Let us begin by establishing that $\phi_h \asymp y_h$ for $h \in \mathfrak{h}$. If not, then we may assume either that $y_u = 0$ or that $\phi_v = 0$. Then, because $\big[[u, v], v\big] \in \mathfrak{z} = 0$, we see from Eq. (2.4) that

$$0 = -(\phi_u y_v - \phi_v y_u) y_v^\dagger + 2i\phi_v \operatorname{Im}(y_u y_v^\dagger) = -\phi_u |y_v|^2 - 0 + 0 \neq 0.$$

This contradiction establishes the claim.

*Case 1. Assume there is a nonzero $w \in \mathfrak{h}$, such that $\phi_w = 0$ and $\mathsf{y}_w = 0$.* Note, from the preceding paragraph, that $y_w = 0$. Then, because $\mathfrak{z} = 0$, we must have $x_w \neq 0$. Therefore, 4.1(6*) and 5.1(2) hold, so $\mu(H) \approx \big[\|h\|, \|h\|^2\big]$, so $H$ is a Cartan-decomposition subgroup.

*Case 2. Assume there does not exist such an element $w \in \mathfrak{h}$.* Then $H$ is described in 4.3(6) and in 5.2(4). ∎

## 5 When is the size of $\rho(h)$ linear?

In this section, Proposition 5.1 is a list of subgroups that contain a sequence $\{h_m\}$ with $\rho(h_m) \asymp h_m$, and Proposition 5.2 is a list of subgroups that do not contain such a sequence. Then Proposition 5.3 shows that both lists are complete.

**Proposition 5.1.** *Assume that $G = \mathrm{SU}(2,n)$. Let $H$ be a closed, connected subgroup of $N$. There is a sequence $h_m \to \infty$ in $H$ with $\rho(h_m) \asymp h_m$ if either*

1. *there is a nonzero element $z$ of $\mathfrak{z}$ with $|\eta_z|^2 = \mathsf{x}_z \mathsf{y}_z$; or*
2. *there is an element $u$ of $\mathfrak{h}$, such that $\phi_u = 0$, $\dim_{\mathbb{C}}\langle x, y\rangle = 1$, and*
$$\mathsf{x}_u |y_u|^2 + \mathsf{y}_u |x_u|^2 + 2\,\mathrm{Im}(x_u y_u^\dagger \overline{\eta_u}) = 0;$$
*or*
3. *there is an element $h$ of $H$ with $y_h = 0$, $\mathsf{y}_h = 0$ and $|x_h|^2 + 2\,\mathrm{Re}(\phi_h \overline{\eta_h}) \neq 0$; or*
4. *there are elements $u$ of $\mathfrak{h}$ and $z$ of $\mathfrak{z}$, such that $\phi_u \neq 0$, $y_u = 0$, $\mathsf{y}_u \neq 0$, $\eta_z \neq 0$, and $\mathsf{y}_z = 0$; or*
5. *there are nonzero elements $u$ of $\mathfrak{h}$ and $z$ of $\mathfrak{z}$, such that $\phi_u \neq 0$, $y_u \neq 0$, $\mathsf{y}_z = 0$, $\phi_u \overline{\eta_z}$ is real, and*
$$\mathsf{x}_z |y_u|^2 - \phi_u \mathsf{y}_u \overline{\eta_z} + 2\,\mathrm{Im}\big(\overline{\eta_z} x_u y_u^\dagger\big) = 0.$$

**Proof.** We separately consider each of the five cases in the statement of the proposition.

(1) From 4.3(1), we have $\rho(h) \asymp h$ for all $h \in \exp(\mathbb{R}z)$.

(2) Replacing $H$ by a conjugate under $\langle U_\alpha, U_{-\alpha}\rangle$, we may assume that $y_u = 0$ (and $x_u \neq 0$). Then, from the assumption of this case, we know that $\mathsf{y}_u$ is also 0. Therefore, 4.3(4) implies that $\rho(h) \asymp h$ for all $h \in \exp(\mathbb{R}u)$.

(3) From 4.3(4), we have $\rho(h) \asymp h$ for all $h \in \exp(\mathbb{R}u)$.

(4). For any large $t$, choose $h \in \exp(tu + \mathfrak{z})$, such that $\mathsf{x}_h \mathsf{y}_h + \frac{1}{12}|\phi_h|^2 \mathsf{y}_h^2 - |\eta_h|^2 = 0$. Note that $\eta_h \asymp |\phi_h y_h| \asymp t^2$, so $h \asymp \mathrm{Re}\, h_{1,n+2} \asymp t^3$, but $h_{i,j} = O(t^2)$ whenever $(i,j) \neq (1, n+2)$, and $h_{i,j} = O(t)$ whenever $i \notin \{1,2\}$ or $j \notin \{n+1, n+2\}$. From the choice of $h$, we have

$$\Delta(h) = 0 + i\left(\tfrac{1}{2}|x_h|^2 \mathsf{y}_h\right) = O(t^3) = O(h),$$

so it is not difficult to see that $\rho(h) \asymp h$.

(5) Replacing $\mathfrak{h}$ by a conjugate, we may assume $u \in \mathfrak{u}_\alpha + \mathfrak{u}_\beta$. (First, conjugate by an element of $U_\beta$ to make $\mathsf{y}_u = 0$. Then conjugate by an element of $U_\alpha$ to make $x_u$ orthogonal to $y_u$. Then conjugate by an element of $U_\beta$ that centralizes $y_u$, to make $x_u = 0$. Then conjugate by an element of $U_{\alpha+\beta}$ to make $\eta_u = 0$. Then conjugate by an element of $U_{\alpha+2\beta}$ to make $\mathsf{x}_u = 0$.) Then, by assumption, we must have $\mathsf{x}_z = 0$, because $\mathsf{y}_u = 0$ and $x_u = 0$.

Furthermore, replacing $\mathfrak{h}$ by a conjugate under a diagonal matrix (that belongs to $G$), we may assume that $\phi_u$ and $y_u$ are real. Then $\eta_z$ must also be real (because $\phi_u \overline{\eta_z}$ is real). Thus, we see that $u, z \in \mathfrak{so}(2,n)$. So [14, Thm. 5.3(1)] implies that $H$ is a Cartan-decomposition subgroup. ∎

**Proposition 5.2.** *Assume that $G = \mathrm{SU}(2,n)$. Let $H$ be a closed, connected, nontrivial subgroup of $N$ such that*

$$|\eta_z|^2 \neq \mathsf{x}_z\mathsf{y}_z, \text{ for every nonzero } z \in \mathfrak{z}. \tag{5.1}$$

1. *If $\mathfrak{h} = \mathfrak{z}$ (so $\dim H \leq 3$), then $\rho(h) \asymp \|h\|^2$ for every $h \in H$.*
2. *If $\phi_h = 0$ and $\dim_{\mathbb{C}}\langle x_u, y_u\rangle \neq 1$ for every $h \in \mathfrak{h}$, then $\rho(h) \asymp \|h\|^2$ for every $h \in H$.*
3. *If $\phi_h = 0$ for every $h \in \mathfrak{h}$, there exist nonzero $u$ and $v$ in $\mathfrak{h}$, such that $\dim_{\mathbb{C}}\langle x_u, y_u\rangle \neq 1$ and $\dim_{\mathbb{C}}\langle x_v, y_v\rangle = 1$, and $\mathsf{x}_v|y_v|^2 + \mathsf{y}_v|x_v|^2 + 2\operatorname{Im}(x_v y_v^\dagger \overline{\eta_v}) \neq 0$ for every such $v \in \mathfrak{h}$, then $\mu(H) \approx \left[\|h\|^{3/2}, \|h\|^2\right]$.*
4. *If $\dim\mathfrak{h} \leq 3$, $\mathfrak{z} = 0$, we have $\phi_v \asymp y_v$ and $v = O\big(|\phi_v| + |\mathsf{y}_v|\big)$ for every $v \in \mathfrak{h}$, and there exists $u \in \mathfrak{h}$, such that $\phi_u \neq 0$, then $\rho(h) \asymp \|h\|^{3/2}$ for every $h \in H$.*
5. *If $\dim\mathfrak{h} \leq 2$ and $\phi_h \neq 0$, $y_h = 0$, $\mathsf{y}_h = 0$, and $|x_h|^2 + 2\operatorname{Re}(\phi_h\overline{\eta_h}) = 0$ for every nonzero $h \in \mathfrak{h}$, then $\rho(h) \asymp \|h\|^2$ for every $h \in H$.*
6. *If $\dim\mathfrak{h} = 2$ and there exist nonzero $u \in \mathfrak{h}$ and $z \in \mathfrak{z}$, such that $\phi_u \neq 0$, $y_u \neq 0$, $\mathsf{y}_z = 0$, $\phi_u\overline{\eta_z}$ is real, and $\mathsf{x}_z|y_u|^2 - \phi_u\mathsf{y}_u\overline{\eta_z} + 2\operatorname{Im}\big(\overline{\eta_z}x_u y_u^\dagger\big) \neq 0$, then $\mu(H) \approx \left[\|h\|^{5/4}, \|h\|^2\right]$.*
7. *If $\dim\mathfrak{h} = 1$, and we have $\phi_h = 0$, $\dim_{\mathbb{C}}\langle x_h, y_h\rangle = 1$, and*

$$\mathsf{x}_h|y_h|^2 + \mathsf{y}_h|x_h|^2 + 2\operatorname{Im}(x_h y_h^\dagger\overline{\eta_h}) \neq 0$$

   *for every nonzero $h \in \mathfrak{h}$, then $\rho(h) \asymp \|h\|^{3/2}$ for every $h \in H$.*
8. *If $\dim\mathfrak{h} = 1$, and $\phi_h \neq 0$, $y_h = 0$, and $\mathsf{y}_h \neq 0$, for every nonzero $h \in \mathfrak{h}$, then $\rho(h) \asymp \|h\|^{4/3}$ for every $h \in H$.*

**Proof.** We separately consider each of the eight cases in the statement of the proposition.

(1) From Eq. (5.1), we know that the quadratic form $|\eta|^2 - \mathsf{x}\mathsf{y}$ is anisotropic on $\mathfrak{z} = \mathfrak{h}$, so

$$\Delta(h) = |\eta_h|^2 - \mathsf{x}_h\mathsf{y}_h \asymp |\eta_h|^2 + \mathsf{x}_h^2 + \mathsf{y}_h^2 \asymp \|h\|^2.$$

(2) Because $\dim_{\mathbb{C}}\langle x_u, y_u\rangle \neq 1$, we have

$$|x_h|^2|y_h|^2 - |x_h y_h^\dagger|^2 \asymp |x_h|^4 + |y_h|^4,$$

so Lemma 5.4 implies $\rho(h) \asymp \|h\|^2$.

(7) From either Proposition 4.3(2) or 4.3(3a) (depending on whether $y_h$ is 0 or not), we have $\rho(h) \asymp \|h\|^{3/2}$ for every $h \in H$.

(3) From Lemma 5.4, we have $\|h\|^{3/2} = O\big(\rho(h)\big)$.

From (2), we see that $\rho(h) \asymp \|h\|^2$ for $h \in \exp(\mathbb{R}u)$.

From (7), we see that $\rho(h) \asymp \|h\|^{3/2}$ for $h \in \exp(\mathbb{R}v)$.

(4) See Proposition 4.3(6).

(5) Because $\operatorname{Re} h_{1,n+2} = 0$, it is easy to see that $\rho(h) \asymp \phi_h^2 \asymp \|h\|^2$.

(6) Replacing $H$ by a conjugate, we may assume $x_u = 0$ and $\mathsf{y}_u = 0$. Therefore, $x_h = 0$ and $\mathsf{y}_h = 0$ for every $h \in H$. Thus

$$x_z|y_u|^2 = \mathsf{x}_z|y_u|^2 - \phi_u \mathsf{y}_u \overline{\eta_z} + 2\,\mathrm{Im}\big(\overline{\eta_z} x_u y_u^\dagger\big) \neq 0,$$

so $\mathsf{x}_z \neq 0$. From Eq. (5.1), we know $\eta_z \neq 0$.

We have $\rho(tz) \asymp \|tz\|^2$ (see 5.2(1)).

Because $\phi_u$ is a real multiple of $\eta_z$, we may let $h$ be a large element of $H$, such that $\eta_h = -|y_h|^2\phi_h/12 + O(\phi_h)$. (So $y_h \asymp \phi_h$ and $\mathsf{x}_h \asymp \eta_h \asymp \phi_h^3$.) Then

$$\begin{aligned}\Delta(h) &= \left(-|\eta_h|^2 - \frac{1}{6}|y_h|^2\eta_h\overline{\phi_h} - \frac{1}{144}|y_h|^4|\phi_h|^2\right) + i\left(\frac{1}{2}\mathsf{x}_h|y_h|^2\right)\\ &= O(\phi_h^4) + i\left(\frac{1}{2}\mathsf{x}_h|y_h|^2\right) \asymp \phi_h^5.\end{aligned}$$

It is clear that all other matrix entries of $\rho(h)$ are $O(\phi_h^5)$. Thus, we have $\rho(h) \asymp \phi_h^5 \asymp \|h\|^{5/4}$.

Now suppose there is a sequence $h_m \to \infty$ in $H$ with $\rho(h_m) = o\big(\|h_m\|^{5/4}\big)$.

*Case 1. Assume* $\eta_m = o(\phi_m^3)$. We have $h_m \asymp \phi_m^4$, so

$$\phi_m^6 \asymp \mathrm{Re}\,\Delta(h_m) = O\big(\rho(h_m)\big) = o(\|h_m\|^{5/4}) = o(\phi_m^5).$$

This is a contradiction.

*Case 2. Assume* $\phi_m^3 = o(\eta_m)$. We have $h_m \asymp \mathrm{Re}\,h_{1,n+2} \asymp \phi_m\eta_m$, so

$$\eta_m^2 \asymp \mathrm{Re}\,\Delta(h_m) = O\big(\rho(h_m)\big) = o(\|h_m\|^{5/4}) = o(\|h_m\|^{3/2}) = o(|\phi_m\eta_m|^{3/2}) = o(\eta_m^2).$$

This is a contradiction.

*Case 3. Assume* $\eta_m \asymp \phi_m^3$. We have $h_m = O(\phi_m^4)$, so

$$\phi_m^5 \asymp \mathsf{x}_m|y_m|^2 \asymp \mathrm{Im}\,\Delta(h_m) = O\big(\rho(h_m)\big) = o\big(\|h_m\|^{5/4}\big) = o(\phi_m^5).$$

This is a contradiction.

(8) See Proposition 4.3(5). ∎

**Proposition 5.3.** *Assume that* $G = \mathrm{SU}(2,n)$. *Let* $H$ *be a closed, connected, nontrivial subgroup of* $N$.

1. *There is a sequence* $h_m \to \infty$ *in* $H$ *with* $\rho(h_m) \asymp h_m$ *if and only if* $H$ *is one of the subgroups described in Proposition* 5.1.
2. *There is* ***not*** *a sequence* $h_m \to \infty$ *in* $H$ *with* $\rho(h_m) \asymp \|h_m\|^2$ *if and only if* $H$ *is one of the subgroups described in Proposition* 5.2.

**Proof.** It suffices to show that $H$ is described in either Proposition 5.1 or Proposition 5.2.

We may assume (5.1) holds (otherwise, Conclusion 5.1(1) holds).

*Case 1. Assume* $\phi_h = 0$ *for every* $h \in H$. We may assume there exists $v \in \mathfrak{h}$, such that $\dim_{\mathbb{C}}\langle x_v, y_v\rangle = 1$ (otherwise 5.2(2) holds). Furthermore, we may assume

$\mathsf{x}_v|y_v|^2+\mathsf{y}_v|x_v|^2+2\operatorname{Im}(x_v y_v^\dagger\overline{\eta_v}) \neq 0$ for every such $v$ (otherwise 5.1(2) holds). Then we may assume $\dim_{\mathbb{C}}\langle x_u, y_u\rangle = 1$ for every nonzero $u \in \mathfrak{h}$ (otherwise 5.2(3) holds).

The argument in Subsubcase 2.1.1 of the proof of Proposition 4.3 implies there exists $\lambda \in \mathbb{C}$, such that, for every $h \in H$, we have $x_h = \lambda y_h$ (or vice-versa: for every $h$, we have $y_h = \lambda x_h$). Thus, replacing $H$ by a conjugate under $\langle U_\alpha, U_{-\alpha}\rangle$, we may assume $x_h = 0$ for every $h \in H$.

If $\dim H > 1$, then there is some nonzero $u \in \mathfrak{h}$, such that $\mathsf{x}_h = 0$. This contradicts the fact that $\mathsf{x}_v|y_v|^2 + \mathsf{y}_v|x_v|^2 + 2\operatorname{Im}(x_v y_v^\dagger\overline{\eta_v}) \neq 0$. Thus, we conclude that $\dim H = 1$, so 5.2(7) holds.

*Case 2. Assume the projection of $\mathfrak{h}$ to $\mathfrak{u}_\alpha$ is one-dimensional.* Replacing $H$ by a conjugate under $A$, we may assume $\phi_h$ is real for every $h \in H$. Fix some $u \in \mathfrak{h}$, such that $\phi_u \neq 0$.

We may assume that $\mathfrak{u}_{2\alpha+2\beta} \not\subset \mathfrak{h}$ (otherwise Conclusion 5.1(1) holds). Therefore $[\mathfrak{h}, u]$ must be zero, so $\mathsf{y}_z = 0$ and $\eta_z$ is a nonzero real, for every nonzero $z \in \mathfrak{z}$. (This implies $\dim \mathfrak{z} \leq 1$.)

*Subcase 2.1. Assume $y_h = 0$ for every $h \in H$.* We may assume Conclusion 5.1(2) does not hold.

We claim that $\mathfrak{h} = \mathbb{R}u + \mathfrak{z}$. Suppose not. Then there is some $v \in \mathfrak{h}$, such that $\phi_v = 0$ and $x_v \neq 0$. Because Conclusion 5.1(2) does not hold, we must have $\mathsf{y}_v \neq 0$. Then $[v, u, u]$ is a nonzero element of $\mathfrak{u}_{2\alpha+2\beta}$. (This can be seen easily by replacing $H$ with a conjugate, so that $u \in \mathfrak{u}_\alpha$.) This contradicts our assumption that $\mathfrak{u}_{2\alpha+2\beta} \not\subset \mathfrak{h}$.

If $\mathsf{y}_u \neq 0$, then either Conclusion 5.2(8) or 5.1(4) holds (depending on whether $\mathfrak{z}$ is 0 or not). If $\mathsf{y}_u \neq 0$, then 5.1(3) or 5.2(6) holds.

*Subcase 2.2. Assume the projection of $\mathfrak{h}$ to $\mathfrak{u}_\beta$ is nontrivial.* Then we may assume $y_u \neq 0$.

*Subsubcase 2.2.1. Assume there are nonzero $v \in \mathfrak{h}$ and $z \in \mathfrak{z}$, such that $\phi_v = 0$, $y_v = 0$, and $x_v \neq 0$.* We may assume that Conclusion 5.1(5) does not hold. Therefore, for every real $t$, we must have

$$\begin{aligned} 0 &\neq \mathsf{x}_z|y_u|^2 - \phi_u(\mathsf{y}_u + t\mathsf{y}_v)\overline{\eta_z} + 2\operatorname{Im}\bigl(\overline{\eta_z}(x_u + tx_v)y_u^\dagger\bigr) \\ &= t\bigl[-\phi_u\mathsf{y}_v\overline{\eta_z} + 2\operatorname{Im}\bigl(\overline{\eta_z}x_v y_u^\dagger\bigr)\bigr] + \text{constant}. \end{aligned}$$

Thus, the coefficient of $t$ must vanish, which (using the fact that $\eta_z$ is real and nonzero) means

$$0 = -\phi_u\mathsf{y}_v + 2\operatorname{Im}\bigl(x_v y_u^\dagger\bigr). \tag{5.2}$$

We have $[u, v] \in \mathfrak{z}$, so $\eta_{[u,v]}$ is real. Thus,

$$0 = \operatorname{Im}\eta_{[u,v]} = \operatorname{Im}\bigl(x_v y_u^\dagger + i\phi_u\mathsf{y}_v\bigr) = \operatorname{Im}\bigl(x_v y_u^\dagger\bigr) + \phi_u\mathsf{y}_v.$$

Comparing this with Eq. (5.2), we conclude that $\phi_u\mathsf{y}_v = 0$. Therefore $\mathsf{y}_v = 0$, so Conclusion 5.1(2) holds (for the element $v$).

*Subsubcase 2.2.2. Assume there do not exist nonzero $v \in \mathfrak{h}$ and $z \in \mathfrak{z}$, such that $\phi_v = 0$, $y_v = 0$, and $x_v \neq 0$.* We must have

$$y_w = 0 \text{ for every } w \in \mathfrak{h}, \text{ such that } \phi_w = 0. \tag{5.3}$$

(Otherwise, we obtain a contradiction by setting $v = [u, w]$ and $z = [u, w, w]$.) We may assume

$$\mathsf{y}_v \neq 0 \text{ for every } v \in \mathfrak{h} \text{ such that } \phi_v = 0,\ y_v = 0, \text{ and } x_v \neq 0. \tag{5.4}$$

(Otherwise, Conclusion 5.1(2) holds.)

We claim $\dim \mathfrak{h} \leq 2$. If not, then there exist linearly independent $v, w \in \mathfrak{h}$, such that $\phi_v = \phi_w = 0$. From (5.3), we know that $y_v = y_w = 0$. By replacing with a linear combination, we may assume $\mathsf{y}_w = 0$. Then, from (5.4), we know that $x_w = 0$, so $w \in \mathfrak{z}$. Because $\mathfrak{z}$ is (at most) one-dimensional, but $v$ and $w$ are linearly independent, we know that $v \notin \mathfrak{z}$, so $x_v \neq 0$. This contradicts the assumption of this subsubcase.

We may now assume $\dim \mathfrak{h} = 2$ (otherwise Conclusion 5.2(5) holds). Choose a nonzero $v \in \mathfrak{h}$, such that $\phi_v = 0$. If $x_v \neq 0$, then Conclusion 5.2(5) holds. If $x_v = 0$, then $v \in \mathfrak{z}$, so either Conclusion 5.1(5) or 5.2(6) holds.

*Case 3. Assume the projection of $\mathfrak{h}$ to $\mathfrak{u}_\alpha$ is two-dimensional.* We may assume $\mathfrak{z} = 0$ (otherwise, $\mathfrak{u}_{2\alpha+2\beta} \subset \mathfrak{h}$, so Conclusion 5.1(1) holds). We may assume $y_h = 0$ for every $h \in H$ (otherwise Lemma 4.5 implies that either 5.2(5) or 5.1(2) applies. Therefore $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{z} = 0$, so $\mathfrak{h}$ is abelian.

Let $u, v \in \mathfrak{h}$ with $\phi_u = 1$ and $\phi_v = i$. Then

$$0 = \eta_{[u,v]} = i\mathsf{y}_v + \mathsf{y}_u,$$

so $\mathsf{y}_u = \mathsf{y}_v = 0$. Then, for every $w \in \mathfrak{h}$, we have $0 = \eta_{[u,w]} = i\mathsf{y}_w$, so $\mathsf{y}_w = 0$. We may assume

$$|x_h|^2 + 2\operatorname{Re}(\phi_h \overline{\eta_h}) = 0 \tag{5.5}$$

for every $h \in \mathfrak{h}$ (otherwise Conclusion 5.1(3) holds). This implies $\dim \mathfrak{h} = 2$ (otherwise, there is some $w \in \mathfrak{h}$ such that $\phi_w = 0$ and $x_w \neq 0$, and then Eq. (5.5) does not hold for $h = u + tw$ when $t$ is sufficiently large). Thus, Conclusion 5.2(5) holds. ∎

**Lemma 5.4.** *Let $H$ be a closed, connected, nontrivial subgroup of $N$. Assume $\phi_h = 0$ for every $h \in \mathfrak{h}$, that (5.1) holds, and that $\mathsf{x}_v|y_v|^2 + \mathsf{y}_v|x_v|^2 + 2\operatorname{Im}(x_v y_v^\dagger \overline{\eta_v}) \neq 0$ for every $v \in \mathfrak{h}$ such that $\dim_{\mathbb{C}}\langle x_v, y_v\rangle = 1$. Then $\|h\|^{3/2} = O\big(\Delta(h)\big)$ for every $h \in H$.*

*Furthermore, $\Delta(h) \asymp \|h\|^2$ whenever $|x_h|^2|y_h|^2 - |x_h y_h^\dagger|^2 \asymp |x_h|^4 + |y_h|^4$.*

**Proof.** We have $h \asymp |x_h|^2 + |y_h|^2 + |\mathsf{x}_h| + |\mathsf{y}_h| + |\eta_h|$. Also, from Eq. (5.1), we have $|\eta_z|^2 - \mathsf{x}_z\mathsf{y}_z \asymp \big(|\mathsf{x}_z| + |\mathsf{y}_z| + |\eta_z|\big)^2$ for every $z \in \mathfrak{z}$. Also, $\mathsf{x}_v|y_v|^2 + \mathsf{y}_v|x_v|^2 + 2\operatorname{Im}(x_v y_v^\dagger \overline{\eta_v}) \asymp |v|^3$ whenever $\dim_{\mathbb{C}}\langle x_v, y_v\rangle = 1$.

*Case 1. Assume $|x_h|^2|y_h|^2 - |x_h y_h^\dagger|^2 = o\big(|x_h|^4 + |y_h|^4\big)$.* Then there is some $v \in \mathfrak{h}$ such that $v - \log h = o\big(|x_h| + |y_h|\big)$ and $|x_v|^2|y_v|^2 - |x_v y_v^\dagger|^2 = 0$. We have

$\dim_{\mathbb{C}}\langle x_v, y_v\rangle = 1$. Therefore

$$\begin{aligned}
\operatorname{Im}\Delta(h) &\asymp \mathsf{x}_h|y_h|^2 + \mathsf{y}_h|x_h|^2 + 2\operatorname{Im}(x_h y_h^\dagger \overline{\eta_h}) \\
&= \mathsf{x}_v|y_v|^2 + \mathsf{y}_v|x_v|^2 + 2\operatorname{Im}(x_v y_v^\dagger \overline{\eta_v}) \\
&\quad + o\big(|\eta_h|^3 + |\mathsf{x}_h|^3 + |\mathsf{y}_h|^3 + |x_h|^3 + |y_h|^3\big) \\
&\asymp |v|^3 + o\big(|\eta_h|^3 + |\mathsf{x}_h|^3 + |\mathsf{y}_h|^3 + |x_h|^3 + |y_h|^3\big) \\
&\asymp |\eta_h|^3 + |\mathsf{x}_h|^3 + |\mathsf{y}_h|^3 + |x_v|^3 + |y_v|^3 \\
&\neq o\big(\|h\|^{3/2}\big).
\end{aligned}$$

Thus, $\|h\|^{3/2} = O\big(\rho(h)\big)$.

*Case 2. Assume* $|x_h|^2|y_h|^2 - |x_h y_h^\dagger|^2 \asymp |x_h|^4 + |y_h|^4$. We may assume $\operatorname{Re}\Delta(h) = o\big(|x_h|^4 + |y_h|^4\big)$ for otherwise it is clear that $\operatorname{Re}\Delta(h) \asymp \|h\|^2$. (So we have $\|h\| \asymp |\eta_h| + |\mathsf{x}_h| + |\mathsf{y}_h| \asymp |x_h|^2 + |y_h|^2$.) Thus, there is some $z \in \mathfrak{z}$, such that $z - \log h = o\big(\log h\big)$ and

$$|\eta_z|^2 - \mathsf{x}_z\mathsf{y}_z = -\frac{1}{4}\big(|x_h|^2|y_h|^2 - |x_h y_h^\dagger|^2\big) + o\big(|x_h|^4 + |y_h|^4\big) < 0.$$

(This implies that $\mathsf{x}_z$ and $\mathsf{y}_z$ must have the same sign.) From (5.1), we conclude that $|\eta_z|^2 - \mathsf{x}_z\mathsf{y}_z < 0$ for every $z \in \mathfrak{z}$. Thus, there is a constant $\epsilon < 1$, such that $|\eta_z| \leq \epsilon\sqrt{\mathsf{x}_z\mathsf{y}_z}$ for every $z \in \mathfrak{z}$. Then

$$|\operatorname{Im}(x_h y_h^\dagger \overline{\eta_z})| \leq |\eta_z||x_h||y_h| \leq \frac{\epsilon}{2}|\mathsf{x}_z|y_h|^2 + \mathsf{y}_z|x_h|^2|,$$

so

$$\operatorname{Im}(x_h y_h^\dagger \overline{\eta_z}) + \frac{1}{2}\mathsf{x}_z|y_h|^2 + \frac{1}{2}\mathsf{y}_z|x_h|^2 \asymp \frac{1}{2}\mathsf{x}_z|y_h|^2 + \frac{1}{2}\mathsf{y}_z|x_h|^2.$$

Therefore

$$\begin{aligned}
\operatorname{Im}\Delta(h) &= \operatorname{Im}(x_h y_h^\dagger \overline{\eta_h}) + \frac{1}{2}\mathsf{x}_h|y_h|^2 + \frac{1}{2}\mathsf{y}_h|x_h|^2 \\
&= \operatorname{Im}(x_h y_h^\dagger \overline{\eta_z}) + \frac{1}{2}\mathsf{x}_z|y_h|^2 + \frac{1}{2}\mathsf{y}_z|x_h|^2 + o\big((|x_h|^2 + |y_h|^2)\log h\big) \\
&\asymp \frac{1}{2}\mathsf{x}_z|y_h|^2 + \frac{1}{2}\mathsf{y}_z|x_h|^2 \\
&\asymp |x_h|^4 + |y_h|^4 \\
&\asymp \|h\|^2.
\end{aligned}$$

■

## 6 Non-Cartan-decomposition subgroups contained in $N$

**Theorem 6.1.** *Assume that* $G = \mathrm{SU}(2,n)$. *Here is a complete list of the closed, connected, nontrivial subgroups* $H$ *of* $N$, *such that* $H$ *is* **not** *a Cartan-decomposition subgroup.*

1. *If* $\dim\mathfrak{h} = 1$, $\mathfrak{h} = \mathfrak{z}$, *and we have* $|\eta_h|^2 = \mathsf{x}_h\mathsf{y}_h$ *for every* $h \in H$, *then* $\rho(h) \asymp h$ *for every* $h \in H$.

2. *If* $\phi_h = 0$ *and* $y_h = 0$ *for every* $h \in \mathfrak{h}$, *there is some* $u \in \mathfrak{h}$, *such that* $\mathsf{y}_u \neq 0$, *and* $\mathfrak{z} \subset \mathfrak{u}_{2\alpha+2\beta}$, *then* $\mu(H) \approx \left[\|h\|, \|h\|^{3/2}\right]$, *unless* $\dim H = 1$, *in which case* $\rho(h) \asymp \|h\|^{3/2}$ *for every* $h \in H$.

3. *Suppose* $\phi_h = 0$ *for every* $h \in \mathfrak{h}$, *and there is some* $\lambda \in \mathbb{C}$, *such that* $x_h = \lambda y_h$ *for every* $h \in H$, *and we have* $\eta_z = i\lambda \mathsf{y}_z$ *and* $\mathsf{x}_z = |\lambda|^2 \mathsf{y}_z$ *for every* $z \in \mathfrak{z}$.

   (a) *If there is some* $u \in \mathfrak{h}$, *such that* $\mathsf{x}_u + |\lambda|^2 \mathsf{y}_u + 2\operatorname{Im}(\lambda\overline{\eta_u}) \neq 0$, *then* $\mu(H) \approx \left[\|h\|, \|h\|^{3/2}\right]$, *unless* $\dim H = 1$, *in which case* $\rho(h) \asymp \|h\|^{3/2}$ *for every* $h \in H$.

   (b) *Otherwise,* $\rho(h) \asymp h$ *for every* $h \in H$.

4. *If* $y_h = 0$, $\mathsf{y}_h = 0$, *and* $|x_h|^2 + 2\operatorname{Re}(\phi_h\overline{\eta_h}) \neq 0$ *for every* $h \in \mathfrak{h} \setminus \mathfrak{u}_{2\alpha+2\beta}$ (*so* $\mathfrak{z} \subset \mathfrak{u}_{2\alpha+2\beta}$), *then* $\rho(h) \asymp h$ *for every* $h \in H$.

5. *If* $\mathfrak{z} = 0$, *there is some* $u \in \mathfrak{h}$ *and some nonzero* $\phi_0 \in \mathbb{C}$, *such that* $\phi_u \neq 0$, *and we have* $\phi_h = \phi_0 \mathsf{y}_h$ *and* $y_h = 0$, *for every* $h \in \mathfrak{h}$, *then* $\mu(H) \approx \left[\|h\|, \|h\|^{4/3}\right]$, *unless* $\dim H = 1$, *in which case,* $\rho(h) \asymp \|h\|^{4/3}$ *for every* $h \in H$.

6. *If* $\phi_h = 0$ *and* $\dim_{\mathbb{C}}\langle x_u, y_u\rangle \neq 1$ *for every* $h \in \mathfrak{h}$, *and* $|\eta_z|^2 \neq \mathsf{x}_z\mathsf{y}_z$, *for every nonzero* $z \in \mathfrak{z}$, *then* $\rho(h) \asymp \|h\|^2$ *for every* $h \in H$.

7. *If* $\phi_h = 0$ *for every* $h \in \mathfrak{h}$, *there exist nonzero* $u$ *and* $v$ *in* $\mathfrak{h}$, *such that* $\dim_{\mathbb{C}}\langle x_u, y_u\rangle \neq 1$ *and* $\dim_{\mathbb{C}}\langle x_v, y_v\rangle = 1$, *and we have* $\mathsf{x}_v|y_v|^2 + \mathsf{y}_v|x_v|^2 + 2\operatorname{Im}(x_v y_v^\dagger \overline{\eta_v}) \neq 0$ *for every such* $v \in \mathfrak{h}$, *and* $|\eta_z|^2 \neq \mathsf{x}_z\mathsf{y}_z$, *for every nonzero* $z \in \mathfrak{z}$ *then* $\mu(H) \approx \left[\|h\|^{3/2}, \|h\|^2\right]$.

8. *If* $\dim \mathfrak{h} \leq 3$, $\mathfrak{z} = 0$, *we have* $\phi_v \asymp y_v$ *and* $v = O\big(|\phi_v| + |\mathsf{y}_v|\big)$ *for every* $v \in \mathfrak{h}$, *and there exists* $u \in \mathfrak{h}$, *such that* $\phi_u \neq 0$, *then* $\rho(h) \asymp \|h\|^{3/2}$ *for every* $h \in H$.

9. *If* $\dim \mathfrak{h} = 2$, $\mathfrak{z} = \mathfrak{u}_{2\alpha+2\beta}$, $\phi_h \neq 0$ *and* $y_h \neq 0$ *for every* $h \in \mathfrak{h} \setminus \mathfrak{z}$, *then* $\mu(H) \approx \left[\|h\|, \|h\|^{3/2}\right]$.

10. *If* $\dim \mathfrak{h} \leq 2$ *and* $\phi_h \neq 0$, $y_h = 0$, $\mathsf{y}_h = 0$, *and* $|x_h|^2 + 2\operatorname{Re}(\phi_h\overline{\eta_h}) = 0$ *for every nonzero* $h \in \mathfrak{h}$, *then* $\rho(h) \asymp \|h\|^2$ *for every* $h \in H$.

11. *If* $\dim \mathfrak{h} = 2$ *and there exist nonzero* $u \in \mathfrak{h}$ *and* $z \in \mathfrak{z}$, *such that* $\phi_u \neq 0$, $y_u \neq 0$, $\mathsf{y}_z = 0$, $\phi_u\overline{\eta_z} \neq 0$ *is real, and* $\mathsf{x}_z|y_u|^2 - \phi_u \mathsf{y}_u \overline{\eta_z} + 2\operatorname{Im}\big(\overline{\eta_z} x_u y_u^\dagger\big) \neq 0$, *then* $\mu(H) \approx \left[\|h\|^{5/4}, \|h\|^2\right]$.

**Proof.** The theorem is obtained by merging the statement of Proposition 4.3 with the statement of Proposition 5.2, and eliminating some redundancy (see 3.4). Specifically:

- 4.3(1) appears here as 6.1(1).
- 4.3(2) appears here as 6.1(2).
- 4.3(3) appears here as 6.1(3).

- 4.3(4) appears here as 6.1(4).
- 4.3(5) appears here as 6.1(5).
- 4.3(6) appears here as 6.1(8).
- 4.3(7) appears here as 6.1(9).
- 5.2(1) is a special case of 6.1(6).
- 5.2(2) appears here as 6.1(6).
- 5.2(3) appears here as 6.1(7).
- 5.2(4) appears here as 6.1(8).
- 5.2(5) appears here as 6.1(10).
- 5.2(6) appears here as 6.1(11).
- 5.2(7) is a special case of 6.1(3a) (with $\dim H = 1$).
- 5.2(8) is a special case of 6.1(5) (with $\dim H = 1$).

■

**Corollary 6.2.** *Assume that* $G = \mathrm{SU}(2,n)$. *Here is a complete list of the closed, connected, nontrivial subgroups* $H$ *of* $N$, *such that* $H$ *is* **not** *a Cartan-decomposition subgroup,* **and** $N_A(H)$ *is nontrivial.*

1. *Suppose* $\dim \mathfrak{h} = 1$, $\mathfrak{h} = \mathfrak{z}$, *and we have* $|\eta_h|^2 = \mathsf{x}_h \mathsf{y}_h$ *for every* $h \in H$.
   (a) *If* $\mathfrak{h} = \mathfrak{u}_{2\beta}$ *or* $\mathfrak{h} = \mathfrak{u}_{2\alpha+2\beta}$, *then* $N_A(H) = A$.
   (b) *Otherwise,* $N_A(H) = \ker(\alpha)$.
2. *Suppose* $\phi_h = 0$ *and* $y_h = 0$ *for every* $h \in \mathfrak{h}$, *there is some* $u \in \mathfrak{h}$, *such that* $\mathsf{y}_u \neq 0$, *and* $\mathfrak{z} \subset \mathfrak{u}_{2\alpha+2\beta}$. *If* $\mathfrak{h} = \big(\mathfrak{h} \cap (\mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{2\beta})\big) + \mathfrak{z}$, *then* $N_A(H) = \ker(\alpha - \beta)$.
3. *Suppose* $\phi_h = 0$ *for every* $h \in \mathfrak{h}$, *and there is some nonzero* $\lambda \in \mathbb{C}$, *such that* $x_h = \lambda y_h$ *for every* $h \in H$, *and we have* $\eta_z = i\lambda \mathsf{y}_z$ *and* $\mathsf{x}_z = |\lambda|^2 \mathsf{y}_z$ *for every* $z \in \mathfrak{z}$. *If* $\mathfrak{h} = \big(\mathfrak{h} \cap (\mathfrak{u}_\beta + \mathfrak{u}_{\alpha+\beta})\big) + \mathfrak{z} \neq \mathfrak{z}$, *then* $N_A(H) = \ker(\alpha)$.
4. *Suppose* $\phi_h = 0$ *and* $x_h = 0$ *for every* $h \in \mathfrak{h}$, *we have* $\mathfrak{z} \subset \mathfrak{u}_{2\beta}$, *and* $\mathfrak{h} \neq \mathfrak{z}$.
   (a) *If* $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{u}_\beta) + \mathfrak{z}$, *then* $N_A(H) = A$.
   (b) *Otherwise:*
      i. *If* $\mathfrak{h} = \big(\mathfrak{h} \cap (\mathfrak{u}_\beta + \mathfrak{u}_{\alpha+2\beta})\big) + \mathfrak{z}$, *then* $N_A(H) = \ker(\alpha + \beta)$.
      ii. *If* $\mathfrak{h} = \big(\mathfrak{h} \cap (\mathfrak{u}_\beta + \mathfrak{u}_{2\alpha+2\beta})\big) + \mathfrak{z}$, *then* $N_A(H) = \ker(2\alpha + \beta)$.
      iii. *If* $\mathfrak{z} = 0$ *and* $\mathfrak{h} \subset \mathfrak{u}_\beta + \mathfrak{u}_{2\beta}$, *then* $N_A(H) = \ker(\beta)$.
5. *Suppose* $y_h = 0$, $\mathsf{y}_h = 0$, *and* $|x_h|^2 + 2\,\mathrm{Re}(\phi_h \overline{\eta_h}) \neq 0$ *for every* $h \in \mathfrak{h} \setminus \mathfrak{u}_{2\alpha+2\beta}$.

(a) *If* $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{u}_{\alpha+\beta}) + \mathfrak{z}$, *then* $N_A(H) = A$.

(b) *If* $\mathfrak{h} \subset \mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{2\alpha+2\beta}$, *but* $\mathfrak{h} \neq (\mathfrak{h} \cap \mathfrak{u}_{\alpha+\beta}) + \mathfrak{z}$, *then* $N_A(H) = \ker(\alpha+\beta)$.

(c) *If* $\mathfrak{h} = \big(\mathfrak{h} \cap (\mathfrak{u}_\alpha + \mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{\alpha+2\beta})\big) + \mathfrak{z}$, *but* $\mathfrak{h} \not\subset \mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{2\alpha+2\beta}$, *then* $N_A(H) = \ker(\beta)$.

6. *Suppose* $\mathfrak{z} = 0$, *there is some nonzero* $\phi_0 \in \mathbb{C}$, *such that* $\phi_h = \phi_0 \mathsf{y}_h$ *and* $y_h = 0$, *for every* $h \in \mathfrak{h}$, *and there is some* $u \in \mathfrak{h}$, *such that* $\phi_u \neq 0$. *If* $\mathfrak{h} = \big(\mathfrak{h} \cap (\mathfrak{u}_\alpha + \mathfrak{u}_{2\beta})\big) + (\mathfrak{h} \cap \mathfrak{u}_{\alpha+\beta})$, *then* $N_A(H) = \ker(\alpha - 2\beta)$.

7. *Suppose* $\phi_h = 0$ *and* $\dim_{\mathbb{C}}\langle x_u, y_u\rangle \neq 1$ *for every* $h \in \mathfrak{h}$, *and* $|\eta_z|^2 \neq \mathsf{x}_z \mathsf{y}_z$, *for every nonzero* $z \in \mathfrak{z}$.

(a) *If* $\mathfrak{h} \subset \mathfrak{u}_{\alpha+2\beta}$, *then* $N_A(H) = A$.

(b) *If* $\mathfrak{h} \not\subset \mathfrak{u}_{\alpha+2\beta}$, *and* $\mathfrak{h} = \big(\mathfrak{h} \cap (\mathfrak{u}_\beta + \mathfrak{u}_{\alpha+\beta})\big) + \mathfrak{z}$, *then* $N_A(H) = \ker(\alpha)$.

8. *Suppose* $\phi_h = 0$ *for every* $h \in \mathfrak{h}$, *there exist nonzero* $u, v \in \mathfrak{h}$, *such that* $\dim_{\mathbb{C}}\langle x_u, y_u\rangle \neq 1$ *and* $\dim_{\mathbb{C}}\langle x_v, y_v\rangle = 1$, $\mathsf{x}_v|y_v|^2 + \mathsf{y}_v|x_v|^2 + 2\operatorname{Im}(x_v y_v^\dagger \overline{\eta_v}) \neq 0$ *for every such* $v \in \mathfrak{h}$, *and* $|\eta_z|^2 \neq \mathsf{x}_z \mathsf{y}_z$, *for every nonzero* $z \in \mathfrak{z}$.

(a) *If* $\mathfrak{h} = \big(\mathfrak{h} \cap (\mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{2\beta})\big) + (\mathfrak{h} \cap \mathfrak{u}_{\alpha+2\beta})$, *then* $N_A(H) = \ker(\alpha - \beta)$ *(and* $\dim H \leq 3$*)*.

(b) *If* $\mathfrak{h} = \big(\mathfrak{h} \cap (\mathfrak{u}_\beta + \mathfrak{u}_{2\alpha+2\beta})\big) + (\mathfrak{h} \cap \mathfrak{u}_{\alpha+2\beta})$, *then* $N_A(H) = \ker(2\alpha + \beta)$ *(and* $\dim H \leq 3$*)*.

9. *Suppose* $\dim \mathfrak{h} \leq 3$, $\mathfrak{h} = \big(\mathfrak{h} \cap (\mathfrak{u}_\alpha + \mathfrak{u}_\beta)\big) + \big(\mathfrak{h} \cap (\mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{2\beta})\big)$, $\mathfrak{h} \cap (\mathfrak{u}_\alpha + \mathfrak{u}_\beta) \neq 0$, *and we have* $\phi_h \asymp y_h$ *and* $x_h \asymp \mathsf{y}_h$ *for* $h \in \mathfrak{h}$, *then* $N_A(H) = \ker(\alpha - \beta)$.

10. *Suppose* $\dim \mathfrak{h} = 2$, $\mathfrak{z} = \mathfrak{u}_{2\alpha+2\beta}$, $\phi_h \neq 0$ *and* $y_h \neq 0$ *for every* $h \in \mathfrak{h} \setminus \mathfrak{z}$. *If* $\mathfrak{h} = \big(\mathfrak{h} \cap (\mathfrak{u}_\alpha + \mathfrak{u}_\beta)\big) + \mathfrak{z}$, *then* $N_A(H) = \ker(\alpha - \beta)$.

11. *Suppose* $\dim \mathfrak{h} \leq 2$ *and* $\phi_h \neq 0$, $y_h = 0$, $\mathsf{y}_h = 0$, *and* $|x_h|^2 + 2\operatorname{Re}(\phi_h \overline{\eta_h}) = 0$ *for every nonzero* $h \in \mathfrak{h}$.

(a) *If* $\mathfrak{h} \subset \mathfrak{u}_\alpha$, *then* $N_A(H) = A$.

(b) *If* $\mathfrak{h} \subset \mathfrak{u}_\alpha + \mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{\alpha+2\beta}$, *but* $\mathfrak{h} \not\subset \mathfrak{u}_\alpha$, *then* $N_A(H) = \ker(\beta)$.

(c) *If* $\mathfrak{h} \subset \mathfrak{u}_\alpha + \mathfrak{u}_{2\alpha+2\beta}$, *but* $\mathfrak{h} \not\subset \mathfrak{u}_\alpha$, *then* $N_A(H) = \ker(\alpha + 2\beta)$.

**Proof.** It is clear that each of the given subgroups is normalized by the indicated torus. We now show that the list is complete, and that no larger subtorus of $A$ normalizes $H$.

Assume $N_A(H)$ is nontrivial. We proceed in cases, determined by Theorem 6.1.

*Case 1. Assume* 6.1(1). We may assume $\mathfrak{h}$ is neither $\mathfrak{u}_{2\beta}$ nor $\mathfrak{u}_{2\alpha+2\beta}$ (otherwise (1a) applies). Then, because $|\eta_u|^2 = \mathsf{x}_u \mathsf{y}_u$ for every $u \in \mathfrak{h}$, we see that $\eta_u \neq 0$ for every nonzero $u \in \mathfrak{h}$. Thus, the projection of $\mathfrak{h}$ to $\mathfrak{u}_{\alpha+2\beta}$ is nontrivial. However, because $|\eta_u|^2 = \mathsf{x}_u \mathsf{y}_u$, we have $\mathfrak{h} \cap \mathfrak{u}_{\alpha+2\beta} = 0$. We know that $\mathfrak{h} \subset \mathfrak{u}_{\alpha+2\beta} + \mathfrak{u}_{2\beta} + \mathfrak{u}_{2\alpha+2\beta}$ (because $\mathfrak{h} = \mathfrak{z}$), so, because each of $2\beta$ and $2\alpha + 2\beta$ differs from $\alpha + 2\beta$ by $\alpha$, we conclude that $N_A(H) = \ker(\alpha)$, so (1b) applies.

*Case 2. Assume* 6.1(2). Let $V$ be the projection of $\mathfrak{h}$ to $\mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{2\beta}$. Because $\mathsf{y}_u \neq 0$, we know that $V$ projects nontrivially to $\mathfrak{u}_{2\beta}$. However, because $\mathfrak{z} \subset \mathfrak{u}_{2\alpha+2\beta}$, we also know that $V \cap \mathfrak{u}_{2\beta} = 0$. Therefore $N_A(H) = \ker(\alpha - \beta)$. Then, because neither $\alpha+2\beta$ nor $2\alpha+2\beta$ differs from $\alpha+\beta$ by a multiple of $\alpha-\beta$, we conclude that $\mathfrak{h} = \big(\mathfrak{h} \cap (\mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{2\beta})\big) + \mathfrak{z}$, so (2) applies.

*Case 3. Assume* 6.1(3). We may assume $\mathfrak{h} \neq \mathfrak{z}$ (otherwise Case 1 applies).

*Subcase 3.1. Assume* $\lambda \neq 0$. Because $\mathfrak{h} \neq \mathfrak{z}$, the projection of $\mathfrak{h}$ to $\mathfrak{u}_\beta + \mathfrak{u}_{\alpha+\beta}$ is nontrivial. However, because $\lambda \neq 0$, this projection intersects neither $\mathfrak{u}_\beta$ nor $\mathfrak{u}_{\alpha+\beta}$. Therefore $N_A(H) \subset \ker(\alpha)$. Then, because neither $2\beta$, $\alpha+2\beta$, nor $2\alpha+2\beta$ differs from $\beta$ by a multiple of $\alpha$, we conclude that $\mathfrak{h} = \big(\mathfrak{h} \cap (\mathfrak{u}_\beta + \mathfrak{u}_{\alpha+\beta})\big) + \mathfrak{z}$, so (3) applies.

*Subcase 3.2. Assume* $\lambda = 0$. This means $x_u = 0$ for every $u \in \mathfrak{h}$, and $\mathfrak{z} \subset \mathfrak{u}_{2\beta}$.

Because $\mathfrak{h} \neq \mathfrak{z}$, we know that $\mathfrak{h}$ projects nontrivially to $\mathfrak{u}_\beta$. Because $\mathfrak{z} \subset \mathfrak{u}_{2\beta}$, we know that $\mathfrak{h} \cap \mathfrak{u}_{\alpha+2\beta} = \mathfrak{h} \cap \mathfrak{u}_{2\alpha+2\beta} = 0$. Thus, it is easy to see that if $\mathfrak{h}$ projects nontrivially to $\mathfrak{u}_{\alpha+2\beta}$ or $\mathfrak{u}_{2\alpha+2\beta}$ then either (4(b)i) or (4(b)ii) applies.

Thus, we may assume $\mathfrak{h} \subset \mathfrak{u}_\beta + \mathfrak{u}_{2\beta}$. If $\mathfrak{z} \neq 0$, then $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{u}_\beta) + \mathfrak{u}_{2\beta}$, so (4a) applies. Otherwise, (4(b)iii) applies.

*Case 4. Assume* 6.1(4).

*Subcase 4.1. Assume the projection of* $\mathfrak{h}$ *to* $\mathfrak{u}_\alpha$ *is trivial.* Because

$$|x_u|^2 = |x_u|^2 + 2\,\mathrm{Re}(\phi_u \overline{\eta_u}) \neq 0$$

for every $u \in \mathfrak{h} \setminus \mathfrak{u}_{2\alpha+2\beta}$, we know that $x_u \neq 0$ for every $u \in \mathfrak{h} \setminus \mathfrak{u}_{2\alpha+2\beta}$. Thus, if the projection of $\mathfrak{h}$ to $\mathfrak{u}_{\alpha+2\beta}$ is nontrivial, then $N_A(H) = \ker(\beta)$, and we see that (5c) applies. If not, then $\mathfrak{h} \subset \mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{2\alpha+2\beta}$, so either (5a) or (5b) applies.

*Subcase 4.2. Assume the projection of* $\mathfrak{h}$ *to* $\mathfrak{u}_\alpha$ *is nontrivial.* Let $V$ be the projection of $\mathfrak{h}$ to $\mathfrak{u}_\alpha + \mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{\alpha+2\beta}$. Because $|x_u|^2 + 2\,\mathrm{Re}(\phi_u \overline{\eta_u}) \neq 0$ for every $u \in \mathfrak{h} \setminus \mathfrak{u}_{2\alpha+2\beta}$, we know that $V \cap \mathfrak{u}_\alpha = 0$. Then, because $\alpha$, $\alpha+\beta$, and $\alpha+2\beta$ all differ by multiples of $\beta$, we conclude that $N_A(H) = \ker(\beta)$. Therefore (5c) applies.

*Case 5. Assume* 6.1(5). Let $V$ be the projection of $\mathfrak{h}$ to $\mathfrak{u}_\alpha + \mathfrak{u}_{2\beta}$. Because $\phi_h = \phi_0 \mathsf{y}_h$, we see that $V \cap \mathfrak{u}_\alpha = 0$ and $V \cap \mathfrak{u}_{2\beta} = 0$. Therefore $N_A(H) = \ker(\alpha - 2\beta)$.

Because no other roots differ by a multiple of $\alpha - 2\beta$ (and $\mathfrak{z} = 0$), we conclude that $\mathfrak{h} = \big(\mathfrak{h} \cap (\mathfrak{u}_\alpha + \mathfrak{u}_{2\beta})\big) + (\mathfrak{h} \cap \mathfrak{u}_{\alpha+\beta})$. Thus, (6) applies.

*Case 6. Assume* 6.1(6).

*Subcase 6.1. Assume* $\mathfrak{h} \neq \mathfrak{z}$. Let $V$ be the projection of $\mathfrak{h}$ to $\mathfrak{u}_\beta + \mathfrak{u}_{\alpha+\beta}$. From the assumption of this subcase, we know $V \neq 0$. However, because $\dim_{\mathbb{C}}\langle x_u, y_u\rangle \neq 1$ for every $u \in \mathfrak{h}$, we know that $V \cap \mathfrak{u}_\beta = 0$ and $\mathfrak{h} \cap \mathfrak{u}_{\alpha+\beta} = 0$. Therefore $N_A(H) = \ker(\alpha)$, so (7b) applies.

*Subcase 6.2. Assume* $\mathfrak{h} = \mathfrak{z}$. We may assume $\mathfrak{h} \not\subset \mathfrak{u}_{\alpha+2\beta}$ (otherwise (7a) applies). Therefore, $\mathfrak{h}$ projects nontrivially to $\mathfrak{u}_{2\beta} + \mathfrak{u}_{2\alpha+2\beta}$. However, because $|\eta_z|^2 \neq \mathsf{x}_z \mathsf{y}_z$, for every nonzero $z \in \mathfrak{z}$, we know that $V \cap \mathfrak{u}_{2\beta} = 0$ and $V \cap \mathfrak{u}_{2\alpha+2\beta} = 0$. Because $2\beta$,

$\alpha+2\beta$, and $2\alpha+2\beta$ all differ by multiples of $\alpha$, we conclude that $N_A(H) = \ker(\alpha)$, so (7b) applies.

*Case 7. Assume* 6.1(7).

*Subcase 7.1. Assume* $N_A(H) = \ker(\alpha)$. Because $\alpha+\beta$ is the only root that differs from $\beta$ by a multiple of $\alpha$, we must have $\mathfrak{h} = \big(\mathfrak{h} \cap (\mathfrak{u}_\beta + \mathfrak{u}_{\alpha+\beta})\big) + \mathfrak{z}$. Thus, there is some $w \in \mathfrak{h}$, such that $x_w = x_v$ and $y_w = y_v$, but the projection of $w$ to $\mathfrak{u}_{2\beta} + \mathfrak{u}_{\alpha+2\beta} + \mathfrak{u}_{2\alpha+2\beta}$ is zero. This contradicts the fact that $\mathsf{x}_w|y_w|^2 + \mathsf{y}_w|x_w|^2 + 2\operatorname{Im}(x_w y_w^\dagger \overline{\eta_w}) \neq 0$.

*Subcase 7.2. Assume* $N_A(H) \neq \ker(\alpha)$. Because $2\beta$, $\alpha+2\beta$, and $2\alpha+2\beta$ all differ by multiples of $\alpha$, we must have $\mathfrak{z} = (\mathfrak{z} \cap \mathfrak{u}_{2\beta}) + (\mathfrak{z} \cap \mathfrak{u}_{\alpha+2\beta}) + (\mathfrak{z} \cap \mathfrak{u}_{2\alpha+2\beta})$. Then, because $|\eta_z|^2 \neq \mathsf{x}_z\mathsf{y}_z$ for every nonzero $z \in \mathfrak{z}$, we conclude that $\mathfrak{z} \subset \mathfrak{u}_{\alpha+2\beta}$.

Let $V$ be the projection of $\mathfrak{h}$ to $\mathfrak{u}_\beta + \mathfrak{u}_{\alpha+\beta}$. Because $\beta$ and $\alpha+\beta$ differ by $\alpha$, we know that $V = (V \cap \mathfrak{u}_\beta) + (V \cap \mathfrak{u}_{\alpha+\beta})$.

*Subsubcase 7.2.1. Assume* $x_v \neq 0$. Because $V = (V \cap \mathfrak{u}_\beta) + (V \cap \mathfrak{u}_{\alpha+\beta})$, there is some $w \in V$, such that $x_w \neq 0$ and $y_w = 0$. For every such $w$, because $\mathsf{x}_w|y_w|^2 + \mathsf{y}_w|x_w|^2 + 2\operatorname{Im}(x_w y_w^\dagger \overline{\eta_w}) \neq 0$, we know that $\mathsf{y}_w \neq 0$. Thus, we see that $N_A(H) = \ker\big((\alpha+\beta) - 2\beta\big) = \ker(\alpha-\beta)$.

We know that $\mathfrak{h} \cap \mathfrak{u}_\beta = 0$, that $\mathfrak{h}$ projects trivially to $\mathfrak{u}_\alpha$, and that $\alpha$ is the only root that differs from $\beta$ by a multiple of $\alpha-\beta$, so we conclude that $y_h = 0$ for every $h \in H$.

We now see that (8a) applies.

*Subsubcase 7.2.2. Assume* $y_v \neq 0$. This is similar to the preceding subsubcase (indeed, they are conjugate under the Weyl reflection corresponding to the root $\alpha$); we see that (8b) applies.

*Case 8. Assume* 6.1(8). By considering the projection of $\mathfrak{h}$ to $\mathfrak{u}_\alpha + \mathfrak{u}_\beta$, and noting that $\phi_h \asymp y_h$ for every $h \in H$, we see that $N_A(H) = \ker(\alpha-\beta)$. The only other pair of roots that differ by a multiple of $\alpha-\beta$ is $\{\alpha+\beta, 2\beta\}$. Thus, we see that (9) applies.

*Case 9. Assume* 6.1(9). By considering the projection of $\mathfrak{h}$ to $\mathfrak{u}_\alpha + \mathfrak{u}_\beta$, we see that $N_A(H) = \ker(\alpha-\beta)$. Because $\phi_u \neq 0$ for every $u \in \mathfrak{h} \setminus \mathfrak{u}_{2\alpha+2\beta}$, but $\beta$ is the only root that differs from $\alpha$ by a multiple of $\alpha-\beta$, we conclude that $\mathfrak{h}$ projects trivially into every root space except $\mathfrak{u}_\alpha$, $\mathfrak{u}_\beta$, and $\mathfrak{u}_{2\alpha+2\beta}$. Thus (10) applies.

*Case 10. Assume* 6.1(10). We may assume $\mathfrak{h} \not\subset \mathfrak{u}_\alpha$ (otherwise (11a) applies). Thus, there is some root $\sigma \neq \alpha$, such that the projection of $\mathfrak{h}$ to $\mathfrak{u}_\sigma$ is nontrivial. However, because $\phi_h \neq 0$ for every nonzero $h \in \mathfrak{h}$, we know that $\mathfrak{h} \cap \mathfrak{u}_\sigma = 0$. Thus, $N_A(H) = \ker(\alpha-\sigma)$.

Because $y_h = 0$ and $\mathsf{y}_h = 0$ for every nonzero $h \in \mathfrak{h}$, we know that $\sigma \neq \beta$ and $\sigma \neq 2\beta$. If $\sigma = \alpha+\beta$ or $\sigma = \alpha+2\beta$, we obtain (11b). If $\sigma = 2\alpha+2\beta$, we obtain (11c).

*Case 11. Assume* 6.1(11). Because $\phi_u \neq 0$ and $y_u \neq 0$, we must have $N_A(H) = \ker(\alpha-\beta)$. Then, because $\alpha+\beta$ does not differ from $\alpha$ by a multiple of $\alpha-\beta$, we conclude that $x_u = 0$.

Because $\eta_z \neq 0$, but no root differs from $\alpha+2\beta$ by a multiple of $\alpha-\beta$, we conclude that $\mathfrak{h} \cap \mathfrak{u}_{\alpha+2\beta} \neq 0$. Because $\mathfrak{z}$ is one-dimensional, this implies $z \in \mathfrak{u}_{\alpha+2\beta}$, so $\mathsf{x}_z = 0$.

Since $\mathsf{x}_z = 0$ and $x_u = 0$, we conclude, from the inequality $\mathsf{x}_z|y_u|^2 - \phi_u \mathsf{y}_u \overline{\eta_z} + 2\,\mathrm{Im}(\overline{\eta_z} x_u y_u^\dagger) \neq 0$, that $\mathsf{y}_u \neq 0$. This is a contradiction, because $2\beta$ does not differ from $\alpha$ by a multiple of $\alpha - \beta$, and $\mathfrak{h} \cap \mathfrak{u}_{2\beta} = 0$ (because, as shown above, $\mathfrak{z} \subset \mathfrak{u}_{\alpha+2\beta}$). ∎

## 7 Subgroups that are not contained in $N$

Let $H$ be a closed, connected subgroup of $G$ that is not contained in $N$. In this section, we determine whether $H$ is a Cartan-decomposition subgroup or not (and, if not, we calculate $\mu(H)$).

Lemma 7.1 shows that we may assume $H \subset AN$, and then Lemma 7.3 shows that we may assume $H$ satisfies the technical condition of being compatible with $A$. (Both of these lemmas are well known.) Furthermore, we may assume that $H \cap N$ is **not** a Cartan-decomposition subgroup, and that $A \not\subset H$ (otherwise, it is obvious that $H$ is a Cartan-decomposition subgroup).

Theorem 7.4 describes $\mu(H)$ for every such subgroup that is a semidirect product $(H \cap A) \ltimes (H \cap N)$; and Proposition 7.6 describes $\mu(H)$ for the other subgroups (except that the one-dimensional case appears in Lemma 7.8).

**Lemma 7.1.** [14, Lem. 2.9] *Let $H$ be a closed, connected subgroup of a connected, almost simple, linear, real Lie group $G$. There is a closed, connected subgroup $H'$ of $G$ and a compact subgroup $C$ of $G$, such that $CH = CH'$, and $H'$ is conjugate to a subgroup of $AN$.*

**Definition 7.2.** Let us say that a subgroup $H$ of $AN$ is *compatible* with $A$ if $H \subset TUC_N(T)$, where $T = A \cap (HN)$, $U = H \cap N$, and $C_N(T)$ denotes the centralizer of $T$ in $N$.

**Lemma 7.3.** [14, Lem. 2.3] *If $H$ is a closed, connected subgroup of $AN$, then $H$ is conjugate, via an element of $N$, to a subgroup that is compatible with $A$.*

**Theorem 7.4.** *Assume that $G = \mathrm{SU}(2,n)$. Here is a list of every closed, connected, nontrivial subgroup $H$ of $AN$, such that $H$ is of the form $H = T \ltimes U$, where $T$ is a one-dimensional subgroup of $A$, and $U$ is a nontrivial subgroup of $N$ that is **not** a Cartan-decomposition subgroup.*

1. *Suppose $\dim \mathfrak{u} = 1$, $\mathfrak{u} = \mathfrak{z}$, and we have $|\eta_h|^2 = \mathsf{x}_h \mathsf{y}_h$ for every $h \in U$.*
   (a) *If $\mathfrak{u} = \mathfrak{u}_{2\beta}$ or $\mathfrak{u} = \mathfrak{u}_{2\alpha+2\beta}$, then $\mu(H)$ is described in [14, Prop. 3.17 or Cor. 3.18].*
   (b) *Otherwise, $T = \ker(\alpha)$, and $H$ is a Cartan-decomposition subgroup.*
2. *Suppose $\mathfrak{u} = \big(\mathfrak{u} \cap (\mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{2\beta})\big) + \mathfrak{z}$, $\mathfrak{z} \subset \mathfrak{u}_{2\alpha+2\beta}$, there is some $v \in \mathfrak{u}$, such that $\mathsf{y}_v \neq 0$, and $T = \ker(\alpha - \beta)$. Then $\mu(H) \approx \big[\|h\|, \|h\|^{3/2}\big]$, unless $\dim H = 2$, in which case $\rho(h) \asymp \|h\|^{3/2}$ for every $h \in H$.*
3. *Suppose $\mathfrak{u} = \big(\mathfrak{u} \cap (\mathfrak{u}_\beta + \mathfrak{u}_{\alpha+\beta})\big) + \mathfrak{z}$, $T = \ker(\alpha)$, and there is some nonzero $\lambda \in \mathbb{C}$, such that we have $x_u = \lambda y_u$ for every $u \in U$, and we have $\eta_z = i\lambda \mathsf{y}_z$ and $\mathsf{x}_z = |\lambda|^2 \mathsf{y}_z$ for every $z \in \mathfrak{z}$. Then $H$ is a Cartan-decomposition subgroup.*

4. *Suppose* $\phi_u = 0$ *and* $x_u = 0$ *for every* $u \in \mathfrak{u}$, *we have* $\mathfrak{z} \subset \mathfrak{u}_{2\beta}$, *and* $\mathfrak{u} \neq \mathfrak{z}$.

   (a) *If* $\mathfrak{u} = (\mathfrak{u} \cap \mathfrak{u}_\beta) + \mathfrak{z}$, *then* $\mu(H)$ *is described in [14, Prop. 3.17 or Cor. 3.18].*

   (b) *Otherwise:*

      i. *If* $\mathfrak{u} = \big(\mathfrak{u} \cap (\mathfrak{u}_\beta + \mathfrak{u}_{\alpha+2\beta})\big) + \mathfrak{z}$, *then* $T = \ker(\alpha + \beta)$, *and* $\rho(h) \asymp h$ *for every* $h \in H$.

      ii. *If* $\mathfrak{u} = \big(\mathfrak{u} \cap (\mathfrak{u}_\beta + \mathfrak{u}_{2\alpha+2\beta})\big) + \mathfrak{z}$, *then* $T = \ker(2\alpha + \beta)$, *and* $\mu(H) \approx \big[\|h\|, \|h\|^{3/2}\big]$, *unless* $\dim H = 2$, *in which case* $\rho(h) \asymp \|h\|^{3/2}$ *for every* $h \in H$.

      iii. *If* $\mathfrak{z} = 0$ *and* $\mathfrak{u} \subset \mathfrak{u}_\beta + \mathfrak{u}_{2\beta}$, *then* $T = \ker(\beta)$, *and* $H$ *is a Cartan-decomposition subgroup.*

5. *Suppose* $y_u = 0$, $\mathsf{y}_u = 0$, *and* $|x_u|^2 + 2\operatorname{Re}(\phi_u \overline{\eta_u}) \neq 0$ *for every* $u \in U \setminus U_{2\alpha+2\beta}$.

   (a) *If* $\mathfrak{u} = (\mathfrak{u} \cap \mathfrak{u}_{\alpha+\beta}) + \mathfrak{z}$, *then* $\mu(H)$ *is described in [14, Prop. 3.17 or Cor. 3.18].*

   (b) *If* $\mathfrak{u} \subset \mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{2\alpha+2\beta}$, *but* $\mathfrak{u} \neq (\mathfrak{u} \cap \mathfrak{u}_{\alpha+\beta}) + \mathfrak{z}$, *then* $T = \ker(\alpha + \beta)$, *and* $H$ *is a Cartan-decomposition subgroup.*

   (c) *If* $\mathfrak{u} = \big(\mathfrak{u} \cap (\mathfrak{u}_\alpha + \mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{\alpha+2\beta})\big) + \mathfrak{z}$, *but* $\mathfrak{u} \not\subset \mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{2\alpha+2\beta}$, *then* $T = \ker(\beta)$, *and* $\rho(h) \asymp h$ *for every* $h \in H$.

6. *Suppose* $\mathfrak{u} = \big(\mathfrak{u} \cap (\mathfrak{u}_\alpha + \mathfrak{u}_{2\beta})\big) + (\mathfrak{u} \cap \mathfrak{u}_{\alpha+\beta})$, $T = \ker(\alpha - 2\beta)$, $\mathfrak{u} \not\subset \mathfrak{u}_{\alpha+\beta}$, *and there is some nonzero* $\phi_0 \in \mathbb{C}$, *such that* $\phi_u = \phi_0 \mathsf{y}_u$ *for every* $u \in U$. *Then* $\mu(H) \approx \big[\|h\|, \|h\|^{4/3}\big]$, *unless* $\dim H = 2$, *in which case,* $\rho(h) \asymp \|h\|^{4/3}$ *for every* $h \in H$.

7. *Suppose* $\phi_u = 0$ *and* $\dim_{\mathbb{C}} \langle x_u, y_u \rangle \neq 1$ *for every* $u \in U$, *and* $|\eta_z|^2 \neq \mathsf{x}_z \mathsf{y}_z$, *for every nonzero* $z \in \mathfrak{z}$.

   (a) *If* $\mathfrak{u} \subset \mathfrak{u}_{\alpha+2\beta}$, *then* $\mu(H)$ *is described in [14, Prop. 3.17 or Cor. 3.18].*

   (b) *If* $\mathfrak{u} \not\subset \mathfrak{u}_{\alpha+2\beta}$, *and* $\mathfrak{u} = \big(\mathfrak{u} \cap (\mathfrak{u}_\beta + \mathfrak{u}_{\alpha+\beta})\big) + \mathfrak{z}$, *then* $T = \ker(\alpha)$, *and* $\rho(h) \asymp \|h\|^2$ *for every* $h \in H$.

8. *Suppose* $\phi_u = 0$ *for every* $u \in U$, *there exist nonzero* $v_1, v_2 \in \mathfrak{u}$, *such that* $\dim_{\mathbb{C}} \langle x_{v_1}, y_{v_1} \rangle \neq 1$ *and* $\dim_{\mathbb{C}} \langle x_{v_2}, y_{v_2} \rangle = 1$, *and we have* $\mathsf{x}_{v_2} |y_{v_2}|^2 + \mathsf{y}_{v_2} |x_{v_2}|^2 + 2\operatorname{Im}(x_{v_2} y_{v_2}^\dagger \overline{\eta_{v_2}}) \neq 0$ *for every such* $v_2 \in \mathfrak{u}$, *and* $|\eta_z|^2 \neq \mathsf{x}_z \mathsf{y}_z$, *for every nonzero* $z \in \mathfrak{z}$.

   (a) *If* $\mathfrak{u} = \big(\mathfrak{u} \cap (\mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{2\beta})\big) + (\mathfrak{u} \cap \mathfrak{u}_{\alpha+2\beta})$, *then* $T = \ker(\alpha - \beta)$ *and* $\mu(H) \approx \big[\|h\|^{3/2}, \|h\|^2\big]$.

   (b) *If* $\mathfrak{u} = \big(\mathfrak{u} \cap (\mathfrak{u}_\beta + \mathfrak{u}_{2\alpha+2\beta})\big) + (\mathfrak{u} \cap \mathfrak{u}_{\alpha+2\beta})$, *then* $T = \ker(2\alpha + \beta)$ *and* $\mu(H) \approx \big[\|h\|^{3/2}, \|h\|^2\big]$.

9. *Suppose* $\dim \mathfrak{u} \leq 3$, $\mathfrak{u} = \big(\mathfrak{u} \cap (\mathfrak{u}_\alpha + \mathfrak{u}_\beta)\big) + \big(\mathfrak{u} \cap (\mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{2\beta})\big)$, $\mathfrak{u} \cap (\mathfrak{u}_\alpha + \mathfrak{u}_\beta) \neq 0$, *and we have* $\phi_u \asymp y_u$ *and* $x_u \asymp \mathsf{y}_u$ *for* $u \in U$. *Then* $T = \ker(\alpha - \beta)$, *and* $\rho(h) \asymp \|h\|^{3/2}$ *for every* $h \in H$.

*10. Suppose* $\dim \mathfrak{u} = 2$, $\mathfrak{z} = \mathfrak{u}_{2\alpha+2\beta}$, $\phi_u \neq 0$ *and* $y_u \neq 0$ *for every* $u \in U \setminus Z$. *If* $\mathfrak{u} = \big(\mathfrak{u} \cap (\mathfrak{u}_\alpha + \mathfrak{u}_\beta)\big) + \mathfrak{z}$, *then* $T = \ker(\alpha - \beta)$, *and* $\mu(H) \approx \big[\|h\|, \|h\|^{3/2}\big]$.

*11. Suppose* $\dim \mathfrak{u} \leq 2$ *and* $\phi_u \neq 0$, $y_u = 0$, $\mathsf{y}_u = 0$, *and* $|x_u|^2 + 2\operatorname{Re}(\phi_u \overline{\eta_u}) = 0$ *for every nontrivial* $u \in U$.

*(a) If* $\mathfrak{u} \subset \mathfrak{u}_\alpha$, *then* $\mu(H)$ *is described in [14, Prop. 3.17 or Cor. 3.18].*

*(b) If* $\mathfrak{u} \subset \mathfrak{u}_\alpha + \mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{\alpha+2\beta}$, *but* $\mathfrak{u} \not\subset \mathfrak{u}_\alpha$, *then* $T = \ker(\beta)$, *and* $H$ *is a Cartan-decomposition subgroup.*

*(c) If* $\mathfrak{u} \subset \mathfrak{u}_\alpha + \mathfrak{u}_{2\alpha+2\beta}$, *but* $\mathfrak{u} \not\subset \mathfrak{u}_\alpha$, *then* $T = \ker(\alpha + 2\beta)$, *and* $\rho(h) \asymp \|h\|^2$ *for every* $h \in H$.

**Proof.** For $h \in H$, we wish to approximately calculate $\|\rho(h)\|$. We write $h = au$ with $a \in T$ and $u \in U$. Writing $a = \operatorname{diag}(a_1, a_2, \dots, a_{n+2})$, we always assume either that $a_1 > 1$ or that $a_1 = 1$ and $a_2 \geq 1$ (perhaps replacing $h$ with $h^{-1}$—because $\|\rho(h)\| = \|\rho(h^{-1})\|$, this causes no harm).

Because $T$ normalizes $U$, we know that $U$ is a subgroup that is listed in Corollary 6.2, and we have $T \subset N_G(U)$. This leads to the various cases listed in the statement of the theorem.

(1b) We have $\rho(u) \asymp u$ for $u \in U$ and $\rho(a) \asymp \|a\|^2$ for $a \in T$, so $H$ is a Cartan-decomposition subgroup.

(2) We have $|\phi_u| + |y_u| + |\eta_u| + |\mathsf{x}_u| = 0$ and $\mathsf{y}_u = O(x_u)$, so $u_{i,j} = O\big(1 + |x_u|\big)$ whenever $(i,j) \neq (1, n+2)$. Then, because $a_1 = a_2^2$, we see that

$$u_{i,j} = O\big[a_2\big(1 + |x_u|\big)\big] = O\big(|h_{1,1}|^{1/2} + |h_{1,n+2}|^{1/2}\big) = O\big(\|h\|^{1/2}\big)$$

whenever $i > 1$. Therefore $\rho(h) = O\big(\|h\|^{3/2}\big)$. This completes the proof if $\dim H > 2$ (that is, if $\dim U > 1$).

If $\dim U = 1$, then $\mathsf{y}_u \asymp x_u$ and $\mathsf{x}_u = 0$. We have $\|h\| = a_1\big(1 + |x_u|^2\big)$,

$$\Delta(h) = a_1 a_2 \left[ i \left( \frac{1}{2} |x_u|^2 \mathsf{y}_u \right) \right] \asymp \big(a_1 |x_u|^2\big)^{3/2}$$

and

$$\det \begin{pmatrix} h_{1,1} & h_{1,2} \\ h_{2,1} & h_{2,2} \end{pmatrix} = a_1 a_2 = a_1^{3/2}.$$

Thus, $\|h\|^{3/2} = O\big(\rho(h)\big)$. We conclude that $\rho(h) \asymp \|h\|^{3/2}$.

(3) Replacing $H$ by a conjugate under $U_\alpha$, we may replace $H$ with a similar subgroup $H'$ with $\lambda = 0$. Thus, $H' = T \ltimes U'$ with $U' \subset U_\beta U_{2\beta}$. Then [14, Prop. 3.17] implies $H$ is a Cartan-decomposition subgroup.

(4(b)i) The Weyl reflection corresponding to the root $\alpha$ conjugates $H$ to a subgroup of type (5c).

(4(b)ii) The Weyl reflection corresponding to the root $\alpha$ conjugates $H$ to a subgroup of type (2).

(4(b)iii) [14, Prop. 3.17] implies $H$ is a Cartan-decomposition subgroup.

(5b) [14, Prop. 3.17] implies $H$ is a Cartan-decomposition subgroup.

(5c) We have

$$h_{i,j} = \begin{cases} O(1) & \text{if } i \neq 1 \text{ and } j \neq n+2 \\ O(a_1 x) & \text{if } i = 1 \text{ and } j \neq n+2 \\ O(x) & \text{if } i \neq 1 \text{ and } j = n+2 \end{cases}$$

and $h_{1,n+2} \asymp a_1\big(|x|^2 + |\mathsf{x}|\big)$. We conclude that $\rho(h) \asymp h$.

(6) From the proof of 4.3(5), we know that $u \asymp u_{1,n+2}$, that $u_{i,j} = O\big(\|u\|^{3/2}\big)$ whenever $(i,j) \neq (1, n+2)$, and that $u_{i,j} = O\big(\|u\|^{1/3}\big)$ whenever $i \neq 1$ and $j \neq n+2$. (In particular, $h \asymp a_1\big(1 + u_{1,n+2}\big)$.) Furthermore, we have $a_1 = a_2^3$. Therefore

$$\rho(h) \asymp a_1 a_2 \rho(u) \asymp a_1^{4/3} \rho(u).$$

The desired conclusion follows.

(7b) From Lemma 5.4, we know $\|u\|^2 = O\big(1 + |\Delta(u)|\big)$. Then, because

$$\det \begin{pmatrix} h_{1,1} & h_{1,2} \\ h_{2,1} & h_{2,2} \end{pmatrix} = a_1 a_2 = a_1^2$$

and $\Delta(au) = a_1^2 \Delta(u)$, we have

$$\|h\|^2 = O\big(a_1^2 \|u\|^2\big) = O\big(a_1^2 + |\Delta(h)|\big) = O\big(\rho(h)\big).$$

(8) Assume (8a). (The other case, (8b), is conjugate to this one by the Weyl reflection corresponding to the root $\alpha$.) From Lemma 5.4, we have $\|u\|^{3/2} = O\big(1 + |\Delta(u)|\big)$. Then, because $a_1 = a_2^2$, we have

$$\|h\|^{3/2} = a_1 a_2 \|u\|^{3/2} = O\big(a_1 a_2 + |\Delta(h)|\big) = O\big(\rho(h)\big).$$

(9) From the proof of 4.3(6), we know $\rho(u) \asymp 1 + \Delta(u) \asymp \|u\|^{3/2}$. The proof is completed as in (8).

(10) Because $\phi_u \asymp y_u$, it is easy to see that

$$h \asymp a_1\big(1 + |\phi_u|^2 |y_u|^2 + |\mathsf{x}_u|\big) \asymp a_1\big(1 + |\phi_u|^4 + |\mathsf{x}_u|\big)$$

and

$$\Delta(h) \asymp a_1 a_2 \big(|y_u|^4 |\phi_u|^2 + |\mathsf{x}_u||y_u|^2\big) \asymp a_1^{3/2}\big(|\phi_u|^6 + |\mathsf{x}_u||\phi_u|^2\big) = O\big(\|h\|^{3/2}\big).$$

Then it is not difficult to see that $\rho(h) = O\big(\|h\|^{3/2}\big)$ for every $h \in H$. So $\mu(H) \approx \big[\|h\|, \|h\|^{3/2}\big]$.

(11b) We have $\rho(a) \asymp a$ for $a \in T$ and $\rho(u) \asymp \|u\|^2$ for $u \in U$, so $H$ is a Cartan-decomposition subgroup.

(11c) $H$ is conjugate (via an element of $U_{\alpha+2\beta}$) to $T \ltimes U_\alpha$. From [14, Prop. 3.18], we have $\rho(h) \asymp \|h\|^2$ for every $h \in T \ltimes U_\alpha$. Therefore $\rho(h) \asymp \|h\|^2$ for every $h \in H$. ∎

**Lemma 7.5.** [14, Lem. 2.4] *Assume $G = \mathrm{SU}(2,n)$, and let $H$ be a closed, connected subgroup of $AN$ that is compatible with $A$. Then either*

1. *$H = (H \cap A) \ltimes (H \cap N)$; or*
2. *there is a positive root $\omega$, a nontrivial group homomorphism $\psi\colon \ker\omega \to U_\omega U_{2\omega}$, and a closed, connected subgroup $U$ of $N$, such that*
   (a) *$H = \{\, a\psi(a) \mid a \in \ker\omega \,\}U$;*
   (b) *$U \cap \psi(\ker\omega) = e$; and*
   (c) *$U$ is normalized by both $\ker\omega$ and $\psi(\ker\omega)$.*

**Proposition 7.6.** *Assume that $G = \mathrm{SU}(2,n)$. Let $H$ be a closed, connected, nontrivial subgroup of $AN$, that is compatible with $A$, such that*

- *$H \cap N$ is not a Cartan-decomposition subgroup;*
- *$H \neq (H \cap A)(H \cap N)$; and*
- *$\dim H > 1$.*

*Then there are positive roots $\omega$ and $\sigma$, and a one-dimensional subspace $\mathfrak{x}$ of $(\ker\omega) + \mathfrak{u}_\omega + \mathfrak{u}_{2\omega}$, such that $\mathfrak{h} = \mathfrak{x} + (\mathfrak{h} \cap \mathfrak{n})$, $\mathfrak{h} \cap \mathfrak{n} \subset \mathfrak{u}_\sigma + \mathfrak{u}_{2\sigma}$, and either:*

1. *$\omega = \alpha$, $\sigma = \alpha + \beta$, and $\mu(H) \approx \big[\|h\|, \|h\|^2/(\log\|h\|)\big]$; or*
2. *$\omega = \alpha$ $\sigma = \alpha + 2\beta$, and $\mu(H) \approx \big[\|h\|^2/(\log\|h\|)^2, \|h\|^2\big]$; or*
3. *$\omega = \beta$, $\sigma = \alpha + 2\beta$, and $\mu(H) \approx \big[\|h\|(\log\|h\|)^{r/2}, \|h\|^2\big]$, where*

$$r = \begin{cases} 1 & \text{if } \mathfrak{x} \subset \mathfrak{u}_{2\beta} \\ 2 & \text{otherwise} \end{cases}$$

   *or*

4. *$\omega = \beta$, $\sigma = \alpha + \beta$, and $\mu(H) \approx \big[\|h\|, \|h\|(\log\|h\|)^r\big]$, where $r$ is defined as above; or*
5. *$\mathfrak{u} \cap (\mathfrak{u}_\omega + \mathfrak{u}_{2\omega}) \neq 0$, in which case $H$ is a Cartan-decomposition subgroup.*

**Proof.** We use the notation of Lemma 7.5: $T = \ker\omega$, $U = H \cap N$, $\psi\colon T \to U_\omega U_{2\omega}$, and $H = \{a\psi(a)\} \ltimes U$.

We need only consider the cases in Corollary 6.2 for which $H$ (now called $U$) is normalized by the kernel of some (reduced) positive root. Here is a list of them.

1. $N_A(U) = \ker(\beta)$: 6.2(4(b)iii), 6.2(5c), and 6.2(11b).
2. $N_A(U) = \ker(\alpha + \beta)$: 6.2(4(b)i) and 6.2(5b).
3. $N_A(U) = \ker(\alpha)$: 6.2(1b), 6.2(3), and 6.2(7b).
4. $N_A(U) = \ker(\alpha + 2\beta)$: 6.2(11c).

5. $N_A(U) = A$: 6.2(1a), 6.2(4a), 6.2(5a), 6.2(7a), and 6.2(11a).

Note that in each of the cases with $N_A(U) = A$, there is a (reduced) positive root $\sigma$, such that $\mathfrak{u} \subset \mathfrak{u}_\sigma + \mathfrak{u}_{2\sigma}$.

*Case 1. Assume* $\omega = \beta$.

*Subcase 1.1. Assume* 6.2(4(b)iii). From (7.7), we know that $H$ is a Cartan-decomposition subgroup.

*Subcase 1.2. Assume* 6.2(5c). There is some $u \in U$, such that $\phi_u \neq 0$. Then, because $\psi(T) \subset U_\beta U_{2\beta}$ normalizes $U$, we must have $U \cap U_{\alpha+2\beta} \neq e$. This is a contradiction.

*Subcase 1.3. Assume* 6.2(11b). Let $u \in \mathfrak{u}$. Because $U$ is normalized by $\psi(T)$, there is some nonzero $v \in \mathfrak{u}_\beta + \mathfrak{u}_{2\beta}$, such that $v$ normalizes $\mathfrak{u}$; thus, $[u, v] \in \mathfrak{u}$. Then, because $\phi_{[u,v]} = 0$, but $\phi_h \neq 0$ for every nontrivial $h \in U$, we conclude that $[u, v] = 0$. However, $\phi_u \neq 0$, and either $y_v \neq 0$ or $\mathbf{y}_v \neq 0$, so either $x_{[u,v]} \neq 0$ or $\eta_{[u,v]} \neq 0$. This is a contradiction.

*Subcase 1.4. Assume* $N_A(U) = A$. There is a positive root $\sigma$, such that $\mathfrak{u} \subset \mathfrak{u}_\sigma + \mathfrak{u}_{2\sigma}$.

If $\sigma = \beta$, then, from (7.7), we know that $H$ is a Cartan-decomposition subgroup.

Suppose $\sigma = \alpha + 2\beta$. Clearly $\|h\| \asymp a_1|\eta_u|$. Also,

$$\rho(h) \asymp a_1|\eta_u|^2 + a_1(\log a_1)^r,$$

where $r = 1$ if $\psi(T) \subset U_{2\beta}$ (i.e., if $y_h = 0$ for every $h \in H$) and $r = 2$ if $\psi(T) \not\subset U_{2\beta}$. The smallest value of $\|\rho(h)\|$ relative to $\|h\|$ is obtained by taking $\eta_u \asymp (\log a_1)^{r/2}$, resulting in $\rho(h) \asymp \|h\|\big(\log\|h\|\big)^{r/2}$. Then, since $\rho(u) \asymp \|u\|^2$ for $u \in U$, we conclude that $\mu(H) \approx \big[\|h\|\big(\log\|h\|\big)^{r/2}, \|h\|^2\big]$.

Because $U$ is normalized by the nontrivial subgroup $\psi(T)$ of $U_\beta U_{2\beta}$, we know that $\sigma \neq \alpha$. Therefore, we may now assume $\sigma = \alpha + \beta$. We show that $\mu(H) \approx \big[\|h\|, \|h\|(\log\|h\|)^r\big]$. For $u \in U$, we have $\rho(u) \asymp u$. For $a \in T$, we have

$$\rho\big(a\psi(a)\big) \asymp \|a\|\big(\log\|a\|\big)^r \asymp \|a\psi(a)\|\big(\log\|a\psi(a)\|\big)^r.$$

All that remains is to show that $\rho(h) = O\big[\|h\|\big(\log\|h\|\big)^r\big]$ for every $h \in H$. Because $\rho(au) \asymp au$ for every $au \in TU$ (see [14, Cor. 3.18]) and $\|\psi(a)\| \asymp \|\psi(a)^{-1}\| \asymp (\log\|h\|)^r$, we have

$$\begin{aligned}\rho(h) &= \rho\big(\psi(a)\big)\rho(au) = O\big[\|\rho\big(\psi(a)\big)\|\|\rho(au)\|\big] \\ &= O\big[\big(\log\|a\|\big)^r\|au\|\big] = O\big[\big(\log\|h\|\big)^r\|h\|\big].\end{aligned}$$

*Case 2. Assume* $\omega = \alpha + \beta$. The Weyl reflection corresponding to the root $\alpha$ conjugates each of 6.2(4(b)i) and 6.2(5b) to a subgroup with $\omega = \beta$.

Thus, we may now assume $N_A(U) = A$. If $\sigma \neq \alpha$, then the Weyl reflection corresponding to the root $\alpha$ conjugates $H$ to a subgroup with $\omega = \beta$. If $\sigma = \alpha$, then the Weyl reflection corresponding to the root $\beta$ does not change $\omega$, but conjugates $H$ to a subgroup $H_1$ with $\sigma = \alpha + 2\beta$. Then (as we already observed)

the Weyl reflection corresponding to the root $\alpha$ conjugates $H_1$ to a subgroup with $\omega = \beta$.

*Case 3. Assume* $\omega = \alpha$. Because $U$ must be normalized by the nontrivial subgroup $\psi(T)$ of $U_\alpha$, we see that $U$ cannot be of type 6.2(1b) or 6.2(3).

*Subcase 3.1. Assume* 6.2(7b). Because $U$ must be normalized by the nontrivial subgroup $\psi(T)$ of $U_\alpha$, we see that $y_u = 0$ for every $u \in U$, so $\mathfrak{u} = \mathfrak{z}$. Thus, again using the fact that $U$ is normalized by $\psi(T)$, we see that $\mathfrak{u} \subset \mathfrak{u}_{\alpha+2\beta} + \mathfrak{u}_{2\alpha+2\beta}$, and the projection of $\mathfrak{u}$ to $\mathfrak{u}_{\alpha+2\beta}$ is one-dimensional. For every $z \in \mathfrak{u}$, we see that $\eta_z \neq 0$ (because $|\eta_z|^2 \neq \mathsf{x}_z\mathsf{y}_z$). Thus, we conclude that $\dim \mathfrak{u} = 1$. Therefore $H$ is conjugate under $U_\alpha$ to a subgroup of type 6.2(7a) (considered in Subsubcase 3.2.2 below).

*Subcase 3.2. Assume* $N_A(U) = A$. If $\sigma = \alpha$, then (7.7) implies that $H$ is a Cartan-decomposition subgroup. Because $U$ is normalized by the nontrivial subgroup $\psi(T)$ of $U_\alpha$, we know that $\sigma \neq \beta$.

*Subsubcase 3.2.1. Assume* $\sigma = \alpha + \beta$. We have

$$h = a\psi(a)u = \begin{pmatrix} a_1 & a_1\phi_{\psi(a)} & a_1 x_u & 0 & -\frac{1}{2}a_1|x_u|^2 + ia_1\mathsf{x}_u \\ & a_1 & 0 & 0 & 0 \\ & & \cdots & & \end{pmatrix}.$$

We have $\|h\| \asymp a_1 \log a_1 + a_1|x_u|^2 + a_1|\mathsf{x}_u|$ and, for $i > 1$, we have $h_{i,j} = O\big(a_1 + |x_u|\big)$. The largest value of $\|\rho(h)\|$ relative to $\|h\|$ is obtained by taking $\log a_1 \asymp |x_u|^2$ (and $\mathsf{x}_u$ small), which yields $\rho(h) \asymp a_1^2 \log a_1 \asymp \|h\|^2/\log\|h\|$. Because $\rho(u) \asymp u$ for $u \in U$, we conclude that $\mu(H) \approx \big[\|h\|, \|h\|^2/\log\|h\|\big]$.

*Subsubcase 3.2.2. Assume* $\sigma = \alpha + 2\beta$. We have

$$h = a\psi(a)u = \begin{pmatrix} a_1 & a_1\phi_{\psi(a)} & 0 & a_1\eta_u & -a_1\phi_{\psi(a)}\overline{\eta_u} \\ & a_1 & 0 & 0 & -a_1\overline{\eta_u} \\ & & \cdots & & \end{pmatrix}.$$

We have $h \asymp \big(1 + a_1\|\psi(a)\|\big)\big(1 + |\eta_u|\big)$ and $\rho(h) \asymp a_1^2\big(1 + |\eta_u|^2\big)$ (note that $\det \begin{pmatrix} h_{1,2} & h_{1,n+2} \\ h_{2,2} & h_{2,n+2} \end{pmatrix} = 0$). The smallest value of $\|\rho(h)\|$ relative to $\|h\|$ is obtained by taking $\eta_u = O(1)$, which results in $\rho(h) \asymp a_1^2 \asymp \|h\|^2/\big(\log\|h\|\big)^2$. Because $\rho(u) \asymp \|u\|^2$ for $u \in U$, we conclude that $\mu(H) \approx \big[\|h\|^2/\big(\log\|h\|\big)^2, \|h\|^2\big]$.

*Case 4. Assume* $\omega = \alpha + 2\beta$. The Weyl reflection corresponding to the root $\beta$ conjugates 6.2(11c) to a subgroup $H'$ with $\omega = \alpha$ (of type 6.2(7b) with $\mathfrak{h}' = \mathfrak{z}' \subset \mathfrak{u}_{\alpha+2\beta} + \mathfrak{u}_{2\alpha+2\beta}$).

Thus, we may now assume $N_A(U) = A$. If $\sigma \neq \beta$, then the Weyl reflection corresponding to the root $\beta$ conjugates $H$ to a subgroup with $\omega = \alpha$. Now assume $\sigma = \beta$. The Weyl reflection corresponding to the root $\alpha$ does not change $\omega$, but conjugates $H$ to a subgroup $H_1$ with $\sigma = \alpha + 2\beta$. Then (as we already observed) the Weyl reflection corresponding to the root $\beta$ conjugates $H_1$ to a subgroup with $\omega = \alpha$. ∎

**Lemma 7.7.** *Assume $G$ is a connected, almost simple, linear, real Lie group of real rank two. Let $H$ be a closed, connected, nontrivial subgroup of $AN$, such that $H$ is compatible with $A$, and $H \neq (H \cap A)(H \cap N)$. We use the notation of [14, Lem. 2.4]: $T = \ker\omega$, $H = T \ltimes U$, $\psi\colon T \to U_\omega U_{2\omega}$, and $H = \{a\psi(a)\} \ltimes U$.*

*If $\mathfrak{u} \cap (\mathfrak{u}_\omega + \mathfrak{u}_{2\omega}) \neq 0$, then $H$ is a Cartan-decomposition subgroup.*

**Proof.** By passing to a subgroup of $H$, there is no harm in assuming $\mathfrak{u} \cap (\mathfrak{u}_\omega + \mathfrak{u}_{2\omega})$. We use the notation of the proof of [14, Prop. 3.17]. For each $a \in T$, clearly $\mu_{MA}\big(a\psi(a)U\big) \supset \mu_{MA}\big(a\psi(a)\big)A_\omega^+$, so $\mu_{MA}(H) \supset \mu_{MA}\big(\{a\psi(a)\}\big)A_\omega^+$. Beause $\mu_{MA}(T) = T$ is a line perpendicular to $A_\omega$, and $\mu_{MA}\big(a\psi(a)\big)$ is logarithmically close to this line, it is clear that $\mu_{MA}\big(a\psi(a)\big)A_\omega^+$ contains all but a bounded subset of the region $\mathcal{C}$. Therefore $\mu(H)$ contains all but a bounded subset of $A^+$, so $H$ is a Cartan-decomposition subgroup. ∎

**Lemma 7.8.** (cf. [14, Prop. 3.16(3)]) *Assume that $G = \mathrm{SU}(2,n)$, and let $H$ be a nontrivial one-parameter subgroup of $AN$, such that $H$ is compatible with $A$, but $H \neq (H \cap A)(H \cap N)$.*

*Then there is a ray $R$ in $A^+$, a ray $R'$ in $A$ that is perpendicular to $R$, and a positive number $k$, such that*

$$\mu(H) \approx \{\, rs \mid r \in R,\ s \in R',\ \|s\| = (\log\|r\|)^k \,\}.$$

## 8 Maximum dimensions of the subgroups

For convenience of reference, Tables 1, 2 and 3 list the (approximate) Cartan projection of each subgroup of $AN$ that is not a Cartan-decomposition subgroup. The maximum possible dimension for a subgroup of each type is also listed. (These dimensions are used in applications to the existence of tessellations.)

**Remark 8.1.** Here are brief justifications of the dimensions listed in Tables 1, 2 and 3.

6.1(1) By assumption, we have $\dim H = 1$.

6.1(2) Let $p\colon \mathfrak{h} \to \mathfrak{u}_{\alpha+\beta}$ be the natural projection. Then $\ker p = \mathfrak{z} \subset \mathfrak{u}_{2\alpha+2\beta}$, so

$$\dim\mathfrak{h} \leq (\dim\mathfrak{u}_{\alpha+\beta}) + (\dim\mathfrak{u}_{2\alpha+2\beta}) = 2(n-2)+1 = 2n-3.$$

6.1(3) We may assume $\lambda = 0$. Then $\mathfrak{h} \subset \mathfrak{u}_\beta + \mathfrak{u}_{2\beta}$. So

$$\dim\mathfrak{h} \leq (\dim\mathfrak{u}_\beta) + (\dim\mathfrak{u}_{2\beta}) = 2(n-2)+1 = 2n-3.$$

It is easy to construct an algebra of this dimension, with or without an element $u$ as described in (3a).

6.1(4) Let $V$ be the projection of $\mathfrak{h}$ to $\mathfrak{u}_\alpha + \mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{\alpha+2\beta}$. Because $\phi\overline{\eta}$ is a form of signature $(2,2)$ on $\mathfrak{u}_\alpha + \mathfrak{u}_{\alpha+2\beta}$, we know that $\dim\big(V \cap (\mathfrak{u}_\alpha + \mathfrak{u}_{\alpha+2\beta})\big) \leq 2$. Thus we have

$$\begin{aligned}\dim\mathfrak{h} &\leq \dim V + \dim\mathfrak{u}_{2\alpha+2\beta} \leq \big(\dim\mathfrak{u}_{\alpha+\beta}+2\big) + \dim\mathfrak{u}_{2\alpha+2\beta}\\ &= \big(2(n-2)+2\big)+1 = 2n-1.\end{aligned}$$

| reference | Cartan projection | maximum dimension |
|---|---|---|
| 6.1(1) | $\rho(h) \asymp h$ | 1 |
| 6.1(2) | $\mu(H) \approx \left[\|h\|, \|h\|^{3/2}\right]$ | $2n-3$ |
| 6.1(2)* | $\rho(h) \asymp \|h\|^{3/2}$ | 1 |
| 6.1(3a) | $\mu(H) \approx \left[\|h\|, \|h\|^{3/2}\right]$ | $2n-3$ |
| 6.1(3a)* | $\rho(h) \asymp \|h\|^{3/2}$ | 1 |
| 6.1(3b) | $\rho(h) \asymp h$ | $2n-3$ |
| 6.1(4) | $\rho(h) \asymp h$ | $2n-1$ |
| 6.1(5) | $\mu(H) \approx \left[\|h\|, \|h\|^{4/3}\right]$ | $2n-3$ |
| 6.1(5)* | $\rho(h) \asymp \|h\|^{4/3}$ | 1 |
| 6.1(6) | $\rho(h) \asymp \|h\|^2$ | $\begin{cases} 2n-1 & n \text{ even} \\ 2n-3 & n \text{ odd} \end{cases}$ |
| 6.1(7) | $\mu(H) \approx \left[\|h\|^{3/2}, \|h\|^2\right]$ | $\begin{cases} n+1 & n \geq 4 \\ 3 & n = 3 \end{cases}$ |
| 6.1(8) | $\rho(h) \asymp \|h\|^{3/2}$ | $\begin{cases} 3 & n \geq 4 \\ 2 & n = 3 \end{cases}$ |
| 6.1(9) | $\mu(H) \approx \left[\|h\|, \|h\|^{3/2}\right]$ | 2 |
| 6.1(10) | $\rho(h) \asymp \|h\|^2$ | 2 |
| 6.1(11) | $\mu(H) \approx \left[\|h\|^{5/4}, \|h\|^2\right]$ | 2 |

Table 1: The subgroups of $N$ that are not Cartan-decomposition subgroups.

6.1(5) Consider $p\colon \mathfrak{h} \to \mathfrak{u}_\alpha$. Because $\mathfrak{z} = 0$, we have $\dim \ker p \leq \dim \mathfrak{u}_{\alpha+\beta} = 2n-4$. Because $p(\mathfrak{h}) \subset \mathbb{R}\phi_0$, we have $\dim p(\mathfrak{h}) \leq 1$. Thus, $\dim \mathfrak{h} \leq 2n-3$.

6.1(6) See Lemma 8.2 below.

6.1(7) See Lemma 8.3 below.

6.1(8) See Lemma 8.4 below.

6.1(9), 6.1(10), 6.1(11) are obvious from the statements.

7.4(1a) Because $\dim \mathfrak{u} = 1$, we have $\dim \mathfrak{h} = \dim \mathfrak{t} + \dim \mathfrak{u} = 2$.

7.4(2) The kernel of the projection from $\mathfrak{u}$ to $\mathfrak{u}_{\alpha+\beta}$ is $\mathfrak{z}$, so $\dim \mathfrak{h} = 1 + \dim U \leq 1 + (1 + \dim \mathfrak{u}_{\alpha+\beta}) = 2n-2$.

7.4(4) $\dim \mathfrak{h} = 1 + \dim \mathfrak{u} \leq 1 + \left(\dim \mathfrak{u}_\beta + \dim \mathfrak{z}\right) = 2n-2$.

7.4(5a) $\dim \mathfrak{h} \leq \dim \mathfrak{t} + \dim \mathfrak{u}_{\alpha+\beta} + \dim \mathfrak{z} \leq 1 + (2n-4) + 1 = 2n-2$.

7.4(5c) Add 1 (the dimension of $T$) to the bound in 6.1(4).

7.4(6) Add 1 (the dimension of $T$) to the bound in 6.1(5).

7.4(7a) $\dim \mathfrak{h} \leq \dim \mathfrak{t} + \dim \mathfrak{u}_{\alpha+2\beta} = 1 + 2 = 3$.

7.4(7b) Add 1 (the dimension of $T$) to the bound in 6.1(6).

7.4(8a) Because $\mathsf{y}_u \neq 0$ for every nonzero $u \in \mathfrak{u} \cap (\mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{2\beta})$, we have $\dim\left(\mathfrak{u} \cap (\mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{2\beta})\right) \leq 1$. Therefore $\dim \mathfrak{h} \leq \dim \mathfrak{t} + 1 + \dim \mathfrak{u}_{\alpha+2\beta} = 4$.

7.4(8b) This is conjugate to 7.4(8a), via the Weyl reflection corresponding to the root $\alpha$.

7.4(9) Add 1 (the dimension of $T$) to the bound in 6.1(8). (To achieve this bound for $n \geq 4$, choose $u, \tilde{u} \in \mathfrak{u} \cap (\mathfrak{u}_\alpha + \mathfrak{u}_\beta)$ in the proof of Lemma 8.4.

7.4(10) and 7.4(11) are obvious from the statements.

| reference | Cartan projection | maximum dimension |
|---|---|---|
| 7.4(1a) | $\mu(H) \approx [\|h\|, \|h\|^s]$ | 2 |
| 7.4(2) | $\mu(H) \approx [\|h\|, \|h\|^{3/2}]$ | $2n-2$ |
| 7.4(2)* | $\rho(h) \asymp \|h\|^{3/2}$ | 2 |
| 7.4(4a) | $\mu(H) \approx [\|h\|, \|h\|^s]$ | $2n-2$ |
| 7.4(4(b)i) | $\rho(h) \asymp h$ | $2n-2$ |
| 7.4(4(b)ii) | $\mu(H) \approx [\|h\|, \|h\|^{3/2}]$ | $2n-2$ |
| 7.4(4(b)ii)* | $\rho(h) \asymp \|h\|^{3/2}$ | 2 |
| 7.4(5a) | $\mu(H) \approx [\|h\|, \|h\|^s]$ | $2n-2$ |
| 7.4(5c) | $\rho(h) \asymp h$ | $2n$ |
| 7.4(6) | $\mu(H) \approx [\|h\|, \|h\|^{4/3}]$ | $2n-2$ |
| 7.4(6)* | $\rho(h) \asymp \|h\|^{4/3}$ | 2 |
| 7.4(7a) | $\mu(H) \approx [\|h\|^s, \|h\|^2]$ | 3 |
| 7.4(7b) | $\rho(h) \asymp \|h\|^2$ | $\begin{cases} 2n & n \text{ even} \\ 2n-2 & n \text{ odd} \end{cases}$ |
| 7.4(8a) | $\mu(H) \approx [\|h\|^{3/2}, \|h\|^2]$ | 4 |
| 7.4(8b) | $\mu(H) \approx [\|h\|^{3/2}, \|h\|^2]$ | 4 |
| 7.4(9) | $\rho(h) \asymp \|h\|^{3/2}$ | $\begin{cases} 4 & n \geq 4 \\ 3 & n = 3 \end{cases}$ |
| 7.4(10) | $\mu(H) \approx [\|h\|, \|h\|^{3/2}]$ | 3 |
| 7.4(11a) | $\mu(H) \approx [\|h\|^s, \|h\|^2]$ | 3 |
| 7.4(11c) | $\rho(h) \asymp \|h\|^2$ | 3 |

Table 2: The subgroups of $AN$ that are not Cartan-decomposition subgroups, and are a nontrivial semidirect product $T \ltimes U$.

| reference | Cartan projection | maximum dimension |
|---|---|---|
| 7.6(1) | $\mu(H) \approx [\|h\|, \|h\|^2/(\log \|h\|)]$ | $2n-2$ |
| 7.6(2) | $\mu(H) \approx [\|h\|^2/(\log \|h\|)^2, \|h\|^2]$ | 2 |
| 7.6(3) $(r=1)$ | $\mu(H) \approx [\|h\|(\log \|h\|)^{1/2}, \|h\|^2]$ | 3 |
| 7.6(3) $(r=2)$ | $\mu(H) \approx [\|h\|(\log \|h\|), \|h\|^2]$ | 3 |
| 7.6(4) $(r=1)$ | $\mu(H) \approx [\|h\|, \|h\|(\log \|h\|)]$ | $2n-2$ |
| 7.6(4) $(r=2)$ | $\mu(H) \approx [\|h\|, \|h\|(\log \|h\|)^2]$ | $2n-3$ |
| 7.8 | $\rho(h) \asymp \|h\|^s(\log \|h\|)^{\pm k}$ | 1 |

Table 3: The subgroups of $AN$ that are not Cartan-decomposition subgroups **and** are not a semidirect product of a torus and a unipotent subgroup.

7.6(1) $\dim \mathfrak{h} \leq 1 + \dim(\mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{2\alpha+2\beta}) = 2n - 2$.

7.6(2) Because $\psi(T)$ normalizes (hence centralizes) $U$, the subgroup $U$ cannot be all of $U_{\alpha+2\beta}$, so $\dim U \leq 1$. Therefore $\dim H = 1 + \dim U \leq 2$.

7.6(3) $\dim \mathfrak{h} \leq 1 + \dim \mathfrak{u}_{\alpha+2\beta} = 3$.

7.6(4) Because $\psi(T)$ normalizes (hence centralizes) $U$, the projection of $\mathfrak{u}$ to $\mathfrak{u}_{\alpha+\beta}$ cannot be all of $\mathfrak{u}_{\alpha+\beta}$ if $\psi(T) \not\subset U_{2\beta}$, that is, if $r = 2$. Therefore $\dim U \leq \dim(\mathfrak{u}_{\alpha+\beta}+\mathfrak{u}_{2\alpha+2\beta})-(r-1) = 2n-2-r$. Therefore $\dim \mathfrak{h} = 1+\dim U \leq 2n-1-r$.

**Lemma 8.2.** *The maximum dimension of a subalgebra of type* 6.1(6) *is as stated in Table* 1.

**Proof.** We begin by showing that $\dim \mathfrak{h} \leq 2n - 1$ (cf. [16, Lem. 5.8]). Let $V$ be the projection of $\mathfrak{h}$ to $\mathfrak{u}_\beta + \mathfrak{u}_{\alpha+\beta}$. Because $\dim \mathfrak{z} \leq 3$, we just need to show that $\dim V \leq 2n - 4$. Because $V$ does not intersect $\mathfrak{u}_\beta$ (or $\mathfrak{u}_{\alpha+\beta}$, either, for that matter), and $\mathfrak{u}_\beta$ has codimension $2n - 4$ in $\mathfrak{u}_\beta + \mathfrak{u}_{\alpha+\beta}$, this is immediate.

When $n$ is even, there is a subgroup of dimension $2n - 1$. (For example, the $N$ subgroup of $\mathrm{Sp}(1, n/2)$. More general examples are constructed in [15, §4].)

Let us show that if $n$ is odd, then $\dim H \leq 2n-3$. (Our proof is topological; we do not know an algebraic proof.) Suppose that $\dim H \geq 2n - 2$ (this will lead to a contradiction). Because $\dim \mathfrak{z} \leq 3$, we have $\dim \mathfrak{h}/\mathfrak{z} \geq 2n - 5$. Thus, there is a $(2n - 5)$-dimensional real subspace $X$ of $\mathbb{C}^{n-2}$ and a real linear transformation $T\colon X \to \mathbb{C}^{n-2}$, such that $x$ and $Tx$ are linearly independent over $\mathbb{C}$, for every nonzero $x \in X$ (cf. [16, Cor. 5.9]). Thus, if we define $U\colon X \to \mathbb{C}^{n-2}$ by $Ux = ix$; then $x$, $Tx$, and $Ux$ are linearly independent over $\mathbb{R}$, for every nonzero $x \in X$. Thus (writing $n = 2k + 3$): there is a $(4k + 1)$-dimensional real subspace $X$ of $\mathbb{R}^{4k+2}$ and real linear transformations $T, U\colon X \to \mathbb{R}^{4k+2}$, such that $x$, $Tx$, and $Ux$ are linearly independent over $\mathbb{R}$, for every nonzero $x \in X$. There is no harm in assuming $X = \mathbb{R}^{4k+1}$ (under its natural embedding in $\mathbb{R}^{4k+2}$).

Let $E = (S^{4k} \times \mathbb{R}^{4k+2})/\!\sim$, where $(x, v) \sim (-x, -v)$, and define a continuous map $\zeta\colon E \to \mathbb{R}P^{4k}$ by $\zeta(x, v) = [x]$, so $(E, \zeta)$ is a vector bundle over $\mathbb{R}P^{4k}$. Then $(E, \zeta) \cong \tau \oplus \epsilon^1 \oplus \gamma^1_{4k}$, where $\tau$ is the tangent bundle of $\mathbb{R}P^{4k}$, $\epsilon^1$ is a trivial line bundle, and $\gamma^1_{4k}$ is the canonical bundle of $\mathbb{R}P^{4k}$. (To see this, note that the subbundle

$$\{\, (x, v) \in S^{4k} \times \mathbb{R}^{4k+1} \mid v \perp x \,\}/\!\sim$$

is the total space of $\tau$ [13, pf. of Lem. 4.4, pp. 43–44], the subbundle

$$\{\, (x, v) \in S^{4k} \times \mathbb{R}^{4k+1} \mid v \in \mathbb{R}x \,\}/\!\sim$$

has the obvious section $x \mapsto (x, x)$, and the subbundle $\big(S^{4k} \times (0 \times \mathbb{R})\big)/\!\sim$ is isomorphic to $\gamma^1_{4k}$ via the bundle map $\big(x, (0, t)\big) \mapsto (x, tx)$.) Therefore, letting $a$ be a generator of the cohomology ring $H^*(\mathbb{R}P^{4k}; \mathbb{Z}_2)$, we see that the total Stiefel-Whitney class of $(E, \zeta)$ is $w = (1+a)^{4k+1}(1)(1+a) = (1+a)^{4k+2}$ [13, Eg. 2, p. 43, and Thm. 4.5, p. 45], so

$$w_{(4k+2)-3+1} = w_{4k} = \binom{4k+2}{4k} a^{4k} = (2k+1)(4k+1)a^{4k} \neq 0$$

(because $(2k + 1)(4k + 1)$ is odd). Therefore, there do not exist three pointwise linearly independent sections of $(E, \zeta)$ [13, Prop. 4, p. 39].

Any linear transformation $Q\colon \mathbb{R}^{4k+1} \to \mathbb{R}^{4k+2}$ induces a continuous function $\hat{Q}\colon S^{4k} \to \mathbb{R}^{4k+2}$, such that $\hat{Q}(-x) = -\hat{Q}(x)$ for all $x \in S^{4k}$; that is, a section of $(E, \zeta)$. Thus, Id, $T$, and $U$ each define a section of $(E, \zeta)$. Furthermore, these three sections are pointwise linearly independent, because $x$, $Tx$, and $Ux$ are linearly independent over $\mathbb{R}$, for every $x \in S^{4k}$. This contradicts the conclusion of the preceding paragraph. ∎

**Lemma 8.3.** *The maximum dimension of a subalgebra of type* 6.1(7) *is as stated in Table* 1.

**Proof.** Replacing $H$ by a conjugate under $\langle U_\alpha, U_{-\alpha} \rangle$, we may assume $x_v = 0$. Therefore $\mathsf{x}_z = 0$ for every $z \in \mathfrak{z}$. (Thus, in particular, we have $\dim \mathfrak{z} \leq 2$.)

For the projection $p\colon \mathfrak{h} \to \mathfrak{u}_{\alpha+\beta}$, we have $\ker p = \mathbb{R}v + \mathfrak{z}$. (There cannot exist a linearly independent $v'$; otherwise, replacing $v'$ by some linear combination with $v$, we could assume $\mathsf{x}_{v'} = 0$, which is impossible.) Thus, $\dim \ker p \leq 3$.

Because $\mathsf{x}_z = 0$ for every $z \in \mathfrak{z}$, $p(\mathfrak{h})$ must be a totally isotropic subspace for the symplectic form $\operatorname{Im}(x\tilde{x}^\dagger)$, so $\dim p(\mathfrak{h}) \leq n-2$. Therefore $\dim \mathfrak{h} \leq (n-2)+3 = n+1$.

For $n \geq 4$, here is an example that achieves this bound:

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & 0 & x_1 & x_2 & \cdots & x_{n-2} & \eta & i\mathsf{x} \\ 0 & 0 & i\mathsf{x} & x_1 & \cdots & x_{n-3} & ix_{n-2} & -\overline{\eta} \end{pmatrix} \;\middle|\; \begin{array}{c} \mathsf{x}, x_1, \ldots, x_{n-2} \in \mathbb{R}, \\ \eta \in \mathbb{C} \end{array} \right\}.$$

For $v \in \mathfrak{h}$, we claim that $\dim_{\mathbb{C}} \langle x_v, y_v \rangle = 1$ only if either $x_v = 0$ or $y_v = 0$. (In either case, it is clear from the definition of $\mathfrak{h}$ that either $\mathsf{x}_u |y_u|^2 \neq 0$ or $\mathsf{y}_u |x_u|^2 \neq 0$, respectively.) Suppose $\dim_{\mathbb{C}} \langle x_v, y_v \rangle = 1$, with $x_v \neq 0$ and $y_v \neq 0$. There is some nonzero $\lambda \in \mathbb{C}$, such that $y_v = \lambda x_v$. We must have $x_1 \neq 0$. (Otherwise, let $i \in \{1, 2, \ldots, n-2\}$ be minimal with $x_i \neq 0$. Then $x_{i-1} = y_i = \lambda x_i \neq 0$, contradicting the minimality of $i$.) Because $y_1 = i\mathsf{x}$ is pure imaginary, but $x_1$ is real, we see that $\lambda$ is pure imaginary. On the other hand, $y_2 = x_1$ is real (and nonzero), and $x_2$ is also real, so $\lambda$ is real. Because $\lambda \neq 0$, this is a contradiction.

Now let $n = 3$, and suppose $\dim \mathfrak{h} = 4$. (This will lead to a contradiction.) Because equality is attained in the proof above, we must have $\dim p(\mathfrak{h}) = n-2 = 1$ and $\dim \mathfrak{z} = 2$. In particular, there exists $w \in \mathfrak{h}$ with $x_w \neq 0$. For $t \in \mathbb{R}$, let $w_t = w + tv$. Then

$$\mathsf{x}_{w_t} |y_{w_t}|^2 + \mathsf{y}_{w_t} |x_{w_t}|^2 + 2 \operatorname{Im}(x_{w_t} y_{w_t}^\dagger \eta_{w_t}) = t^3 \mathsf{x}_v |y_v|^2 + O(t^2) \to \begin{cases} +\mathsf{x}_v \infty & \text{as } t \to \infty \\ -\mathsf{x}_v \infty & \text{as } t \to -\infty. \end{cases}$$

Thus, this expression changes sign, so it must vanish for some $t$. This is a contradiction, because $\dim_{\mathbb{C}} \langle x_{w_t}, y_{w_t} \rangle = 1$ for every $t$. ∎

**Lemma 8.4.** *The maximum dimension of a subalgebra of type* 6.1(8) *is as stated in Table* 1.

**Proof.** For $n \geq 4$, here is the construction of 3-dimensional subalgebras of $\mathfrak{n}$ of this type. Let $\phi = 1$ and $\tilde{\phi} = i$. Choose $y, \tilde{y}, x, \tilde{x} \in \mathbb{C}^{n-2}$, $\eta, \tilde{\eta} \in \mathbb{C}$, and $\mathsf{x}, \tilde{\mathsf{x}} \in \mathbb{R}$, such that

$$|y|^2 = |\tilde{y}|^2 = 3iy\tilde{y}^\dagger \neq 0. \tag{8.1}$$

Now, choose $\mathsf{y}, \tilde{\mathsf{y}} \in \mathbb{R}$, such that

$$\mathrm{Im}(\tilde{y}x^\dagger - iyx^\dagger + \tilde{y}x^\dagger - y\tilde{x}^\dagger + i\tilde{\mathsf{y}}) = 0 \tag{8.2}$$

and

$$\mathrm{Im}(\tilde{y}\tilde{x}^\dagger - iy\tilde{x}^\dagger + i\tilde{y}x^\dagger - iy\tilde{x}^\dagger + i\mathsf{y}) = 0. \tag{8.3}$$

Define $u, \tilde{u}$ as in Eq. (2.3), and let $v = [u, \tilde{u}]$. Then $\mathsf{y}_v \neq 0$ and $x_v \neq 0$, but, from Eq. (8.1), Eq. (8.2) and Eq. (8.3), we have $[v, u] = [v, \tilde{u}] = 0$. Thus, we may let $\mathfrak{h}$ be the subalgebra generated by $u$ and $\tilde{u}$. (So $\{u, \tilde{u}, v\}$ is a basis of $\mathfrak{h}$ over $\mathbb{R}$.)

Note that, because $|y\tilde{y}^\dagger| = |y|^2/3 \neq |y|^2$, we know that $y$ and $\tilde{y}$ must be linearly independent over $\mathbb{C}$. Thus, these 3-dimensional examples do not exist when $n = 3$. ∎

## References


[1] Benoist, Y., *Actions propres sur les espaces homogènes réductifs*, Ann. Math. **144** (1996), 315–347.

[2] Borel, A., *Compact Clifford-Klein forms of symmetric spaces*, Topology **2** (1963), 111–122.

[3] Helgason, S., "Differential Geometry, Lie Groups, and Symmetric Spaces", Academic Press, New York, 1978.

[4] Iozzi, A., and D. Witte, Tessellations of homogeneous spaces of classical groups of real rank two, (in preparation).

[5] Kobayashi, T., *On discontinuous groups acting on homogeneous spaces with non-compact isotropy groups*, J. Geom. Physics **12** (1993), 133–144.

[6] ———, *Proper action on a homogeneous space of reductive type*, Math. Ann. **285** (1989), 249–263.

[7] ———, *Discontinuous groups and Clifford-Klein forms of pseudo-Riemannian homogeneous manifolds*, in: B. Ørsted and H. Schlichtkrull, eds., "Algebraic and Analytic Methods in Representation Theory", Academic Press, New York, 1997, pp. 99–165.

[8] ———, *Criterion of proper actions on homogeneous spaces of reductive groups*, J. Lie Th. **6** (1996), 147–163.

[9] Kostant, B., *On convexity, the Weyl group, and the Iwasawa decomposition*, Ann. Sc. ENS. **6** (1973), 413–455.

[10] Kulkarni, R., *Proper actions and pseudo-Riemannian space forms*, Adv. Math. **40** (1981) 10–51.

[11] Labourie, F., *Quelques résultats récents sur les espaces localement homogènes compacts*, in: P. de Bartolomeis, F. Tricerri and E. Vesentini, eds., "Manifolds and Geometry", Symposia Mathematica, v. XXXVI, Cambridge U. Press, 1996.

[12] Margulis, G. A., *Existence of compact quotients of homogeneous spaces, measurably proper actions, and decay of matrix coefficients*, Bull. Soc. Math. France **125** (1997) 447–456.

[13] Milnor, J. W., and J. D. Stasheff, "Characteristic Classes", Princeton U. Press, Princeton, 1974.

[14] Oh, H., and D. Witte, *Cartan-decomposition subgroups of* $\mathrm{SO}(2,n)$, Trans. Amer. Math. Soc. (to appear).

[15] ———, *New examples of compact Clifford-Klein forms of homogeneous spaces of* $\mathrm{SO}(2,n)$, Internat. Math. Res. Not. **2000** (8 March 2000), no. 5, 235-251.

[16] ———, *Compact Clifford-Klein forms of homogeneous spaces of* $\mathrm{SO}(2,n)$, (preprint).

[17] Raghunathan, M. S., "Discrete Subgroups of Lie Groups", Springer-Verlag, New York, 1972.

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