GLOBAL THEOREMS ON VERTICES AND FLATTENINGS OF CLOSED CURVES

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A *vertex* of a curve in the Euclidean plane is a point where the curvature is extremal. Equivalently a *vertex* is a point where the order of contact of the osculating circle with the curve is higher than usual (this will be precised later). The classical four–vertex theorem [14] states that: *Any convex curve in the Euclidean plane has at least four vertices.* For example, the points of intersection of an ellipse with its principal axes are the vertices of this curve.

Various higher dimensional generalizations of the four–vertex theorem are given and some properties of closed curves related to its vertices and its *flattenings* are studied. In particular we introduce a class of curves, which we call *spherically convex*, in the Euclidean space $\mathbb{R}^n$, the sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, and the Lobachevskian space $\mathbb{L}^n$. We prove the following theorems: *Any spherically convex curve in* $\mathbb{R}^{2k}$ *(respectively in* $\mathbb{S}^{2k} \subset \mathbb{R}^{2k+1}$, $\mathbb{R}P^{2k}$ *and* $\mathbb{L}^{2k}$*) has at least* $2k+2$ *vertices.* We also prove that these three theorems are equivalent for our class of curves.

In [8], Barner introduced a class of curves (called below Barner curves) in the projective space $\mathbb{R}P^n$ and proved that these curves have at least $n+1$ points in which the osculating hyperplane is stationary. We introduce a class of curves in the odd dimensional Lobatchevskian spaces (the class analogue to Barner curves) and prove that Barner's theorem also holds in odd dimensional Lobatchevskian spaces.

We prove that *the vertices are extrema of the radius of the osculating hypersphere* and that *the converse is not true.* We give a formula to calculate the vertices of a curve in $\mathbb{R}^n$ as the zeros of a determinant. Our formula does not depend on a spetial parametrization. With our formula we calculate the number of vertices of the *generalized ellipses* introduced in [6].

A convex curve has no flattening and its osculating hyperplane intersects it only at the point of osculation. A curve in $\mathbb{R}P^2$ ($\mathbb{R}^2$) is convex if and only if it has these two properties. To answer a question of V. Arnol'd ([6]), we show that this two properties don't imply convexity for curves in $\mathbb{R}P^n$, for $n > 2$.

We prove that *any small enough generic perturbation in* $\mathbb{R}^{2k+1}$ *(taking the derivatives into account) of a spherically convex curve in* $\mathbb{S}^{2k} \subset \mathbb{R}^{2k+1}$ *has at least* $2k+2$ *extrema of the radius of the* $(2k-1)$*-osculating sphere.* We also show that *any small enough generic perturbation of a closed curve embedded in* $\mathbb{S}^2 \subset \mathbb{R}^3$ *has at least* 4 *points with extremal curvature.*

The conditions defining classes of closed curves in $\mathbb{R}^n$ that guarantee a minimum number of flattenings (or vertices) on each curve of that class has been a classical object of study. The interest on this subject was revived by the recent progres in symplectic and contact geometries and the relations of this problems with Sturm theory (see [6], [4], [5], [7], [13], [1], [20], [24]). We study three classes of curves in $\mathbb{R}^3$ all whose elements have at least four flattenings ([6], [21], [19]) and give a four–flattening conjecture for a closed curve γ in $\mathbb{R}^3$ in terms of the 1–dimensional Legendrian knot in $ST^*\mathbb{S}^2$ associated to the tangent indicatrix $\mathbf{T}_\gamma \subset \mathbb{S}^2$ of the curve γ in $\mathbb{R}^3$.

Acknowledgements. The author is gratiful to V.I. Arnol'd for helpful remarks and for setting up some problems considered in this work, to M.E. Kazarian for helpful discutions and the attention given to this work, and to V.D. Sedykh for helpful coments.

§1. Higher Dimensional Four-Vertex Theorems for Curves in the Euclidean Space $\mathbb{R}^n$, in the Sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, in the Projective Space $\mathbb{R}P^n$ and in the Lobachevskian Space $\mathbb{L}^n$

A curve embedded in the Euclidean space $\mathbb{R}^n$ is called *spherically convex* if for any n–tuple of points of the curve there exists a hypersphere through these points that does not intersect the curve elsewhere. We also introduce the class of spherically convex curves in the following spaces: the sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, the projective space $\mathbb{R}P^n$ and the Lobachevskian space $\mathbb{L}^n$. We prove: *Any spherically convex curve in $\mathbb{R}^{2k}$ (respectively in $\mathbb{S}^{2k} \subset \mathbb{R}^{2k+1}$, $\mathbb{R}P^{2k}$ and $\mathbb{L}^{2k}$) has at least $2k+2$ vertices.*

1. Introduction and Results

Below, a curve in the Euclidean space $\mathbb{R}^n$ always means a smooth immersion $\gamma : \mathbb{S}^1 \to \mathbb{R}^n$. We will always assume that the derivatives of γ of order $1, \dots, n-1$, are linearly independent at any point (this is true for generic curves). We will often identify the immersion with its image and use the abbreviation γ to denote $\gamma(\mathbb{S}^1)$. In this chapter we will consider curves in the Euclidean space $\mathbb{R}^n$, in the n–dimensional sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ and in the Lobachevskian space $\mathbb{L}^n$ modeled on a ball in the Euclidean space $\mathbb{R}^n$.

We state the following conventions:

a) A curve $\gamma \subset \mathbb{S}^n \subset \mathbb{R}^{n+1}$ is refered as a spatial curve when it is considered as a curve in $\mathbb{R}^{n+1}$, otherwise it is regarded as a curve in S^n.

b) Let $\mathbb{L}^n$ be the open unit ball in $\mathbb{R}^n$, (the interior of the sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$) considered as the Poincaré's model of the n-dimensional Lobatchevskian space. A *hyperbolic hyperplane* in $\mathbb{L}^n$ is the intersection of $\mathbb{L}^n$ with a hypersphere of $\mathbb{R}^n$ orthogonal to $\mathbb{S}^{n-1}$. The spheres, horospheres and equidistant spheres will be called *generalized spheres*.

We will systematically use the notion of *order of contact*:

Definition – Let M be a d–dimensional submanifold of $\mathbb{R}^n$, considered as a complet intersection: $M = \{x \in \mathbb{R}^n : g_1(x) = \dots = g_{n-d}(x) = 0\}$. We say that k is the *order of contact* of a curve $\gamma : t \mapsto \gamma(t) \in \mathbb{R}^n$ with the submanifold M, at a point of intersection $\gamma(t_0)$, if each function $g_1 \circ \gamma, \dots, g_{n-d} \circ \gamma$ has a zero of multiplicity at least k at $t = t_0$, and at least one of them has a zero of multiplicity k at $t = t_0$.

Roughly spiking, this definition means, in the former language of geometers, that the curve γ and the submanifold M "meet at k consecutive points", or that γ and M "meet at k infinitely close points".

Remark – In the most part of cases considered here M will be a d–dimensional affine subspace or a d–dimensional sphere.

Example – The order of contact of a smooth curve in $\mathbb{R}^n$ with its tangent line (at the point of tangency) is two for the generic points of the curve. The order of

contact of the curve $y = x^3$ with the line $y = 0$ is 3: the equation $x^3 = 0$ has a root of multiplicity 3.

By convention, the set of k–dimensional spheres of the Euclidean space $\mathbb{R}^n$ contains the k–dimensional affine subspaces, considered as spheres of infinite radius.

DEFINITION – For $k = 1, \ldots, n-1$, the *k-osculating sphere* at a point of a curve in the Euclidean space $\mathbb{R}^n$ (in $\mathbb{S}^n$ or in $\mathbb{L}^n$) is the k–dimensional sphere (generalized sphere in $\mathbb{L}^n$), whose order of contact with the curve at that point is at least $k+2$. For $k = n-1$ we will simply write osculating hypersphere.

Example – The order of contact of a plane curve and its osculating circle at a generic point of the curve is 3.

We observe that the k–osculating spheres of a spatial curve $\gamma \subset \mathbb{S}^n \subset \mathbb{R}^{n+1}$ also lie in $\mathbb{S}^n$: They are the intersection of the $k+1$–osculating subspaces of the curve with $\mathbb{S}^n$.

DEFINITION – A *vertex* of a curve in $\mathbb{R}^n$ (in $\mathbb{S}^n$ or in $\mathbb{L}^n$) is a point where the order of contact with the osculating hypersphere is no less than $n+2$.

Example – An ellipse in the plane $\mathbb{R}^2$ has 4 vertices. They are the points at which the ellipse intersects its principal axes.

The following definition, classical for curves in $\mathbb{R}^n$ and $\mathbb{R}P^n$, is extended to curves in $\mathbb{S}^n$ and L^n.

DEFINITION – An embedded curve in $\mathbb{R}^n$ (or $\mathbb{R}P^n$, or $\mathbb{S}^n$ or $\mathbb{L}^n$) is called *convex* if it intersects any hyperplane (or projective hyperplane, or maximal hypersphere or hyperbolic hyperplane, respectively) at no more than n points, taking multiplicities into account.

Example – A plane curve is convex if it intersects any straight line in at most two points, taking multiplicities into account.

Example – For $n = 2k$, the *generalized ellipse*, given by $(\cos t, \sin t, \cos 2t, \sin 2t, \ldots, \cos kt, \sin kt)$, is convex.

The following theorem was proved in [8] and [22]. In the next chapter we give a new proof based on Sturm theory:

THEOREM – *Any convex curve in $\mathbb{R}^{2k}$ has at least $2k+2$ vertices.*

In [22] and [23] we proved that this theorem holds for the convex curves in the sphere $\mathbb{S}^{2k} \subset \mathbb{R}^{2k+1}$, in the projective space $\mathbb{R}P^{2k}$ and in the Lobachevskian space $\mathbb{L}^{2k}$. These theorems are a direct consequence of our theorems R, S and L stated and proved below.

We introduce a class of curves generalizing the convex ones:

DEFINITION – A curve embedded in $\mathbb{R}^n$ ($\mathbb{S}^n$ or $\mathbb{L}^n$) is called *spherically convex* if for each k–tuple of points of the curve, $k \leq n$, with positive multiplicities satisfying $m_1 + \cdots + m_k = n$, there exists at least one hypersphere of $\mathbb{R}^n$ (or hypersphere of $\mathbb{S}^n$ or hyperbolic hypersphere of $\mathbb{L}^n$, respectively) intersecting the curve at these points, with corresponding multiplicities, that does not intersect the curve elsewhere. The hyperspheres of infinite radius are not excluded.

Remark – For any point of a spherically convex curve there exists a hypersphere containing the codimension 2 osculating sphere through this point which does not intersect the curve elsewhere.

Remark – Spherically convex curves exist in Euclidean spaces, spheres and Lobachevskian spaces of even dimension only.

Remarks – Any convex curve is spherically convex. The affine transfomations of $\mathbb{R}^n$ preserve convex curves but don't preserve vertices. Moreover, the conformal transformations of $\mathbb{R}^n$ (respectively of $\mathbb{S}^n$ or of $\mathbb{L}^n$) preserve vertices and preserve spherically convex curves but don't preserve convex curves. So to study global problems about vertices it seems to be more natural to consider spherically convex curves instead of convex ones.

Example – Let γ be a closed convex curve in the Euclidean space $\mathbb{R}^2$ (or $\mathbb{R}^{2k}$). Let $p \in \gamma$ be a point which is not a vertex of γ. Consider an inversion σ centered at a point not belonging to γ and belonging to the osculating circle (hypersphere, respectively) of γ at p. Then the image of γ by the inversion σ is a non–convex curve which is spherically convex. In particular, the order of contact of $\sigma(\gamma)$ with its tangent line (hyperplane, respectively) at the point $\sigma(p)$ is 3 ($2k+1$, respectively).

Our main results in this paragraph are theorems R, S and L below ([24]).

THEOREM R – *Any spherically convex curve in the Euclidean space $\mathbb{R}^{2k}$ has at least $2k+2$ vertices.*

THEOREM S – *Any spherically convex curve in the sphere $\mathbb{S}^{2k} \subset \mathbb{R}^{2k+1}$ has at least $2k+2$ vertices.*

THEOREM L – *Any spherically convex curve in $\mathbb{L}^{2k}$ has at least $2k+2$ vertices.*

Theorems R, S and L are direct corollaries of the following theorem:

THEOREM 1 – *If a spherically convex curve in $\mathbb{R}^{2k}$ (respectively in $\mathbb{S}^{2k}$ or in $\mathbb{L}^{2k}$) transversally intersects a hypersphere at l points then it has at least l distinct vertices.*

Convex curves in $\mathbb{R}^n$ (in $\mathbb{S}^n$ or $\mathbb{L}^n$) exist only for even dimensions. However, convex curves exist in projective spaces of any dimension.

Example 1 – The projective line $\mathbb{R}P^1$ is a convex curve in $\mathbb{R}P^1$ itself. The curve $\theta \mapsto (\cos\theta, \sin\theta, \cos 3\theta, \sin 3\theta)$ is a convex curve in $\mathbb{R}P^3$: the antipodal points are identified, i.e. θ is identified with $\theta + \pi$.

To consider vertices of curves in the projective space, fix the spherical metric in $\mathbb{R}P^n$ by the double covering, $\pi : \mathbb{S}^n \to \mathbb{R}P^n$, identifying antipodal points. A *vertex* of a curve in γ in $\mathbb{R}P^n$ is a point where the lifted curve $\pi^{-1}(\gamma)$ has a vertex as a spherical curve.

DEFINITION – An embedded curve in $\mathbb{R}P^n$ is called *spherically convex* if its lift $\pi^{-1}(\gamma)$ is spherically convex in $\mathbb{S}^n$. Antipodal points are counted as one point.

As a consequence of theorem S we have

THEOREM P^{2k} – *Any spherically convex curve in the projective space* $\mathbb{R}P^{2k}$ *has at least* $2k+2$ *vertices.*

Barner's Theorem on flattenings holds for Barner curves in $\mathbb{R}P^n$ for any $n \geq 1$. However, for odd–dimensional projective spaces we have the following result [23] on vertices:

THEOREM P^{2k+1} – *There exist convex curves in* $\mathbb{R}P^{2k+1}$ *(and thus spherically convex curves) having no vertex.*

2. Proofs

First, we prove that the three versions of theorem 1 (for $\mathbb{R}^{2k}$, for $\mathbb{S}^{2k}$, and for L^{2k}) are equivalent (In contrast to the corresponding theorems for convex curves). Next, we prove Theorem 1 for curves in $\mathbb{S}^{2k}$.

Let H^n be a hyperplane of $\mathbb{R}^{n+1}$. Consider any inversion σ with respect to a point exterior to H^n. The image of a convex curve in H^n may be non–convex in $\sigma(H^n) = \mathbb{S}^n$. Conversely, the image of a convex curve in $\mathbb{S}^n$ may be non–convex in H^n. However, we have the

PROPOSITION 1 – *A curve in* $\mathbb{S}^n = \sigma(H^n)$ *not containing the center of the inversion is spherically convex in* $\mathbb{S}^n$ *if and only if its image is spherically convex in* $H^n = \sigma(\mathbb{S}^n)$.

Proof – The hyperspheres of $\mathbb{R}^n$ (including those of infinite radius) are sent onto the hyperspheres of $\mathbb{S}^n$, and vice versa. □

DEFINITION – For $k = 1, \ldots, n-1$, the *k-osculating subspace* at a point of a curve in $\mathbb{R}^n$ is the k-dimensional affine subspace spanned by the first k derivatives of the curve at that point.

Remark – The order of contact of the k-osculating subspace with the curve is at least $k+1$. For $k = 1, \ldots, n-2$, the $(k+1)$-osculating subspace contains the *k-osculating sphere.*

DEFINITION – A *flattening* of a curve γ in $\mathbb{R}^n$ ($\mathbb{R}P^n$) is a point where the derivatives of γ of order $1, \ldots, n$, are linearly dependent.

Remark – The order of contact of a curve with its osculating hyperplane, at a flattening is at least $n+1$, whereas at an ordinary point it is n.

Example – The flattenings of a plane curve are their inflections. The flattenings of a curve in $\mathbb{R}^3$ are those at which the torsion vanishes.

In [17] and [18] there is a proposition equivalent to the following lemma, proved in [22]:

LEMMA 1 – *Any inversion whose centre does not belong to a hyperplane* H *of* $\mathbb{R}^{n+1}$ *sends the vertices of any curve* γ *of* H *onto the flattenings of its image.*

To prove lemma 1 we need the following two lemmas:

LEMMA– *The image of a sphere* S^{n-1} *lying in a hyperplane* H *of* $\mathbb{R}^{n+1}$ *under an inversion belongs to a hyperplane of* $\mathbb{R}^{n+1}$ *and it still is a* $(n-1)$*–dimensional sphere.*

Proof. – The n-dimensional spheres containing S^{n-1} cover all the space. Hence, one of them goes through the centre of the inversion. The inversion sends this sphere to a hyperplane and the hyperplane H to a hypersphere. So the image of S^{n-1} is the intersection of a hyperplane and a hypersphere. □

LEMMA– *The image of the osculating hypersphere of a curve γ lying in a hyperplane of $\mathbb{R}^{n+1}$ under an inversion σ whose centre does not belongs to the hyperplane is the $(n-1)$-osculating sphere of the image curve $\sigma(\gamma)$ and is contained in the osculating hyperplane of $\sigma(\gamma)$.*

Proof. – By the preceding lemma, the image of the osculating hypersphere of γ belongs is a sphere of dimension $n-1$. It is osculating since the inversion preserves order of contact. So the hyperplane containing it is the osculating hyperplane. □

Proof of lemma 1 – By the preceding lemma, the inversion σ sends the osculating hyperspheres of a curve in a hyperplane H onto the $(n-1)$-osculating spheres of the image curve in $\mathbb{R}^{n+1}$. Since the order of contact is preserved by the inversion, the vertices of the hyperplane curve γ are sent onto the points at which the order of contact of the image curve with its $(n-1)$-osculating sphere (and with the osculating hyperplane) is at least $n+2$. So the vertices of the hyperplane curve γ are sent onto the flattenings of the spatial curve $\sigma(\gamma)$. □

LEMMA 2 *(see* [22]*)* – *The vertices of a spherical curve $\gamma \subset \mathbb{S}^n \subset \mathbb{R}^{n+1}$ are the flattenings of γ regarded as a spatial curve.*

Proof – The osculating hyperplane at a point of the spatial curve $\gamma \subset \mathbb{S}^n \subset \mathbb{R}^{n+1}$ contains the $(n-1)$-osculating sphere at that point. So at any point of γ the order of contact with its $(n-1)$-osculating sphere and with its osculating hyperplane is the same. □

Lemma 2, proposition 1 and lemma 1 imply that theorem 1 for $\mathbb{R}^{2k}$ is equivalent to theorem 1 for $\mathbb{S}^{2k}$.

Consider the Lobatchevskian space $\mathbb{L}^n \subset \mathbb{R}^n$. A convex curve in $\mathbb{L}^n$ may be non–convex considered as a curve of $\mathbb{R}^n$. Reciprocally, a curve contained in $\mathbb{L}^n \subset \mathbb{R}^n$ which is convex in $\mathbb{R}^n$ may be non–convex in $\mathbb{L}^n$. However we have the

PROPOSITION 2 – *A curve in $\mathbb{L}^n \subset \mathbb{R}^n$ is spherically convex in $\mathbb{L}^n$ if and only if it is spherically convex in $\mathbb{R}^n$.*

Proof – The generalized hyperspheres of $\mathbb{L}^n$ are intersections of $\mathbb{L}^n$ with hyperspheres of $\mathbb{R}^n$ (may be of infinite radius). □

Proposition 2 implies that theorem 1 for $\mathbb{R}^{2k}$ and theorem 1 for $\mathbb{L}^{2k}$ are equivalent.

To prove theorem 1, for $\mathbb{S}^{2k}$, we need introduce a definition and state a result of [8].

DEFINITION – A curve in $\mathbb{R}^n$ (in $\mathbb{R}P^n$) is called a *Barner curve* if for every $(n-1)$–tuple of points of the curve there exists a hyperplane through these points that does not intersect the curve elsewhere.

In [8], is proved that *Any Barner curve in $\mathbb{R}P^n$ transversally intersected by a hyperplane at l points has at least l distinct flattenings.*

We will use the following version of the preceding statement:

BARNER'S THEOREM – *Any Barner curve in $\mathbb{R}^{2k+1}$ transversally intersected by a hyperplane at l points has at least l distinct flattenings.*

Proof of theorem 1 – We will prove that any spherically convex curve γ of $\mathbb{S}^{2k} \subset \mathbb{R}^{2k+1}$ is a Barner curve considered as a spatial curve. Let $q_1, \ldots, q_{2k}$ be $2k$ points of γ. By hypothesis there is a hypersphere $S^{2k-1} \subset \mathbb{S}^{2k}$ through these points not intersecting γ elsewhere. The hyperplane of $\mathbb{R}^{2k+1}$ containing S^{2k-1} meets γ at the points $q_1, \ldots, q_{2k}$ and does not intersect it elsewhere. So γ is a Barner curve of $\mathbb{R}^{2k+1}$. If γ is transversally intersected by a hypersphere Γ of $\mathbb{S}^{2k}$ at l points then the hyperplane of $\mathbb{R}^{2k+1}$ containing Γ intersects γ transversally at the same l points. By Barner's theorem γ has at least l distinct flattenings. By lemma 2, the spherical curve $\gamma \subset \mathbb{S}^{2k} \subset \mathbb{R}^{2k+1}$ has at least l distinct vertices. This proves theorem 1.

Proof of theorem P^{2k+1} – We will prove that the convex curve in $\mathbb{R}P^{2k+1}$ given by

$$\gamma : \theta \mapsto (\cos\theta, \sin\theta, \cos 3\theta, \sin 3\theta, \ldots, \cos(2k+1)\theta, \sin(2k+1)\theta),$$

(identifying antipodal points) has no vertex. The curve γ lies in a hypersphere of $\mathbb{R}^{2k+2}$. The vertices of the spherical curve γ are its flattenings, considering γ as a spatial curve. So we must show that γ has no flattenings. All curvatures of γ are constant; thus it suffices to check that $\gamma(\theta)|_{\theta=0}$ is not a flattening. So it suffices (and it is easy) to check that the Wronskian of γ at $\theta = 0$ does not vanish. (The Wronskian of γ is the determinant of the matrix whose colums are the first $2k+2$ derivatives of γ). □

§2. Barner's Theorem in Lobatchevskian Spaces

We consider the natural generalization of Barner curves in Lobatchevskian spaces and prove a generalization of the Barner's theorem: *Any Barner curve in the Lobatchevskian space* $\mathbb{L}^{2k+1}$ *has at least* $2k+2$ *hyperbolic flattenings.*

1. Introduction and Results

Let $\mathbb{L}^n$ denote the open unit ball in $\mathbb{R}^n$, (the interior of the sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$) considered as Poincaré's model of the n-dimensional Lobatchevskian space. A *hyperbolic hyperplane* in $\mathbb{L}^n$ is the intersection of $\mathbb{L}^n$ with a hypersphere of $\mathbb{R}^n$ orthogonal to $\mathbb{S}^{n-1}$.

Definition – The *osculating hyperbolic hyperplane* at a point of a curve in $\mathbb{L}^n$ is the hyperbolic hyperplane whose order of contact with the curve at that point is at least n.

We recall the definition of flattening given in §1 and generalize it to curves in Lobatchevskian spaces:

Definition – A *flattening* (*hyperbolic flattening*) of a curve in the Euclidean or affine space $\mathbb{R}^n$ ($\mathbb{L}^n$, respectively) is a point where the order of contact of the curve with its osculating hyperplane (hyperbolic hyperplane, respectively) is at least $n+1$, whereas at an ordinary point it is n.

We generalize the definition of Barner curves to curves in Lobatchevskian spaces:

Definition – A curve embedded in $\mathbb{L}^n$ is called a *Barner curve* if for each k–tuple of points of the curve, $k \leq n-1$, with positive multiplicities satisfying $m_1 + \cdots + m_k = n-1$, there exists at least one hyperbolic hyperplane of $\mathbb{L}^n$ intersecting the curve at these points, with corresponding multiplicities, that does not intersect the curve elsewhere.

Barner curves exist only in odd-dimensional Lobatchevskian spaces.

The main results of this paragraph are theorems 1 and 2 below ([24]):

Theorem 1– *If a Barner curve in* $\mathbb{L}^{2k+1}$ *transversally intersects a hyperbolic hyperplane in* l *points then it has at least* l *distinct hyperbolic flattenings.*

Corollary – Any Barner curve in $\mathbb{L}^{2k+1}$ has at least $2k+2$ distinct hyperbolic flattenings.

Definition – Let p be a point of $\mathbb{R}^n$. A curve in $\mathbb{R}^n$ is called a *p–Barner curve* if for each k–tuple of points of the curve ($k \leq n-1$) with positive multiplicities satisfying $m_1 + \cdots + m_k = n-1$, there exists a hypersphere intersecting the curve at these points, with corresponding multiplicities, that does not intersect the curve elsewhere and that contains p.

Definition – Let p be a point of $\mathbb{R}^n$. A point q of a curve in $\mathbb{R}^n$ is called a *$p-$flattening* of the curve if there exists a hypersphere containing p and whose order of contact with the curve at q is at least $n+1$.

Theorem 2 – *Let* $p \in \mathbb{R}^{2k+1}$. *If a p–Barner curve transversally intersects in* l *points a hypersphere containing* p *then it has at least* l *$p-$flattenings.*

COROLLARY – *Any Barner curve with respect to a point $p \in \mathbb{R}^{2k+1}$ has at least $2k+2$ $p-$flattenings.*

2. Proofs

We will prove that theorem 1, theorem 2 and Barner's theorem (for curves in $\mathbb{R}^{2k+1}$) are equivalent.

First, we prove that theorem 2 implies Barner's theorem for curves in $\mathbb{R}^{2k+1}$: Barner's theorem for curves in $\mathbb{R}^{2k+1}$ is obtained as a particular case of theorem 2 when the point p is at infinity.

LEMMA 1 – *Let σ be an inversion with respect to a hypersphere in $\mathbb{R}^{n+1}$, and let H^n be a hyperplane not containing its centre. Let S^{n-1} be a hypersphere of H^n. Then all the hyperplanes containing the image under σ of some hypersphere of H^n orthogonal to S^{n-1} have a common point O.*

Remark 1 – The inversion sends the hyperplane H^n onto a hypersphere. If the image of S^{n-1} under σ is an equator of this hypersphere then the point O of lemma 1 is at infinity.

Proof of lemma 1. – Let C be the centre of the hypersphere $S^{n-1} \subset H^n$. Consider a point R in S^{n-1}. Consider a hypersphere $\hat{S}^n$ in $\mathbb{R}^{n+1}$ containing both the centre Q of the inversion and a hypersphere of H^n orthogonal to S^{n-1}. The power of C with respect to the hypersphere $\hat{S}^n$ is $CR \cdot CR$. Hence the line through the points C and Q intersects the hypersphere $\hat{S}^n$ at Q and at a point P such that $CQ \cdot CP = CR \cdot CR$. So all hyperspheres containing both the centre Q of the inversion and some hypersphere of H^n orthogonal to S^{n-1} must also contain the point P. The point O of lemma 1 is $\sigma(P)$. □

COROLLARY – *Let σ, $H^n \subset \mathbb{R}^{n+1}$ and O be like in lemma 1. Let $\mathbb{L}^n \subset H^n$ be the n-dimensional Lobatchevskian space and let H' be any hyperplane not going through O, parallel to the hyperplane containing $\sigma(\mathbb{S}^{n-1}) = \sigma(\partial\mathbb{L}^n)$. Consider the projection $\pi : \sigma(H^n) \to H'$, from O. Then the image of each hyperbolic hyperplane of $\mathbb{L}^n$ under $\pi \circ \sigma$ is the intersection of a Euclidean hyperplane of H' with $\pi \circ \sigma(\mathbb{L}^n)$.*

We prove that Barner's theorem for curves in $\mathbb{R}^{2k+1}$ implies theorem 1:

Proof of theorem 1. – The restriction to $\mathbb{L}^n$ of the map $\pi \circ \sigma$ used in the corollary, sends Poincaré's model of Lobatchevkian space to Klein's model. The hyperbolic flattenings of a curve in $\mathbb{L}^n$ are sent onto the flattenings of its image in H'. In particular, for $n = 2k+1$ Barner's curves in $\mathbb{L}^{2k+1}$ are sent onto Barner's curves in $H' = \mathbb{R}^{2k+1}$. Applying Barner's theorem we prove Theorem 1. □

Finally, we will prove that theorem 1 implies theorem 2:

Proof of theorem 2. – Consider the exterior of a hypersphere S^{2k} as a model of the Lobatchevskian space $\mathbb{L}^{2k+1}$. Theorem 1 works also here, in particular when the hypersphere S^{2k} has infinitely small radius (that is, when S^{2k} becomes a point of $\mathbb{R}^{2k+1}$). □

§3. Generating Family of the Normal Map of a Curve in $\mathbb{R}^n$ Some Properties of Vertices and a Formula for Calculate Them

We prove that the vertices of a curve $\gamma \subset \mathbb{R}^n$ are extrema of the radius of the osculating hypersphere. Using Sturm Theory, we give a proof of the $2k+2$–Vertex Theorem for convex curves in the Euclidean space $\mathbb{R}^{2k}$. As a by–product of this proof we obtain a formula to calculate the vertices of a curve in $\mathbb{R}^n$. Applying Sturm theory and our formula to calculate vertices we obtain the number of vertices of the *generalized ellipses* introduced by Arnol'd in [6].

1. Statement of Results on Vertices

Vertices and Flattenings of curves in $\mathbb{R}^n$ are related to Sturm Theory. In point **3** of this §, we give a proof of the $2k+2$–Vertex Theorem for convex curves in the Euclidean space $\mathbb{R}^{2k}$ based on Sturm Theory. This proof allows us to give a formula to calculate the vertices of a curve in $\mathbb{R}^n$ as the zeroes of a determinant:

THEOREM 1 – *The vertices of any curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^n$ (or $\gamma : \mathbb{R} \to \mathbb{R}^n$), $\gamma : s \mapsto (\varphi_1(s), \ldots, \varphi_n(s))$ are given by the zeroes of*

$$\det(R_1, \ldots, R_n, G)$$

where R_i (G) is the column vector defined by the first $n+1$ derivatives of φ_i (of $g = \frac{\gamma^2}{2}$, respectively).

Remark – Theorem 1 says that the vertices of any curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^n$ (or $\gamma : \mathbb{R} \to \mathbb{R}^n$), $\gamma : s \mapsto (\varphi_1(s), \ldots, \varphi_n(s))$ are given by the flattenings of the curve $\Gamma : \mathbb{S}^1 \to \mathbb{R}^{n+1}$ (or $\Gamma : \mathbb{R} \to \mathbb{R}^{n+1}$),

$$\Gamma : s \mapsto \left(\varphi_1(s), \ldots, \varphi_n(s), \frac{\gamma^2(s)}{2}\right).$$

This means that the vertical projection of a curve $\gamma \subset \mathbb{R}^n$ on a paraboloid 'of revolution' $z = \frac{1}{2}(x_1^2 + \cdots + x_n^2)$ sent the vertices of the curve γ onto the flattenings of its image. We will discuss the properties of this and other projections related to Lagrangian and Legendrian singularities in another paper.

Noticing that the formula of Theorem 1 can be 'simplified', we obtain the following

THEOREM 1^{bis} – *The vertices of any curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^n$ (or $\gamma : \mathbb{R} \to \mathbb{R}^n$), $\gamma : s \mapsto (\varphi_1(s), \ldots, \varphi_n(s))$ are given by the zeros of the following determinant*[1]*:*

$$\begin{vmatrix} \varphi_1' & \cdots & \varphi_n' & 0 \\ \varphi_1'' & \cdots & \varphi_n'' & h_1 \\ \vdots & & \vdots & \vdots \\ {\varphi_1}^{(n+1)} & \cdots & {\varphi_n}^{(n+1)} & h_n \end{vmatrix} = 0,$$

where $h_1 = \gamma' \cdot \gamma'$ and $h_k = h_{k-1}' + \gamma' \cdot \gamma^k$.

[1] I discovered the formula of Theorem 1 in May 1995 and calculated vertices of many curves with it. In July 1999 J.J. Nuño Ballesteros told me that he knew the formula of theorem 1^{bis}. I don't know when he discovered it. In August 2000 he told me that he will publish it in some preprint.

Proof – The column vector G in the determinant of theorem 1 is the sum of various column vectors, n of which can be eliminated by substracting the column vectors $\varphi_i R_i,\ i = 1, \ldots, n$. □

For a curve in the Euclidean plane, to have a vertex is equivalent to to have an extremum of the radius of the osculating circle. In higher dimensional spaces this is not the case. However we have the following theorem proved in point **2** of this §.

THEOREM 2 ([22]) – *The vertices of a curve $\gamma \subset \mathbb{R}^n$ are extrema of the radius of the osculating hypersphere.*

Remark – The converse is not true for $n > 2$. For example, all the points of the circular helix $t \mapsto (\cos t, \sin t, t)$ are extrema of the radius of the osculating hypersphere. However it has no vertex. A more generic example is given by the curve $t \mapsto (a\cos t, b\sin t, t)$ which has no vertex for any $a, b \in \mathbb{R} \setminus \{0\}$ such that $|a^3 - b^2| < 1/3$.

Proof of remark – It suffices to use our formula from theorem 1. Writing out the equation, we obtain

$$\begin{vmatrix} -a\sin t & b\cos t & 1 & 1/2(b^2-a^2)\sin 2t + t \\ -a\cos t & -b\sin t & 0 & (b^2-a^2)\cos 2t + 1 \\ a\sin t & -b\cos t & 0 & -2(b^2-a^2)\sin 2t \\ a\cos t & b\sin t & 0 & -4(b^2-a^2)\cos 2t \end{vmatrix} = 0,$$

which gives $ab(1 - 3(b^2 - a^2)\cos 2t) = 0$. This equation has no real solution for $|a^3 - b^2| < 1/3$. □

The ellipse is the simplest closed convex curve in the plane having the minimum number of vertices: 4.

A *generalized ellipse* in $\mathbb{R}^{2k}$ is a convex curve given by the following parametrization ([6]): $\theta \mapsto (a_1\cos\theta, b_1\sin\theta, a_2\cos 2\theta, b_2\sin 2\theta, \ldots, a_k\cos k\theta, b_k\sin k\theta)$. We can expect that generalized ellipses are convex curves in $\mathbb{R}^{2k}$ having the minimum number of vertices, i.e. $2k + 2$. However, the following example shows that the generalized ellipses in $\mathbb{R}^{2k}$ can have more than $2k + 2$ vertices.

Example 1 – The generalized ellipse in $\mathbb{R}^4$, $\gamma(\theta) = (a_1\cos\theta, b_1\sin\theta, a_2\cos 2\theta, b_2\sin 2\theta)$, with $a_2^2 \neq b_2^2$ and $a_1b_1a_2b_2 \neq 0$ has 8 vertices. If $a_2^2 = b_2^2$ then γ is a spherical curve and all its points are thus vertices.

Denote $C_k = \cos k\theta$ and $S_k = \sin k\theta$.

THEOREM 3– *Consider the generalized ellipse in $\mathbb{R}^{2k}$ given by*

$$\gamma(\theta) = (a_1C_1, b_1S_1, a_2C_2, b_2S_2, \ldots, a_kC_k, b_kS_k),$$

with $a_1b_1a_2b_2\cdots a_kb_k \neq 0$. Then, for even k, γ can have $2k + 4$, $2k + 8$,...,$4k$ or an infinity of vertices depending on the values of the parameters a_j and b_j, for $j \geq \frac{k}{2} + 1$. For odd k, γ can have $2k + 2$, $2k + 6$,...,$4k$ or an infinity of vertices depending on the values of the parameters a_j and b_j, for $j \geq \frac{k+1}{2}$.

We will construct a convex curve in $\mathbb{R}^{2k}$ having the minimum number of vertices, i.e. $2k + 2$. Consider the generalized ellipse of Theorem 3 with coefficients $a_1 =$

$b_1 = \cdots = a_k = b_k = 1$ and denote it by γ_0. Obviously γ_0 is a spherical curve and all its points are vertices. In order to obtain the desired convex curve, we will perturb γ_0 in the "radial direction". Let $\gamma_\varepsilon = (1 + \varepsilon \cos(k+1)\theta)\gamma_0$.

THEOREM 4 – *For $\varepsilon \neq 0$ small enough the curve γ_ε has exactly $2k+2$ vertices.*

Example 1 and Theorems 3 and 4 are proved in point **4** of this §.

2. Proof of Theorem 2 and Description of the Focal Set of a Curve

Proof of theorem 2 – The generating family $F : \mathbb{R}^n \times S^1 \to \mathbb{R}$ associated to the focal set of the curve γ is given by

$$F(q,s) = \frac{1}{2} \parallel q - \gamma(s) \parallel^2 .$$

We shall write $\Sigma(i) = \{(q,s)/\partial_s F(q,s) = 0, ..., \partial_s^i F(q,s) = 0\}$. Thus $\Sigma(1)$ is the set of pairs (q,s) such that q is the center of some hypersphere of $\mathbb{R}^n$ whose order of contact with γ at s is at least 2 (this means that q is in the normal hyperplane to γ at s). So $\Sigma(2)$ is the set of pairs (q,s) such that q is the center of some hypersphere of $\mathbb{R}^n$ whose order of contact with γ at s is at least 3. From the equations can be seen that these points generate a plane of dimension $n-2$ contained in the normal hyperplane to γ at s. So $\Sigma(n)$ is the set of pairs $(q(s),s)$ such that $q(s)$ is the centre of the osculating hypersphere at $\gamma(s)$. Hence the value of F at the point $(q(s),s)$ in $\Sigma(n)$ is one half of the square of the radius of the osculating hypersphere at $\gamma(s)$. The condition for a point $p = \gamma(s)$ to be a vertex is equivalent to the fact that the first $n+1$ derivatives of F with respect to s vanish at s. Hence $\Sigma(n+1)$ is the set of vertices of the curve. It is a well-known fact of singularity theory [2] that a point belonging to $\Sigma(n+1)$ is a critical point of the restriction of F to $\Sigma(n)$. So a vertex is a critical point of the radius of the osculating hypersphere. □

Remark – The centers of the osculating hyperspheres at the vertices of γ are given by the $q \in \mathbb{R}^n$ for which there exists a solution s of the $n+1$–system of equations

$$\begin{aligned} F_q'(s) &= 0 \\ F_q''(s) &= 0 \\ &\vdots \\ F_q^{(n+1)}(s) &= 0. \end{aligned}$$

For a fixed s, the first equation gives the normal hyperplane to the curve at the point $\gamma(s)$. The first two equations give a codimension 1 subspace of the normal hyperplane to the curve at the point $\gamma(s)$. Following this process we obtain a complete flag at each point of the curve. The *focal curve* $q(s)$, formed by the centers of the osculating hyperspheres, is determined by the n first equations. The complete flag is the osculating flag of the focal curve. In particular, the osculating hyperplane of the focal curve at the point $q(s)$ is the normal hyperplane to the curve γ at the point $\gamma(s)$. As the point moves along the curve γ, the corresponding flag (starting with the codimension 2 subspace) generates a hypersurface which is stratified in a natural way by the components of the flag. This stratified hypersurface is a component of the focal set of the curve γ. The other component of

the focal set is the curve itself. The stratum of dimension 1 (generated by the 0–dimensional subspace of the flag, i.e. generated by center of the osculating hypersphere at the moving point) is the focal curve of γ. The equation $F_q^{(n+1)}(s) = 0$ gives a finite number of isolated points on the focal curve. These points correspond to the vertices.

The focal set is also a component of the caustic of the Lagrangian map (normal map) defined by the generating family $F(q, s)$ (For the notions of caustic, Lagrangian map, Lagrangian singularity and generating family, we refere the reader to chapter 1). Thus the vertices of a curve in $\mathbb{R}^n$ correspond to a Lagrangian singularity A_{n+1} of the normal map.

3. A Proof of the $2k+2$–Vertex Theorem in $\mathbb{R}^{2k}$ by Sturm Theory

We begin this paragraph with some definitions and results of Sturm theory, taken from [6] and [11].

A set of functions $\{\varphi_1, \ldots, \varphi_{2k+1}\}$ with $\varphi_i : \mathbb{S}^1 \to \mathbb{R}$ is a *Chebishev system* if any linear combination $a_1\varphi_1 + \cdots + a_{2k+1}\varphi_{2k+1},\ a_i \in \mathbb{R}$, with $a_1^2 + \ldots + a_{2k+1}^2 \neq 0$ has at most $2k$ zeros on $\mathbb{S}^1$.

Example 1 – The system of functions $\{1, \cos\theta, \sin\theta\}$ is a Chebishev system.

Remark – Any convex curve $\theta \mapsto (\varphi_1(\theta), \ldots, \varphi_{2k}(\theta))$ in $\mathbb{R}^{2k}$ defines a Chebishev system: $\{1, \varphi_1, \ldots, \varphi_{2k}\}$.

Definition – A linear differential operator $L : C^\infty(\mathbb{S}^1) \to C^\infty(\mathbb{S}^1)$ is called *disconjugate* if it has a fundamental system of solutions for the equation $Lg = 0$ which are defined on the circle and form a Chebishev system.

Example 2 – The operator $L = \partial(\partial^2 + 1)$ is disconjugate. The Chebishev system $\{1, \cos\theta, \sin\theta\}$ is a fundamental system of solutions for it.

Example 3 – Any convex curve $\gamma : \theta \mapsto (\varphi_1(\theta), \ldots, \varphi_{2k}(\theta))$ in $\mathbb{R}^{2k}$ defines a $2k+1$–order disconjugate operator L_γ defined by

$$L_\gamma g = \det(R_1, \ldots, R_{2k}, G),$$

where R_i (G) is the column vector defined by the first $2k+1$ derivatives of φ_i (of g, respectively). Evidently the Chebishev system $\{1, \varphi_1, \ldots, \varphi_{2k}\}$ is a fundamental system of solutions of the equation $L_\gamma g = 0$.

Example 4 – The generalized ellipse ([6])

$$\gamma : \theta \mapsto (a_1 \cos\theta, b_1 \sin\theta, a_2 \cos 2\theta, b_2 \sin 2\theta, \ldots, a_k \cos k\theta, b_k \sin k\theta),$$

defines, up to a constant factor, the $2k+1$–order disconjugate operator

$$L_\gamma = \partial(\partial^2 + 1)\cdots(\partial^2 + n^2).$$

Some proofs of 4–vertex type theorems are based on the following theorem due to Hurwitz ([12]):

Hurwitz's Theorem – *Any function $f \in C^\infty(\mathbb{S}^1)$ whose Fourier series begins with the harmonics of order N, $f = \sum_{k \geq N} a_k \cos k\theta + b_k \sin k\theta$, has at least $2N$ zeroes.*

In fact any function $f \in C^\infty(\mathbb{S}^1)$ without harmonics up to order n is orthogonal to the solutions of the equation $\partial(\partial^2+1)\cdots(\partial^2+n^2)\varphi = 0$, and such solutions form a Chebishev system.

The following theorem generalizes Hurwitz's theorem.

STURM–HURWITZ THEOREM ([6],[11]) – Let $f : \mathbb{S}^1 \to \mathbb{R}$ be a C^∞ function such that $\int_{\mathbb{S}^1} f(\theta)\varphi_i(\theta)d\theta = 0$, $\{\varphi_i\}_{i=1,\ldots,2k+1}$ being a Chebishev system. Then f has at least $2k+2$ sign changes.

COROLLARY – ([11]) *Any function in the image of a disconjugate operator ($f = Lg$, where $g \in C^\infty(\mathbb{S}^1)$ is any function) of order $2k+1$ has at least $2k+2$ sign changes.*

PROOF OF THE $2k$+2–VERTEX THEOREM IN $\mathbb{R}^{2k}$ – Let $\gamma : \theta \mapsto (\varphi_1(\theta), \ldots, \varphi_{2k}(\theta))$ be a convex curve in $\mathbb{R}^{2k}$. Consider the family of functions on the circle $F : \mathbb{S}^1 \times \mathbb{R}^{2k} \to \mathbb{R}$ defined by

$$F_q(\theta) = \frac{1}{2} \parallel q - \gamma(\theta) \parallel^2 .$$

In the proof of theorem 1 we saw that the centers of the osculating hyperspheres at the vertices of γ are given by the $q \in \mathbb{R}^n$ for which there exists a solution θ of the $2k+1$–system of equations:

$$\begin{aligned} F_q'(\theta) &= 0 \\ F_q''(\theta) &= 0 \\ &\vdots \\ F_q^{(2k+1)}(\theta) &= 0 \end{aligned}$$

The *focal curve* $q(\theta)$ of centers of the osculating hyperspheres is determined by the first $2k$ equations. The last equation is the condition on this curve determining the vertices. Write $g = \frac{\gamma^2}{2}$. Using the fact that $-F = \gamma \cdot q - \frac{\gamma^2}{2} - \frac{q^2}{2}$, the system of equations can be written as

$$\begin{aligned} \gamma' \cdot q - g' &= 0 \\ \gamma'' \cdot q - g'' &= 0 \\ &\vdots \\ \gamma^{(2k+1)} \cdot q - g^{(2k+1)} &= 0 \end{aligned}$$

This means that the vector $(q, -1)$ in $\mathbb{R}^{2k+1}$ is orthogonal to the $2k+1$ vectors (γ', g'), $(\gamma'', g''), \ldots, (\gamma^{(2k+1)}, g^{(2k+1)})$. So the vertices of γ are given by the zeros of the determinant of the matrix whose lines are these $2k+1$ vectors. This determinant is equal to $\det(R_1, \ldots, R_{2k}, G)$ where R_i (G) is the column vector defined by the first $2k+1$ derivatives of φ_i (of $g = \frac{\gamma^2}{2}$, respectively). This is the image of $g = \frac{\gamma^2}{2}$ under the operator L_γ (see example 3). So corollary 1 implies that this determinant has at least $2k+2$ sign changes. This proves the theorem.

Proof of Theorem 1

In the above proof of the $2k+2$–vertex theorem for convex curves in $\mathbb{R}^{2k}$, the convexity of the curve and the parity of the dimension were used only in the last

step. So the determinant obtained in the proof gives a formula to calculate the vertices of a curve in $\mathbb{R}^n$. This proves theorem 1.

4. On the Number of Vertices of Generalized Ellipses

We will prove example 1 and Theorem 3 given in the begining of this §.

Example 1 – The generalized ellipse in $\mathbb{R}^4$,

$$\gamma(\theta) = (a_1\cos\theta, b_1\sin\theta, a_2\cos 2\theta, b_2\sin 2\theta), \text{ with } a_2^2 \neq b_2^2 \text{ and } a_1b_1a_2b_2 \neq 0$$

has 8 *vertices. If* $a_2^2 = b_2^2$ *then* γ *is a spherical curve and all its points are thus vertices.*

Proof – Denote $C_k = \cos k\theta$, $S_k = \sin k\theta$ and $g = a_1^2C_1^2 + b_1^2S_1^2 + a_2^2C_2^2 + +b_2^2S_2^2$. By example 4 of point **2** and the formula of Theorem 1 the vertices of γ correspond to the roots of the equation $\partial(\partial^2+1)(\partial^2+2^2)g = 0$. The trigonometric identity

$$a^2\cos^2\theta + b^2\sin^2\theta = \frac{1}{2}(a^2 + b^2 + (a^2 - b^2)\cos 2\theta)$$

allows us to write

$$g = (a_1^2 - b_1^2)C_2 + (a_2^2 - b_2^2)C_4 + a_1^2 + b_1^2 + a_2^2 + b_2^2.$$

The operator ∂ kills the constant terms (i.e. the harmonics of order zero), and the operator $(\partial^2 + 2^2)$ kills the second order harmonics. Thus

$$\partial(\partial^2+1)(\partial^2+2^2)g = K(a_2^2 - b_2^2)S_4,$$

where K is a non zero constant. Thus the vertices of γ correspond to the solutions of the equation $K(a_2^2 - b_2^2)S_4 = 0$, i.e. γ has 8 vertices for $a_2^2 \neq b_2^2$ and all its points are vertices for $a_2^2 = b_2^2$. □

We keep the notation $C_k = \cos k\theta$ and $S_k = \sin k\theta$.

Example 2 – The generalized ellipse in $\mathbb{R}^6$, $\gamma(\theta) = (a_1\cos\theta, b_1\sin\theta, a_2\cos 2\theta, b_2\sin 2\theta, a_3\cos 3\theta, b_3\sin 3\theta)$, *with* $a_1b_1a_2b_2a_3b_3 \neq 0$ *can have* $8, 12$ *or an infinity of vertices, depending on the values of the parameters* a_2, b_2, a_3, b_3. *In particular, if* $a_2^2 = b_2^2$ *and* $a_3^2 \neq b_3^2$ *then* γ *has* 12 *vertices, and if* $a_2^2 \neq b_2^2$ *and* $a_3^2 = b_3^2$ *then* γ *has* 8 *vertices. If* $a_2^2 = b_2^2$ *and* $a_3^2 = b_3^2$ *then* γ *is a spherical curve and all its points are thus vertices.*

Proof – As in example 1, the vertices of γ are the roots of the equation given by $\partial(\partial^2+1)(\partial^2+2^2)(\partial^2+3^2)g = 0$ where

$$g = (a_1^2 - b_1^2)C_2 + (a_2^2 - b_2^2)C_4 + (a_3^2 - b_3^2)C_6 + \sum_{i=1}^{3}(a_i^2 + b_i^2).$$

The operator $\partial(\partial^2+1)(\partial^2+2^2)(\partial^2+3^2)$ kills the harmonics of orders zero, one, two and three. Thus

$$\partial(\partial^2+1)(\partial^2+2^2)(\partial^2+3^2)g = K_2(a_2^2 - b_2^2)S_4 + K_3(a_3^2 - b_3^2)S_6,$$

where K_2 and K_3 are non zero constants. □

THEOREM 3– *Consider the generalized ellipse in* $\mathbb{R}^{2k}$

$$\gamma(\theta) = (a_1C_1, b_1S_1, a_2C_2, b_2S_2, \ldots, a_kC_k, b_kS_k),$$

with $a_1b_1a_2b_2\cdots a_kb_k \neq 0$. *Then, for even* k, γ *can have* $2k+4$, $2k+8$,...,$4k$ *or an infinity of vertices depending on the values of the parameters* a_j *and* b_j, *for* $j \geq \frac{k}{2}+1$. *For odd* k, γ *can have* $2k+2$, $2k+6$,...,$4k$ *or an infinity of vertices depending on the values of the parameters* a_j *and* b_j, *for* $j \geq \frac{k+1}{2}$.

Proof of Theorem 3. – As in examples 1 and 2, the vertices of γ are the roots of the equation given by

$$\partial(\partial^2+1)(\partial^2+2^2)\cdots(\partial^2+k^2)g = 0,$$

where $g = \sum_{i=1}^{k}(a_i^2 - b_i^2)C_{2i} + \sum_{i=1}^{k}(a_i^2+b_i^2)$. The operator

$$\partial(\partial^2+1)(\partial^2+2^2)\cdots(\partial^2+k^2)$$

kills the harmonics from the order zero until order k. Thus, for even k,

$$\partial(\partial^2+1)(\partial^2+2^2)\cdots(\partial^2+k^2)g = \sum_{i\geq\frac{k}{2}+1} K_i(a_i^2-b_i^2)S_{2i},$$

where K_i is a non zero constant, for $i \geq \frac{k}{2}+1$. For odd k

$$\partial(\partial^2+1)(\partial^2+2^2)\cdots(\partial^2+k^2)g = \sum_{i\geq\frac{k+1}{2}} K_i(a_i^2-b_i^2)S_{2i},$$

where K_i is a non zero constant, for $i \geq \frac{k+1}{2}$. This proves Theorem 3.

Proof of Theorem 4. – Applying our formula of Theorem 1 we obtain that the number of vertices of the curve $\gamma_\varepsilon = (1+\varepsilon\cos(k+1)\theta)\gamma_0$ is given by the number of solutions $\theta \in \mathbb{S}^1$ of an equation of the form

$$0 = \varepsilon K\sin(k+1)\theta + \varepsilon^2 f(\theta,\varepsilon),$$

where $K \neq 0$ is a constant and $f(\theta,\varepsilon)$ is a bounded function. Thus for $\varepsilon \neq 0$ small enough this equation has exactly $2k+2$ solutions. □

§4. Weakly Convex Curves in $\mathbb{R}^n$ and $\mathbb{R}P^n$

A convex curve has no flattening and his osculating hyperplane intersects it only at the point of osculation. A curve in $\mathbb{R}P^2$ ($\mathbb{R}^2$) is convex if and only if it has these two properties. To answer a question of V. Arnol'd ([6]), we show that this two properties don't imply convexity for curves in $\mathbb{R}P^n$, for $n > 2$.

1. Statement of Results

We recall that a smooth closed curve in $\mathbb{R}P^n$ ($\mathbb{R}^n$) is called *convex* if any hyperplane intersects it in at most n points, taking multiplicities into account.

A convex curve has no flattening and its osculating hyperplane intersects it only at the point of osculation. A curve in $\mathbb{R}P^2$ ($\mathbb{R}^2$) is convex if and only if it has these two properties. In [6], V. Arnol'd put the problem to know whether these two properties imply convexity (for dimensions greater than 2). In this section we answer this question.

We say that a curve in $\mathbb{R}P^n$ ($\mathbb{R}^n$) is *weakly convex* if it has no flattening and its osculating hyperplane intersects it only at the point of osculation.

For $n > 2$ the answer to Arnol'd's question is negative. In [1], S. Anisov gave an example of a weakly convex curve in $\mathbb{R}P^3$. For $n > 2$ we give examples of curves in $\mathbb{R}P^n$ ($\mathbb{R}^n$ for n even) which are weakly convex but are no convex.

Remark – Any weakly convex curve in $\mathbb{R}P^n$ is affine for even n, i.e. there exists a hyperplane of $\mathbb{R}P^n$ not intersecting the curve. For odd n any weakly convex curve in $\mathbb{R}P^n$ is not contractible, i.e. it intersects any hyperplane in an odd number of points, counting multiplicities.

PROPOSITION 1– *The curve in $\mathbb{R}P^{2k}$, with $k \geq 2$, given in affine coordinates by*

$$\theta \mapsto (\cos\theta, \sin\theta, \cos 2\theta, \sin 2\theta, \ldots, \cos(k-1)\theta, \sin(k-1)\theta, \cos(k+1)\theta, \sin(k+1)\theta),$$

is weakly convex but not convex.

PROPOSITION 2– *The curve γ in $\mathbb{R}P^{2k-1}$, $k \geq 2$, given in homogeneous coordinates by*

$$\theta \mapsto [\cos\theta : \sin\theta : \cos 3\theta : \ldots : \cos(2k-3)\theta : \sin(2k-3)\theta : \cos(2k+1)\theta : \sin(2k+1)\theta],$$

is weakly convex but not convex.

Remark – The curve in Proposition 2 can be considered as a curve in $\mathbb{S}^{2k-1} \subset \mathbb{R}^{2k}$, where the points $\gamma(\theta)$ and $\gamma(\theta + \pi) = -\gamma(\theta)$ are identified.

We constructed many other examples of weakly convex curves which are not convex. In particular, for the Euclidean space $\mathbb{R}^{2k}$ we calculate the number of vertices for many examples in which the convexity is "slightly broken": Weakly convex curves in $\mathbb{R}^{2k}$ which intersect any hyperplane in at most $2k+2$ points and intersect at least one hyperplane in exactly $2k+2$ points.

In all examples of this kind of weakly convex curves in $\mathbb{R}^{2k}$ we obtained that the number of vertices was always greater or equal to $\sqrt{2k+2}$. Moreover we constructed weakly convex curves of this kind in $\mathbb{R}^{2k}$ for which the number of vertices

is the smallest even number greater or equal to $\sqrt{2k+2}$. From this information we formulate the following

CONJECTURE– *Let γ be a weakly convex curve in $\mathbb{R}^{2k}$ which intersect any hyperplane in at most $2k+2$ points and intersect at least one hyperplane in exactly $2k+2$ points. Then γ has an even number of vertices greater or equal to $\sqrt{2k+2}$.*

Example – The curve $\gamma : \theta \mapsto (a_1 \cos\theta, b_1 \sin\theta, a_2 \cos 3\theta, b_2 \sin 3\theta)$ in $\mathbb{R}^4$ has 4 vertices for a_2 and b_2 small enough. Suppose $\mathbb{R}^4 \subset \mathbb{R}^5$. An inversion in $\mathbb{R}^5$ centered at a point exterior to $\mathbb{R}^4$ sends the curve γ into a spherical curve $\hat{\gamma} \subset \mathbb{R}^5$. The curve $\hat{\gamma}$ lies on the boundary of its convex hull and has only 4 flattenings. This example shows that Sedykh's theorem (see §3 of this chapter and [19]) can't be extended to higher dimensions.

2. Proof of proposition 1

The proof consists of various simple steps:

0 – Consider the standard coordinates $(x_1, x_2, \ldots, x_{2k})$ in $\mathbb{R}^{2k}$. The curve of proposition 1 is not convex because it intersects the hyperplane $x_{2k} = 0$ at the $2k+2$ points which correspond to the solutions of the equation $\sin(k+1)\theta = 0$.

1 – Observe that all curvatures of the curve in proposition 1 (regarded as a curve in the Euclidean space $\mathbb{R}^{2k}$) are constant. Observe also that for each pair of points $\gamma(\theta_0), \gamma(\theta_1)$ of the curve there is an orthogonal transformation of $\mathbb{R}^{2k}$ preserving the curve and sending the point $\gamma(\theta_0)$ in the point $\gamma(\theta_1)$. This orthogonal tranformation is obtained by a rotation of an angle $(\theta_1 - \theta_0) \cdot j$ on the 2–plane of coordinates x_{2j-1}, x_{2j} for $j < k$ and an angle $(\theta_1 - \theta_0) \cdot (k+1)$ in the 2–plane of coordinates x_{2k-1}, x_{2k}. Thus it suffices to calculate the osculating hyperplane for $\theta = 0$ and to show that this hyperplane does not meet the curve elsewhere.

2 – The equation of osculating hyperplane at $\theta = 0$ involves only odd index variables: It is of the form $a_1 x_1 + a_3 x_3 + \cdots + a_{2k-1} x_{2k-1} + b = 0$.

3 – Substitute the odd components of the curve in the preceding equation to find the points at which the curve intersects the osculating hyperplane. This gives $a_1 \cos\theta + a_3 \cos 3\theta + \cdots + a_{k-1} \cos(k+1)\theta + b = 0$.

4 – Make the change of variables $\theta = 2\varphi$ and introduce the following notation: $C = \cos\varphi$, $S = \sin\varphi$ and $C_k = \cos k\varphi$ and $S_k = \sin k\varphi$ for $k \geq 2$. The equation of step 3 becomes $a_1 C_2 + a_3 C_6 + \cdots + a_{k-1} C_{2(k+1)} + b = 0$.

5 – Use the identities
$C_{2k} = 1 - 2S_k^2$ and $S_n^2 = n^2 S^2 + \cdots + 2n(-4)^{n-1} S^{2(n-1)} + (-4)^{n-1} S^{2n}$, for $n \geq 2$. Equation of step 4 becomes an equation of degree $2k+2$ in S.

6 – The osculating hyperplane at $2\varphi = \theta = 0$ intersects the curve with multiplicity $2k$. Thus the equation is of the form $b_1 S^{2k}(b_2 + b_3 S^2)$, where b_1, b_2 and b_3 are constants. We only need to known the constants b_2 and b_3.

7 – Observe that the terms S^{2k} and S^{2k+2} may only come from the term S_{k+1}^2. Thus the equation to solve is $2(k+1)(-4)^{k-1} S^{2k} + (-4)^k S^{2(k+1)}$, which is equivalent

to $2(-4)^{k-1}S^{2k}(k+1-2S^2)=0$. This equation has a root of multiplicity $2k$ at $S=0$, which corresponds to the intersection of the osculating hyperplane with the curve at $2\varphi=\theta=0$. The equation $k+1-2S^2=0$ has no real solution for $k\geq 2$ because $S^2=\sin^2\varphi\leq 1$. This proves proposition 1.

3. Proof of proposition 2

Proof of proposition 2 – Let γ be the parametrization of proposition 2. The proof of proposition 2 is similar to the proof of proposition 1; let us just point out the differences. The parametrization given in proposition 2 is in $\mathbb{R}^{2k}\setminus\{0\}$ where the points belonging to a straight line through the origin of $\mathbb{R}^{2k}$ are identified. In particular $\gamma(\theta)=-\gamma(\theta+(2m+1)\pi),\ m\in\mathbb{Z}$. The osculating hyperplane of the curve is determined by γ and its first $2k-2$ derivatives.

1 – Use the fact the parametrization also gives a curve in the Euclidean space $\mathbb{R}^{2k}$ having all curvatures constant. Thus it suffices to calculate the osculating hyperplane for $\theta=0$ and to show that this hyperplane does not meet the curve elsewhere.

2 – Consider the standard coordinates $(x_1,x_2,\ldots,x_{2k})$ in $\mathbb{R}^{2k}$. The equation of the osculating hyperplane at $\theta=0$ involves only even variables: It is of the form $a_2x_2+a_4x_4+\cdots+a_{2k}x_{2k}=0$.

3 – Introduce the notation: $C=\cos\theta$, $S=\sin\theta$ and $C_k=\cos k\theta$ and $S_k=\sin k\theta$ for $k\geq 2$. Substitute the even components of the curve in the preceding equation to find the points at which the curve intersects the osculating hyperplane. This gives: $a_2S+a_4S_3+\cdots+a_{2k}S_{2k+1}=0$.

4– Use the identity $S_{2k+1}=(2k+1)S+\cdots+(-4)^{k-1}(2k+1)S^{2k-1}+(-4)^kS^{2k+1}$ to obtain an equation of degree $2k+1$ in S.

The arguments of steps 6 and 7 of the proof of proposition 1 can be applied here and lead to the equation $2k+1-4S^2=0$. This equation has no real solution for $k\geq 2$ because $S^2=\sin^2\theta\leq 1$. This proves proposition 2.

Remark – After a generic linear transformation in $\mathbb{R}^{2k}$, the curves in propositions 1 and 2 will not have constant curvatures. However, the non–convexity and the weak convexity will be preserved.

§5. A Non–standard 4–Vertex Theorem

We prove that any small enough generic perturbation in $\mathbb{R}^{2k+1}$ (taking the derivatives into account) of a spherically convex curve in $\mathbb{S}^{2k} \subset \mathbb{R}^{2k+1}$ has at least $2k+2$ extrema of the radius of the $(2k-1)$-osculating sphere. We also show that any small enough generic perturbation of a closed curve embedded in $\mathbb{S}^2 \subset \mathbb{R}^3$ has at least 4 points with extremal curvature.

1. Statement of Results

THEOREM 1– *Any small enough generic perturbation in $\mathbb{R}^{2k+1}$ (taking the derivatives into account) of a generic spherically convex curve in $\mathbb{S}^{2k} \subset \mathbb{R}^{2k+1}$ has at least $2k+2$ extrema of the radius of the $(2k-1)$-osculating sphere.*

In the particular case of curves in $\mathbb{R}^3$ we have a stronger theorem:

THEOREM 2– *Any small enough generic perturbation in $\mathbb{R}^3$ (taking the derivatives into account) of any embedded curve in $\mathbb{S}^2 \subset \mathbb{R}^3$ has at least 4 points of extremal curvature.*

2. Proofs

LEMMA 1– *The vertices of a spherical curve $\gamma \subset \mathbb{S}^n \subset \mathbb{R}^{n+1}$ are extrema of the radius of the $(n-1)$-osculating sphere.*

Proof – Let $\gamma \subset \mathbb{S}^n \subset \mathbb{R}^{n+1}$ be a spherical curve. Apply an inversion σ with respect to a hypersphere of $\mathbb{R}^{n+1}$ centered at a point of $\mathbb{S}^n \setminus \gamma$. The inversion σ sends the curve $\gamma \subset \mathbb{S}^n$ onto a hyperplane curve $\hat{\gamma} \subset \mathbb{R}^n \subset \mathbb{R}^{n+1}$ and the $(n-1)$-osculating spheres of γ on the osculating hypersheres of $\hat{\gamma} \subset \mathbb{R}^n$. The vertices of the spherical curve γ are sent by σ onto the vertices of $\hat{\gamma}$. The vertices of $\hat{\gamma}$ are extrema of the radius of the osculating hypershere. Thus the vertices of γ are extrema of the radius of the $(n-1)$-osculating sphere. □

Proof of theorems 1 and 2 – Let γ be a spherically convex curve in $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. By theorem S of §1 and lemma 1, γ has at least $2k+2$ extrema of the radius of the $(n-1)$-osculating sphere. These extrema are nondegenerate because γ is a generic spherical curve. Let Γ be a small enough generic perturbation of γ in $\mathbb{R}^{n+1}$ (taking the derivatives into account). This pertubation does not destroy the nondegenerate extrema of the radius of the $(n-1)$-osculating sphere. This proves theorem 1. To prove theorem 2 we use the same arguments and the fact that any embedded spherical curve $\gamma \subset \mathbb{S}^2$ has at least 4 vertices. □

3. The Flattenings of a Curve Lying in $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ Are Not Generic

The following remarks show that the flattenings of a curve lying in a sphere are not generic. We recall that the vertices of a spherical curve $\gamma \subset \mathbb{S}^n \subset \mathbb{R}^{n+1}$ are the flattenings of γ considered as a spatial curve.

Remark – At a generic point of a curve in $\mathbb{R}^{n+1}$, the osculating hypersphere is uniquely determined. If the point is a generic flattening then the osculating hypersphere is also unique and coincides with the osculating hyperplane.

Example – The osculating circle of a plane curve at an inflection point coincides with the tangent line at that point.

Remark– The flattenings of a curve lying on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ are not generic from the point of view of the geometry of curves in $\mathbb{R}^{n+1}$ (see remark above).

Proof of remark – At a flattening p of a curve lying on a sphere $\mathbb{S}^n$, the order of contact with its $(n-1)$-osculating sphere $\hat{S}^{n-1}(p)$ is at least $n+2$. Thus, at this point, the multiplicity if intersection of the curve with all the hyperspheres containing $\hat{S}^{n-1}(p)$ is at least $n+2$. Hence the osculating hypersphere is not uniquely determined. □

Remark – In the proof of theorems 1 and 2 we saw that the flattenings of a curve lying in $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ are extrema of the radius of the $(n-1)$-osculating sphere. The flattenings of a generic curve in $\mathbb{R}^{n+1}$ are no extrema of the radius of the $(n-1)$-osculating sphere.

4. Problem

How much can a closed curve embedded in $\mathbb{S}^2$ be deformed inside $\mathbb{R}^3$ such that the deformed curve keeps at least 4 extrema of the curvature?

§6. On Three Classes of Closed Curves in $\mathbb{R}^3$ Having at Least 4 Flattenings and a 4–Flattening Conjecture

We discuss three classes of closed curves in the Euclidean space $\mathbb{R}^3$ which have non–vanishing curvature and at least 4 *flattenings* (points with torsion zero). Calling these classes (defined below) *Barner*, *Segre* and *Sedykh*, we prove that Barner$\subset$(Segre$\cap$Sedykh). We also prove that (Segre)\(Segre$\cap$Sedykh) and (Sedykh)\(Segre$\cap$Sedykh) are open sets in the space of closed smooth curves with the C^∞–topology. Finally, we conjecture that the curves of a class containing Segre (defined below) have at least 4 flattenings.

1. Introduction and Main Results.

As in the previous sections, by a *closed curve* in the Euclidean space $\mathbb{R}^n$ (projective space $\mathbb{R}P^n$) we mean a C^∞ mapping $\gamma : \mathbb{S}^1 \to \mathbb{R}^n$ ($\gamma : \mathbb{S}^1 \to \mathbb{R}P^n$, respectively). We consider the space of all closed curves in the Euclidean space (projective space) equipped with the C^∞–topology.

We recall that a *flattening* of a curve γ in $\mathbb{R}^n$ ($\mathbb{R}P^n$) is a point where the derivatives of γ of order $1, \ldots, n$, are linearly dependent.

We also recall that a curve embedded in $\mathbb{R}^n$ (in $\mathbb{R}P^n$) is called *convex* if it intersects no hyperplane at more than n points, counting multiplicities.

If a closed curve γ in $\mathbb{R}P^n$ ($\mathbb{R}^n$) can be projected, from a point exterior to it, into a convex curve of $\mathbb{R}P^{n-1}$ ($\mathbb{R}^{n-1}$) then γ has at least $n+1$ flattenings [6].

Finally, we recall that a closed curve in $\mathbb{R}^n$ (in $\mathbb{R}P^n$) is called *Barner curve* if for every $(n-1)$–tuple of (not necesarily geometrically different) points of the curve there exists a hyperplane through these points that does not intersect the curve elsewhere.

In the Euclidean case, Barner curves exist only in odd dimensions. *Any Barner curve in $\mathbb{R}^n$ ($\mathbb{R}P^n$) has at least $n+1$ flattenings* [8].

A closed curve γ in $\mathbb{R}P^n$ ($\mathbb{R}^n$) which can be projected, from a point exterior to it, into a convex curve of $\mathbb{R}P^{n-1}$ ($\mathbb{R}^{n-1}$) is a Barner curve. Answering the question about the relation between these two classes of curves (V.I. Arnol'd 1996) V.D. Sedykh ([20]) proved: *There is an open set of embedded closed curves in $\mathbb{R}P^n$ which are Barner curves and have no convex projections into any hyperplane.*

The conditions defining classes of closed curves in $\mathbb{R}^n$ that guarantee a minimum number of flattenings (or vertices) on each curve of that class has been a classical object of study. The interest on this subject was revived by the recent progres in simplectic and contact geometries and the relations of this problems with Sturm theory (see [6], [4], [5], [7], [13], [1], [20], [24]). We consider three classes of closed curves in the three–dimensional Euclidean space $\mathbb{R}^3$ all whose elements have at least four flattenings. In particular any Barner curve of $\mathbb{R}^3$ has at least 4 flattenings.
C. Romero–Fuster (for the generic case [15][2]) and V.D. Sedykh (for the general case [19]) proved the following theorem:

Sedykh's Theorem– *A closed C^3–smooth curve in $\mathbb{R}^3$ lying on the boundary of its convex hull with non–vanishing curvature has at least four flattenings.*

[2]In [9], Blaschke atributes an equivalent result to Charatheodory, but he does not give the reference.

In [21], Segre proved the

SEGRE'S THEOREM– *Any closed curve in* $\mathbb{R}^3$ *with non–vanishing curvature and no parallel tangents with the same orientation has at least four flattenings.*

We call *Sedykh curves* the closed curves in $\mathbb{R}^3$ lying on the boundary of its convex hull with never vanishing curvature. We call *Segre curves* the closed curves in $\mathbb{R}^3$ with non–vanishing curvature and no parallel tangents having the same orientation.

The natural problems arises:

a) Are there Sedykh curves which are no Segre curves? (C. Romero–Fuster [16]).

b) Are there Segre curves which are no Sedykh curves?

c) How are the Barner curves in $\mathbb{R}^3$ related to the Sedykh and Segre curves?

The answer to these questions is given by the following three theorems:

THEOREM A– *There is an open set of Sedykh curves in* $\mathbb{R}^3$ *which are not Segre.*

THEOREM B– *There is an open set of Segre curves in* $\mathbb{R}^3$ *which are not Sedykh.*

THEOREM C– *Any Barner curve in* $\mathbb{R}^3$ *is a Sedykh curve and also a Segre curve.*

To prove theorems A and B we give methods to construct generic examples.

A Four–Flattening Conjecture. When the unit tangent vector $\mathbf{t}$ of a curve γ in the Euclidean Space $\mathbb{R}^3$ is translated to a fixed point O, the end point of the translated vectors describe a curve $\mathbf{T}$ on the unit sphere $\mathbb{S}^2$, called the *tangent indicatrix* of γ. The points of γ at which the curvature vanishes corresponds to the cusps of the tangent indicatrix. So Segre theorem can be reformulated in the following way: *Any closed curve in* $\mathbb{R}^3$ *whose tangent indicatrix is embedded in* $\mathbb{S}^2$ *has at least four flattenings.*

We say that curve on $\mathbb{S}^2$ has *direct self–tangency* if it has self–tangency and the tangent branches have the same orientation at the point of tangency. Let $\gamma_t : \mathbb{S}^1 \to \mathbb{R}^3$, $0 \leq t \leq 1$, be a one–parameter family of immersed curves. Suppose that γ_0 is a Segre curve (for instance a plane convex curve in $\mathbb{R}^3$) and that for all $t \in [0,1]$ the tangent indicatrix $\mathbf{T}_t$ of γ_t is an immersed curve of $\mathbb{S}^2$ having no direct self–tangencies.

CONJECTURE 1.– *The curve* γ_1 *(and each curve* γ_t*) has at least* 4 *flattenings.*

We stated conjecture 1 in terms of the curves $\gamma_t \in \mathbb{R}^3$ and its tangent indicatrices in $\mathbb{S}^2$, but it comes from a conjecture about some class of Legendrian knots in $ST^*\mathbb{S}^2$ and the number of spherical inflections of the fronts in $\mathbb{S}^2$ of these Legendrian knots (see, [5] for more information about the Legendrian knots associated to curves in $\mathbb{S}^2$). More precisely, to each smoothly immersed co–oriented curve $\alpha : \mathbb{S}^1 \to \mathbb{S}^2$ is associated a Legendrian knot $L_\alpha \subset ST^*\mathbb{S}^2$ consisting of the co–oriented contact elements of $\mathbb{S}^2$ tangent to α with corresponding co-orientation. Conversely, to each closed Legendrian knot in $ST^*\mathbb{S}^2$ corresponds a co–oriented

curve in $\mathbb{S}^2$ which is not necessarily smooth and is called the front of the Legendrian knot. We recall (see [10]) that *a curve $\alpha : \mathbb{S}^1 \to \mathbb{S}^2$ is the tangent indicatrix of some smoothly immersed curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ if and only if it intersects each great circle of $\mathbb{S}^2$*. So we formulate the

CONJECTURE 1'.– *Let L_0 be the Legendrian knot associated to a co–oriented great circle of $\mathbb{S}^2$. Let L_1 be any Legendrian knot which can be joined to L_0 by a Legendrian isotopy L_t (i.e. a homotopy of Legendrian knots for which the knot type does not change) satisfying the following condition: The front α_t of each Legendrian knot L_t is a smooth curve of $\mathbb{S}^2$ which intersects every great circle of $\mathbb{S}^2$. Then the front γ_1 (and each front γ_t, $0 \leq t \leq 1$) has at least four spherical inflections.*

The relatrion between both conjectures comes from the fact that the spherical inflections of the tangent indicatrix of a curve in $\mathbb{R}^3$ correspond to the flattenings of the original curve in $\mathbb{R}^3$.

In [7], V. Arnol'd gave the first step towards a Legendrian Sturm theory of space curves. He imposed some conditions to the curves in terms of the 2–dimensional Legendrian knot, of the space $PT^*\mathbb{R}^3$ of contact elements of $\mathbb{R}^3$, associated to each curve in $\mathbb{R}^3$ (or $\mathbb{R}P^3$). This Legendrian 2–dimensional knot consists of the contact elements of $\mathbb{R}^3$ (or $\mathbb{R}P^3$) tangent to the curve.

Even inside the class of Barner curves, it is easy to go outside the class of curves considered in [7], (in [7] there is one example). With our conjecture we try to follow the Arnol'd–Chekanov's philosophy. But instead of consider the 2–dimensional Legendrian knot in $PT^*\mathbb{R}^3$ (or $PT^*\mathbb{R}P^3$) associated to a curve in $\mathbb{R}^3$ (or $\mathbb{R}P^3$), we consider the 1–dimensional Legendrian knot in $ST^*\mathbb{S}^2$ associated to the tangent indicatrix of a curve in $\mathbb{R}^3$. The class of curves considered in our conjecture contains the whole class of Segre curves which, by Theorem C, contains the whole classe of Barner curves.

Acknowledgements. The author is grateful to Carmen Romero–Fuster for setting up problem a).

2. Proof of theorem C:

A Barner Curve is a Sedykh Curve. The Barner curves have no points with vanishing curvature. Let γ be a Barner curve. The definition of Barner curves implies that for any point $p \in \gamma$ there is a plane tangent to the curve at p not intersecting the curve elsewhere. This plane determines a closed half–space H_p containing the curve. The convex hull of γ is contained in H_p and the point p lies on the boundary of the convex set H_p. So p lies on the boundary of the convex hull of γ. Thus γ is a Sedykh curve. □

A Barner Curve is a Segre Curve. We will prove that any curve which is not a Segre curve cannot be a Barner curve. Let γ be a closed curve with non–vanishing curvature. Suppose that γ has two parallel tangents with the same orientation at the points p_1 and p_2 of γ. We will prove that any plane containing the points p_1 and p_2 must intersect the curve at least at 4 points, taking multiplicities into account. Consider the projection $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ parallel to the line p_1p_2. The projection π sends the curve γ onto a plane curve $\hat{\gamma} = \pi(\gamma)$. The point $p = \pi(p_1) = \pi(p_2)$ is a point of self–tangency with the same orientation as the curve $\hat{\gamma}$. So the curve $\hat{\gamma}$ can be decomposed into two closed curves having a

tangency at the point p. Any line of $\mathbb{R}^2$ containing p intersects each one of these curves at least at two points, taking multiplicities into account. Each line of $\mathbb{R}^2$ containing p is the image (by the projection π) of a plane of $\mathbb{R}^3$ containing the points p_1 and p_2. So any plane containing p_1 and p_2 intersects γ at least at 4 points, taking multiplicities into account. □

3. Proof of theorem A:

Sedykh Curves which are no Segre Curves. Consider a smooth, closed and strictly convex smooth surface S (for instance an ellipsoid) in the Euclidean space $\mathbb{R}^3$. Consider a bundle of parallel lines of $\mathbb{R}^3$. Let Γ denote the set of points of S at which a line of the bundle is tangent to S. The set Γ is a closed curve of S, which separates S in two parts S_1 and S_2. Any embedded curve of S is a Sedykh curve.

PROPOSITION 1– *Let $\gamma : \theta \mapsto \gamma(\theta)$ be a closed embedded curve of S crossing the curve Γ transversally at $2k > 2$ points $\theta_1, \ldots, \theta_{2k}$. If the tangent lines of γ at two crossings θ_i and θ_j with the same parity ($i = j$ (mod 2)) are lines of the bundle then γ is not a Segre curve.*

Proof – The tangents of the curve γ at θ_i and θ_j are parallel. We must only prove that the tangents at these points have the same orientation. Suppose that at θ_1 the curve γ traverses from S_1 to S_2. Then at the odd (even) crossings the curve γ traverses from S_1 to S_2 (from S_2 to S_1). So if both i and j are odd (even) then the crossing from S_1 to S_2 (from S_2 to S_1) gives to the tangents at the points θ_i and θ_j the same orientation (See Fig. 1). □

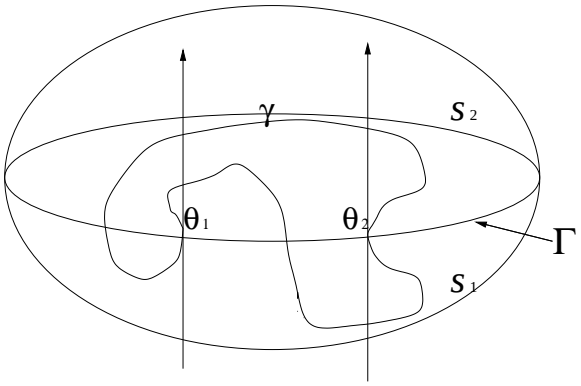

Figure 1: A Sedykh curve which is not a Segre curve.

A closed curve in $\mathbb{R}^3$ is a Segre curve if and only if its tangent indicatrix on $\mathbb{S}^2$ has no double points (self–intersections). If the tangent indicatrix of a closed curve γ has transversal self–intersections, then any small enough perturbation of γ (taking the derivatives into account) is no Segre curve: transversality is an open condition. The tangent indicatrix $\mathbf{T} : \mathbb{S}^1 \to \mathbb{S}^2$ of a closed curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ has a transversal self–intersection at the point $\mathbf{T}(\theta_1) = \mathbf{T}(\theta_2)$ if the tangents to γ at θ_1 and θ_2 are parallel with the same orientation, but the osculating planes at these points are not parallel.

So the curves in the proposition 1 can be constructed in such a way that the osculating planes at the points θ_i and θ_j be not parallel. This proves theorem A.

4. Proof of theorem B:

Segre Curves which are no Sedykh Curves. We give a method of constructing Segre curves. Let γ be an oriented convex curve, with two axes of symmetry l_1, l_2, in the Euclidean plane $\mathbb{R}^2 \subset \mathbb{R}^3$ (for instance an ellipse). Deform the plane $\mathbb{R}^2$ in $\mathbb{R}^3$ on a right cylinder C with the following conditions:

a) The base of the cylinder C can be any smooth immersed plane curve.

b) The lines of $\mathbb{R}^2$ parallel to one of the axes of symmetry of γ, say l_1, must become the generatrices of C.

c) The image, under the deformation, of the lines of $\mathbb{R}^2$ parallel to the axes of symmetry l_2 must be orthogonal to the generatrices of C.

Write $\tilde{\gamma}$ for the image of γ by this deformation, and $\tilde{p} \in \tilde{\gamma}$ for the image of $p \in \gamma$ by this deformation.

PROPOSITION 2– *The deformed curve $\tilde{\gamma}$ is a Segre curve.*

Proof – We will use the fact that two unit tangent vectors are parallel and have the same orientation if and only if for any orthogonal projection on a plane (or on a line) their images are parallel with the same orientation and the same length. Let $\mathbf{t}(\tilde{p})$ be the unit tangent vector of $\tilde{\gamma}$ (given by the orientation of $\tilde{\gamma}$) at the point $\tilde{p} \in \tilde{\gamma}$. Let P_2 be the plane orthogonal to the generatrices of C and containing the image of the axis of symmetry l_2 of γ. By construction, the curve $\tilde{\gamma}$ is symmetric with respect to the plane P_2. Let π_1 (or π_2) be the orthogonal projection of the unit tangent vectors of $\tilde{\gamma}$ on a plane orthogonal to the generatrices (respectively on a line parallel to the generatrices). The projections by π_1 (respectively by π_2) of two unit tangent vectors of $\tilde{\gamma}$ have the same length if and only if the corresponding points of the plane curve γ are symmetric with respect to any one of the axes of symmetry l_1, l_2 of γ. If two points p and q of γ are symmetric with respect to l_1 or l_2 and lie on one of these axis of symmetry of γ, then $\mathbf{t}(\tilde{p})$ and $\mathbf{t}(\tilde{q})$ have opposite orientation. Let $\tilde{p_1}, \tilde{p_2}, \tilde{p_3}, \tilde{p_4}$ be four points of $\tilde{\gamma}$, such that the corresponding points p_1, p_2, p_3, p_4 of γ are symmetric and don't lie in the axes of symmetry of γ. We will prove that $\mathbf{t}(\tilde{p_1}) \neq \mathbf{t}(\tilde{p_i})$, $i = 2, 3, 4$. Suppose that $\tilde{p_1}$ and $\tilde{p_2}$ (and consequently $\tilde{p_3}$ and $\tilde{p_4}$) are symmetric with respect to P_2. The projections $\pi_1(\mathbf{t}(\tilde{p_1}))$ and $\pi_1(\mathbf{t}(\tilde{p_2}))$ have the same length but different orientation. So $\mathbf{t}(\tilde{p_1}) \neq \mathbf{t}(\tilde{p_2})$. The projections $\pi_2(\mathbf{t}(\tilde{p_3}))$ and $\pi_2(\mathbf{t}(\tilde{p_4}))$ are oriented in opposite direction with respect to $\pi_2(\mathbf{t}(\tilde{p_1}))$, Thus $\mathbf{t}(\tilde{p_3}) \neq \mathbf{t}(\tilde{p_1}) \neq \mathbf{t}(\tilde{p_4})$. This proves proposition 2.

The curves of proposition 2 can be constructed in such a way that the curve $\tilde{\gamma}$ does not lie on the boundary of its convex hull. This proves theorem B.

Realizations of this construction are given by the families of curves of the following examples.

Example 1 – The curves of the family $\tilde{\gamma}_\varepsilon : \mathbb{S}^1 \to \mathbb{R}^3$ given by the parametrization

$$\tilde{\gamma}_\varepsilon(\theta) = ((2\cos\theta + \varepsilon)^3 - (2\cos\theta + \varepsilon),\ \sin\theta,\ (2\cos\theta + \varepsilon)^2)$$

are Segre curves for any value of ε but are no Sedykh curves for any small enough ε. The curve $\tilde{\gamma}_0$ is not embedded (it has two points of self–intersection), so it is no Sedykh curve. For any small enough $\varepsilon \neq 0$, the curve $\tilde{\gamma}_\varepsilon$ is embedded and does not lie in the boundary of its convex hull. This family of curves lies in the cylinder given by the following parametrization: $(s, t) \mapsto (t^3 - t, s, t^2 - 1)$. In Figure 2 we

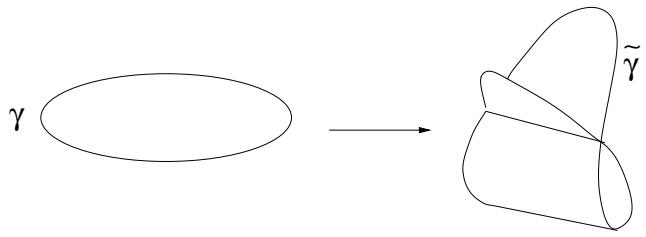

Figure 2: A Segre curve which is not a Sedykh curve.

have considered the plane curve γ as the boundary of a disc, and the spatial curve $\tilde{\gamma}$ as the image of γ by the deformation of the disc.

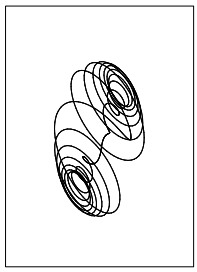

Figure 3: A Segre curve which is not a Sedykh curve.

Example 2 – The curves of the family $\tilde{\gamma}_\lambda : \mathbb{S}^1 \to \mathbb{R}^3$ given by the parametrization

$$\tilde{\gamma}_\lambda(\theta) = (e^{\cos\theta}\sin(\lambda\pi\cos\theta),\ \sin\theta,\ -e^{\cos\theta}\cos(\lambda\pi\cos\theta))$$

are Segre curves and for any $\lambda \geq 0.7$ they are no Sedykh curves. In Fig.3 we have the curve $\tilde{\gamma}_\lambda$ for $\lambda = 10$. The curve $\tilde{\gamma}_\lambda$ of this family lies in the cylinder C_λ given by the following parametrization: $(s,t) \mapsto (e^{2t/\lambda}\sin(2\pi t),\ s,\ -e^{2t/\lambda}\cos(2\pi t))$.

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