MONODROMY CONJECTURE FOR SOME SURFACE SINGULARITIES

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ABSTRACT. In this work we give a formula for the local topological zeta function of a superisolated singularity of hypersurface in terms of the local topological zeta function of the singularities of its tangent cone. We apply it to prove the Monodromy Conjecture for some surfaces singularities.

Introduction. Troughout this paper we work over the complex numbers. The local topological zeta function $Z_{top,0}(f,s) \in \mathbb{Q}(s)$ is an analytic subtle invariant associated with any germ of an analytic function $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$. This rational function was first introduced by J. Denef and F. Loeser as a kind of limit of the p-adic Igusa zeta function, see [DL1,DL3]. Its former definition was in terms of any embedded resolution of its zero locus germ $(V,0) = (f^{-1}(0), 0) \subset (\mathbb{C}^{n+1}, 0)$ (although it does not depend on any particular resolution). In [DL3], J. Denef and F. Loeser gave an intrinsic definition of $Z_{top,0}(f,s)$ using arc spaces and the motivic zeta function, see also [DL4] and the Sèminaire Bourbaki talk of E. Looijenga [Loo].

Each exceptional divisor of an embedded resolution $\pi\colon (Y, \mathcal{D}) \to (\mathbb{C}^{n+1}, 0)$ of the germ $(V,0)$ gives a pole candidate of the rational function $Z_{top,0}(f,s)$. Nevertheless only a few of them truly give a pole of $Z_{top,0}(f,s)$. There are several conjectures related to topological zeta functions. In this paper we are interested in the *Monodromy Conjecture*, see [D],[DL1].

Given $x \in f^{-1}(0)$ it is known that the *Milnor fibration* of the holomorphic function f at x is the C^∞ locally trivial fibration $f| : B_\varepsilon(x) \cap f^{-1}(\mathbb{D}^*_\eta) \to \mathbb{D}^*_\eta$, where $B_\varepsilon(x)$ is the open ball of radius ε centred at x, $\mathbb{D}_\eta = \{z \in \mathbb{C} : |z| < \eta\}$ and $\mathbb{D}^*_\eta$ is the open punctured disk ($0 < \eta << \varepsilon$ and ε small enough). Any fibre $F_{f,x}$ of this fibration is the *Milnor fibre* of f at x. The *monodromy transformation* $h : F_{f,x} \to F_{f,x}$ is the well defined (up to isotopy) diffeomorphism of $F_{f,x}$ induced by a small loop around $0 \in \mathbb{D}_\eta$. The *complex algebraic monodromy of f at x* is the corresponding linear transformation $h_* : H_*(F_{f,x}, \mathbb{C}) \to H_*(F_{f,x}, \mathbb{C})$ on the homology groups.

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The *local Monodromy Conjecture* states that *if* s_0 *is a pole of the topological zeta function* $Z_{top,0}(f,s)$ *of the local singularity defined by* f, *then* $\exp(2i\pi s_0)$ *is an eigenvalue of the complex algebraic monodromy around* $f^{-1}(0)$. Note that if f defines an isolated hypersurface singularity then $\exp(2i\pi s_0)$ has to be an eigenvalue of the complex algebraic monodromy of the germ $(f^{-1}(0),0)$.

There are three general problems to consider when one tries to prove (or disprove) the conjecture using resolution of singularities:

(i) Explicit computation of an embedded resolution of the hypersurface $(V,0)\subset(\mathbb{C}^{n+1},0)$.

(ii) Elimination of the pole candidates which are not really poles of $Z_{top,0}(f,s)$.

(iii) Explicit computation of the eigenvalues of the complex algebraic monodromy (or computing the characteristic polynomials of the corresponding action of the complex algebraic monodromy).

The Monodromy Conjecture has been proved for curve singularities by F. Loeser [Lo1], see also [V2,V3]. F. Loeser proved that any pole of the topological zeta function give a root of the Berstein polynomial of the singularity which is a stronger version of the Monodromy Conjecture. The behaviour of the topological zeta function for germs of curves is rather well understood after the knowledge of an explicit embedded resolution $\pi\colon(Y,\mathcal{D})\to(\mathbb{C}^2,0)$ of curve singularities, e.g. the minimal one. W. Veys proved *essentially* that any irreducible component E of the exceptional divisor $\mathcal{D}=\pi^{-1}(0)$ which intersects the total transform $\pi^{-1}(V)$ in at most two points verifies that its contribution to the residue of $Z_{top,0}(f,s)$ at the pole candidate is zero. Then one eliminates the pole candidates where it is well known that the eigenvalue candidates of the complex algebraic monodromy are not actual eigenvalues.

There are other classes of singularities where the embedded resolution is known. For example, for any singularity of hypersurface defined by an analytic function which is non-degenerated with respect to its Newton polytope, problems (*i*) and (*iii*) above are solved. Nevertheless problem (*ii*) seems to be a hard combinatorial problem. This problem was partially solved by F. Loeser whenever f has non-degenerated Newton polytope and verifies some technical extra conditions, [Lo2].

Embedded resolution is also known for superisolated surface singularities, SIS for short, see [Ar]. This kind of singularities, named by I. Luengo in [L], was used to prove that the μ-constant stratum of an isolated hypersurface singularity is not smooth, see also [S]. E. Artal used them for disproving a conjecture of S.S.T. Yau.

Even in one of the simplest case of f with non-isolated singularities, namely homogeneous surfaces case, problems (*i*) and (*iii*) are solved but problem (*ii*) is still open. For any degree d and any homogeneous polynomial $f_d\in\mathbb{C}[x_1,x_2,x_3]$ the first pole candidate is $s_0=-3/d$. It appears when one blows-up once at the origin. A sufficient condition for the pole candidate $s_0=-3/d$ of $Z_{top,0}(f,s)$ verifies the Monodromy Conjecture is the following topological condition about its Euler-Poincaré characteristic: $\chi(\mathbb{P}^2\setminus\{f_d=0\})\neq 0$.

B. Rodrigues and W. Veys in [RV] have proved the Monodromy Conjecture for any homogeneous polynomial $f_d\in\mathbb{C}[x_1,x_2,x_3]$ with $\chi(\mathbb{P}^2\setminus\{f_d=0\})\neq 0$. They exclude the case $\chi(\mathbb{P}^2\setminus\{f_d=0\})=0$ because they couldn't solved problem (*ii*) for the pole candidate $s_0=-3/d$ in this case.

Essentially in this paper we prove the Monodromy Conjecture for SIS singularities and also complete the proof of the Monodromy Conjecture for homogeneous polynomials in three variables. The results of this paper are the following.

Let f be a germ of a superisolated hypersurface singularity defined by $f = f_d+f_{d+1}+\ldots \in \mathbb{C}\{x_0, x_1, \ldots, x_n\}$. Let us denote by $C_m \subset \mathbb{P}^n$ the divisor associated with the homogeneous polynomial f_m. By definition, the hypersurface singularity $(V,0) = (f^{-1}(0), 0) \subset (\mathbb{C}^{n+1}, 0)$ is *superisolated*, SIH for short, if and only if the projective set $C_{d+1} \cap \operatorname{Sing}(C_d)$ is empty. For each $P \in \operatorname{Sing}(C_d)$ we choose analytic coordinates centered at the origin and we denote by g^P the equation of C_d in these coordinates.

The point is to get a formula for the topological zeta function of a SIH singularity in terms of similar invariants of its tangent cone. In Section 1 such a formula is given.

Corollary 1.12. *Let $f := f_d+f_{d+1}+\cdots \in \mathbb{C}\{x_0, x_1, \ldots, x_n\}$ define a SIH singularity $(V,0) \subset (\mathbb{C}^{n+1}, 0)$. Then its local topological zeta function satisfies the following equality*

$$Z_{top,0}(V,s) = \frac{\chi(\mathbb{P}^n \setminus C_d)}{t-s} + \frac{\chi(\check{C}_d)}{(t-s)(s+1)} +$$

$$+ \sum_{P \in \operatorname{Sing}(C_d)} \left(\frac{1}{t} + (t+1)\left(\frac{1}{(t-s)(s+1)} - \frac{1}{t} \right) Z_{top,0}(g^P, t) \right),$$

where $t := n+1+(d+1)s$, $\check{C}_d = C_d \setminus \operatorname{Sing}(C_d)$ and $Z_{top,0}(g^P, s)$ stands for the local topological zeta function for the germ g^P at the singular point $P \in \operatorname{Sing}(C_d)$.

Using the formula above, the Alexander polynomial formula of the complex algebraic monodromy of a SIS singularity (e.g. see [Ar]), and the Monodromy Conjecture for curves, see [Lo1], we prove the following for a SIS singularity of multiplicity d:

- If $\chi(\mathbb{P}^2 \setminus C_d) > 0$ then the Monodromy Conjecture holds for $(V,0) \subset (\mathbb{C}^3, 0)$.
- If $\chi(\mathbb{P}^2 \setminus C_d) \leq 0$, then any pole of the local topological zeta function of $(V,0) \subset (\mathbb{C}^3, 0)$, but $s_0 = -3/d$, verifies the Monodromy Conjecture. Furthermore, if $s_0 = -3/d$ is a pole of the local topological zeta function of the germ of plane curve C_d at some singular point then the Monodromy Conjecture for $(V,0) \subset (\mathbb{C}^3, 0)$ also holds. This fact motivates the following definition.

We say that a degree d effective divisor D on $\mathbb{P}^2$ ($d > 3$) is a *bad divisor* if $\chi(\mathbb{P}^2 \setminus D) \leq 0$ and $s_0 = -3/d$ is not a pole of $Z_{top,P}(g_D^P, s)$ for any singular point P in its support D_{red}, where g_D^P is the local equation of the divisor D at P.

In order to prove the Monodromy Conjecture for SIS singularities it only remains to deal with the pole $s_0 = -3/d$ whenever the tangent cone C_d is a bad divisor. In such a case, $s_0 = -3/d$ can only be (at most) a simple pole of $Z_{top,0}(f,s)$ and its residue is equal to

$$\rho(C_d) := \chi(\mathbb{P}^2 \setminus C_d) + \chi(C_d \setminus \operatorname{Sing}(C_d)) \frac{d}{d-3} + \sum_{P \in \operatorname{Sing}(C_d)} Z_{top,P}(g^P, -3/d) \in \mathbb{Q}.$$

It turns out that the residue $\rho(C_d)$ agrees with the value of $z(C_d, s)$ at $s_0 = -3/d$, where $z(D,s)$ stands for the topological zeta function associated with a divisor D

on $\mathbb{P}^2$; this invariant was recently introduced by W. Veys, see [V5]. This residue has also another meaning, $\rho(C_d)$ coincides with an invariant ζ_K associated with the $\mathbb{Q}$-canonical divisor $K := (-3/d)C_d$ on the rational surface $\mathbb{P}^2$. In this paper we use these meanings to extend the notion of the residue $\rho(D)$ to bad divisors D on $\mathbb{P}^2$ (not only for reduced curves C_d) and to some canonical divisors on rational surfaces.

The main part of Section 2 is devoted to determine bad divisors D on $\mathbb{P}^2$ such that $\rho(D) \neq 0$. Note that the Euler-Poincaré characteristic condition on a bad divisor D implies that D has at least two irreducible components, all of them being rational curves, see [JS], [GP], [Ko] and [ALM].

Our second main result is the following theorem.

Theorem 2.21. *Let D be a bad divisor on $\mathbb{P}^2$. If $\rho(D) \neq 0$, then the irreducible components of D are in a pencil of rational curves having only one base point and such that any fibre minus the base point is isomorphic to $\mathbb{C}$; at least one generic fibre (resp. two) is contained in C_d if the pencil has two exceptional fibres (resp. one).*

The proof of this result is quite elaborate. We use the following result of W. Veys [V4].

Veys' Theorem. *Let D be a curve in $\mathbb{P}^2$ such that $\chi(\mathbb{P}^2 \setminus D) \leq 0$. Then D can be extended to a configuration $D' \supset D$ with still $\chi(\mathbb{P}^2 \setminus D') \leq 0$ for which there exists a diagram*

$$\Sigma \xleftarrow{g} X \xrightarrow{f} \mathbb{P}^2$$

where f is a composition of blowing-ups with center in D' and g is a composition of blowing-downs whose exceptional curve is contained in $f^{-1}(D')$ and such that Σ is a ruled surface. Moreover, one can require the configuration $g(f^{-1}(D'))$ to be one of the following two types:

(A) One section C_1 and at least two fibres, or

(B) Two disjoints sections C_1 and C_2 and at least one fibre.

The proof of the theorem consists of studying the behaviour of the invariant ζ_K when one applies blow-up and blow-down processes. This step has been partially studied in [V1] in a slightly different context.

The residue $\rho(D)$ does not change after the blow-up process except for one type of blowing-up which will be the key point. Since on the ruled surface Σ the configuration $g(f^{-1}(D'))$ has residue zero, then if $\rho(D)$ is not zero on $\mathbb{P}^2$ it is because one makes at least one blow-up which changes the residue. The most serious obstacle to characterize bad divisor on $\mathbb{P}^2$ whose residue $\rho(D) \neq 0$ is to prove that the components of such divisors appear as members of pencils of type $(0,1)$ on $\mathbb{P}^2$, see Appendix. Then we use the classification of T. Kizuka [Ki] of pencils of rational curves of type $(0,2)$ to show that arrangements of rational curves in which the residue changes can be put in a rational pencil of type $(0,1)$.

The last step in the proof of the Monodromy Conjecture for SIS singularities consists of computing the Alexander polynomial of the curves which satisfy the theorem above. For this purpose we use the classification of H. Kashiwara [K] of pencils of rational curves of type $(0,1)$. We prove, case by case, that the $\exp(2i\pi(-3/d))$ is a root of the Alexander polynomial of the curve at its only singular point. Again using the computation in [Ar], $\exp(2i\pi(-3/d))$ is a root of the Alexander polynomial of the corresponding SIS singularity.

This work allows also to extend the proof given by B. Rodrigues and W. Veys of the Monodromy Conjecture for homogeneous polynomials $f_d \in \mathbb{C}[x_1, x_2, x_3]$ with $\chi(\mathbb{P}^2 \setminus \{f_d = 0\}) = 0$.

We would like to point out that W. Veys in his work has found that in the case of curves, if an exceptional divisor E_i has $\chi(\check{E}_i) = 0$ ($\check{E}_i = E_i \setminus \bigcup_{j\neq i} E_j$) then E_i does not contribute to the pole candidate $\frac{-\nu_i}{N_i}$ of $Z_{top,0}(f,s)$. He suggests that this fact could be due to the Monodromy Conjecture. It is remarkable that if one considers a SIS singularity or a homogeneous surface whose tangent cone D is a bad divisor with residue $\rho(D) \neq 0$ then the exceptional component $E_0 = \mathbb{P}^2$, obtained when one blows-up once at the origin, verifies $\chi(\check{E}_0) = \chi(\mathbb{P}^2 \setminus D) = 0$ and $s_0 = -3/d$ is a pole of the local topological zeta function. In our opinion this fact gives strong evidence for the Monodromy Conjecture in the following sense. Given a bad divisor D the condition $s_0 = -3/d$ is a pole of the topological zeta function is a generic condition (s_0 is not a root of certain polynomial) and the condition $\exp(2i\pi(-3/d))$ is an eigenvalue of the complex monodromy is non-generic ($\exp(2i\pi(-3/d))$ has to be a root of another polynomial), see §5 for details.

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§1.- General formula for the topological zeta function

We first recall the definition of the topological zeta functions associated to a polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]$, see [DL1,DL2]. Let $\pi\colon Y \to \mathbb{C}^{n+1}$ be an embedded resolution of the hypersurface V defined by the zero locus of f. Let $E_i, i \in I$, be the irreducible components of the divisor $\pi^{-1}(f^{-1}(0))$. For each subset $J \subset I$ we set

$$E_J := \bigcap_{j\in J} E_j \text{ and } \check{E}_J := E_J \setminus \bigcup_{j\notin J} E_{J\cup\{j\}}.$$

For each $j \in I$, we denote by N_j the multiplicity of E_j in the divisor of the function $f \circ \pi$ and we denote by $\nu_j - 1$ the multiplicity of E_j in the divisor of $\pi^*(\omega)$ where ω is a non-vanishing holomorpic $(n+1)$-form in $\mathbb{C}^{n+1}$. Then the *local topological zeta function* of f is:

$$Z_{top,0}(f,s) := \sum_{J\subset I} \chi(\check{E}_J \cap \pi^{-1}(0)) \prod_{j\in J} \frac{1}{\nu_j + N_j s} \in \mathbb{Q}(s),$$

and the *topological zeta function* of f is:

$$Z_{top}(f,s) := \sum_{J\subset I} \chi(\check{E}_J) \prod_{j\in J} \frac{1}{\nu_j + N_j s} \in \mathbb{Q}(s),$$

where χ denotes Euler-Poincaré characteristic.

In fact the definition can be extended to an effective divisor D on a nonsingular $n+1$-dimensional complex variety X instead of just a morphism $f : X \to \mathbb{C}$, see [V5]. If $\pi\colon Y \to X$ is an embedded resolution of the support of D and $E_i, i \in I$, are the irreducible components of the divisor $\pi^{-1}(SuppD)$ with associated multiplicities $N_i, i \in I$ and $\nu_i - 1$, where $\pi^*(D) = \sum N_i E_i$ and the divisor $\operatorname{div}(\pi^*\omega_{|W}) = \sum(\nu_i - 1)E_i$ (ω is a non-vanishing holomorpic $(n+1)$-form in X in a neighbourhood W of $\operatorname{Sing}(D_{red})$) then the *topological zeta function* of D is defined by

$$z(D,s) := \sum_{J\subset I} \chi(\check{E}_J) \prod_{j\in J} \frac{1}{\nu_j + N_j s} \in \mathbb{Q}(s).$$

We shall compute the topological zeta function of a SIH singularity in $\mathbb{C}^{n+1}$. We will make use of three general principles which are at least implicitely known. We begin recalling the generalization of this zeta function by J. Denef et F. Loeser, [DL2], see also [V5].

Let X be an algebraic $(n+1)$-manifold, $f\colon X \to \mathbb{C}$ an algebraic function and ω an $(n+1)$-holomorphic form (algebraically defined) on X. One can replace *algebraic* by *analytic* if we are in the germ case or if we add some natural hypothesis about finiteness. Then one can define the topological zeta function $Z_{top}(f,\omega,s)$ in the same way as the original zeta function; in this case the ν-invariant is associated with the form ω instead with a non-vanishing form. We state the three main principles:

PBM Principle 1.1. *(See* [V5, Theorem 5.6]*) Let $\pi\colon V \to X$ be a proper birational morphism. Then*

$$Z_{top}(f,\omega,s) = Z_{top}(f\circ\pi, \pi^*(\omega), s).$$

Stratum Principle 1.2. *Let $X = \coprod_{S\in\mathcal{S}} S$ be a finite prestratification of X such that for each $x \in X$, the local topological zeta function $Z_{top,x}(f,\omega,s)$ at x, depends only on the stratum S containing x. Let us denote $Z_{top,S}(f,\omega,s)$ the common zeta function associated with the stratum S. Then,*

$$Z_{top}(f,\omega,s) = \sum_{S\in\mathcal{S}} \chi(S) Z_{top,S}(f,\omega,s).$$

The key point in this principle is that one may construct a resolution of both, f and ω, such that one can distribute the terms for the left-hand side of the formula in several terms of the right-hand side of the formula.

Fubini's Principle 1.3. *Let us consider two germs of functions $f_i\colon (\mathbb{C}^{n_i+1},0) \to (\mathbb{C},0)$ and two germs of (n_i+1)-holomorphic forms ω_i, $i = 1,2$. We consider $f := f_1 f_2$ and $\omega := \omega_1 \wedge \omega_2$ as germs of function and form in $(\mathbb{C}^{n_1+n_2+2},0)$. Then,*

$$Z_{top,0}(f,\omega,s) = \prod_{i=1}^{2} Z_{top,0}(f_i,\omega_i,s).$$

In order to prove this Fubini's Principle, it is enough to consider a proper birational mapping obtained by the resolution of f_1 and ω_1 in the first variables and the identity in the second variables; the zeta function does not change because of PBM Principle. Then we have a prestratification such that for any stratum, we have some power of coordinate functions in the first variables and f_2 and ω_2 in the second variables. On each stratum, we now consider the proper birational mapping associated with the second variables and the result easily follows.

Example 1.4. Let us take a germ $f\colon (\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ and a germ of holomorphic form ω. We can choose a good representative W (where f and ω are defined); W comes with a finite prestratification as in (1.2). All the strata but the origin have Euler characteristic zero. Then the zeta function of the germ is the same as the zeta function of the good representative.

Example 1.5. Let us take a germ $f\colon (\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ and a germ of holomorphic form ω. Fix a good representative W where f and ω are defined. Let us consider the blowing-up $\pi\colon \hat{W}\to W$ along a smooth subvariety of W containing 0. Consider $D:=\pi^{-1}(0)$ and let $\mathcal{S}_D$ be a finite prestratification of D satisfying the property (1.2). Then,

$$Z_{top,0}(f,\omega,s)=\sum_{S\in\mathcal{S}_D}\chi(S)Z_{top,S}(f\circ\pi,\pi^*(\omega),s).$$

We apply these facts to a SIH singularity defined by $f:=f_d+f_{d+1}+\cdots\in\mathbb{C}\{x_0,x_1,\ldots,x_n\}$. Let us denote by $C_m\subset\mathbb{P}^n$ the divisor associated with the homogeneous polynomial f_m. By definition, the projective set $C_{d+1}\cap\operatorname{Sing}(C_d)$ is empty. In particular the set $\operatorname{Sing}(C_d)$ of singular points of the projective hypersurface C_d is finite. Let us denote by $\check{C}_d$ the regular part of $C_d\subset\mathbb{P}^n$. For each $P\in\operatorname{Sing}(C_d)$ we choose analytic coordinates centered at the origin and we denote by g^P the equation of C_d in these coordinates. Then from the PBM and Stratum Principles one gets

$$Z_{top,0}(f,s)=\frac{\chi(\mathbb{P}^n\setminus C_d)}{t-s}+\frac{\chi(\check{C}_d)}{(t-s)(s+1)}+$$
$$+\sum_{P\in\operatorname{Sing}(C_d)}Z_{top,P}\big(\big(z-g^P(x_1,\ldots,x_n)\big)\,z^d,\omega,s\big),$$

where $\omega:=z^n dx_1\wedge\cdots\wedge dx_n\wedge dz$ and $t:=n+1+(d+1)s$. Let us fix a singular point P and set $g:=g^P$. We also assume that in the local coordinates $x_1,\ldots,x_n,z$ the point P is the origin, and $z=0$ is the equation of the exceptional divisor. The main point is to consider a birational map π which is an embedded resolution of $g^{-1}(0)$ in coordinates $x_1,\ldots,x_n$ and which is the identity in z.

We fix some notation about the embedded resolution of $g^{-1}(0)$. With each irreducible component $D_0,D_1,\ldots,D_r$ of the total transform of $g^{-1}(0)$, we associate the numbers N_i,ν_i, as usual. For each subset $J\subset\{0,1,\ldots,r\}$, we define the number χ_J, which is the Euler characteristic of the *complement* of the singular part of the intersection of the components parametrized by J. We recall that:

$$Z_{top,0}(g,s)=\sum_{J\subset\{0,1,\ldots,r\}}\chi_J\prod_{i\in J}\frac{1}{\nu_i+N_is}.$$

The following formula can be easily deduced from the two first Principles:

$$Z_{top,0}((z-g(x_1,\dots,x_n))\,z^d,\omega,s)=\sum_{J\subset\{0,1,\dots,r\}}\chi_J Z_{top,0}((z-x^J)z^d,z^n\omega^J,s),$$

where if $J=\{j_1,\dots,j_l\}$, then

$$x^J:=\prod_{k=1}^{l}x_k^{N_{j_k}},\qquad \omega^J:=\left(\prod_{k=1}^{l}x_k^{\nu_{j_k}-1}\right)dx_1\wedge\dots\wedge dx_n\wedge dz.$$

Notation 1.6. Given $n_j:=a_j+b_js$, $a_j,b_j\in\mathbb{Z}_{\geq 0}$, $j=1,\dots,l$ and $m_1,\dots,m_l\in\mathbb{Z}_{>0}$, $\tau:=as+b$, $a,b\in\mathbb{Z}$, we denote the local topological zeta function associated with the germs of function qz^ah and of form $z^b\left(\prod_{k=1}^{l}x_k^{a_k-1}\right)dx_1\wedge\dots\wedge dx_n\wedge dz$ by

$$Z(n_1,\dots,n_l;m_1,\dots,m_l;\tau,s):=Z_{top,0}(qz^ah,z^b\eta,s),$$

where

$$q:=\prod_{k=1}^{l}x_k^{b_k},\qquad h:=z-\prod_{k=1}^{l}x_k^{m_k},\qquad \eta:=\left(\prod_{k=1}^{l}x_k^{a_k-1}\right)dx_1\wedge\dots\wedge dx_n\wedge dz.$$

We next compute this local topological zeta function blowing-up along the coordinate subspace $z=x_l=0$ and apply the above mentioned principles.

Formula 1.7. *Assume that $l>1$. If $m_l>1$, then using the PBM and Stratum Principles one gets*

$$Z(n_1,\dots,n_l;m_1,\dots,m_{l-1},m_l;\tau,s)=\left(\prod_{j=1}^{l}\frac{1}{n_j}\right)\frac{1}{n_l+\tau+s+1}+$$
$$+Z(n_1,\dots,n_{l-1},n_l+\tau+s+1;m_1,\dots,m_{l-1},m_l-1;\tau,s),$$

and using Fubini's Priciple

$$Z(n_1,\dots,n_l;m_1,\dots,m_{l-1},1;\tau,s)=$$
$$=\left(\prod_{j=1}^{l}\frac{1}{n_j}\right)\frac{1}{n_l+\tau+s+1}+\frac{1}{n_l+\tau+s+1}Z(n_1,\dots,n_{l-1};m_1,\dots,m_{l-1};\tau,s),$$

Then, by induction on m_l:

$$Z(n_1,\dots,n_l;m_1,\dots,m_{l-1},m_l;\tau,s)=$$
$$=\frac{1}{n_l+(\tau+s+1)m_l}\left(m_l\prod_{j=1}^{l}\frac{1}{n_j}+Z(n_1,\dots,n_{l-1};m_1,\dots,m_{l-1};\tau,s)\right).$$

And by induction on l, if $u:=\tau+s+1$:

$$Z(n_1,\dots,n_l;m_1,\dots,m_{l-1},m_l;\tau,s)=$$
$$=\sum_{k=2}^{l}m_k\left(\prod_{j=1}^{k}\frac{1}{n_j}\right)\left(\prod_{j=k}^{l}\frac{1}{n_j+m_ju}\right)+\left(\prod_{j=2}^{l}\frac{1}{n_j+m_ju}\right)Z(n_1;m_1;\tau,s).$$

Formula 1.8. *It is easily seen that*

$$Z(n_1;m_1;\tau,s)=\frac{1}{n_1+m_1u}\left(\frac{m_1}{n_1}+\frac{1}{s+1}+\frac{1}{\tau+1}-1\right).$$

Formula 1.9. *Combining last formulæ we obtain*

$$\begin{aligned}Z(n_1,\dots,n_l;m_1,\dots,m_l;\tau,s)&=\\&=\frac{1}{u}\prod_{j=1}^{l}\frac{1}{n_j}+(u+1)\left(\frac{1}{(\tau+1)(s+1)}-\frac{1}{u}\right)\prod_{j=1}^{l}\frac{1}{n_j+m_ju}.\end{aligned}$$

Applied to the case of SIH singularities, i.e., $\tau=n+ds$, $t:=(n+1)+(d+1)s$, $m_i=N_i$ *and* $n_i=\nu_i$, *one gets*

$$\begin{aligned}Z(\nu_1,\dots,\nu_l;Nm_1,\dots,N_l;n+ds,s)=&\frac{1}{t}\prod_{j=1}^{l}\frac{1}{\nu_j}+\\&+(t+1)\left(\frac{1}{(t-s)(s+1)}-\frac{1}{t}\right)\prod_{j=1}^{l}\frac{1}{\nu_j+N_jt}.\end{aligned}$$

Remark 1.10. In [DL1], J. Denef and F. Loeser using p-adic integration and the Grothendieck-Lefschetz trace formula showed that the local topological zeta function of the non-vanishing function germ g^P verifies the equality

$$Z_{top,0}(g^P,0)=\sum_{J\subset\{0,1,\dots,r\}}\chi_J\prod_{j\in J}\frac{1}{\nu_i}=1.$$

Theorem 1.11. *If* $P\in\operatorname{Sing}(C_d)$ *then*

$$Z_{top,0}((z-g^P(\bar{x}))\,z^d,\omega,s)=\frac{1}{t}+(t+1)\left(\frac{1}{(t-s)(s+1)}-\frac{1}{t}\right)Z_{top,0}(g^P,t).$$

Corollary 1.12. *Let* $f:=f_d+f_{d+1}+\cdots\in\mathbb{C}\{x_0,x_1,\dots,x_n\}$ *define a SIH singularity* $(V,0)\subset(\mathbb{C}^{n+1},0)$. *Then its local topological zeta function satisfies the following equality*

$$\begin{aligned}Z_{top,0}(V,s)=&\frac{\chi(\mathbb{P}^n\setminus C_d)}{t-s}+\frac{\chi(\check{C}_d)}{(t-s)(s+1)}+\\&+\sum_{P\in\operatorname{Sing}(C_d)}\left(\frac{1}{t}+(t+1)\left(\frac{1}{(t-s)(s+1)}-\frac{1}{t}\right)Z_{top,P}(g^P,t)\right),\end{aligned}$$

where g^P *is a local equation of* C_d *at* P *and* $t:=n+1+(d+1)s$.

§2.- The pole $s_0 = -3/d$ for $n = 2$

Let $f := f_d + f_{d+1} + \cdots \in \mathbb{C}\{x_0, x_1, x_2\}$ be an analytic function such that its zero locus $(V, 0) \subset (\mathbb{C}^3, 0)$ defines a SIS singularity. It means that $C_{d+1} \cap \operatorname{Sing}(C_d) = \emptyset$.

The formula in (1.12) for the local topological zeta function can be rewritten in the form

$$Z_{top,0}(V,s) = \frac{3 + \chi(\mathbb{P}^2 \setminus C_d)s}{(t-s)(s+1)} + \\ + \frac{(t+1)s(1-(t-s))}{t(t-s)(1+s)} \sum_{P \in \operatorname{Sing}(C_d)} \left(Z_{top,P}(C_d, t) - \frac{1}{1+t} \right),$$

where $t = 3 + (d+1)s$ and $Z_{top,P}(C_d, s)$ means the local topological zeta function of the germ at the point $P \in \operatorname{Sing}(C_d)$ of the plane curve singularity $C_d \subset \mathbb{P}^2$ defined by $\{f_d = 0\}$.

Lemma 2.1. *The pole candidates of $Z_{top,0}(V,s)$ are $s = -1$, $s = -3/(d+1)$, $s = -3/d$, and the poles $-(\nu + 3N)/(d+1)N$ whenever $-\nu/N$ is a pole of the local topological zeta function of the germ of C_d at some point $P \in \operatorname{Sing}(C_d)$.*

We know that $Z_{top,P}(C_d, 0) = 1$, see Remark (1.12). Hence $t = 0$, i.e. $s = -3/(d+1)$, is not a pole of $Z_{top,0}(V,s)$.

The germ $(V,0) \subset (\mathbb{C}^3, 0)$ is an isolated surface singularity. Hence $H_0(F, \mathbb{C})$ and $H_2(F, \mathbb{C})$ are the only non vanishing homology vector spaces on which the monodromy acts, (we denote the Minor fibre by F). The only eigenvalue of the action of the monodromy on $H_0(F, \mathbb{C})$ is equal to 1. The characteristic polynomial of the action of the complex monodromy on $H_2(F, \mathbb{C})$ is given by the formula

$$\Delta_V(t) = \frac{(t^d - 1)^{\chi(\mathbb{P}^2 \setminus C_d)}}{(t-1)} \prod_{P \in \operatorname{Sing}(C_d)} \Delta^P(t^{d+1}),$$

where $\Delta^P(t)$ is the characterisitic polynomial (or Alexander polynomial) of the action of the complex monodromy of the germ (C_d, P) on $H_1(F_{g^P}, \mathbb{C})$, (F_{g^P} denotes de corresponding Milnor fibre), e.g. see [Ar].

The following lemma is an easy consequence of the formulæ above and the proved Monodromy Conjecture for curves [Lo].

Lemma 2.2. *Let $(V,0) \subset (\mathbb{C}^3, 0)$ be a SIS singularity of multiplicity d. Let C_d be its tangent cone. Then:*

- *If $\chi(\mathbb{P}^2 \setminus C_d) > 0$ then the Monodromy Conjecture for $(V,0)$ holds.*
- *If $\chi(\mathbb{P}^2 \setminus C_d) \leq 0$ then $s_0 = -3/d$ is the only pole candidate which might not verify the Monodromy Conjecture for $(V,0)$.*

Proposition 2.3. *Suppose that $\chi(\mathbb{P}^2 \setminus C_d) \leq 0$. If $s_0 = -3/d$ is a pole of $Z_{top,0}(V,s)$ of order greater than one then $s_0 = -3/d$ is a pole of $Z_{top,P}(C_d, s)$ at some point $P \in \operatorname{Sing}(C_d)$.*

Corollary 2.4. *In such a case, the Monodromy Conjecture for germs of curves implies that $\exp(2i\pi(-3/d))$ is a root of $\Delta^P(t)$ (and also it is a root of $\Delta^P(t^{d+1})$). Hence the Monodromy Conjecture for $(V,0)$ also holds in this case.*

Example 2.5. Consider two lines L_1 and L_2 on $\mathbb{P}^2$ and $(V_D, 0) \subset (\mathbb{C}^3, 0)$ a SIS singularity whose tangent cone is $D = L_1 \cup L_2$. Hence the invariants $Z_{top,0}(V, s)$ and $\Delta_V(t)$ are

$$Z_{top,0}(V,s) = \frac{4+s}{(1+s)(4+3s)}, \qquad \text{and} \qquad \Delta_V(t) = \frac{t^3-1}{t-1}.$$

Assume that D has degree 3 and $\chi(\mathbb{P}^2 \setminus D) \leq 0$. Then:

(A) If $D = L_1 \cup L_2 \cup L_3$ is an arrangement of three lines meeting at only one point then the corresponding SIS singularity has invariants:

$$Z_{top,0}(V,s) = \frac{11}{(1+s)(11+12s)}, \qquad \text{and} \qquad \Delta_V(t) = \frac{(t^{12}-1)(t^4-1)}{(t^3-1)(t-1)}.$$

(B) If $D = L_1 \cup L_2 \cup L_3$ consits of three lines in general position, i.e. $\mathrm{Sing}(D)$ are three nodes, then the invariants of the SIS singularity are:

$$Z_{top,0}(V,s) = \frac{3s^2+6s+4}{4(1+s)^3}, \qquad \text{and} \qquad \Delta_V(t) = \frac{(t^4-1)^3}{(t-1)}.$$

(C) If D is a conic with a tangent line then in such a case the invariants of the SIS singularity are

$$Z_{top,0}(V,s) = \frac{3s+15}{(1+s)(15+16s)}, \qquad \text{and} \qquad \Delta_V(t) = \frac{(t^{16}-1)(t^4-1)}{(t^8-1)(t-1)}.$$

With these examples the Monodromy Conjecture for any SIS singularity of multiplicity $d = 2$ or 3 is proved.

The discussion above suggests the following definition which we need to extend to divisors on the projective plane.

Definition 2.6. We say that a degree d effective divisor D on $\mathbb{P}^2$ $(d > 3)$ is a *bad divisor* if $\chi(\mathbb{P}^2 \setminus D) \leq 0$ and $s_0 = -3/d$ is not a pole of $Z_{top,P}(g_D^P, s)$, for any singular point P in its support D_{red}, where g_D^P is the local equation of the divisor D at P.

We are concerned with the (at most simple) pole $s_0 = -3/d$ of $Z_{top,0}(V, s)$ when the tangent cone C_d is a bad divisor on $\mathbb{P}^2$.

Lemma 2.7. *If the tangent cone C_d of $(V, 0)$ is a bad divisor on $\mathbb{P}^2$ then the residue of $s_0 = -3/d$ in $Z_{top,0}(V, s)$, will be denoted by $\rho(C_d)$, and it is equal to*

$$\chi(\mathbb{P}^2 \setminus C_d) + \chi(\check{C}_d)\frac{d}{d-3} + \sum_{P \in \mathrm{Sing}(C_d)} Z_{top,P}(C_d, -3/d) \in \mathbb{Q}.$$

In such a way, the Monodromy Conjecture will be true for SIS singularities if and only if $\rho(C_d) \neq 0$ implies that $\exp(2i\pi(-3/d))$ is a root of $\Delta_V(t)$.

Remark 2.8. The residue $\rho(C_d)$ is related to the topological zeta function associated with an effective divisor $D = a_1 D_1 + \ldots + a_r D_r$ of degree d on $\mathbb{P}^2$, $d > 3$, introduced by W. Veys. We set $\check{D}_i := D_i \setminus \operatorname{Sing}(D_{red})$. From Section 1, the topological zeta function of the divisor D on $\mathbb{P}^2$ can be rewritten as follows

$$z(D,s) = \chi(\mathbb{P}^2 \setminus D) + \sum_{i=1}^{r} \frac{\chi(\check{D}_i)}{1+a_i s} + \sum_{P\in \operatorname{Sing}(D_{red})} Z_{top,P}(D,s).$$

Then $\rho(C_d)$ is equal to the value $z(C_d, \frac{-3}{d})$. The value $z(D, \frac{-3}{d}) \in \mathbb{Q}$ is defined to be the residue of a bad divisor D on $\mathbb{P}^2$; we are also going to use another interpretation of this rational number.

Since D is an effective divisor then $(-3/d)D$ is a $\mathbb{Q}$-canonical divisor on $\mathbb{P}^2$. Let $\pi : X \to \mathbb{P}^2$ be the minimal embedded resolution for the support of D in $\mathbb{P}^2$. The map π is a sequence of blowing-ups centered at infinitely near points of points in $\operatorname{Sing}(D_{red})$ such that the divisor $\pi^* D$ is a normal crossing divisor. Let K_X be the $\mathbb{Q}$-canonical divisor on the surface X obtained from the pull-back of $(-3/d)D$, (see Remark 2.9). The irreducible components of K_X are the strict transforms of the irreducible components of D and the exceptional components over each singular point of D_{red}. The corresponding multiplicities in K_X are:

- $-3a_i/d$ for the strict transform of the irreducible component D_i of D.
- $\frac{-3}{d}N_i + \nu_i - 1$, for any exceptional component E_i, associated with a point $P \in \operatorname{Sing}(D_{red})$, where N_i and ν_i are defined as in §1.

Remark 2.9. For instance, if $\pi_1 : Y \to \mathbb{P}^2$ is the blow-up at some point P then $K_Y = \pi^*(-3/dD) + E$. In general $\pi : X \to \mathbb{P}^2$ is a composition of blow-ups. By canonical pull-back of a $\mathbb{Q}$-divisor K we mean that if K_X is the divisor of a multivaluated meromorphic 2-form ω, then its canonical pull-back is the divisor of the pull-back $\pi^*\omega$ of ω.

Definition 2.10. Let X be a rational surface and let K be a $\mathbb{Q}$-canonical divisor with normal crossings. We construct a weighted stratification $(\mathcal{S}_K, w_K)$ of X as follows:

- There is a unique stratum of dimension 2 which is $\check{X} := X \setminus K_{red}$. The weight of this stratum is $w_{\check{X}} = 1$.
- The 1-dimensional strata are the connected components of $K_{red} \setminus \operatorname{Sing}(K_{red})$. If $K = \sum_{i=1}^r k_i E_i$, and
$$\check{E}_i := E_i \setminus \bigcup_{j\neq i} E_j,$$
then $\check{E}_1, \ldots, \check{E}_r$ are these strata. We set $w_{\check{E}_i} := k_i + 1$.
- The strata of dimension 0 are the connected components of $\operatorname{Sing}(K_{red})$, i.e, the ordinary double points of K_{red}. If such a stratum S is in $E_i \cap E_j$, then $w_S := (k_i+1)(k_j+1)$.

Definition 2.11. Let X be a rational surface and let K be a $\mathbb{Q}$-canonical divisor with normal crossings as above. We say that K is *admissible* if 0 is not a weight in w_K, i.e., if no k_i is equal to -1. For an admissible $\mathbb{Q}$-divisor K we define

$$\zeta_K^1 := \sum_{S\in\mathcal{S}_K} \frac{\chi(S)}{w_S} \in \mathbb{Q}.$$

In principle we assume that all the k_i are different from zero. Nevertheless, the invariant ζ_K^1 does not change if we drop this additional hypothesis.

In order to relate this invariant with the residue $\rho(C_d)$ we need some notation.

Definition 2.12. We say that $s_0 = -3/d$ *is not a pole candidate of* D *for any singular point in* D_{red} if and only if $s_0 = -3/d$ is not a pole candidate of the local topological zeta function $Z_{top,P}(D,s)$ for any $P \in D_{red}$, i.e. for any $P \in D_{red}$, a pair (N_i, ν_i), with $-\nu_i/N_i = -3/d$, does not appear in any exceptional divisor of the minimal resolution of the germ of D at its singular point P.

Lemma 2.13. *Let D be an effective divisor of degree d on $\mathbb{P}^2$, $d > 3$. Let $\pi\colon X \to \mathbb{P}^2$ be the minimal resolution of* $\operatorname{Sing}(D_{red})$. *Let us suppose that $-3/d$ is not a pole candidate of D for any singular point in D_{red}, and let K be the canonical pull-back of $(-3/d)D$ by π. Then, K is admissible and*

$$\rho(D) = \zeta_K^1.$$

The following result says that ζ_K^1 is invariant by almost all proper birational mappings.

Lemma 2.14. *Let X be a rational surface and $\pi\colon Y \to X$ a blow-up at some point. Let K_X be an admissible $\mathbb{Q}$-canonical divisor such that the canonical pull-back K_Y is also admissible. Then,*

$$\zeta_{K_X}^1 = \zeta_{K_Y}^1.$$

Proof. Let P be the centre of the blow-up and E the exceptional divisor. If P is not in the support of K_X, then the multiplicity of E in K_Y is 1 and the formula holds because the contribution of P in $\zeta_{K_X}^1$ is one and it coincides with the contribution of E in $\zeta_{K_Y}^1$.

What we mean by *contribution* is that there exists $\eta \in \mathbb{Q}$ such that $\zeta_{K_X}^1 = \eta +$ (contribution in $\zeta_{K_X}^1$) and $\zeta_{K_Y}^1 = \eta +$ (contribution in $\zeta_{K_Y}^1$).

If P is a smooth point of $\operatorname{supp}(K_X)$, say $P \in \check{E}_i$, then its contribution to $\zeta_{K_X}^1$ is $\dfrac{1}{k_i+1}$. The multiplicity of E in K_Y is k_i+1; then by hypothesis, $k_i \neq -2$. Its contribution to $\zeta_{K_Y}^1$ is

$$\frac{1}{k_i+2}\left(1+\frac{1}{k_i+1}\right) = \frac{1}{k_i+1}.$$

If P is a double point of K_X, say $P \in E_i \cap E_j$, then its contribution to the invariant $\zeta_{K_X}^1$ is $\dfrac{1}{(k_i+1)(k_j+1)}$. The exceptional divisor E appears in K_Y with multiplicity k_i+k_j+1; then by hypothesis, $k_i+k_j \neq -2$. Its contribution to $\zeta_{K_Y}^1$ is

$$\frac{1}{k_i+k_j+2}\left(\frac{1}{k_i+1}+\frac{1}{k_j+1}\right) = \frac{1}{(k_i+1)(k_j+1)}.$$

This completes the proof. $\square$

Consider an admissible $\mathbb{Q}$-canonical divisor $K = \sum_{i=1}^{r} k_i E_i$ on a rational surface X. Let G be the dual graph of K; we consider G as a weighted graph, such that if v_i is the vertex associated with E_i, then its weight is $w_{v_i} := k_i + 1$. We denote by $V(G)$ the set of vertices of G and by $E(G)$ its set of edges. Recall that the *valency* of a vertex $v \in V(G)$ is the number of edges from v. Given $e \in E(G)$ we also denote by $V(e)$ the set of extremities of the edge e. Then we can rewrite:

$$\zeta_K^1 = \chi(\check{X}) + \sum_{j=1}^{r} \frac{\chi(\check{E}_j)}{w_{v_j}} + \sum_{e \in E(G)} \prod_{v \in V(e)} \frac{1}{w_v}.$$

This graph is also weighted by the self-intersections numbers $a_i := E_i^2$ of the irreducible components E_i of K on the surface X. A subgraph G_1 of G is a graph such that $V(G_1) \subset V(G)$ and any edge in G, with extremities in $V(G_1)$, is an edge in G_1.

Definition 2.15. We say that a subgraph G_1 of G is a *set of bamboos* if the following conditions hold

- any connected component of the graph G_1 is linear;
- the irreducible components of K associated with the vertices of G_1 are rational curves;
- if $v \in V(G)$ is an extremity of G_1, then its valency in G is less than 3.

In such a case, each connected component of G_1 is called a *bamboo*. A bamboo is of type 1 (resp. 2) if it has one (resp. two) neighbour vertex (resp. vertices) in G.

Let $V(B) := \{v_{i_1}, \dots, v_{i_r}\}$ be the set of vertices of the bamboo B. The intersection matrix of B is the integer matrix $A = (a_{ij}) \in M(r, \mathbb{Z})$ such that:

- If $j \neq k$, then a_{jk} is the number of edges between v_{i_j} and v_{i_k}, i.e. the intersection number between E_{i_j} and E_{i_k},
- $a_{jj} := a_j$.

The *determinant* of the bamboo is $\det(B) := \det(-A)$ (which does not depend on the order of the vertices of B, e.g. see [V3]).

Definition 2.16. Choose K and G as above. Let $G_1 \neq G$ be a set of bamboos of G. We define the graph G/G_1 which has weighted vertices, weighted edges and weighted arrows as follows.

- The set of vertices $V(G/G_1)$ is nothing but $V(G) \setminus V(G_1)$ and they are weighted as in G.
- The set of edges $E(G/G_1)$ has two types of elements. Edges of G not intersecting G_1 produce edges of G/G_1; theses edges are weighted by 1. Each bamboo of type 2 produces also one edge with the obvious extremities and weighted by the determinant of the bamboo.
- The set $A(G/G_1)$ of arrows of the graph G/G_1 is in one-to-one correspondence with the set of bamboos of G_1 of type 1. It is weighted by the determinant of the corresponding bamboo. Note that each arrow a in G/G_1 has only one neighbour vertex v_a.

The adjunction formula is the key point of the following result which is a generalization of a Veys' result in [V3, Theorem 3.3].

Proposition 2.17. *Let X be a rational surface and K an admissible $\mathbb{Q}$-canonical divisor. Let G be the dual graph and let G_1 be a set of bamboos. Then:*

$$\zeta_K^1 = \chi(\check{X}) + \sum_{v\in V(G/G_1)} \frac{\chi(\check{E}_v)}{w_v} + \sum_{e\in E(G/G_1)} w_e \prod_{v\in V(e)} \frac{1}{w_v} + \sum_{a\in A(G/G_1)} \frac{w_a}{w_{v_a}}.$$

Proof. Let us consider a bamboo of length k, weights $w_1,\dots,w_k$ and self-intersections $-a_1,\dots,-a_k$. Assume that the weight of the neighbours vertices are w_0 and w_{k+1}; if the bamboo is of type 1 everything works if we suppose $w_{k+1}=1$ whenever the k^{th} vertex has valency one in G.

We denote by B_j the determinant of the matrix of the first j vertices. It is easily cheked that we have:

$$B_j = a_j B_{j-1} - B_{j-2}.$$

The adjunction formula implies

$$a_j w_j = w_{j-1} + w_{j+1}.$$

We must prove that the sum in the definition of the topological zeta function of the terms involving the bamboo is equal to $\dfrac{B_k}{w_0 w_{k+1}}$.

We prove this fact by induction on k. The case $k=0$ is trivial. For the general case, the sum associated with the bamboo is:

$$\frac{B_{k-1}}{w_0 w_k} + \frac{1}{w_k w_{k+1}} = \frac{1}{w_0 w_{k+1}}\left(\frac{B_{k-1} w_{k+1}}{w_k} + \frac{w_0}{w_k}\right).$$

Then, it is enough to prove that

$$B_k w_k - B_{k-1} w_{k+1} = w_0.$$

This is straightforward by induction. □

This proposition suggests how the ζ_K^1-invariant of an admissible $\mathbb{Q}$-canonical divisor K on X can be extended to the following more general canonical divisors.

Definition 2.18. Let X be a rational surface and let $K=\sum_{i=1}^r k_i E_i$ be a $\mathbb{Q}$-canonical divisor on it with normal crossings; let K_0 be the reduced subdivisor of K consisting of irreducible components E_i such that $k_i=-1$. Let G be the dual graph and let G_0 be the dual graph of K_0. We say that K is *g-admissible* if G_0 is a set of bamboos. In this case, we define

$$\zeta_K := \chi(\check{X}) + \sum_{v\in V(G/G_0)} \frac{\chi(\check{E}_v)}{w_v} + \sum_{e\in E(G/G_0)} w_e \prod_{v\in V(e)} \frac{1}{w_v} + \sum_{a\in A(G/G_0)} \frac{w_a}{w_{v_a}}.$$

Properties 2.19. We state some properties of this invariant.

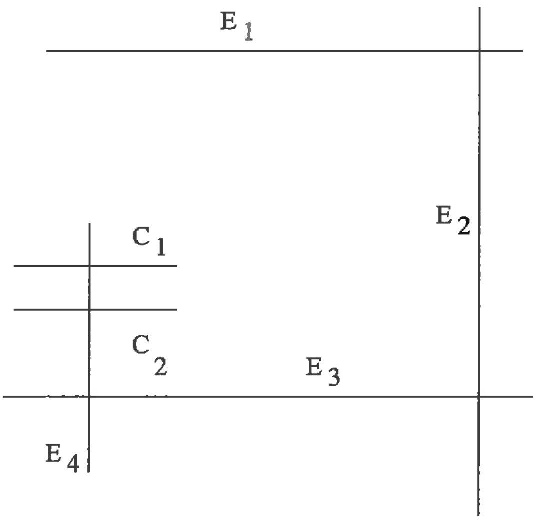
(i) Of course, when K is admissible then $\zeta_K = \zeta_K^1$.

(ii) The formula in (2.17) is also true for g-admissible divisors.

(iii) If $v \in V(G)$ is a vertex with weight zero then the adjunction formula implies that the sum of the neighbour weights equals the valency of the vertex minus 2. It turns out that each connected component of K_0 consists only of one rational curve E. This implies that:

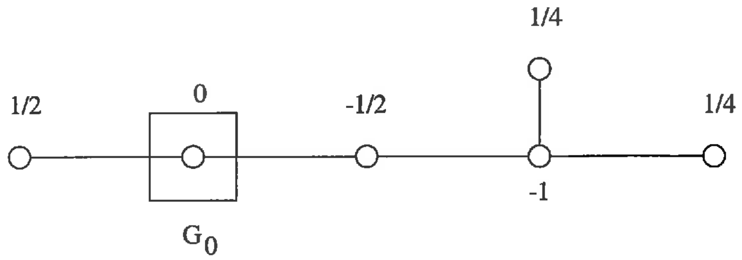
- if the bamboo is of type 2 and gives an edge $e \in E(G/G_0)$, then the weights of the vertices in $V(e)$ are opposite to each other,
- and if it is of type 1 and gives an arrow $a \in A(G/G_0)$, then its weight in $V(a)$ is equal to -1.

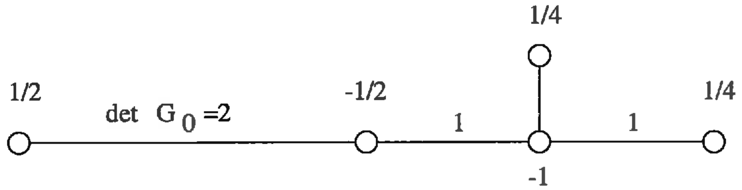
Example 2.20. Let $D \subset \mathbb{P}^2$ be the union of two smooth conics C_1 and C_2 which meet at only one point $\{P\} = C_1 \cap C_2$. We consider D as divisor of degree 4 and $K := (-3/4)D$ as a $\mathbb{Q}$-rational divisor on $\mathbb{P}^2$. Let $\pi : X \to \mathbb{P}^2$ be the minimal embedded resolution of the singularity of D at the point P. In the rational surface X we have the following configuration of curves and the corresponding associated invariants:



	(N_i, ν_i)	ω_i	E_i^2
E_1	(2,2)	1/2	-2
E_2	(4,3)	0	-2
E_3	(6,4)	-1/2	-2
E_4	(8,5)	-1	-1
C_1 C_2	(1,1)	1/4	0

The dual graph of the resolution G has only one bamboo G_0 and the corresponding graph G/G_0 is given as follows





Then the ζ_K-invariant is non-zero because

$$\zeta_K = \frac{1}{\frac{1}{2}} + (-1)(-1) + 2\frac{1}{\frac{1}{4}} + 2\frac{1}{\frac{-1}{2}\frac{1}{2}} + \frac{1}{\frac{-1}{2}(-1)} + 2\frac{1}{\frac{1}{4}(-1)} \neq 0.$$

Below we compute $Z_{top,0}(V_D, s)$ and $\Delta_V(t)$ for a SIS singularity $(V_D, 0) \subset (\mathbb{C}^3, 0)$ whose tangent cone is D. In this case $\chi(\mathbb{P}^2 \setminus D) = 0$ and $s_0 = -3/4$ is not a pole of $Z_{top,P}(D, s)$ for the germ of curve D at P. Hence D is a bad divisor on $\mathbb{P}^2$. Since the residue $\rho(D) = \zeta_K \neq 0$ then $s_0 = -3/4$ is a simple pole of $Z_{top,0}(V_D, s)$ and as one can easily check that $\exp(2i\pi(-3/4))$ is a root of $\Delta_V(t)$.

$$Z_{top,P}(D, s) = \frac{3s + 5}{(1 + s)(5 + 8s)},$$

$$Z_{top,P}(V_D, s) = \frac{130s + 20s^2 + 87}{(1 + s)(3 + 4s)(29 + 40s)},$$

$$\Delta_V(t) = \frac{t^{35} - t^{30} + t^{25} - t^{20} + t^{15} - t^{10} + t^5 - 1}{t - 1}.$$

Compare the following result with (2.13).

Lemma 2.21. *Let D be an effective divisor on $\mathbb{P}^2$, $d > 3$. Let $\pi\colon X \to \mathbb{P}^2$ be the minimal resolution of* $\mathrm{Sing}(D_{red})$. *Let us suppose that $-3/d$ is not a pole for any singular point in D_{red} and let K be the canonical pull-back of $(-3/d)D$ by π. Then K is g-admissible and*

$$\rho(D) = \zeta_K.$$

We could expect ζ_K to be invariant for blowing-ups. Nevertheless, this is not the case.

Proposition 2.22. *Let X be a rational surface and let K_X be a g-admissible $\mathbb{Q}$-canonical divisor. Let $\pi\colon Y \to X$ be the blowing-up of a point $P \in X$ and let K_Y be the canonical pull-back of K_X. Then, K_Y is g-admissible and:*

(i) *If P does not belong to $K_0 \setminus \mathrm{Sing}(K_{X,red})$, then $\zeta_{K_X} = \zeta_{K_Y}$.*

(ii) *If P belongs to $K_0 \setminus \operatorname{Sing}(K_{X,red})$, let B_1 be the bamboo of K_0 containing P. Let us suppose that the self-intersection of B_1 is $-a$ and the neighbours of B_1 in G have weights w and $-w$ (or $w=-1$ if it is of type 1). Then, the corresponding bamboo B_2 in $\tilde{K}$ has self-intersection $-a-1$ and*

$$\zeta_{K_X} + 1 - \frac{1}{w^2} = \zeta_{K_Y}.$$

Proof. Let E be the exceptional divisor of the blowing-up $\pi\colon Y \to X$ of X at the point $P \in X$. It is easily seen that if K_X is g-admissible, it is also the case for K_Y. We restrict ourselves to the cases which are not covered by the proof of (2.14).

The proof of the remaining cases is based on the study of the contribution of the point P (and its neighbours) to ζ_{K_X} and the exceptional curve E (and its neighbours) to ζ_{K_Y}.

Case 1. *The point P is smooth in $K_{X,red}$, $P \in \check{E}_i$ and $k_i = -2$.*

The contribution of P to ζ_{K_X} is equal to -1. The exceptional divisor E produces a bamboo of type 1 with determinant 1. The result holds easily.

Case 2. *The point P is a double point of $K_{X,red}$, $P \in E_i \cap E_j$ and $k_i + k_j = -2$.*

Let us consider the weights w_i, w_j. We have $w_j = -w_i (\neq 0)$. In this case the contribution of P to ζ_{K_X} is equal to $\dfrac{-1}{w_i^2}$. The exceptional divisor E produces a bamboo of type 2 with determinant 1 and the result holds easily.

Case 3. *The point P is a double point of $K_{X,red}$, $P \in E_i \cap E_j$, $k_i = -1$.*

Let us consider again weights $w_i (= 0), w_j$. The curve E_i gives a bamboo in K_X, with determinant a and neighbour weights $-w_j$ and w_j. The contribution of this bamboo to ζ_{K_X} is $\dfrac{-a}{w_j^2}$. The strict transform of E_i in K_Y is a bamboo of determinant $a+1$ and neighbour weights $-w_j, w_j$. The intersection point between the new exceptional divisor E and E_j contributes with $1/w_j^2$. Hence its contribution to ζ_{K_Y} is:

$$-\frac{a+1}{w_j^2} + \frac{1}{w_j^2}.$$

Case 4. *The point P is smooth in K_{red}, $P \in \check{E}_i$ and $k_i = -1$.*

Let us assume the notation of (ii). The contribution of the bamboo E_i in ζ_{K_X} is equal to $\dfrac{-a}{w^2}$. The strict transform of E_i is also a bamboo in K_Y whose contribution is $\dfrac{-a-1}{w^2}$. But in this case the exceptional divisor E has $k=0$, then it is not in the support of K_Y and the Euler characteristic of the complement of K_Y in Y differs from 1 from the one of K_X in X.

The formula is proved. $\square$

Corollary 2.23. *Let X be a rational surface and let K be a g-admissible $\mathbb{Q}$-canonical divisor. Let $\pi\colon Y \to X$ be the blowing-up at a point $P \in X$ and let $\tilde{K}$ be the canonical pull-back of K. If P belongs to $K_0 \setminus \operatorname{Sing}(K_{red})$ then*

$$\chi(X \setminus K) < \chi(Y \setminus \tilde{K}).$$

In particular the unique blow-up proccess in which the following holds: $\zeta_K \neq \zeta_{\tilde{K}}$ *and* $\chi(X \setminus K) < \chi(Y \setminus \tilde{K})$ *is the blowing-up of* X *at* $P \in K_0 \setminus \mathrm{Sing}(K_{red})$ *having valency 2 and whose weights of the neighbour vertices are* $\pm w$, $w \neq 1$.

The Euler-Poincaré characteristic condition on a bad divisor D implies that D has at least two irreducible components, all of them being rational curves, e.g. see [JS]. In the above definition we impose that s_0 will not be a pole (although s_0 might be a pole candidate) of $Z_{top,P}(D,s)$ for a singularity P in D_{red}.

Let us apply the Veys' Theorem to the curve D. It means that we can extend the curve D to another curve $D' \supset D$, also having $\chi(\mathbb{P}^2 \setminus D') \leq 0$, such that we have the following diagram

$$\Sigma \xleftarrow{\pi_3} X_2 \xrightarrow{\pi} \mathbb{P}^2$$

where π is a composition of blowing-ups with center in D' and π_3 is a composition of blowing-downs with exceptional curve contained in $\pi^{-1}(D')$ and such that Σ is a ruled surface. Moreover the configuration $T := \pi_3(\pi^{-1}(D'))$ is either

(A) one section C_1 and at least two fibres, or

(B) two disjoints sections C_1 and C_2 and at least one fibre.

In fact, the Veys' process is based on a theorem of R.V. Gurjar and A.J. Parameswaran which, roughly speaking, states that if $\chi(\mathbb{P}^2 \setminus D) \leq 0$ then there exists a map $\varphi : \mathbb{P}^2 \setminus D \to B$ to a smooth curve B, see [GP, Theorem 3].

Considering φ as a rational map $\mathbb{P}^2 \dashrightarrow \overline{B}$ ($\overline{B}$ being a smooth compactification of B) and resolving indeterminacies one gets a morphism $\overline{\varphi} := \varphi \circ \pi : X_2 \to \overline{B}$, where $\pi : X_2 \to \mathbb{P}^2$ is a composition of blow-ups centered at points on D and its consecutive total transform. W. Veys also proved that $\overline{B} \simeq \mathbb{P}^1$ and the generic fibre of $\overline{\varphi}$ is $\mathbb{P}^1$.

It implies that there exists a pencil Λ on (or a rational function R on) $\mathbb{P}^2$ whose generic member is a rational curve. Moreover, looking at the two possible configurations (A) and (B) obtained by W. Veys, the generic fibre of the rational map $R : \mathbb{P}^2 \setminus \{\text{base points}\} :\to \mathbb{P}^1$ is an open Riemann surface of genus 0 minus at most two points. These pencils have been studied by H. Kashiwara in [K] and T. Kizuka [Ki], see Appendix.

More precisely, Veys' process can be decompose in three stages.

- In the first step we consider the minimal embedded resolution $\pi_1 \colon X_1 \to \mathbb{P}^2$ of the local singularities of $D_{red} \subset \mathbb{P}^2$. At this stage, the residue of C_d is already computed as ζ_K, where K is the canonical pull-back of $-3/dC_d$. Since $-3/d$ is not a pole for $\zeta_{top,P}(C_d, s)$ for any $P \in \mathrm{Sing}(C_d)$, then K is g-admissible.
- Secondly, we consider the resolution $\pi_2 \colon X_2 \to X_1$ of the indeterminacy locus of the pencil $\Lambda^* \subset X_1$ defined by the function $\pi_1^* R$; it might happen that π_2 is the identity map.
- Finally, let $\pi_3 \colon X_2 \to \Sigma$ be the contraction onto a rational ruled surface Σ, which can be supposed to be $\mathbb{P}^1 \times \mathbb{P}^1$, see [V4, Thm 4.1].

$$\begin{array}{ccc} & X_2 & \\ \pi_3 \swarrow & & \searrow \pi_2 \\ \Sigma & & X_1 \\ & & \downarrow \pi_1 \\ & & \mathbb{P}^2 \end{array}$$

The irreducible components of the strict transform of D in X_2 have to be either irreducible components of members of the total transform of the pencil Λ or dicritical divisors of the map $\overline{\varphi}: X_2 \to \mathbb{P}^1$, (a curve E is *dicritical* if the restricted map $\overline{\varphi}|: E \to \mathbb{P}^1$ is surjective).

Note that several pencils may match for a given curve $D \subset \mathbb{P}^2$ and then several different constructions can be reached.

We have proved in the previous lemma that the residue $\rho(D)$ of a bad divisor D is equal to the ζ-invariant of any g-admissible $\mathbb{Q}$-canonical divisor on X_1. The following result shows that in the final configurations the ζ_K-invariant is zero.

Proposition 2.24. *Let K be a g-admissible $\mathbb{Q}$-canonical divisor on a ruled rational surface such that its support is of type* (A) *or* (B). *Then $\zeta_K = 0$.*

Proof. Let us suppose $\Sigma = \mathbb{P}^1 \times \mathbb{P}^1$, see [V4, Theorem 4.1]. Let us denote by S and by F the general 0-section and the general fibre. It is known that canonical divisors are linearly equivalent to $-2S - 2F$.

If we are in case (A), let us denote by S_0 the section and by $F_1, \dots, F_r$ the fibres, $r \geq 2$. Let K be a g-admissible canonical divisor with support contained in the curves above. Then $K = -2S_0 + \sum_{j=1}^r k_i F_i$, with $\sum_{j=1}^r k_i = -2$. Applying the formula in the definition of the invariant ζ_K, we have the result.

If we are in case (B), let us denote by S_0, S_1 the sections and $F_1, \dots, F_r$ the fibres; we can suppose $r \geq 2$. Let K be a g-admissible canonical divisor with support contained in the curves above. Since we are in $\Sigma = \mathbb{P}^1 \times \mathbb{P}^1$ we can interchange fibres and sections if necessary.

In order to be g-admissible, we can suppose that $K = a_0 S_0 + a_1 S_1 + \sum_{j=1}^r k_i F_i$, with $\sum_{j=1}^r k_i = -2$, $a_0 + a_1 = -2$, $a_0, a_1 \neq -1$. Applying the definition of the invariant ζ_K the result is proved. □

We show in the following theorem that for any bad divisor D with $\rho(D) \neq 0$ one can always start the Veys' process with a rational pencil Λ on $\mathbb{P}^2$ of type $(0, 1)$, see Appendix. We are looking for bad divisors having $\rho(D)$ non-zero. For this purpose we must understand the behaviour of the maps π_2 and π_3 and find out when it is possible that at least one of the blow-ups in π_2 or π_3 matches the hypothesis of (2.22)(ii), with $1 \neq w^2$.

Theorem 2.25. *Let D be a bad divisor on $\mathbb{P}^2$. If $\rho(D) \neq 0$, then D can be sent by means of Veys' process to a curve T of type* (A) *with at least three fibres.*

Moreover the irreducible components of D are in a pencil of rational curves having only one base point and such that any fibre minus the base point is isomorphic to $\mathbb{C}$; at least one generic fibre (resp. two) is contained in D if the pencil has two exceptional fibres (resp. one).

Proof. Since the ζ-invariant of the surfaces X_1 and Σ do not coincide at least one of the blow-ups in $\pi_2 : X_2 \to X_1$ or $\pi_3 : X_2 \to \Sigma$ is as in (2.22)(ii).

First of all we deal with the case where no component of D is sent to a section in Σ.

($\bullet$) If the changes of the ζ-invariant happen in $\pi_3 : X_2 \to \Sigma$ then the exceptional curve which is blown-down to a point will give either an exceptional component for

$\pi = \pi_2 \circ \pi_1 : X_2 \to \mathbb{P}^2$ or an additional projective curve in order to extend the curve D, see the Veys' theorem.

(i) In the first case we will have in the exceptional divisor of the map $\pi : X_2 \to \mathbb{P}^2$, an irreducible component with more than 2 neighbours and multiplicity -1. Since the map π is an embedded resolution of D we get $-3/d$ as a pole of the local topological zeta function of at least one local singularity of D_{red} which implies that D is not a bad divisor.

It is not possible for this component to become a component of valency 2 in the minimal resolution if π_2 is not trivial, because in this case the neighbour component which is not blown-down has weight equal to 1 and in this case ζ_K does not change.

(ii) In the second case the Euler characteristic of the complement changes. This implies that D' is sent to a curve T of type (A) with at least three fibres. We have seen in the above properties that in this case the Euler characteristic of the complement is increased by one from the Euler characteristic on the curve of type (A) or (B). Since we must keep this invariant non-positive, in the curve of type (A) or (B) we must have negative Euler characteristic. This is only possible if the type is (A) with at least three fibres, (see Corollary 2.17).

(••) If the changes of the ζ-invariant occur in a blowing-up in $\pi_2 : X_2 \to X_1$ we have the following situation. We are blowing-up at a point P which is base point of the strict transform of the pencil Λ and belongs to an irreducible exceptional component E of weight 0, valency 2 in the dual graph and so that its neighbours have weights $\pm a$ ($a \neq 1$). Since any curve of this pencil goes through P, the exceptional curve E is in a non-generic member L_v of the pencil (because it is different from the generic one); we are assuming that this member corresponds to the value v in $\mathbb{P}^1$.

Because of the valency 2 condition for E, the strict transform of D is not going through the base point P. It turns out that generic members of the pencil Λ are not in D. More precisely, none of the irreducible components of the members of the pencil with strict transform going through P is in D. But the strict transforms of any member of the pencil, but L_v, goes through the base point P. This implies that the pencil Λ has reduced members and hence the pencil Λ is of type $(0,2)$ (see e.g. [K]: any member of a pencil of type $(0,1)$ is irreducible).

Remember that, in pencils of type $(0,2)$, all the members but one are irreducible. Moreover, the reduced member, say L_1, has only two components, see [Ki, Proposition 4] and Appendix.

We have shown that components of D are either in the reduced member L_1 (which must be equal to L_1) or chosen between a non-generic irreducible member L_v and the irreducible component S of L_1 which is not going through the base point P. Since $\chi(\mathbb{P}^2 \setminus D) \leq 0$ then D has exactly two irreducible components. There are two possibilities for D.

Case 1: D is formed by the two irreducible components of the fibre $L_v = L_1$ or

Case 2: D is the union of S and an non-generic irreducible member L_v, S being one of the irreducible components of the reduced fibre L_1 of the pencil Λ.

We have also shown that if the ζ-invariant changed in the map $\pi_2 : X_2 \to X_1$ then the components of D have to be in a pencil Λ of type $(0,2)$. Next we will prove that the components of D are also in a new pencil of type $(0,1)$ for which the condition (••) *"the ζ-invariant has changed in the map $\pi_2 : X_2 \to X_1$"* is not possible as we have pointed out above. Then we use this new pencil to conclude that the components of D can always be transformed as in the above case $(\bullet)$ (ii).

T. Kizuka classified pencils of type $(0,2)$ in CLASSES A,B,C,D and E, see [Ki]. According to the above, every CLASS has Cases 1 and 2.

(i) Pencils with two base points $\{p_1, p_2\}$.

CLASS A. All the members are irreducible and of $\mathbb{C}^*$-type but L_1 consisting of two irreducible curves S_1 and S_2 both of which of $\mathbb{C}$-type such that they intersect transversally at one point in $\mathbb{P}^2 \setminus \{p_1, p_2\}$. Let t_i be the irreducible homogeneous polynomial defining S_i and let t_v be the irreducible homogeneous polynomial defining L_v.

A1. In case 1, the curve D has components S_1 and S_2. T. Kizuka showed that the rational functiong $= t_2^{\alpha_2}/t_1^{\alpha_1}$ defines a pencil of type $(0,1)$ which has the components of D as irreducible members ($\alpha_2 \deg(t_1) = \deg(t_2)\alpha_1$).

A2. In case 2, the components of D are L_v and either S_1 or S_2. Then T. Kizuka showed, p. 170, that for $i = 1, 2$ the rational functions $g_{v,i} = t_v/t_i^{\beta_i}$ in $\mathbb{P}^2$ ($\beta_i = \deg(t_v)$) define pencils of type $(0,1)$ which have the components of D as irreducible members.

CLASS B. All the members are irreducible and of $\mathbb{C}^*$-type but L_1 consisting of two irreducible curves S_1 and S_2 disjoint in $\mathbb{P}^2 \setminus \{p_1, p_2\}$ and one of them, say S_1, is of $\mathbb{C}$-type and the other is of $\mathbb{C}^*$-type.

B1. As before, D has components S_1 and S_2 the rational function $g = t_2^{\alpha_2}/t_1^{\alpha_1}$ defines a pencil of type $(0,1)$ which has the components of D as irreducible members ($\alpha_2 \deg(t_1) = \deg(t_2)\alpha_1$).

B2. D has components L_v and S_1. The rational function $g_v = t_v/t_1^{\beta_1}$ on $\mathbb{P}^2$ defines a pencil of type $(0,1)$ which has the components of D as members.

(ii) Pencils with one base point $\{p_1\}$.

CLASS C. All the members are irreducible and of $\mathbb{C}^*$-type but two members L_1 and L_2. The level curve L_1 consists of two irreducible curves S_1 and S_2 disjoint in $\mathbb{P}^2 \setminus \{p_1\}$ and one of them, say S_1, is of $\mathbb{C}$-type and the other one is of $\mathbb{C}^*$-type. The level curve L_2 is irreducible and of $\mathbb{C}$-type.

C1. The Euler-Poincaré characteristic $\chi(\mathbb{P}^2 \setminus D) = 1$ so D is not a bad divisor.

C2. The irreducible components of D are S_1 and L_2 (i.e. L_2 must be L_v). T. Kizuka showed (p. 167) that the rational function $R = t_v^{\beta_v}/t_1^{\beta_1}$ on $\mathbb{P}^2$ is of type $(0,1)$.

CLASS D. All the members but L_1 and L_2 are irreducible and of $\mathbb{C}^*$-type. The level curve L_1 consists of two irreducible curves S_1 and S_2 both of which of $\mathbb{C}$-type such that they intersect transversally at one point in $\mathbb{P}^2 \setminus \{p_1\}$. The level curve L_2 is irreducible and of $\mathbb{C}$-type.

D1. As above, the Euler-Poincaré characteristic $\chi(\mathbb{P}^2 \setminus D) = 1$ and D is not a bad divisor.

D2. We have two possible configurations for D. The components of D are L_2 and either S_1 or S_2. T. Kizuka showed, see p. 167 and the proof of his Proposition 3, that the rational functions $R_i = t_i^{\beta_i}/t_v^{\beta_v}$ on $\mathbb{P}^2$ ($\beta_i \deg(t_i) = \beta_v \deg(t_v)$) define pencils of type $(0,1)$ which have the components of D as irreducible fibres.

CLASS E. All the level curves are irreducible and of $\mathbb{C}^*$-type but one level curve L_1 consisting of two irreducible curves S_1 and S_2 disjoint in $\mathbb{P}^2 \setminus \{p_1\}$ both of which of $\mathbb{C}$-type.

E1. The components of D are S_1 and S_2 and then the rational functions $R = t_1^{\beta_1}/t_2^{\beta_2}$ in $\mathbb{P}^2$ ($\beta_1 \deg(t_2) = \beta_2 \deg(t_1)$) defines a pencil of type $(0,1)$ which has the components of D as irreducible fibres.
E2. The Euler-Poincaré characteristic $\chi(\mathbb{P}^2 \setminus D) = 1$ and D is not a bad divisor.

It turns out that in all the previous cases, the components of D can be put in a pencil of type $(0,1)$ for which the condition $(\bullet\bullet)$ is not possible.

The remaining case, i.e. when a component of the curve D is sent to a section in Σ, results in case (B) and the pencil Λ is of type $(0,2)$ because each base point of the pencil gives at least one section in Σ. Since the ζ-invariant of X_1 and Σ do not coincide then at least one of the blow-ups in $\pi_2 : X_2 \to X_1$ or $\pi_3 : X_2 \to \mathbb{F}_n$ is as in (2.22)(ii) and we are as in $(\bullet)$ (i) or (ii). Note that case (ii) is not possible because the pencil is of type $(0,2)$. Hence D either is not a bad divisor or does not verify $\rho(D) \neq 0$. This proves the claims of the Theorem. $\square$

Example 2.26. As we have seen in the previous example there are curves matching the hypothesis of the theorem: two smooth conics with only one intersection point.

§3.- Monodromy conjecture for SIS

Theorem 3.1. *Let D be a bad divisor of degree d on $\mathbb{P}^2$ associated with a $(0,1)$-pencil of rational curves. Let us suppose that at least one generic fibre (resp. two) is contained in D if the pencil has two exceptional fibres (resp. one). If $\rho(D) \neq 0$ then D verifies that* $\exp(2i\pi(-3/d))$ *is an eigenvalue of the complex monodromy of its only singular point.*

In particular, no counter-example exists for the Monodromy Conjecture for SIS singularities.

As we have already shown if $\rho(D) \neq 0$ then its components are in a pencil of type $(0,1)$. This pencil is defined by a rational function R_l on $\mathbb{P}^2$. Pencils of type $(0,1)$ have at most two special members, we denote them by $\{P_l = 0\}$ and $\{Q_l = 0\}$, see Appendix.

Then the above theorem is equivalent to the following one.

Theorem 3.2. *Let D be a bad divisor whose support is the curve $C_1 \cup C_2$, where (1) C_1 is the union of any number of different generic members $\{R_l = \mu_i\}$ of the pencil defined by R_l and (2) C_2 is one of the curves $\{P_l = 0\}$,$\{Q_l = 0\}$, $\{P_lQ_l = 0\}$ or $\{R_l = \mu\}$,$\mu \neq \mu_i$. Let d be the degree of D. If $\rho(D) \neq 0$ then* $\exp(2i\pi(-3/d))$ *is a root of the Alexander polynomial of the germ of D at its singular point.*

The Alexander polynomial of the complex monodromy of a germ of a plane curve singularity is equal to the Alexander polynomial of its splice diagram $\mathcal{D}$. Let l_v be the multiplicity of the vertex v and δ_v its valence, then by [EN, p.96], one knows that the Alexander polynomial of a diagram $\mathcal{D}$ is

$$\Delta_{\mathcal{D}}(t) = (t-1)\prod_{v}(t^{l_v} - 1)^{\delta_v - 2}$$

the product being taken over all the vertices of the diagram.

Remark 3.3. Let v be a vertex of valence 1 connected to a vertex v' of valence greater than or equal to 3, then

$$\frac{t^{l_{v'}}-1}{t^{l_v}-1}$$

is a polynomial, because l_v divides $l_{v'}$.

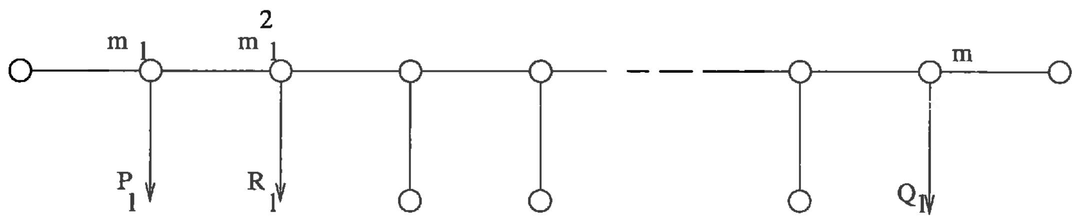
Proof of the Theorem. We have four different cases to consider depending on the irreducible components of the curve C_2.

(1) The irreducible component of C_2 is $\{P_l = 0\}$.
(2) The irreducible component of C_2 is $\{Q_l = 0\}$.
(3) The irreducible component of C_2 is $\{R_l = \mu\}$.
(4) The irreducible components of C_2 are $\{P_l = 0\}$ and $\{Q_l = 0\}$.

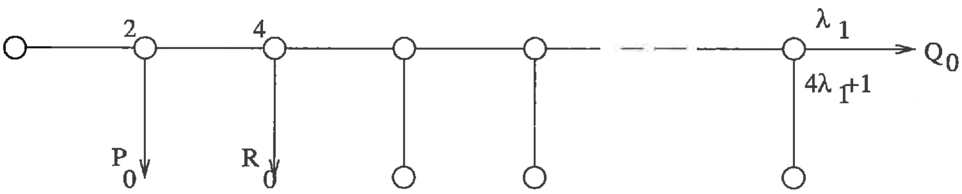
We use the classification of Kashiwara of pencils of type $(0,1)$ and divide the proof of the theorem in several steps.

Rational pencils of type $(0,1)$ belonging to $\mathcal{F}_{II}$

We begin studying the set $\mathcal{F}_{II}$ of pencils of type $(0,1)$. From the Appendix, in the case $\mathcal{F}_{II}$ one has the splice diagram **1**

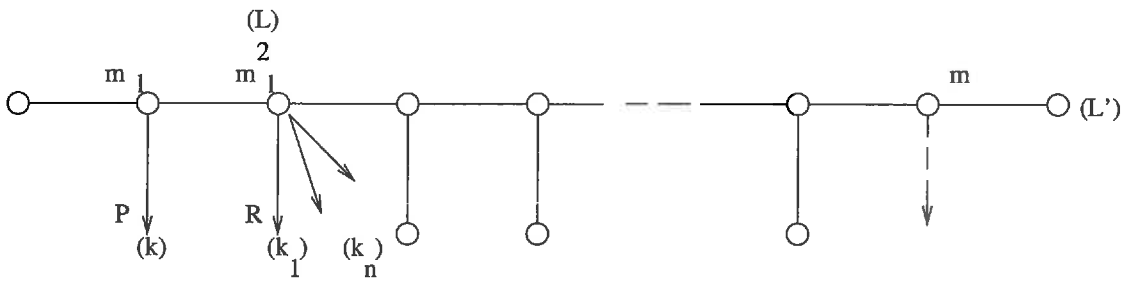


where $m \in \{m_{l+1}, m_{l-1}\}$ and the splice diagram **2**



For each one of the above types of splice diagrams we have four different cases to consider, depending on the irreducible components of the curve C_2.

Case 1: The irreducible component of C_2 is $\{P_l = 0\}$. The splice diagram **1** is



Because of the preceding remark there exists a polynomial $H(t) \in \mathbb{C}[t]$ such that

$$\Delta(t) = H(t)\frac{(t^L-1)^n}{t^{L'}-1},$$

where L is the multiplicity of the vertex at which the generic fibres $\{R_l = \mu_i\}$ separate, L' is the multiplicity at the right end of the diagram and n is the number of irreducible curves $\{R_l = \mu_i\}$ in C_1.

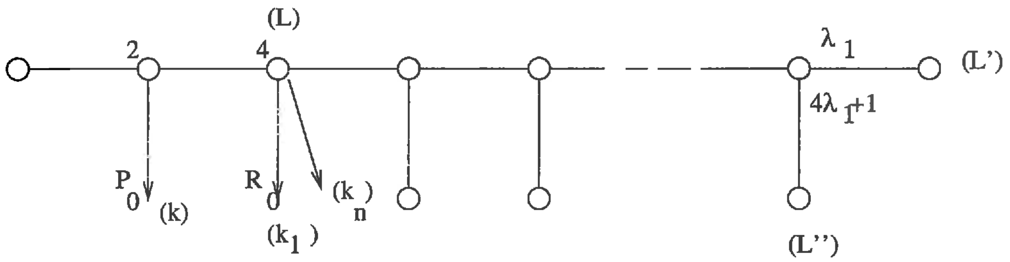
We have the equalities $d = (k_1 + \cdots + k_n)\deg(R_l) + k\deg(P_l) = \deg(P_l)((k_1 + \cdots + k_n)m_l + k)$. To compute L we use the fact that the intersection multiplicity of two generic curves of the pencil is $m_l^2 \deg(P_l)^2$ and the intersection multiplicity of a generic curve of the pencil with $\{P_l = 0\}$ is $m_l \deg(P_l)^2$. Then $L = (k_1 + \cdots + k_n)m_l^2 \deg(P_l)^2 + km_l \deg(P_l)^2$. It turns out that $L = dm_l \deg(P_l)$. Now $L' = cm_l((k_1 + \cdots + k_n)m_l + k)$ and c can be computed using $m_l \deg(P_l) = m_l mc$. Then $L' = m_l((k_1 + \cdots + k_n)m_l + k)\deg(P_l)/m$.

Assume that $d/3$ divides L', then

$$m_l((k_1 + \cdots + k_n)m_l + k)\deg(P_l)/m = h((k_1 + \cdots + k_n)m_l + k)\deg(P_l)/3,$$

that is, $3m_l = hm$. But $\gcd(m_l, m) = 1$ and $\gcd(m, 3) = 1$. This is impossible and it proves the theorem in this case.

For the splice diagram **2** one has

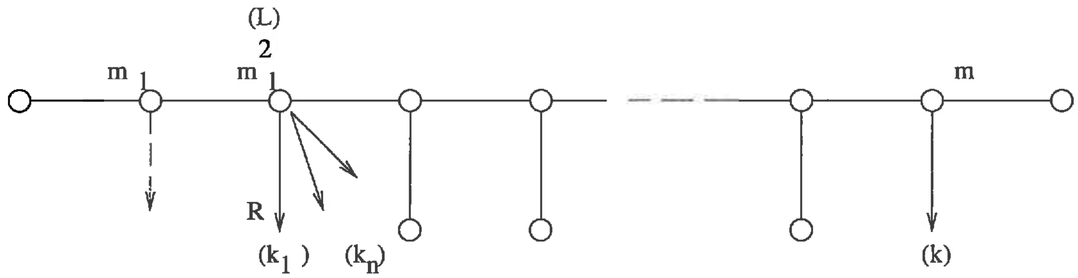


The first part of the computation is analogous. We still have $L = 2d\deg(P_0)$. Now if we compute L', we get $L' = (4\lambda_1 + 1)c(2k + 4(k_1 + \cdots + k_n))$, and computing the intersection number between Q_0 and P_0 one has $2(4\lambda_1 + 1)c = 2\deg(P_0)$, then $L' = 2d$ and we can't conclude. But we also have

$$\Delta(t) = H(t)\frac{(t^L - 1)^n}{t^{L''} - 1}$$

and $L'' = \lambda_1 c(2k + 4(k_1 + \cdots + k_n)) = 2\lambda_1 d/(4\lambda_1 + 1)$. Finally if we assume that $d/3$ divides L'' we get an equation $h(4\lambda_1 + 1) = 6\lambda_1$ which is impossible.

Case 2: The irreducible component of C_2 is $\{Q_l = 0\}$. The splice diagram **1** is



In this case we have

$$\Delta(t) = H(t)\frac{(t^L - 1)^n}{t^{L/m_l^2} - 1}$$

where again $H(t) \in \mathbb{C}[t]$ is a polynomial.

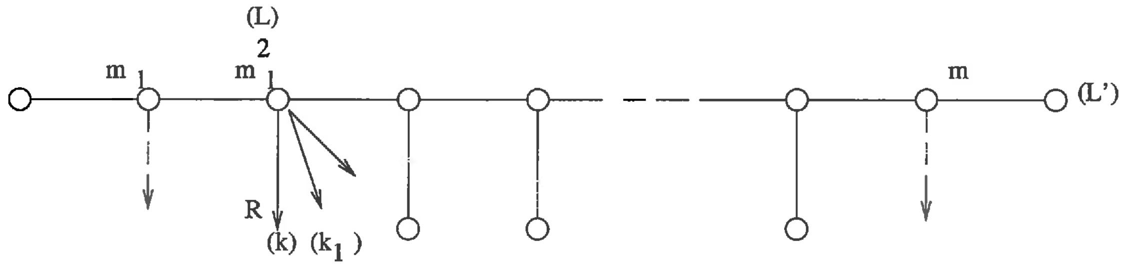
We have the equalities $d = (k_1 + \cdots + k_n)\deg(P_l)m_l + km_l = m_l((k_1 + \cdots + k_n)\deg(P_l) + k)$, $L = (k_1 + \cdots + k_n)\deg(P_l)^2 m_l^2 + k\deg(P_l)m_l^2$.

If $d/3$ divides L/m_l^2 then

$$\deg(P_l)((k_1 + \cdots + k_n)\deg(P_l) + k) = hm_l((k_1 + \cdots + k_n)\deg(P_l) + k)/3,$$

that is, $3\deg(P_l) = hm_l$. Note again that $\gcd(\deg(P_l), m_l) = 1$ and $\gcd(m_l, 3) = 1$. The computation is analogous for the splice diagram **2**.

Case 3: The irreducible component of C_2 is $\{R_l = \mu\}$. For the splice diagram **1**



the Alexander polynomial is

$$\Delta(t) = H(t)\frac{(t^L - 1)^{n+1}}{(t^{L/m_l^2} - 1)(t^{L'} - 1)}.$$

Again we obtain the equalities $d = (k+k_1+\cdots+k_n)m_l\deg(P_l)$, $L = (k+k_1+\cdots+k_n)m_l^2\deg(P_l)^2$ and $L' = (k+k_1+\cdots+k_n)cm_l^2 = (k+k_1+\cdots+k_n)\deg(P_l)m_l^2/m_{l+1}$. Assume that $d/3$ divides L/m_l^2. Then

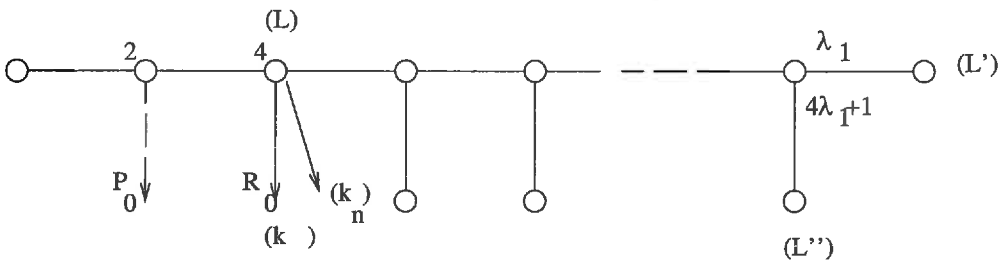
$$(k + k_1 + \cdots + k_n)\deg(P_l)^2 = h(k + k_1 + \cdots + k_n)m_l\deg(P_l)/3,$$

that is, $3\deg(P_l) = hm_l$ which is impossible. If $d/3$ divides L' then

$$(k + k_1 + \cdots + k_n)\deg(P_l)m_l^2/m_{l+1} = h(k + k_1 + \cdots + k_n)m_l\deg(P_l)/3,$$

which gives $3m_l = hm_{l+1}$ which is impossible. The theorem is proved in this case.

For the splice diagram **2**

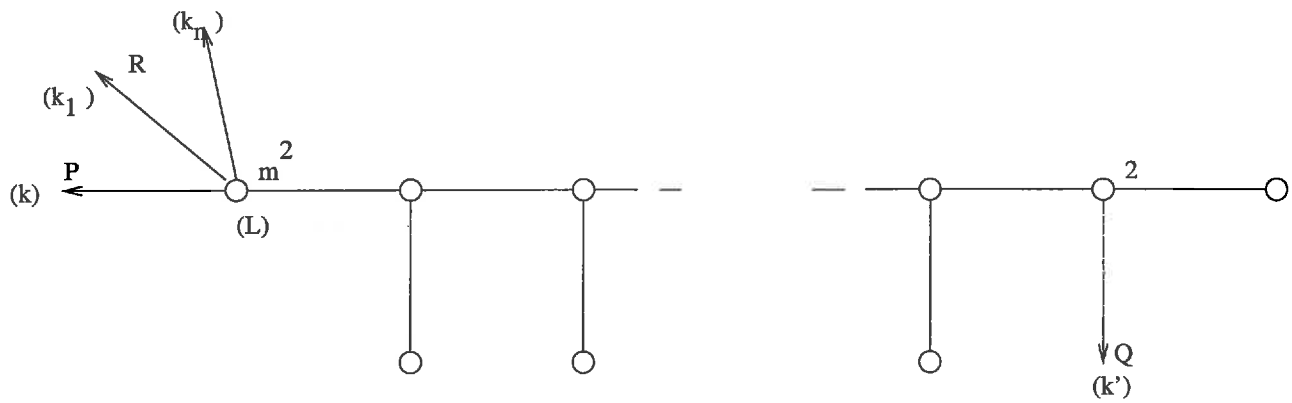


we can write the Alexander polynomial as

$$\Delta(t) = H(t)\frac{(t^L - 1)^{n+1}}{(t^{L/4} - 1)(t^{L''} - 1)}.$$

The computation of L is the same as before. The computation of L'' gives $L'' = 4\lambda_1 c(k + k_1 + \cdots + k_n)$. But one has $2(4\lambda_1 + 1)c = 2\deg(P_0)$. Then $L'' = 4\lambda_1(k + k_1 + \cdots + k_n)/(4\lambda_1 + 1)$. If $d/3$ divides L'', one has $3\lambda_1 = h(4\lambda_1 + 1)$ with $h \in \mathbb{N}$, which is impossible.

Case 4: The irreducible components of C_2 are $\{P_l = 0\}$ and $\{Q_l = 0\}$. For the splice diagram **1**



the Alexander polynomial of the local singularity is

$$\Delta(t) = H(t)(t^L - 1)^n$$

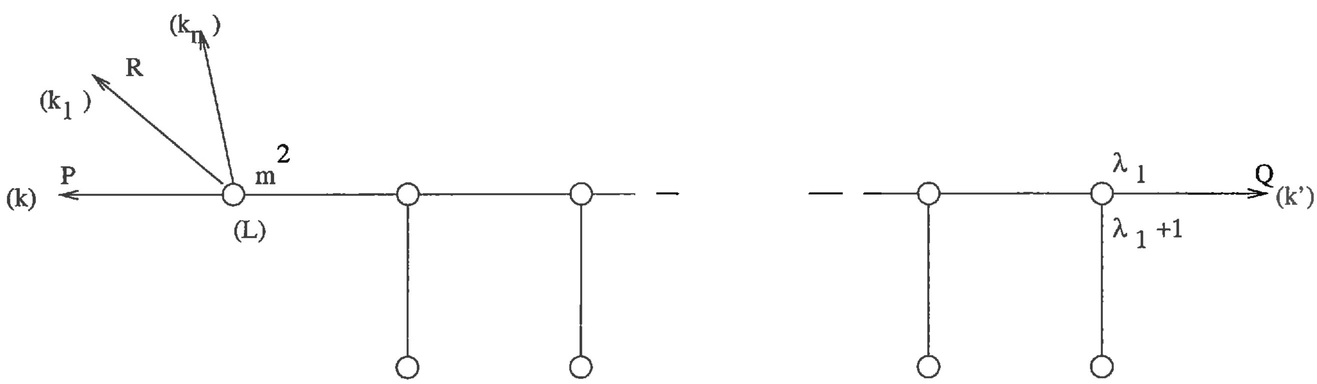
where $H \in \mathbb{C}[t]$ is a polynomial. One has the equalities

$$L = m_l^2 \deg(P_l)^2(k_1 + \cdots + k_n) + km_l \deg(P_l)^2 + k'm_l^2 \deg P_l$$

$$d = k\deg(P_l) + (k_1 + \cdots + k_n)m_l \deg(P_l) + k'm_l.$$

Then $L = m_l \deg(P_l)d$.

For the splice diagram **2**



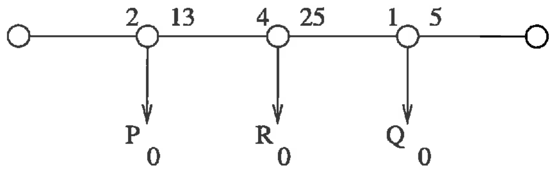
the computation is the same. Then we have finished the prove of the theorem if the corresponding rational function is in $\mathcal{F}_{II}$.

Example 3.4. Using the computation of the local topological Zeta function from the splice diagrams for curves, as explained in [ACLM], one can compute the value for $s_0 = -3/d$ in the examples we give in the Appendix and one sees that in fact $s_0 = -3/d$ is a pole of the local topological zeta function of the corresponding SIS singularity.

For example, for

$$Q_0 = y - x^2, P_0 = (y - x^2)^2 - 2xy^2(y - x^2) + y^5$$

with splice diagram



the local topological zeta function of the degree $d = 15$ curve $D = \{P_0 = 0\} \cup \{R_0 = \mu\}$ at the origin is

$$Z_{top,0}(D,s) = \frac{25}{29+150s} + \frac{2}{(29+150s)(15+76s)} + \frac{2}{15+76s} - \frac{1}{29+150s} - \frac{1}{15+76s}$$
$$+ \frac{1}{(29+150s)(1+s)} + \frac{1}{(15+76s)(1+s)}$$

and $Z_{top,0}(D, -3/d) + 2d/(d-3) = -51/4$. The Alexander polynomial of the curve D at its only singular point is

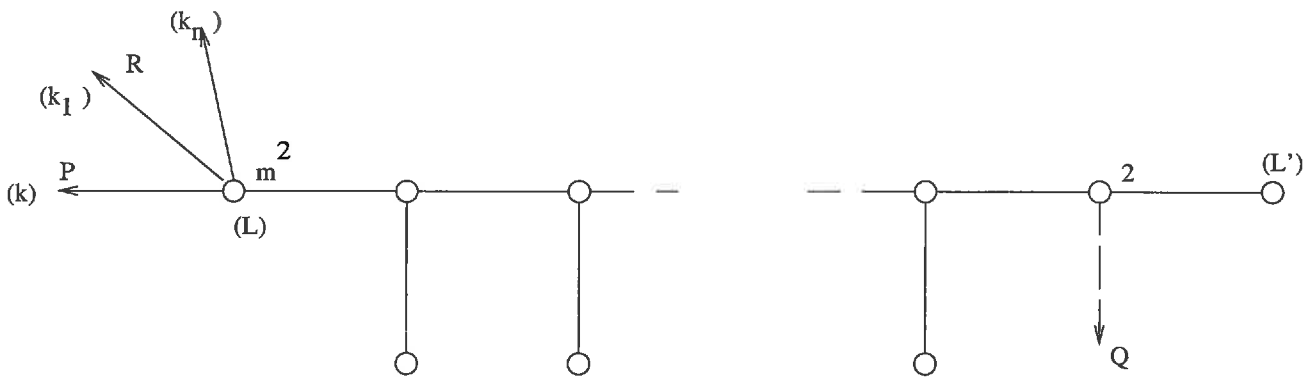
$$\Delta(t) = (t-1)\frac{(t^{150}-1)(t^{76}-1)}{(t^6-1)(t^{38}-1)}.$$

We will come back to this example in §5.

Rational pencils of type $(0,1)$ belonging to $\mathcal{F}_I$

We next compute the Alexander polynomial in case $\mathcal{F}_I$, see Appendix. For this purpose, families $I(0)$ and $I^+(N; \lambda_1, \ldots, \lambda_N)$ are considered the same.

(•) In **Case 1** and **Case 3** the splice digram is

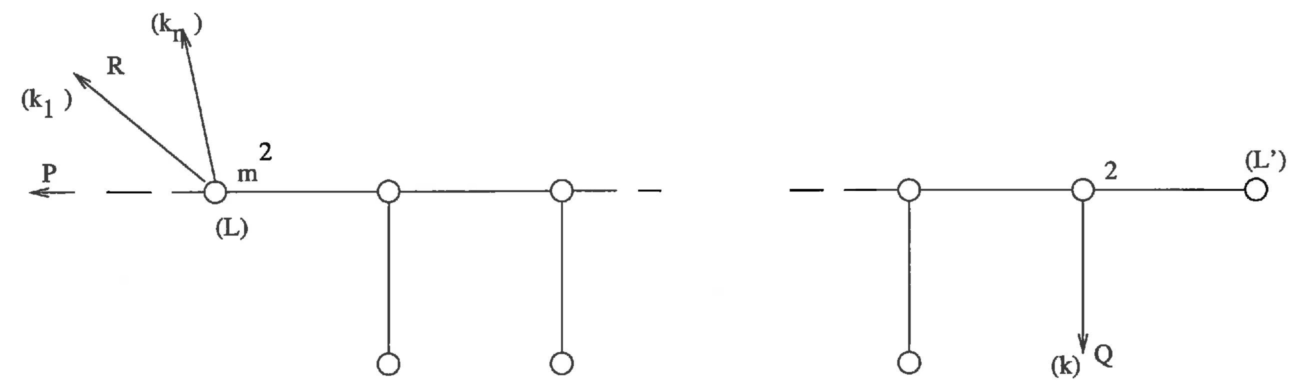


Its Alexander polynomial is

$$\Delta(t) = H(t)\frac{t^L - 1}{t^{L'} - 1},$$

where $H(t)$ is a polynomial. We have $L = md$ and $L' = d/2$. Then $\exp(-6i\pi/d)$ is a root of the Alexander polynomial.

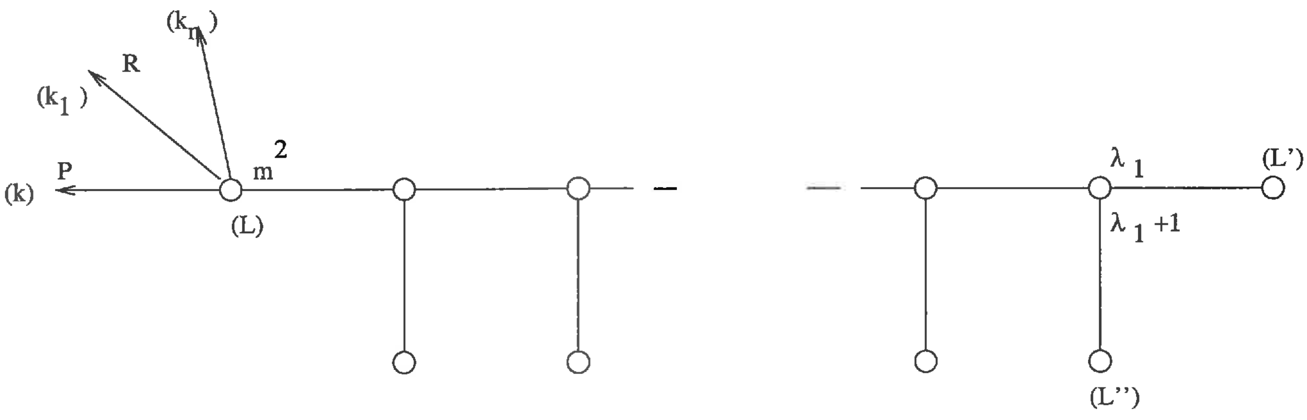
(•) For **Case 2** and **Case 4** the splice diagram is



and we have $\Delta(t) = H(t)(t^L - 1)$. Then $L = md$. It implies that the conjecture is verified in this case.

The remaining family in $\mathcal{F}_I$ is $I^-(N; \lambda_1, \dots, \lambda_N)$.

(•) For **Case 1** and **Case 3** the splice diagram is



We can write

$$\Delta(t) = H(t)\frac{(t^L - 1)^n}{t^{L'} - 1},$$

or

$$\Delta(t) = H_1(t)\frac{(t^L - 1)^n}{t^{L''} - 1}.$$

We still have $L = md$, but we have $L' = d$ and $L'' = d(\lambda_1/(\lambda_1 + 1))$. If $\lambda_1 = 2$ and we only have $n = 1$ component then $\exp(-6i\pi/d)$ is not a root of the Alexander polynomial.

In order to prove the Monodromy Conjecture in this particular case we have to prove that the bad divisor D verifies $\rho(D)=0$. The curve D is formed by (at least) two generic members of a pencil of type $I^-(N;2,\ldots,\lambda_N)$.

Looking at its resolution graph of the Appendix, one checks that in these cases the only possible contraction to $\mathbb{P}^1\times\mathbb{P}^1$ is blowing-down $\{Q=0\}$. The curve $\{Q=0\}$ is the only one special fibre of these pencils, it is also a multiple fibre. In any case, it is cheched that map π_2 is the identity (we resolve the pencil when we resolve the curve, since there are at least two generic fibres). The blowing-ups of type (ii) in (2.22) produce special fibres of the pencil. Then, only one such a blowin-up can take place. It is easily seen that the blowing-up giving $\{Q=0\}$ is produced in a component of valency one, and in this case the ζ-invariant does not change and the residue $\rho(D)$ is zero.

(•) For **Case 2** and **Case 4** the arguments of the proof are analogous and we leave the proof to the reader.

§4.- Monodromy conjecture for homogeneous polynomials

B. Rodrigues and W. Veys have proved the Monodromy Conjecture for any homogeneous polynomial $f_d\in\mathbb{C}[x_1,x_2,x_3]$ with $\chi(\mathbb{P}^2\setminus\{f_d=0\})\neq 0$, see [RV].

In fact in the proof of their Theorem 4.2 they show that for any homogeneous polynomial $f_d\in\mathbb{C}[x_1,x_2,x_3]$ of degree d and for any pole $s_0\neq -3/d$ of $Z_{top,0}(f,s)$, $\exp(2i\pi s_0)$ is an eigenvalue of the local monodromy of f_d at some complex point of the effective divisor $D=f_d^{-1}(0)$.

In this homogeneous case, one of the key points in their proof is the following equality, see [RV, (3.6)],

$$\begin{aligned} Z_{top,0}(f_d,s) &= \frac{1}{3+ds}z(D,s)+\\ &= \frac{1}{3+ds}\left(\chi(\mathbb{P}^2\setminus D)+\sum_{i=1}^{r}\frac{\chi(\check{D}_i)}{1+a_is}+\sum_{P\in\operatorname{Sing}(D_{red})} Z_{top,P}(D,s)\right), \end{aligned}$$

where $D=a_1D_1+\ldots+a_rD_r$ and $\check{D}_i:=D_i\setminus\operatorname{Sing}(D_{red})$.

We are interested in the remaining case $\chi(\mathbb{P}^2\setminus\{f_d=0\})=0$ and the pole candidate $s_0=-3/d$. If it is a pole of order greater than 1 then either $s_0=-3/d$ is a pole of $Z_{top,P}(D,s)$ (and then Monodromy Conjecture for curves implies that $\exp(2i\pi s_0)$ is a root of the local monodromy of f_d at some complex point of D) or $s_0=-3/d$ is the pole $-1/a_i$, for some a_i. In such a case, if $P\in D_i\cap\operatorname{Sing}(D)\neq\emptyset$ then the branches of D at P have multiplicity a_i and W. Veys showed that $Z_{top,P}(D,s)$ has $-1/a_i$ as a pole. Again, the Monodromy Conjecture for curves implies that $\exp(2i\pi s_0)$ is a root of the local monodromy of f_d at some complex point of D.

The discussion above means that $s_0=-3/d$ is a simple pole of $Z_{top,0}(f_d,s)$ if and only if D is a bad divisor on $\mathbb{P}^2$ and $z(D,-3/d)=\rho(D)\neq 0,\infty$. However, after Theorem (3.2) $\exp(2i\pi(-3/d))$ is eigenvalue of the monodromy of its only singular point. Then the Monodromy Conjecture is also proved in this remaining case. Then the results of B. Rodrigues and W. Veys and these facts show the following theorem.

Theorem 4.1. *For any homogeneous polynomial $f_d\in\mathbb{C}[x_1,x_2,x_3]$ the Monodromy Conjecture holds.*

§5.- Rational arrangements of plane curves

The results of this paper can be applied to prove the non-existence of arrangements of rational curves in $\mathbb{P}^2$. Therefore, we restrict ourselves to arrangements whose complement in the plane has Euler-Poincaré characteristic 0.

Let $D = \cup C_i$ be an arrangement of reduced rational curves. The dual graph of the minimal embedded resolution of D is determined by the following data:

(1) The degrees d_i of the irreducible components of D,
(2) The list of the topological types of the local singularities of D,
(3) The global component of D which contains each branch Γ of D at a singular point.

We call these data *the combinatorial type* of the curve D in $\mathbb{P}^2$. We also call the data in (2) together with the total degree d of D *the local combinatorial data* of D in $\mathbb{P}^2$.

For any $(V_D, 0) \subset (\mathbb{C}^3, 0)$ SIS singularity whose tangent cone is D, the local topological zeta function $Z_{top,0}(V_D, s)$ and the eigenvalues of the complex algebraic monodromy of $(V_D, 0)$ are determined by the local combinatorial data of D. Hence the Monodromy Conjecture gives necessary conditions on the local combinatorial data of D for D to exist.

We have developed a program with Maple (available upon request) such that calculates the local embedded resolution of the singularities of a curve D, the local topological zeta function $Z_{top,0}(V_D, s)$ and the eigenvalues of the complex algebraic monodromy of $(V_D, 0)$ from the local combinatorial data of D. Thus, given local combinatorial data of D the above necessary conditions can be easily verified. Let us present some few examples.

Example 1. Let D consist of two conics which only meet at one point and a line which is tangent to each conic in different points. Using elementary properties of pencils of conics it is easy to see that D does not exist. In this case, the topological zeta function of $(V_D, 0)$ would be

$$Z_{top,0}(V_D, s) = \frac{(680s^3 + 1839s^2 + 1582s + 435)}{(1+s)(3+5s)(29+48s)(5+8s)},$$

and the characteristic polynomial of the monodromy of $(V_D, 0)$ would turn out to be

$$\Delta(t) = \frac{(t^6-1)^3(t^{48}-1)(t^{24}-1)^2}{(t-1)(t^{12}-1)^3}.$$

Thus $\rho(D) = -3/5$, $s_0 = -3/5$ would be a pole whereas $\exp(2i\pi(-3/5))$ would not be a root of $\Delta(t)$.

Example 2. Consider a rational curve C of degree six with only one singular point P which is a simple singularity. Then P can be either an $\mathbb{A}_{19}$ or $\mathbb{A}_{20}$ singularity (in Arnold classification). It is known that the $\mathbb{A}_{19}$ case exits, e.g. see [P]. The double covering of $\mathbb{P}^2$ ramified along C is a K3-surface. Using K3-surfaces theory one shows that the $\mathbb{A}_{20}$ case is not possible.

Let $D = C \cup C_2$ be the curve whose components are the sextic C with the $\mathbb{A}_{20}$ singularity at P and C_2 the unique conic going through the first five infinitely near point of C at P. Using Cremona transformations one can show that this conic in fact has to go through the sixth infinitely near point of C at P. Hence the conic only meets C at its singular point. The topological zeta function and the characteristic polynomial of the monodromy of $(V_D, 0)$ would be

$$Z_{top,0}(V_D, s) = \frac{49248s^3 + 267332s^2 + 183554s + 33855}{(1+s)(3+8s)(61+162s)(185+486s)},$$

$$\Delta_V(t) = \frac{(t^9-1)(t^{162}-1)(t^{486}-1)}{(t-1)(t^{27}-1)(t^{243}-1)}.$$

Hence D does not exist because the residue at the value $s_0 = -3/8$ is different from zero and $\exp(2i\pi(-3/8))$ is not an eigenvalue of the complex monodromy of the SIS singularity.

Example 3. We present several examples which show the power of the Monodromy Conjecture. Consider C a rational curve of degree 10 with only one singular point P whose multiplicity sequence is $[4,4,4,4,4,4,1,1,1,1] = [4_6]$, (this curve exists and it appears in the classification of H. Kashiwara, see Appendix).

Let $D = C \cup C_2$ be the curve whose components are C and C_2 the unique conic going through the first five infinitely near point of C at P. In this case the residue $\rho(D) = -3$ and $\exp(2i\pi(-3/12))$ is a root of the characteristic polynomial of the monodromy of the SIS singularity $(V_D, 0)$. The computations are the following:

$$Z_{top,0}(V_D, s) = \frac{127140s^3 + 559954s^2 + 258079s + 31509}{(1+s)(1+4s)(81+325s)(389+1560s)},$$

$$\Delta_V(t) = \frac{(t^{13}-1)(t^{325}-1)(t^{1560}-1)}{(t-1)(t^{65}-1)(t^{390}-1)}.$$

Now we give a list of several possible cuspidal rational curves of degree 10 which might exist. We give each singularity as sequence of multiplicities.

$$\begin{array}{cccc}
[4_5, 2_6], & [4_5, 2_5] + 1\mathbb{A}_2, & [4_5, 2_4] + 2\mathbb{A}_2, & [4_5, 2_4] + 1\mathbb{A}_4, \\
{[4_5, 2_3] + 3\mathbb{A}_2}, & [4_5, 2_3] + 3\mathbb{A}_2, & [4_5, 2_3] + 1\mathbb{A}_2 + 1\mathbb{A}_4, & [4_5, 2_3] + 1\mathbb{A}_6, \\
{[4_5, 2_3] + 1\mathbb{E}_6}, & [4_5, 2_2] + 4\mathbb{A}_2, & [4_5, 2_2] + 2\mathbb{A}_2 + 1\mathbb{A}_4, & [4_5, 2_2] + 2\mathbb{A}_4, \\
{[4_5, 2_2] + 1\mathbb{A}_2 + 1\mathbb{A}_6}, & [4_5, 3] + 3\mathbb{A}_2, & [4_5, 2_2] + 1\mathbb{A}_2 + 1\mathbb{E}_6, & [4_5, 3] + 1\mathbb{A}_2 + 1\mathbb{A}_4, \\
{[4_5, 3] + 1\mathbb{A}_6}, & [4_5, 3] + 1\mathbb{E}_6, & &
\end{array}$$

If one considers the corresponding curve D as the union of the curve of degree 10 with these singularities and the conic as before, then all of them give counter-examples to the Monodromy Conjecture. Hence they do not exist.

Appendix

Since we are concerned with rational functions on $\mathbb{P}^2$ with rational fibres we review some facts related to this theory. All asertions in this Appendix are explained in much greater details in [K] and [Ki].

Let Λ be a pencil on the projective plane $\mathbb{P}^2$ defined by a rational function R. Let $\{p_1, \ldots, p_s\}$ be the set of base points of the pencil. Then R defines a well-defined map $R : \mathbb{P}^2 \setminus \{p_1, \ldots, p_s\} \to \mathbb{P}^1$.

We say that Λ (or R) is of type (g, n) if the irreducible components of a generic fibre of the map R are open Riemann surfaces of genus g with n points at the boundary. A pencil (or a rational function) of type $(0, n)$ is called *rational.* If moreover it is of type $(0,1)$ or $(0,2)$ then we say that Λ (or R) is *of special type.*

H. Kashiwara in [K] (resp. T. Kizuka in [Ki]) classified the pencils of type $(0,1)$ (resp. pencils of type $(0,2)$). Pencils of type $(0,1)$ (resp. $(0,2)$) are called $\mathbb{C}$-type pencils (resp. $\mathbb{C}^*$-type pencils).

The pencils of special type on $\mathbb{P}^2$ are classified in two classes: (1) class $\mathcal{F}_I$, all pencils of special type such that there exists a fibre which is a projective line and (2) class $\mathcal{F}_{II}$, the complement of the class $\mathcal{F}_I$.

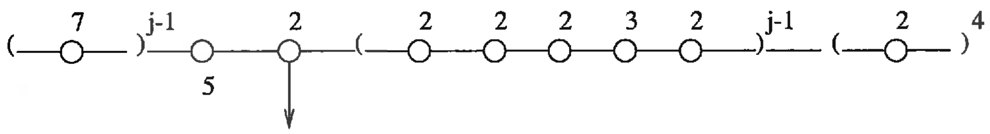
In this Appendix we collect the graphs for $\mathbb{C}$-type pencils obtained by H. Kashiwara. We translate her graphs into splice diagrams of D. Eisenbud and W. Neumann [EN].

A $\mathbb{C}$-type rational function on $\mathbb{P}^2$ has only one indeterminacy point. Any fibre of the function is irreducible and all fibres but at most two are reduced. Any fibre with its reduced structure is non-singular away from its indeterminacy point.

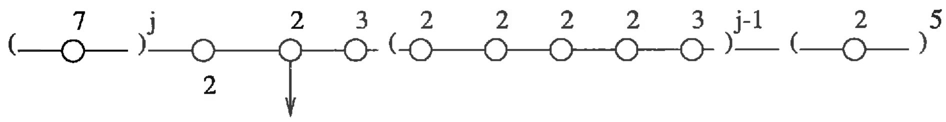
If R is a rational function on $\mathbb{P}^2$ of $\mathbb{C}^*$-type then R has at most two indeterminacy points, only one level curve with two irreducible components and the other level curves of R are irreducible. All the fibres of R are of either $\mathbb{C}$- or $\mathbb{C}^*$-type. The most important fact for us is that some fibres of any $\mathbb{C}^*$-type pencil can be put in a pencil of type $(0,1)$. T. Kizuka classified the pencils of type $(0,2)$ in [Ki] and there is a summary inside the proof of Theorem 2.21.

We next recall the H. Kashiwara's results that we need. They are expressed in terms of the resolution graphs, we will give them also in terms of splice diagrams which are more convenient for us.

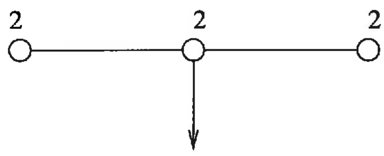
Let $\overrightarrow{G_l}$ denote the graph



if $l = 2j-1, j \geq 1$,



if $l = 2j, j \geq 1$, and

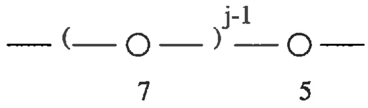


if $l = 0$.

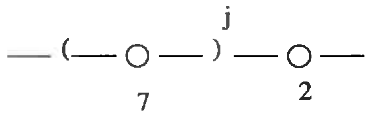
Lemma 6.1. *Define* m_l*,* $l \in \mathbb{N}$ *by*

$$m_0 = 2, \quad m_1 = 5, \quad m_l = 3m_{l-1} - m_{l-2}.$$

Then, if $l = 2j-1$*,* $j \geq 1$*,* m_l *is the determinant of the graph*



If $l = 2j$, $j \geq 1$, m_l is the determinant of the graph



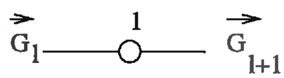
Proof. Denote by n_l the determinant of the above graphs. The computation of these determinants is due to N. Duchon and is explained in [EN p. 153]. For $j = 1$, it is easy to compute that $n_1 = 5$ and $n_2 = 13$. Then one has the recurrence formula $n_l = 7n_{l-2} - n_{l-4}$. The lemma is proved.

H. Kashiwara decomposes $\mathcal{F}_{II}$ in different sets that we will study one by one. For $l \in \mathbb{N}$, let $R_l \in \mathcal{F}_{II}$ be a rational function given by

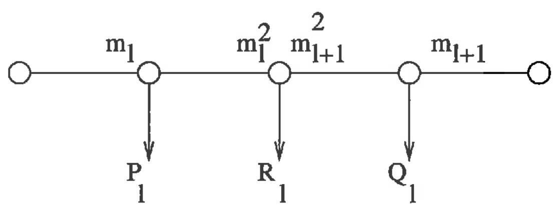
$$R_l = \frac{P_l^{m_l}}{Q_l^{degP_l}}.$$

Let Σ be the resolution graph of the pencil R_l, $\hat{S}_0$ and $\hat{S}_\infty$ the strict transforms of $\{P_l = 0\}$ and $\{Q_l = 0\}$. The graph $\Sigma \cup \hat{S}_0 \cup \hat{S}_\infty$ is given in Theorem 6.1, p.536.

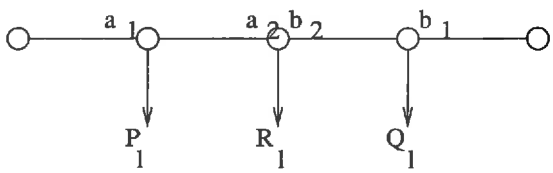
1) Case 1: II (l), $l \geq 0$. For $l \geq 0$, the graph $\Sigma \cup \hat{S}_0 \cup \hat{S}_\infty$ is:



Lemma 6.2. *The splice diagram of the germ $\{P_l = 0\} \cup \{R_l = \mu\} \cup \{Q_l = 0\}$ at its singular point is*



Proof. The strict transform of $\{R_l = \mu\}$ is transversal to the unique component, in the resolution graph, with self-intersection -1. Then using the relation between resolution graphs and splice diagrams as explained in [EN], one sees that the splice diagram is

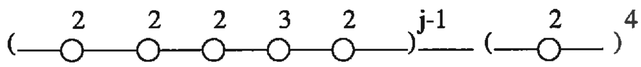


By (6.1), we know that $a_1 = m_l$. We also know that $\deg(P_l) = m_{l+1}$. Since the curves $\{P_l = 0\}$ and $\{Q_l = 0\}$ only meet at $(0, 0)$ then $a_1 b_1 = m_l m_{l+1}$. Then $b_1 =$

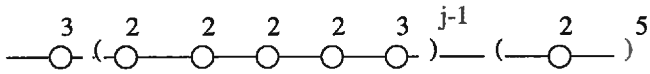
m_{l+1}. Moreover as $\{R_l = 0\}$ is a generic member of the pencil, $a_2 b_1 m_{l+1} = a_1 b_2 m_l$. As $\gcd(m_l, m_{l+1}) = 1$ we deduce that $a_2 = m_l^2$ and $b_2 = m_{l+1}^2$. Then the lemma is proved.

Note that, along the lines, we have also proved the following

Lemma 6.3. *If $l = 2j-1, j \geq 1$, the integer m_l is the determinant of the graph*



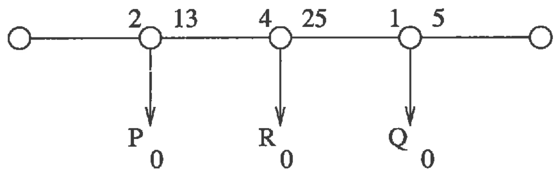
and if $l = 2j, j \geq 1$, m_l is the determinant of the graph o



The simplest example of rational functions in II (l) is given by

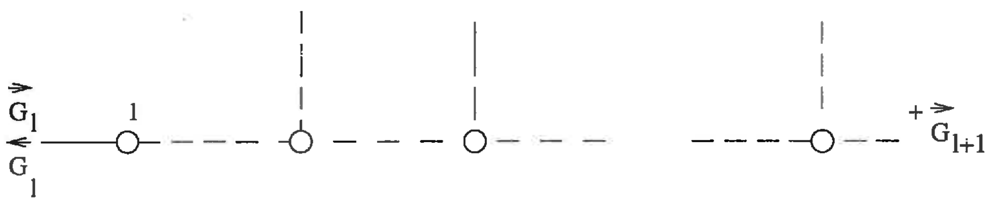
$$Q_0 = y - x^2, P_0 = (y-x^2)^2 - 2xy^2(y-x^2) + y^5$$

with splice diagram



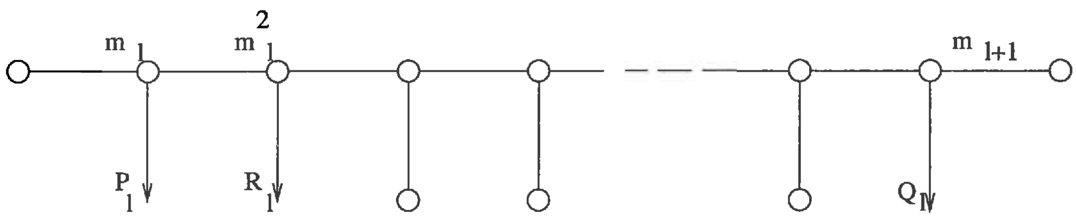
2) Case 2: $II^+(l, N; \lambda_1, \ldots, \lambda_N)$, $l \geq 0$.

We denote by $\overleftarrow{G_l}$ the graph $\overrightarrow{G_l}$ when is read from right to left and by $^+\overrightarrow{G_l}$ the graph $\overrightarrow{G_l}$ whose weight at the left extremity is increased by one. The graph is the following

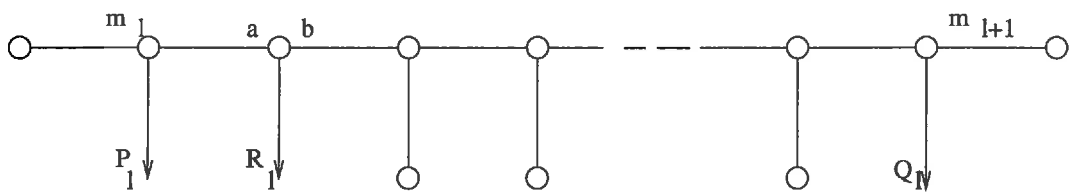


where $\lambda_1, \ldots, \lambda_N$ belong to $\mathbb{Z}_{\geq 0}$ if $l \geq 1$ and to $\mathbb{Z}_{>0}$ if $l = 0$.

Lemma 6.4. *The splice diagram of the germ $\{P_l = 0\} \cup \{R_l = \mu\} \cup \{Q_l = 0\}$ is*



Proof. ¿From (6.1) and (6.3), we know that the splice diagram will be of the form



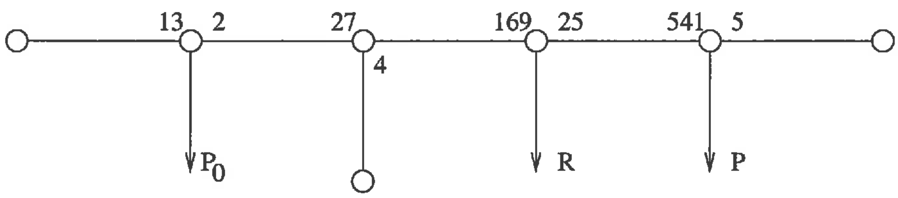
Since $\{R_l = 0\}$ is a generic fibre of the pencil, we have $bm_l^2 = am_{l+1}\deg(P_l)$. Since $\gcd(m_l, m_{l+1}\deg(P_l)) = 1$, and $\gcd(a,b) = 1$, we have $a = m_l^2$.

An example of such a rational function is given by the following formulae. Let

$$\phi = xy - x^3 - y^3,\ P_{-1} = y - x^2,\ P_1 = (\phi^5 + P_0^3)/P_{-1},$$

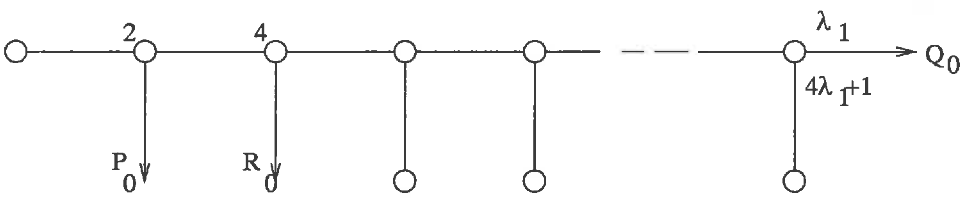
$$F = \phi P_0^2 + aP_1,\ P = (F^5 + P_0^{13})/P_1.$$

Then P, which is of degree 52, is in a $(0,1)$-pencil with P_0. Its splice diagram is



3) Case 3: $II^-(0, N; \lambda_1, \ldots, \lambda_N)$. The resolution graph is

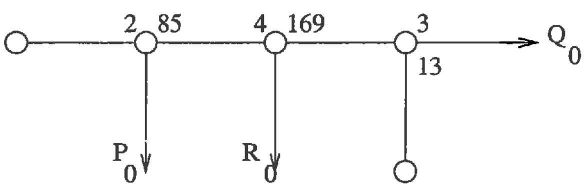
Lemma 6.5. *The splice diagram of the germ* $\{P_l = 0\} \cup \{R_l = \mu\} \cup \{Q_l = 0\}$ *is*



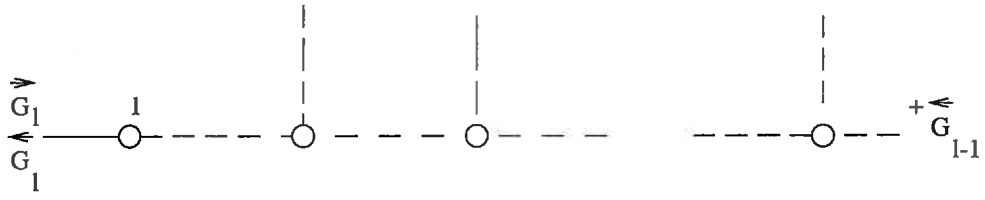
The proof uses the same argument as above.
One example of such a rational function is

$$F_1 = \phi P_{-1}^2 + a_3 y P_{-1}^3 + a_2 y^3 P_{-1}^2 + a_1 y^5 P_{-1} + a_0 y^7, P = (P_{-1}^7 + F_1^2)/y$$

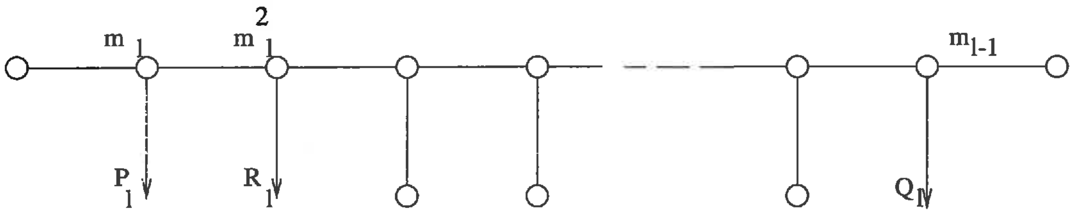
Then P is a polynomial of degree 13 in a pencil $(0,1)$ with P_{-1}. Its splice diagram is



4) Case 4: $II^{-}(l, N; \lambda_1, \ldots, \lambda_N), l \geq 1$. The resolution graph is



Lemma 6.6. *The splice diagram of the germ* $\{P_l = 0\} \cup \{R_l = \mu\} \cup \{Q_l = 0\}$ *is*

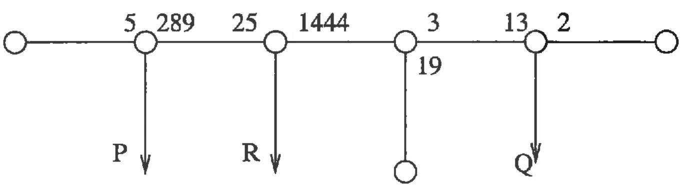


The proof uses the same argument as above.

One example of such rational function is

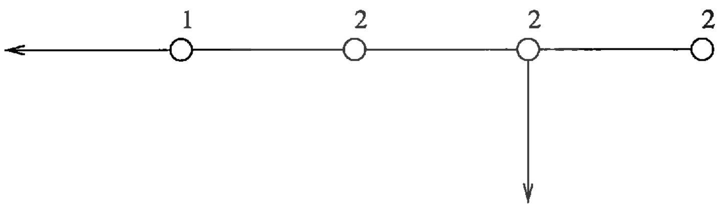
$$F_2 = \phi P_0 + aQ_0^4, P = (P_0^8 + F_2^5)/Q_0$$

Then P is a polynomial of degree 38, which is in a pencil $(0,1)$ with P_0. The splice diagram is

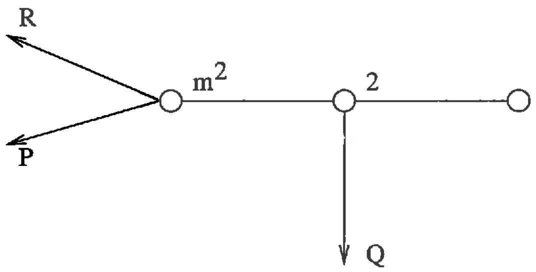


We next recall the graphs that H. Kashiwara gives for the case $\mathcal{F}_I$. There are 3 cases.

1) Case 1: I (0)

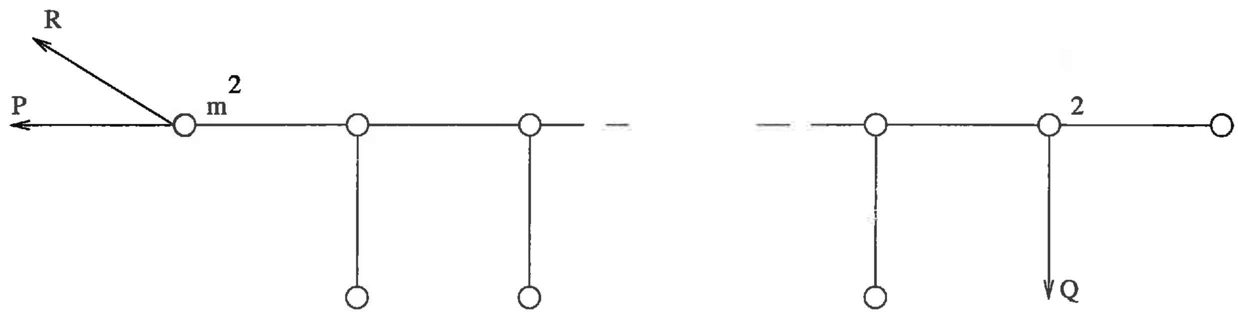


Then we have the following splice diagram

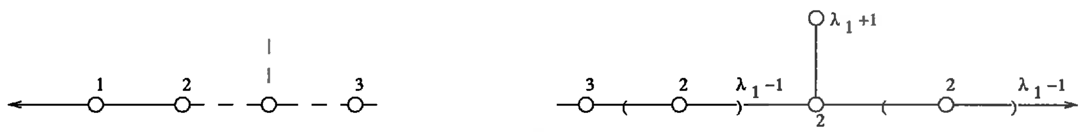


Here we find the case of the two conics we mentioned before.
2) Case 2: $I^+(N;\lambda_1,\ldots,\lambda_N)$. In this case we have the following graph

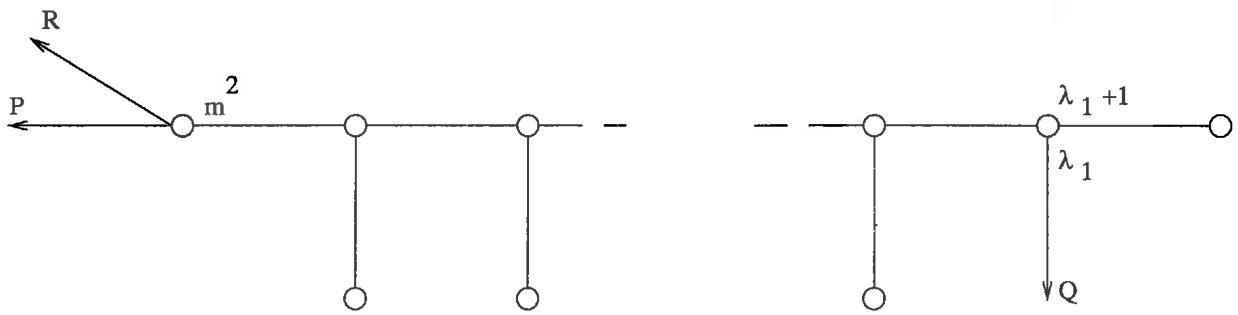
which gives the following splice diagram



3) Case 3: $I^-(N;\lambda_1,\ldots,\lambda_N)$. In this case we have



with the following splice diagram:



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