On the classification and topology of complex map-germs of corank one and $\mathcal{A}_e$-codimension one

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Abstract

Corank one map-germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, $n < p$, of $\mathcal{A}_e$-codimension one are classified and their vanishing topology is shown to be homotopically equivalent to a sphere.

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1 Introduction

In his classic paper [10] Mather classified the $\mathcal{A}$-stable map-germs. The next target for classification, the $\mathcal{A}_e$-codimension one germs, appears to be considerably more difficult, as one does not have an equivalent of Mather's result that $\mathcal{K}$-equivalent $\mathcal{A}$-stable maps are $\mathcal{A}$-equivalent. For example, the two real maps, $(x, y) \to (x, y^2, y^3 \pm x^2 y)$, have $\mathcal{A}_e$-codimension one, are $\mathcal{K}$-equivalent but not $\mathcal{A}$-equivalent, see [11]. However, this problem does not occur in the complex situation for this example.

In his Ph.D. thesis, [1], Cooper classified corank 1 $\mathcal{A}_e$-codimension 1 map-germs $\mathbb{C}^n$ to $\mathbb{C}^{n+1}$ by using explicit changes in source and target to reduce the map to a normal form. A more elementary proof of the classification is given in [2]. Surprisingly, just as in the stable case the situation comes down to dealing with the $\mathcal{K}$-equivalence class of the germ mainly because if the map is not an augmentation then the $\mathcal{A}$-orbit is open in the $\mathcal{K}$-orbit.

In this paper we generalise to the case of corank 1 $\mathcal{A}_e$-codimension 1 map-germs $\mathbb{C}^n$ to $\mathbb{C}^p$, $n < p$, i.e. the dimension of the target space is increased.

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2 The results

The main theorem is the following.

Theorem 2.1 *Suppose that $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, $n < p$, is a corank 1 $\mathcal{A}_e$-codimension 1 map-germ, then the following are true.*

1. *f is $\mathcal{A}$-equivalent to a map of the form,*

$$(u_1,\dots,u_{l-1},v_1,\dots,v_{l-1},w_{11},w_{12},\dots,w_{rl},x_1,\dots,x_{n-l(r+2)+1},y)$$
$$\mapsto (u_1,\dots,u_{l-1},v_1,\dots,v_{l-1},w_{11},w_{12},\dots,w_{rl},x_1,\dots,x_{n-l(r+2)+1},$$
$$y^{l+1}+\sum_{i=1}^{l-1}u_iy^i,y^{l+2}+\sum_{i=1}^{l-1}v_iy^i+y^l\sum_{i=1}^{n-l(r+2)+1}x_i^2,\sum_{i=1}^{l}w_{1i}y^i,\dots,\sum_{i=1}^{l}w_{ri}y^i),$$

   where $r=p-n-1$ and $l+1$ is the multiplicity of the germ. Conversely, any such germ has $\mathcal{A}_e$-codimension 1.

2. *The germ is precisely $l+2$-determined.*

3. *An $\mathcal{A}_e$-versal unfolding is given by unfolding with the addition of the term λy^l to the $(p-rl-1)$th component function.*

One immediately deduces the following.

Corollary 2.2 *Corank 1 $\mathcal{A}_e$-codimension 1 map-germs from $\mathbb{C}^n$ to $\mathbb{C}^p$ which are $\mathcal{K}$-equivalent are $\mathcal{A}$-equivalent.*

To every finitely $\mathcal{A}$-determined corank 1 map-germ there exists a unique stabilisation, see [7]. The image of this stabilisation is called the disentanglement of f. One can also investigate the multiple points in this image.

Definition 2.3 *Let $h: X \to Y$ be a continuous map. The kth image multiple point space of h, denoted $M_k(h)$, is defined to be,*

$$M_k(h) := closure\{y\in Y | \#h^{-1}(y)\geq k\}.$$

Definition 2.4 *We define the kth disentanglement of f, denoted $\mathrm{Dis}_k(f)$ to be the kth multiple point space of the stabilisation of f.*

Suppose that $f_{\mathbb{R}}:(\mathbb{R}^n,0)\to(\mathbb{R}^p,0)$ is a finitely $\mathcal{A}$-determined map-germ, with a real stabilisation $f_{\mathbb{R},t}$ and that the complexification of $f_{\mathbb{R}}$, denoted $f_{\mathbb{C}}$ has stabilisation arising from complexifying $f_{\mathbb{R},t}$. We can denote the kth image multiple point spaces of these maps by $\mathrm{Dis}(f_{\mathbb{R}})$ and $\mathrm{Dis}(f_{\mathbb{C}})$.

Definition 2.5 *The map $f_{\mathbb{R},t}$ is a good real perturbation if $\dim H_i(\mathrm{Dis}_1(f_{\mathbb{R}});\mathbb{Z}) = \dim H_i(\mathrm{Dis}_1(f_{\mathbb{C}});\mathbb{Z})$ for all $i=p-(p-n-1)k-1$, with $2\leq k\leq p/(p-n)$.*

This is a generalisation of the notion given in [12] and [9]. The idea is that the complex topology is visible over the reals.

Theorem 2.6 *Suppose that $f:(\mathbb{C}^n,0)\to(\mathbb{C}^p,0)$, $n<p$, is a corank 1 $\mathcal{A}_e$-codimension 1 map-germ.*

1. *The disentanglement $\mathrm{Dis}_1(f)$ is homotopically equivalent to a (n-l(p-n-1))-sphere. The higher disentanglements are empty or contractible.*

2. *It is obvious that f is the complexification of a real map-germ. This map has a good real perturbation and in fact the natural inclusion for this perturbation $\mathrm{Dis}_k(f_{\mathbb{R}})\hookrightarrow\mathrm{Dis}_k(f_{\mathbb{C}})$ is a homotopy equivalence for all $k\geq 1$.*

These results are analogous to the case of a quadratic isolated complete intersection singularity. For then the Milnor fibre is homotopically equivalent to a single sphere and it is possible to define a real Milnor fibre with the same topology. (In fact the above theorem is a consequence of these results).

When an isolated complete intersection singularity has Milnor number equal to one then it is $\mathcal{K}$-equivalent to a quadratic singularity. One may ask for corank 1 maps in the range $n<p$, if the disentanglement is homotopically a sphere, then is the map $\mathcal{A}_e$-codimension 1?

3 Classification

3.1 Proof of Theorem 2.1 part 1

Firstly we define the augmentation of a map-germ.

Definition 3.1 *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a map with a 1-parameter stable unfolding $F : (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}^p \times \mathbb{C}, 0)$, where $F(x, \lambda) = (f_\lambda(x), \lambda)$. Then the augmentation of f by F is the map $A_F(f) := (f_{\lambda^2}(x), \lambda)$.*

If f has $\mathcal{A}_e$-codimension 1 then $A_F(f)$ has $\mathcal{A}_e$-codimension 1 and the equivalence class of $A_F(f)$ is independent of the choice of miniversal unfolding of f. See Proposition 2.1 and Theorem 2.4 of [2]. Thus we can produce new codimension 1 maps from old codimension 1 maps. If f is not the augmentation of another germ then f is called primitive.

One can also generalise this definition so that the unfolding parameter is replaced by a function, see [4].

To prove part 1 of Theorem 2.1 we use results from the classification in the $p = n + 1$ case given in [2]. Let us follow them and begin by defining a map $f^l : (\mathbb{C}^{2l-1}, 0) \to (\mathbb{C}^{2l}, 0)$ by

$$f^l(u, v, y) = (u, v, y^{l+1} + \sum_{i=1}^{l-1} u_i y^i, y^{l+2} + \sum_{i=1}^{l-1} v_i y^i).$$

By Lemma 4.1 of [2] the $\mathcal{A}_e$-codimension is 1. If we label the last two coordinates of $\mathbb{C}^{2l}$ Y_1 and Y_2 then the $\mathcal{A}_e$-tangent space is

$$\begin{aligned} T\mathcal{A}_e f^l &= \theta(f^l) \backslash \left\{ y^l \partial/\partial Y_2, y^{l-1}\partial/\partial v_1, \ldots, y\partial/\partial v_{l-1} \right\} + \\ &\quad \left\langle y^{l-1}\partial/\partial v_1 + y^l \partial/\partial Y_2, \ldots, y\partial/\partial v_{l-1} + y^l \partial/\partial Y_2 \right\rangle. \end{aligned}$$

Let us now define an extension of this map, $f^{l,r} : (\mathbb{C}^{2l-1+rl}, 0) \to (\mathbb{C}^{2l+r(l+1)}, 0)$:

$$f^{l,r}(u, v, y, w) = (u, v, y^{l+1} + \sum_{i=1}^{l-1} u_i y^i, y^{l+2} + \sum_{i=1}^{l-1} v_i y^i, w, \sum_{i=1}^{l} w_{1i} y^i, \ldots, \sum_{i=1}^{l} w_{ri} y^i).$$

Through augmentation we get a map of the form in Theorem 2.1. By the proof of Proposition 3.7 of [6] it is known that $f^{l,r}$ is finitely determined. However we can do better than this as the following shows.

Theorem 3.2 *The map $f^{l,r}$ has $\mathcal{A}_e$-tangent space equal to*

$$\begin{aligned} T\mathcal{A}_e f^{l,r} &= \theta(f^{l,r}) \backslash \left\{ y^l \partial/\partial Y_2, y^{l-1}\partial/\partial v_1, \ldots, y\partial/\partial v_{l-1} \right\} + \\ &\quad \left\langle y^{l-1}\partial/\partial v_1 + y^l \partial/\partial Y_2, \ldots, y\partial/\partial v_{l-1} + y^l \partial/\partial Y_2 \right\rangle. \end{aligned}$$

Hence $f^{l,r}$ has $\mathcal{A}_e$-codimension equal to 1. To prove the above theorem let us investigate what the effect of extension is.

Suppose we have a finitely determined map $h : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ such that

$$\begin{aligned} & h(w_1, \ldots, w_l, y, u_1, \ldots, u_{l-1}, x) = (w_1, \ldots, w_l, \\ & \sum_{i=1}^{l} w_i y^i, y^{l+1} + \sum_{i=1}^{l-1} u_i y^i, u_1, \ldots, u_{l-1}, x, f_1(u, x, y), \ldots, f_t(u, x, y)). \end{aligned}$$

Let $\mathcal{O}_X$ denote the ring of function germs at 0 for the germ $(X, 0)$. The tangent space $T\mathcal{A}_e$ is a $h^*(\mathcal{O}_{\mathbb{C}^p})$ submodule of $(\mathcal{O}_{\mathbb{C}^n})^p$. Let e_i denote the standard basis vector for the ith copy of $\mathcal{O}_{\mathbb{C}^n}$.

Lemma 3.3 *$\mathcal{O}_{\mathbb{C}^n} e_i \in T\mathcal{A}_e h$ for all $1 \leq i \leq l+1$.*

Proof. It is evident that we can reduce the requirement to $y^k e_i \in T\mathcal{A}_e$ for all $i = 1, \ldots, l+1$.

Note that

$$y^k e_{l+1} \in T\mathcal{A}_e \iff y^{k-1} e_i \in T\mathcal{A}_e, \; k - i \geq 0.(*)$$

This follows from the fact that $y^j(e_i + y^i e_{l+1}) \in T\mathcal{R}_e$ and it implies that it suffices to show that $y^k e_{l+1} \in T\mathcal{A}_e$ for all k.

For $1 \leq s \leq l$ $e_i + y^s e_{l+1} \in T\mathcal{R}_e$ and $e_s \in T\mathcal{L}_e$ so $y^s e_{l+1} \in T\mathcal{A}_e$. We will now use induction: Suppose $y^s e_{l+1} \in T\mathcal{A}_e$ for all $s < k$ then $y^k e_{l+1} \in T\mathcal{A}_e$

The number k will be of the form $k = r(l+1) + i$ with $r \geq 1$ (assuming $k < l+1$ already dealt with as above) and $0 \leq i \leq l$.

Case $i = 0$: Clearly $(y^{l+1} + \sum_{j=1}^{l-1} u_j y^j)^r e_{l+1} \in T\mathcal{L}_e$ so $y^{r(l+1)} e_{l+1} \in T\mathcal{A}_e$ as the other terms in y in the expansion have order less than $r(l+1)$.

Case $i > 0$: The assumption $y^s e_{l+1} \in T\mathcal{A}_e$ for all $s < r(l+1) + i$ implies that $y^{s-i} e_i \in T\mathcal{A}_e$ for all $i \leq s < r(l+1) + i$ by $(*)$, i.e.

$$y^s e_i \in T\mathcal{A}_e \text{ for all } s < r(l+1).(**)$$

Obviously $(y^{l+1} + \sum_{j=1}^{l-1} u_j y^j)^r e_i \in T\mathcal{L}_e$ and this with $(**)$ implies that $y^r(l+1) e_i \in T\mathcal{A}_e$. Thus as $y^{r(l+1)}(e_i + y^i e_{l+1}) \in T\mathcal{R}_e$ we deduce that $y^{r(l+1)+i} e_{l+1} \in T\mathcal{A}_e$. □

After applying this lemma to the map $f^{l,r}$ all that is required is to check that if g is a function in variables w_1 to w_l then $g y^l \partial/\partial Y_2$ is in the tangent space. This is easy to check.

The maps $f^{l,r}$ have a very interesting property which will be very useful.

Lemma 3.4 *The $\mathcal{A}$-orbit of $f^{l,r}$ is open in its $\mathcal{K}$-orbit.*

Proof. Let the dimension of the source be n and that of the target be p. and denote the normal space of the $\mathcal{G}_e$-orbit by $N\mathcal{G}_e$. It is easy to calculate that $\dim N\mathcal{K}_e(f^{l,r}) = p + 1$ (It should be noted that this is not true for augmentations of $f^{l,r}$ as then we have $e_i \in T\mathcal{K}_e$ for at least one i.) Thus we find that $\dim N\mathcal{A}_e = \dim N\mathcal{K}_e - p$. But $\dim N\mathcal{A}_e = \dim N\mathcal{A} - n$ (as $f^{l,r}$ is not $\mathcal{A}$-stable, see [14] p.510) and $\dim N\mathcal{K}_e = \dim N\mathcal{K} + (p - n)$ ([14] p.509). So $\dim N\mathcal{A} = \dim N\mathcal{K}$, implying that the $\mathcal{A}$-orbit is open in the $\mathcal{K}$-orbit. □

Proof (of Theorem 2.1). We now generalise the proof of Proposition 4.3 of [2]. Suppose that $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is a corank 1 $\mathcal{A}_e$-codimension 1 map-germ, $n < p$ with multiplicity $l + 1$. The versal unfolding $G : (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}^p \times \mathbb{C}, 0)$ is a $n - l(p - n + 1) + 1$-fold prism on a minimal stable map-germ of multiplicity $l + 1$. Thus by Theorem 2.7 of [2] f is the $n - l(p - n + 1) + 1$-fold augmentation of an $\mathcal{A}_e$-codimension 1, corank 1, multiplicity $l + 1$ map-germ $f' : (\mathbb{C}^{2l + l(p-n+1)-1}, 0) \to (\mathbb{C}^{2l+(p-n-1)(l+1)}, 0)$. Such a map is obviously $\mathcal{K}$-equivalent to the map $f^{l,p-n-1}$. The $\mathcal{A}$-orbit of $f^{l,p-n-1}$ is open in its $\mathcal{K}$-orbit by Lemma 3.4 and by Lemma 3.12 of [2] there is at most one open $\mathcal{A}$-orbit in a given complex contact class, thus we conclude that f' and $f^{l,p-n-1}$ are $\mathcal{A}$-equivalent.

The $n - l(p - n + 1) + 1$-fold augmentation of $f^{l,p-n-1}$ is $\mathcal{A}$-equivalent to f as the $\mathcal{A}$-equivalence class of the augmentation of codimension 1 map-germ g depends only on the $\mathcal{A}$-equivalence class of g. □

3.2 Order of determinacy

To find the order of determinacy we use the techniques of [13], in particular his Proposition 3.8, which we summarise as the following. Denote the maximal ideal of $\mathcal{O}_{\mathbb{C}^d}$ by $\mathbf{m}_d$ and use the standard tf and wf notation of Singularity Theory, see[14].

Proposition 3.5 *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a map-germ. Let*

$$D \subset tf(\theta_{\mathbb{C}^n}) + wf(\theta_{\mathbb{C}^p}) + \mathbf{m}_n^s \theta_f$$

be an $\mathcal{O}_{\mathbb{C}^n}$-module such that

$$\mathbf{m}_n^s \theta_f \subset tf(\mathbf{m}_n \theta_{\mathbb{C}^n}) + f^*(\mathbf{m}_p) \cdot D + \mathbf{m}_n^{s+1} \theta_f.$$

Then f is s-determined.

Let f be as in Theorem 2.1. Then by calculation one can see that $T\mathcal{A}_e f$ has the same type of structure as $T\mathcal{A}_e(f^{l,r})$: Let $m = p - rl - 1$ then $y^l e_m$, and $y^{l-i} e_{l+i-1}$, $i = 1, \ldots, l-1$ are missing from $T\mathcal{A}_e f$, but $y^l e_m + y^{l-i} e_{l+i-1}$ is included. Thus if we let $\mathbf{m}_{n-1}$ denote the ideal generated by the variables other than y and

$$\begin{aligned} D \;=\; & \langle \mathcal{O}_n, \ldots, \mathcal{O}_n, \mathbf{m}_{n-1}\mathcal{O}_n + \langle y^l \rangle \mathcal{O}_n, \ldots, \mathbf{m}_{n-1}\mathcal{O}_n + \langle y^2 \rangle \mathcal{O}_n, \\ & \mathcal{O}_n, \mathbf{m}_{n-1}\mathcal{O}_n + \langle y^{l+1} \rangle \mathcal{O}_n, \mathcal{O}_n, \ldots, \mathcal{O}_n \rangle \end{aligned}$$

where the $\mathbf{m}_{n-1}\mathcal{O}_n + \langle y^j \rangle \mathcal{O}_n$ terms begin at position l, then D is an $\mathcal{O}_n$-module contained in $T\mathcal{A}_e f$.

The non-trivial problem is to show that, for all i, $y^{l+2} e_i$ is in the right hand side of the second inclusion in the proposition. For the positions corresponding to the functions $u_1, \ldots, u_{l-1}$, $v_1, \ldots, v_{l-1}$ and $w_{11}, \ldots, w_{rl}$ we can use elements of $tf(\mathbf{m}_n \theta_{\mathbb{C}^n})$ modulo $\mathbf{m}_n^{l+3}$. For the r extension terms and position $2l-1$ we use $y^{l+2} + \sum_{i=1}^{l-1} v_i y^i$, elements of tf and $f^*(\mathbf{m}_p) \cdot D$. For the remaining position we use $y \partial f / \partial y$ and terms in tf and $f^*(\mathbf{m}_p) \cdot D$.

So f is at least $(l+2)$-determined. This is in fact exact. The $(l+1)$-jet is not finitely $\mathcal{A}$-determined as can be seen by showing (using the method of [8]) that $(l+1)$th multiple point space has dimension greater than that of a finitely determined map-germ.

4 Topology

Theorem 2.6 part 1 on the topology of the kth disentanglement has been proved for the $p = n+1$ in Corollary 5.3 of [5], though note that this was first proved in this case for $k = 1$ in [1], see [2]. For more general p that $\mathrm{Dis}_1(f)$ is homotopically equivalent to a sphere can be deduced from the proof of Proposition 3.7 of [6] and Theorem 4.24 of [3] but the following, which investigates higher disentanglements, also shows it.

We begin with noting from Theorem 3.2 of [5] that for an augmentation $\mathrm{Dis}_m(A_F f)$ is homotopically equivalent to the suspension of $\mathrm{Dis}_m(f)$. Thus we can assume our map is primitive.

Define $f_t : \mathbb{C}^{2l-1} \times \mathbb{C}^{rl} \to \mathbb{C}^{2l} \times \mathbb{C}^{r(l+1)}$ by

$$f_t(u, v, y, w) = (u, v, y^{l+1} + \sum_{i=1}^{l-1} u_i y^i, y^{l+2} + \sum_{i=1}^{l-1} v_i y^i + t y^l, w, \sum_{i=1}^{l} w_{1i} y^i, \ldots, \sum_{i=1}^{l} w_{ri} y^i).$$

Then, for $t \neq 0$ we can produce the disentanglement map for f_0.

Define $g_t : \mathbb{C}^{2l-1} \to \mathbb{C}^{2l}$ by $g_t := f_t | f_t^{-1}(\mathbb{C}^{2l} \times \{0\})$, then g_t for $t \neq 0$ gives the disentanglement map for g_0, a corank 1 map-germ of $\mathcal{A}_e$-codimension 1. The space $\mathrm{Dis}_m(g)$ is homotopically equivalent to a $2l-1$ sphere if $m = 1$, and contractible or empty for $m > 1$ by Corollary 5.3 of [5]. We shall show that $\mathrm{Dis}_m(f_0)$ is homotopically equivalent to this space. In the following we assume $t \neq 0$ defines the disentanglement maps.

For a continuous map $h : X \to Y$ of topological spaces let $D^k(h)$ denote the kth multiple point space as defined in [8].

From the natural inclusion of $\mathbb{C}^{2l}$ into $\mathbb{C}^{2l+r(l+1)}$ we induce a natural map $\phi^k : D^k(g_t) \to D^k(f_t)$.

It is shown in the proof of Proposition 3.7 of [6] that $D^k(g_t)$ and $D^k(f_t)$ are non-singular for $k < l+1$, and from the description there we can deduce that $D^k(f_t)$ contracts equivariantly onto $D^k(g_t)$. The only other non-trivial spaces are $D^{l+1}(f_t)$ and $D^{l+1}(g_t)$ and from the description in [6] it follows that these are S_k-equivariantly homeomorphic Milnor fibres of what is effectively the same isolated complete intersection singularity.

To conclude that the natural map $\mathrm{Dis}_m(g_0) \to \mathrm{Dis}_m(f_0)$ induces an isomorphism on integer homology for all m we use Theorem 3.2 of [6]:

Lemma 4.1 *Suppose that $h_i : X_i \to Y_i$, $i = 1, 2$, are finite and proper continuous maps for which the image computing spectral sequence exists (this is a technical condition which is true for the maps under consideration here) and that there exist continuous maps ϕ and ψ such that the diagram*

$$\begin{array}{cccc} h_1: & X_1 & \to & Y_1 \\ & \phi \downarrow & & \downarrow \psi \\ h_2: & X_2 & \to & Y_2 \end{array}$$

commutes. Then if the map $\phi^k : D^k(h_1) \to D^k(h_2)$ is an S_k-homotopy equivalence for all $k \geq 1$, then $\psi|_{M_m(h_1)} : M_m(h_1) \to M_m(h_2)$ induces an isomorphism on integer homology groups for all $m \geq 1$.

We turn our attention to the fundamental groups of the image multiple point spaces and to this end we prove the following.

Lemma 4.2 *Suppose that $f : X \to Y$ is a finite and proper continuous map, $D^m(f)$ is path connected and that there exists a point $(x_1, \ldots, x_m) \in D^m(f)$ such that $x_c = x_d$ for $c \neq d$.*

1. *If $D^{m-1}(f)$ is path connected then the natural map of fundamental groups*

$$\pi_1(D^{m-1}(f)) \to \pi_1(M_{m-1}(f))$$

is surjective.

2. *If $D^{m+1}(f)$ is empty then*

$$\pi_1(D^m(f)) \to \pi_1(M_m(f))$$

is surjective.

Proof. (i) For a continuous map h we can define $\varepsilon^k : D^k(h) \to D^{k-1}(h)$ by projecting through omission of the last copy of the source of h. Let $D^k_j(h)$ be the image of $D^k(h)$ in $D^j(h)$ (through composition of maps ε^i). Then $M_r(h)$ is the image of $h_r := h|D^r_1$. We have

$$D^j(f_r) = \begin{cases} D^r_j(h), & \text{for } j < r, \\ D^j(h), & \text{for } j \geq r. \end{cases}$$

As $D^{m-1}(f)$ is path connected, $D^j(f_{m-1})$ is path connected for $j < m-1$ as it is the image of $D^{m-1}(f)$ in $D^j(f)$. As $D^m(f)$ has a point with $x_c = x_d$, $c \neq d$, then so does $D^j(f_{m-1})$ for all $2 \leq j < m$. These two facts imply that every point in $D^j(f_m)$ is connected by a path to a point with $x_c = x_d$, $c \neq d$.

Now, for any continuous map h, $D^{j+1}(h) = D^2\left(\varepsilon^j : D^j(h) \to D^{j-1}(h)\right)$. From this and Theorem 4.18 of [3] we deduce that for $j \leq m+1$ that

$$\pi_1(D^j(f_{m-1})) \to \pi_1(\varepsilon^j(D^j(f_{m-1}))) = \pi_1(D^{j-1}(f_{m-1}))$$

is surjective and produce a chain of maps to get

$$\pi_1(D^{m-1}(f)) = \pi_1(D^{m-1}(f_{m-1})) \to \pi_1(M_{m-1}(f))$$

surjective.

(ii) One can follow a similar argument to show that $\pi_1(D^{m-1}(f_m)) \to \pi_1(M_m(f))$ is surjective. As $D^{m+1}(f)$ is empty then $\varepsilon^m : (D^m(f)) \to \varepsilon^m(D^m(f)) = D^{m-1}(f_m)$ is a bijective and proper map so is a homeomorphism. □

Proposition 4.3 *The inclusion* $\mathrm{Dis}_m(g_0) \to \mathrm{Dis}_m(f_0)$ *is a homotopy equivalence for all* $m \geq 1$ *and hence Theorem 2.6 part 1 is proved.*

Proof. Note that $M_m(f_t)$ and $M_m(g_t)$ are Stein spaces and so are homotopy equivalent to CW-complexes of dimension equal to their complex dimension.

If $\dim_{\mathbb{C}} M_m(f_t) \leq 1$ then the statement is elementary to prove. If $\dim_{\mathbb{C}} M_m(f_t) > 1$ then it is enough to show that $M_m(g_t)$ and $M_m(f_t)$ are simply connected because a map between simply connected CW-complexes that induces an isomorphism on integer homology is a homotopy equivalence by Whitehead's theorem, [15], p220. In our given range we know that $M_m(g_t)$ is simply connected.

Note that $D^j(f_t)$ is contractible for $j < l+1$ and $D^{l+1}(f_t)$ is the Milnor fibre of an isolated complete intersection singularity and so is homotopically equivalent to a wedge of spheres. Higher multiple point spaces are empty.

Case $\dim D^{l+1}(f_t) > 0$: Here $D^{l+1}(f_t)$ is connected and since the restriction to a reflecting hyperplane in the ambient space is the Milnor fibre of an isolated complete intersection singularity, see [8] Theorem 2.14, there exists a point $(x_1, \dots, x_{l+1})$ such that $x_c = x_d$ for some $c \neq d$. From Lemma 4.2 we deduce that $\pi_1(D^m(f_t)) \to \pi_1(M_m(f_t))$ is surjective for all $m \leq l+1$. For $m < l+1$ the result is then true. For the $l+1$ case we note that we have are only concerned with $\dim_{\mathbb{C}} M_{l+1}(f_t) \geq 2$, i.e. $D^{l+1}(f_t)$ is simply connected.

Case $\dim D^{l+1}(f_t) = 0$: As $\dim D^{l+1}(f_t) = l-1$ the only situations to check are for $M_1(f_t)$, which is simple, it is homotopically a circle, and for $M_2(f_t)$ which has dimension 0. □

Proof (of Theorem 2.6 part 2). From Proposition 3.7 of [6] we see that a good real perturbation exists, (use $t < 0$ in f_t) and that the natural map $\mathrm{Dis}_m(f_{\mathbb{R}}) \to \mathrm{Dis}_m(f_{\mathbb{C}})$ induces an isomorphism of integer homology groups.

If $\dim M_m(f_{\mathbb{C}}) \leq 1$ then the statement is trivial. For the other situations we must show that $\mathrm{Dis}_m(f_{\mathbb{R}})$ is simply connected. Calculations show that $D^k(f_{\mathbb{R},t})$ and $D^k(f_{\mathbb{C},t})$ are connected, non-singular and contract onto the diagonal for $k < l+1$. The space $D^{l+1}(f_{\mathbb{C},t})$ is simply connected when its dimension is greater than 1, and $D^{l+1}(f_{\mathbb{R},t})$ is S_{l+1}-homotopically equivalent to it. Thus by Lemma 4.2 the image multiple point sets for $f_{\mathbb{R},t}$ are simply connected.

Again using Whitehead's theorem we conclude that the spaces are homotopically equivalent. □

We finish with a theorem on augmentations.

Theorem 4.4 *Suppose that $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is the augmentation by the isolated hypersurface singularity $g : (\mathbb{C}^q, 0) \to (\mathbb{C}, 0)$ of the corank 1 $\mathcal{A}_e$-codimension 1, multiplicity $l+1$ map-germ. Let g have Milnor number $\mu(g)$.*

Then $\mathrm{Dis}_1(A_{F,g}(f))$ is homotopically equivalent to a wedge of $\mu(g)$ $n - l(p-n-1) + q$-spheres. Higher disentanglements are contractible or empty. Furthermore,

$$\mu(g) \leq \mathcal{A}_e - cod(A_{F,g}(f)),$$

with equality if g is quasihomogenous.

Proof. The result on homotopy follows from Theorem 3.2 of [5].

Note that f is quasihomogeneous and hence so is the unfolding F. Then, (denoting Tyurina number of g by $\tau(g)$ and Milnor number by $\mu(g)$),

$$\begin{aligned} \mathcal{A}_e - cod(A_{F,g}(f)) &= \tau(g)\mathcal{A}_e - cod(f), \text{ by Theorem 3.3 of [4]}, \\ &= \tau(g) \\ &= \leq \mu(g), \text{ with equality if } g \text{ quasihomogeneous.} \end{aligned}$$

□

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