Some invariants of admissible homotopies of space curves

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Abstract

A regular homotopy of a generic curve in the projective 3-space is called admissible if it defines a generic one-parameter family of curves, in which every curve has no self-intersections, no inflection points, is not tangent to a smooth part of its evolvent and has no tangent planes osculating to the curve at two different points. We introduce a number of invariants of admissible homotopies of space curves and prove, in particular, that in the class of such homotopies the curve $x = \cos t, y = \sin t, z = \cos 3t$ cannot be deformed into a curve without flattening points.

1 Introduction

For every point of a C^∞-smooth curve in the three-dimensional real projective space P^3, there exists a plane intersecting the curve at this point at least three times. Such a plane is called an *osculating plane* to the curve at a given point. A point of a curve is a *flattening point* if the multiplicity of the intersection of the curve with an osculating plane at this point is more than three. A flattening point of a curve where the osculating plane is not unique is called an *inflection point*.

Suppose that for any two points (taking the multiplicities into account) of a smooth closed curve in P^3, there is a plane which passes through these points and does not intersect the curve anymore. Then this curve has at least four geometrically different flattening points ([3]).

Notice that a curve satisfying the above condition has no inflection points, is embedded and affine. Moreover, such a curve lies on the boundary of its affine convex hull. It has been shown in [6] that any smooth closed curve embedded into the affine three-dimensional space R^3 without inflection points and lying on the boundary of its convex hull has at least four geometrically different flattening points.

It would be interesting to extend the theorem on four flattening points to a wider class of space curves. One of the approaches to this problem was suggested by V. I. Arnold in [1]. It is based on methods of contact geometry.

Namely, consider a closed generic front in P^3. It is a singular surface with cuspidal edges, swallowtails and transversal self-intersections. The union of cuspidal edges and

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vertices of swallowtails is called the *cuspidal line* of a front. This curve has no self-intersections but has cusps at vertices of swallowtails.

The closure of the set of self-intersection points of a front is the union of curves having only cusps, generic double or triple self-intersections and end-points at vertices of swallowtails. These curves are called *double lines* of a front. Vertices of swallowtails form a *connected pair* if they are end-points of a front double line.

Take a smooth closed generic curve in P^3. The set of tangent planes to this curve is a closed generic front in the dual space. The cuspidal line of this front is the *dual curve* (the set of osculating planes to the initial curve). Flattening points of the curve one to one correspond to vertices of swallowtails of the front (that is, to cusps of the dual curve).

1.1. Definition. Two flattening points of a smooth closed generic curve in P^3 form a *connected pair* if the corresponding vertices of swallowtails on the front of tangent planes to this curve form a connected pair.

Orient a given curve and the ambient space. A germ of the curve at a flattening point has a parametrization $x = t, y = t^2 + \dots, z = t^4 + \dots$, in a suitable system of affine coordinates. The flattening point is called *positive (negative)* if this coordinate system orients the ambient space positively (negatively).

The number of connected pairs of flattening points having the same sign does not depend on the choice of orientations. This number is called the *sturmianity* of a curve. According to [1], the sturmianity is an invariant of so-called admissible homotopies of a curve.

1.2. Definition. A regular homotopy of a smooth closed generic curve in P^3 is called *admissible* if it defines a generic 1-parameter family of curves where every curve has no

1) self-intersections (that is, its front has no self-tangencies);

2) inflection points (that is, the dual curve is irreducible);

3) tangencies with a smooth part of its evolvent (that is, cuspidal edges of the front are not tangent to its smooth part);

4) tangent planes osculating to the curve at two different points (that is, the dual curve has no self-intersections).

In an admissible homotopy, the number of flattening points of a curve can not become less than twice its sturmianity. But the sturmianity can be 0. Let us consider, for example, the curve

$$\Gamma : x = \cos t, y = \sin t, z = \cos 3t, \quad t \bmod 2\pi.$$

The set of singular points of its front (the cuspidal line and self-intersections) is represented on Fig.1 (see details in Section 6). Generic curves sufficiently close to Γ (in C^∞-topology) have 6 flattening points. Some of them have zero sturmianity. V. I. Arnold raised the following question (see [2]; the problem 1998-6): is it possible to annihilate all 6 flattening points of such curves by admissible homotopies?

We give below the negative answer on this question, namely, we show that the number of flattening points of any generic curve sufficiently close to Γ cannot vanish in an admissible homotopy. For the proof, we define a new invariant of admissible homotopies of space curves – the number of closed double lines of a front. Besides, we construct an invariant generalising the sturmianity of a curve. It is a chord diagram in which the number of chords intersecting odd number of other chords coincides with the sturmianity of a curve.

2 Closed double lines of a front

Let $\gamma : S^1 \to P^3$ be a smooth closed generic curve in P^3 and l be a double line of its front. Orient the dual space P^{3*}, the dual curve γ^* and the line l. A cusp c on l is *positive (negative)* if the following set of three vectors in T_cP^{3*} is positively (negatively) oriented:

1) a tangent vector to γ^*;

2) a direction vector of the one-sided half-tangent line to l at c;

3) a vector in the tangent plane to the smooth part of the front of γ at c showing the direction of the deviation of the branch of l going out of c from the line tangent to l at c.

2.1. Definition. The absolute value of the difference between the number of positive and the number of negative cusps on a line l is called the *weight* of l.

The weight of a line l does not depend from the choice of orientations.

2.2. Theorem. *The (unordered) set of weights of closed double lines on the front of a smooth closed generic curve in P^3 is an invariant of admissible homotopies. In particular, the number of these lines is not changed in such homotopies.*

The proof is given in Section 4.

2.3. Definition. Two smooth closed curves embedded into P^3 are called *isotopic* if they are homotopic in the space of embedded curves.

Admissible homotopies preserve the isotopy class of a curve.

2.4. Theorem. *Let a smooth closed generic curve in P^3 be isotopic to affine. Then its front has double lines (that is, there exist projective planes in P^3 tangent to the curve at two different points).*

If a curve is affine, then this is evident. Indeed, such a curve has affine support planes tangent to it at two different points (see [5]). The proof in a general case is given in Section 5.

2.5. Remark. If a curve in P^3 is not isotopic to affine, then its front can have no double lines. The curve $(\cos t : \sin t : \cos 3t : \sin 3t)$, $t \bmod \pi$ is an example of this (any plane intersects it at most at three points).

Theorems 2.2 and 2.4 imply

2.6. Corollary. *Let a smooth closed generic curve in P^3 be isotopic to affine and its front have no closed double lines. Then this curve has flattening points as well as any other curve obtained from it by admissible homotopies.*

Consider the curve Γ from Section 1. It is affine.

2.7. Proposition. *The front of any generic curve sufficiently close to Γ has no closed double lines.*

The proof is given in Section 6. Proposition 2.7 and Corollary 2.6 imply the answer to the mentioned question of Arnold:

2.8. Corollary. *The number of flattening points of any generic curve sufficiently close to Γ cannot vanish in an admissible homotopy.*

3 Principal flattenings diagram of a curve

Consider flattening points of a smooth closed generic curve γ in P^3.

3.1. Definition. The *weight* of a connected pair of flattening points of a curve γ is the weight of the double line of its front which connects the corresponding vertices of swallowtails.

Two connected pairs of flattening points are *alternate* if going around a curve a flattening point of one connected pair follows a flattening point of the other one. Flattening points of alternate connected pairs are *basic* flattening points of a curve.

3.2. Definition. A connected pair of flattening points of a curve is called *non-principal* if its weight is 2 and points of this pair separate the curve onto two open arcs one of which has no basic flattening points. All other connected pairs are called *principal.*

The curve γ defines a weight chord diagram D_γ of unordered pairs of points on S^1 which are preimages of principal connected pairs of flattening points and equipped with weights of these pairs. Two such diagrams are *equivalent* if one of them can be transferred to another by an orientation-preserving diffeomorphism of S^1.

3.3. Definition. The equivalence class of the diagram D_γ is called the *principal flattenings diagram* of a curve γ.

3.4. Remark. The sturmianity of a curve is equal to the number of connected pairs of flattening points which alternate with an odd number of other connected pairs, that is, to the number of chords of the principal flattenings diagram which intersect an odd number of other chords. In particular, the sturmianity of a curve does not exceed the number of all chords of the principal flattenings diagram, that is does not exceed the number of principal connected pairs of flattening points.

3.5. Theorem. *The principal flattenings diagram of a smooth closed generic curve in P^3 is an invariant of admissible homotopies. In particular, the number of principal connected pairs of flattening points of a curve is not changed in such homotopies.*

The proof is given in Section 4.

3.6. Corollary. *In an admissible homotopy, the number of flattening points of a smooth closed generic curve in P^3 cannot become less than twice the number of principal connected pairs of flattening points of the initial curve.*

Consider the curve Γ from Section 1. It is not generic (Γ has three pairs of points where the osculating planes coincide).

3.7. Proposition. *The principal flattenings diagram of any generic curve γ sufficiently close to Γ has one of the following two types:*

1) *the empty diagram (without chords; the curve γ has three non-principal connected pairs of flattening points; for every pair, the front double line connecting the corresponding vertices of swallowtails has two cusps of the same sign);*

2) *the nonempty diagram having two intersecting chords of weight* 0 *(the curve γ has two alternate and one non-principal connected pairs of flattening points; the front double line connecting the vertices of the swallowtails corresponding to the non-principal pair has two cusps of the same sign; the front double line connecting the vertices of the swallowtails corresponding to any principal pair has two cusps of opposite signs).*

The proof is given in Section 6. Proposition 3.7 and Corollary 3.6 imply

3.8. Corollary. *The number of flattening points of any generic curve which is sufficiently close to Γ and has nonempty principal flattenings diagram can not become less than* 4 *in an admissible homotopy.*

4 Proof of Theorems 2.2 and 3.5

Consider an admissible homotopy of a smooth closed generic curve in P^3. It defines a deformation of the front of this curve. Perestroikas of the front which can happen in such a deformation were listed in [1]. They are as follows:

A_4: a birth or a death of a connected pair of vertices of swallowtails on a front cuspidal line which are close to each other along this line (the front double line which connects them has two cusps of the same sign);

$A_3 + A_1$: a passage of the smooth part of the front through a vertex of a swallowtail (a pair of cusps of opposite signs appears or disappears on a front double line);

$A_2 + 2A_1$: a passage of a cuspidal edge through a front double line;

$4A_1$: a passage of the smooth part of the front through a point of its triple self-intersection;

$2A_1 \parallel A_1$: a tangency of a front double line to the smooth part of the front.

Perestroikas $A_3 + A_1$, $A_2 + 2A_1$, $4A_1$ and $2A_1 \parallel A_1$ do not change the number of front double lines, their types (closed or unclosed) and weights. An A_4 perestroika changes the number of front double lines which connect vertices of swallowtails corresponding to non-principal connected pairs of flattening points of the curve. Theorems 2.2 and 3.5 are proved.

5 Proof of Theorem 2.4

Let γ be a smooth closed generic curve in P^3.

5.1. Lemma. *If the front of a curve γ has no double lines, then any plane in P^3 intersects this curve at most at four points (taking the multiplicities into account).*

Proof. Consider an arbitrary plane π intersecting the curve γ at $n \geq 3$ points. Only one multiple point can be among them and its multiplicity is not greater than three.

If there is a multiple intersection point, then the osculating plane to the curve γ at this point intersects the curve n times. This follows from the fact that the number of intersection points of the curve γ with any plane tangent to it at a given point is the same.

If all intersections are transversal, then one can find a plane which is tangent to the curve γ and intersects it at n points. Such a plane is obtained by a deformation of the plane π in which two points of the intersection $\pi \cap \gamma$ move along the curve γ towards each other.

Thus, there exists a plane osculating to the curve γ at some point O and intersecting it exactly n times. If we rotate this plane around the tangent line to the curve γ at the point O, then its $n-2$ intersection points with γ will move along the curve in the same direction. At the moments of their passing through O, the rotating plane will be an osculating plane to the curve. But the curve γ has no inflection points. Hence, $n-2 \leq 2$, that is $n \leq 4$. Lemma 5.1 is proved.

Suppose that there exists a plane Π in P^3 transversally intersecting a curve γ at exactly two geometrically different points. We denote by R^3 the affine chart in P^3 for which the plane Π is infinite. In this chart, the curve γ is presented by two branches γ_1 and γ_2 for which there exist two parallel planes π_1, π_2, satisfying the following conditions ($i = 1, 2$):

1) the plane π_i transversally intersects the branch γ_i at exactly two points and has no common points with the other branch;

2) any plane obtained from π_i by a parallel translation into the half-space which does not contain the second plane transversally intersects the branch γ_i at exactly two points and has no common points with the other branch as well.

Let P_i, Q_i be intersection points of the curve γ_i with the plane π_i. Denote by $\tilde{\gamma}_i$ the closed curve in R^3 which is the union of the segment P_iQ_i and the arc of the curve γ_i bounded by points P_i, Q_i.

5.2. Lemma. *If the front of a curve γ has no double lines, then 1) any plane in R^3 which is parallel to the planes π_1, π_2 and lies between them intersects one of the curves γ_1, γ_2; 2) such a plane can intersect every curve γ_1, γ_2 only two times (taking the multiplicities into account) 3) the linking number of the curves $\tilde{\gamma}_1, \tilde{\gamma}_2$ is ± 1.*

PROOF. 1) Consider any plane $\pi \subset R^3$ which is parallel to the planes π_1, π_2 and lies between them. If it does not intersect the curves γ_1, γ_2, then γ is an affine curve and hence, its front has double lines (see [5]).

2) Assume that π intersects γ_1. Then the number of their intersection points is not greater than four by Lemma 5.1. But this number is even. Hence it is either two or four.

Suppose that the number of points of the intersection $\pi \cap \gamma_1$ is four. Then the plane π does not intersect the curve γ_2 and there exists a plane $\tilde{\pi}$ satisfying the following conditions: it is parallel to π, does not intersect γ_2, is tangent to γ_1 at some point O such that a germ of γ_1 at O and the curve γ_2 lie on different sides of $\tilde{\pi}$.

Denote by L_1 and L_2 the affine parts of the tangent line to the curve γ at the points of the intersection $\gamma \cap \Pi$. Since these lines are crossed, there exists a projection $p : R^3 \to R^2$ sending the plane $\tilde{\pi}$ onto the line $p(\tilde{\pi})$ and the lines L_1, L_2 into a pair of parallel lines $p(L_1), p(L_2)$ transversal to the line $p(\tilde{\pi})$.

Consider the projections $p(\gamma_1), p(\gamma_2)$ of the curves γ_1, γ_2. They are non-closed C^∞-parametrised curves in R^2 having only finitely many cusps and simple double self-intersections. Each of the curves $p(\gamma_i)$ goes to infinity asymptotically approaching one of the lines $p(L_1), p(L_2)$ when the parameter increases, and the other one when it decreases. Moreover, both distant pieces of the same curve lie on the same side of the line $p(\tilde{\pi})$ and distant pieces of the different curves lie on different sides of this line.

This implies that lines in R^2, tangent to the curve $p(\gamma_1)$ or passing through its cusps, sweep the entire connected component of $R^2 \setminus p(\tilde{\pi})$ containing $p(\gamma_2)$. At least two of these lines have common points with the curve $p(\gamma_2)$ and are supporting lines for it (distant ends of the curve $p(\gamma_2)$ asymptotically approach parallel lines at the infinity!). It is clear that the p-preimage of each of the indicated supporting lines is a plane in R^3 tangent to both curves γ_1, γ_2. But this contradicts the fact that the front of the curve γ has no double tangent planes.

Thus, the number of intersection points of the plane π with the curve γ_1 is 2.

3) Fix an arbitrary sphere S^2 in R^3 containing the curves $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ inside. This sphere intersects the curve γ_1 at two geometrically different points P', Q' whose order $P' \to Q'$ defines an orientation of γ_1. It is easy to see that the linking number of the curves $\tilde{\gamma}_1, \tilde{\gamma}_2$ is equal to the linking number of the curve $\tilde{\gamma}_2$ with the closed curve γ_1' which is the union of the segment $P'Q'$ and the arc of the curve γ_1 bounded by the points P', Q'.

Consider the plane π' in R^3 which is a supporting plane of the curve γ_1 and is parallel to the planes π_1, π_2. It lies between these planes, is tangent to the curve γ_1 at some point O' and has no other common points with γ_1. The tangent line L' to the curve γ_1 at O' intersects the sphere S^2 at points P'', Q'' such that P'' lies on the negative half-tangent line and Q'' is on the positive.

Denote by δ_- (δ_+) the set of intersection points of the sphere S^2 with negative (positive) half-tangent lines to the curve γ_1 at points of the arc $P'O'$ ($O'Q'$, respectively). Since the front of the curve γ has no double lines, the sets δ_-, δ_+ are non-intersecting arcs of C^∞-parametrised curve on S^2 without self-intersections but with cusps. Hence, the union Δ of the segments $P'Q'$,$P''Q''$ and arcs δ_-, δ_+ is a simple closed curve in R^3.

Clearly, the curve Δ is isotopic to the curve γ_1': an isotopy is defined by a retraction of arcs δ_-, δ_+ along the half-tangent lines indicated above. But the curve γ_2 does not intersect tangent lines to γ_1. Therefore, the linking number of the curves $\tilde{\gamma}_1, \tilde{\gamma}_2$ is equal to the linking number of the curves $\tilde{\gamma}_2, \Delta$. Thus, to prove claim 3 of the theorem, it is sufficient only to show that the curve γ_2 intersects the plane π' at exactly two geometrically different points lying in this plane on different sides of the line L'.

Let us check that the latter is true. Firstly, $\pi' \cap \gamma_2 \neq \emptyset$. Indeed, otherwise any plane $\tilde{\pi}'$ obtained from π' by a sufficiently small parallel translation in the direction to the plane π_2 would not intersect the curves γ_1, γ_2 that (as we saw in the proof of the first claim) contradicts our condition. Hence, the plane π' intersects the curve γ_2 at exactly two points and no-one of them lies on the line L'.

Secondly, the plane π' cannot be tangent to the curve γ_2. In fact, if the intersection points $\pi' \cap \gamma_2$ coincide, then the curves γ_1, γ_2 lie on different sides of the plane π'. But distant pieces of these curves do not lie in the region between the planes π_1, π_2. Therefore, the convex hulls of the curves γ_1, γ_2 do not intersect each other and hence there exists a plane in R^3 which does not intersect γ_1, γ_2 which again contradicts the condition.

Thirdly, the intersection points of the plane π' with the curve γ_2 cannot lie in this plane on the same side of the line L'. Indeed, otherwise one could rotate the plane π' around L' so that its intersection points with γ_2 would move along the curve towards each other. Hence, there would exist a plane tangent to both curves γ_1, γ_2 which contradicts the absence of double lines on the front of the curve γ.

Lemma 5.2 is completely proved.

Now, let us formulate the condition under which there exists a plane in P^3 that transversally intersects the curve γ at exactly two geometrically different points.

5.3. Lemma. *If a curve γ is isotopic to affine and its front has no double lines, then there exists a plane in P^3 transversally intersecting this curve at exactly two geometrically different points.*

PROOF. By Lemma 5.1, any osculating plane to the curve γ intersects it exactly four times and, moreover, three times at the point of osculation. The required plane is obviously obtained by a small rotation of the osculating plane around the point of osculation in the tame direction. Lemma 5.3 is proved.

Lemmas 5.3 and 5.2 imply that if a curve γ is isotopic to affine and its front has no double lines, then, in the space of smooth immersions of a circle into P^3, this curve can be connected with unknotted affine curve by a generic homotopy containing exactly one self-intersecting curve which is homeomorphic to a pair of intersecting projective lines.

Thus, to prove Theorem 2.4, it is sufficiently only to check the following statement:

5.4. Lemma. *Assume that a smooth closed generic curve in P^3 can be connected, in the space of smooth immersions $S^1 \to P^3$, with an unknotted affine curve by a generic homotopy containing exactly one self-intersecting curve which is homeomorphic to a pair of intersecting projective lines. Then it is not isotopic to an affine curve.*

This fact must be well-known to specialists. We will deduce it from a more general statement describing an invariant of contractible curves in the projective space. The construction of this invariant is realized according to the standard scheme of knot theory stated in [4].

Let γ be a smooth closed contractible curve embedded into P^3. At isolated moments of a generic retraction of this curve to a point, there appear curves with one simple double self-intersection homeomorphic to the union of two smooth closed non-contractible curves embedded into P^3. The number of such moments will be called the *projectivity coefficient* of the curve γ.

5.5. Proposition. *The projectivity coefficient of a smooth closed generic contractible curve in P^3 does not depend on its retraction, that is, this number is an invariant of a curve. In particular, the projectivity coefficient is not changed in isotopies of a curve.*

PROOF. Consider the set Ω of all smooth closed contractible curves in P^3. Non-embedded curves form a singular co-oriented stratified submanifold Υ in this space ([4]).

Let us co-orient this manifold and consider the subset $\Upsilon_0 \subset \Upsilon$ consisting of curves which are homotopic to the union of two smooth closed non-contractible curves in P^3. Since no one-point curve in P^3 belongs to the manifold Υ_0, the projectivity coefficient of any curve $\gamma \in \Omega \setminus \Upsilon$ is equal (up to a sign) to the algebraic number of points at which a generic path in the space Ω connecting γ with one-point curve intersects the smooth part of the manifold Υ_0. Thus our statement claims that the projectivity coefficient satisfies the cocycle condition, that is, the algebraic number $I(s)$ of intersection points of any closed generic path s in the space Ω with the manifold Υ_0 is equal to 0.

For the proof, notice at first that the manifold Υ_0 does not have a boundary. This follows from the fact that a generic point of a boundary would correspond to a curve in P^3 having one simple cusp and no self-intersections. But no curve sufficiently close to such a curve belongs to the manifold Υ_0. Hence, the number $I(s)$ depends only on the homotopic class of the path s in the space Ω, that is it is sufficient to check the cocycle condition on an arbitrary element of the fundamental group of this space.

Let O be an arbitrary point in P^3 and $\{O\} \in \Omega$ be the one-point curve. Mark a point on every curve $\gamma \in \Omega$ so that the bundle $p : \Omega \to P^3$ which associates the marked point to a curve is smooth. Since $\pi_0(p^{-1}(O), \{O\}) = 0$ (the space Ω is connected) and $\pi_1(p^{-1}(O), \{O\}) = \pi_2(P^3, O) = 0$ ([7], p. 78), then the exact homotopic sequence of the bundle p

$$\cdots \to \pi_1(p^{-1}(O), \{O\}) \to \pi_1(\Omega, \{O\}) \to \pi_1(P^3, O) \to \pi_0(p^{-1}(O), \{O\})$$

implies $\pi_1(\Omega, \{O\}) = \pi_1(P^3, O) = Z_2$.

Thus, it is sufficient to check the cocycle condition only for a path $s \subset \Omega$ such that $p(s) = \delta$ where δ is a projective line in P^3. For such a path, one can take the path s_δ consisting of the one-point curves defining by points of the curve δ. Since the path s_δ does not intersect the manifold Υ_0, then $I(s_\delta) = 0$. Proposition 5.5 is proved.

PROOF OF LEMMA 5.4. The projectivity coefficient of an affine curve is 0. The projectivity coefficient of a curve satisfying the conditions of Lemma 5.4 is 1. Hence, such curves are not isotopic according to Proposition 5.5.

Lemma 5.4 and Theorem 2.4 as well are proved.

6 Proof of Propositions 2.7 and 3.7

Consider the curve Γ from Section 1. It lies in the affine space $R^3 = \{(x, y, z)\}$ and has a strictly convex projection onto the plane $z = 0$. Hence, any osculating plane to Γ or a plane tangent to it at least at two different points is transversal to the line $x = y = 0$ and is defined by the equation $z = ax + by + c$.

Let us take the coefficients (a, b, c) as an affine coordinate system in the dual space R^{3*} and make some calculations.

6.1. Lemma. *For any two points t_1 and t_2 of the curve Γ,*

$$\det \begin{pmatrix} x'(t_1) & x'(t_2) & x(t_1) - x(t_2) \\ y'(t_1) & y'(t_2) & y(t_1) - y(t_2) \\ z'(t_1) & z'(t_2) & z(t_1) - z(t_2) \end{pmatrix} = 32 \left(\sin \frac{t_2 - t_1}{2}\right)^4 \cos \frac{t_2 - t_1}{2} \sin \frac{3}{2}(t_2 + t_1).$$

It follows from Lemma 6.1 that any two points t_1,t_2 (taking the multiplicities into account) at which the curve Γ has a common tangent plane satisfy one of the following four equations (mod 2π):

$$t_2 + t_1 = \frac{2\pi k}{3} \quad (k = 0, 1, 2) \qquad \text{and} \quad t_2 - t_1 = \pi.$$

6.2. Lemma. *The set of planes tangent to the curve Γ at least at two points (taking multiplicities into account) is the union of four curves in R^{3*}:*

$$\xi_k : \begin{cases} a = 3(-1)^k \cos \frac{\pi k}{3} (4 \cos^2 t - 1) \\ b = 3(-1)^k \sin \frac{\pi k}{3} (4 \cos^2 t - 1) \\ c = -8(-1)^k \cos^3 t \end{cases}, \quad t \in [0, \pi], \quad k = 0, 1, 2;$$

$$\eta : \begin{cases} a = 2 \cos 2t - \cos 4t \\ b = -2 \sin 2t - \sin 4t \\ c = 0 \end{cases}, \quad t \bmod \pi.$$

The curves $\xi_1, \xi_2, \xi_3, \eta$ are flat. Namely, the curve ξ_k lies in the vertical plane $a \sin \frac{\pi k}{3} = b \cos \frac{\pi k}{3}$, and η lies in the horizontal plane $c = 0$.

We denote the front of the curve Γ by Σ.

6.3. Lemma. *Every curve $\xi_k, k = 0, 1, 2,$ is a part of a semicubical parabola with a cusp*

$$C_k = \left(3(-1)^{k+1} \cos \frac{\pi k}{3}, 3(-1)^{k+1} \sin \frac{\pi k}{3}, 0\right).$$

Its end-points

$$V_k^m = \left(9(-1)^k \cos \frac{\pi k}{3}, 9(-1)^k \sin \frac{\pi k}{3}, 8(-1)^{m+1}\right), \quad m = 0, 1,$$

are vertices of swallowtails of the front Σ. *The curve* η *is a hypocycloid with three cusps* C_0, C_1, C_2.

Notice that the curves ξ_1, ξ_2, ξ_3 have two common points $T_0 = (0,0,-1)$ and $T_1 = (0,0,1)$. They are triple self-intersection points of the front Σ. The union of the curves $\xi_1, \xi_2, \xi_3, \eta$ is presented on Fig.2 (for simplicity, we present cusps by angles).

6.4. Lemma. *The osculating plane to the curve* Γ *at a point* t *has the following coordinates in* R^{3*}:

$$a = 6\cos 2t + 3\cos 4t, \quad b = 3\sin 4t - 6\sin 2t, \quad c = -8\cos 3t.$$

The formulas of Lemma 6.4 define a parametrisation of the dual curve Γ^*.

6.5. Lemma. *The curve* Γ^* *is a closed curve with six cusps* $V_k^m, k \in \{0,1,2\}, m \in \{0,1\}$, *and three self-intersection points* C_0, C_1, C_2. *Every point* C_k *is an intersection point of two branches of the curve* Γ^*; *every branch transversally intersects at this point the plane* $c = 0$ *and the osculating plane to the other branch.*

The dual curve Γ^* is presented on Fig.3. We will suppose that it is oriented by the order of its cusps: $V_0^0 \to V_1^1 \to V_2^0 \to V_0^1 \to V_1^0 \to V_2^1 \to V_0^0$.

6.6. Remark. Fig.1 is obtained by a superposition of Fig.2 and Fig.3.

Consider germs of the front Σ at the points C_0, C_1, C_2. By Lemmas 6.2 and 6.5, they are diffeomorphic to a germ at 0 of the set $\{y^2 = x^3\} \cup \{z^2 = x^3\}$ (a germ of the front Σ at the point C_2 is presented on Fig.4). A small generic deformation of the curve Γ, creates two cusps on double lines of the front of a curve at each of the points C_0, C_1, C_2. In addition, a double line going into a cusp along the curve η leaves it along one of the curves ξ_1, ξ_2, ξ_3.

This shows that a germ of the front Σ of the curve Γ at any point $C_k, k = 0,1,2$, can be deformed in two different ways. Hence, the set of singular points of the front of any generic curve sufficiently close to Γ can be obtained from the set $S_\Sigma = \xi_1 \cup \xi_2 \cup \xi_3 \cup \eta \cup \Gamma^*$ by one of 8 deformations. But the set S_Σ is invariant with respect to the reflection $(a,b,c) \to (a,b,-c)$ and rotations around the line $a = b = 0$ through angles which are multiples of $2\pi/3$. Therefore it is sufficient to consider only two small deformations of the front Σ (they are realized by suitable small deformations of the curve Γ):

Θ_1: the intersection points of branches $V_0^0V_1^1, V_2^0V_0^1, V_1^0V_2^1$ $(V_0^1V_1^0, V_2^1V_0^0, V_1^1V_2^0)$ of the curve Γ^* with the plane $c = 0$ go outside of the triangle $C_0C_1C_2$ (inside the triangle, respectively);

Θ_2: the intersection points of branches $V_0^1V_1^0, V_2^0V_0^1, V_1^0V_2^1$ $(V_0^0V_1^1, V_2^1V_0^0, V_1^1V_2^0)$ of the curve Γ^* with the plane $c = 0$ go outside of the triangle $C_0C_1C_2$ (inside the triangle, respectively);

A front obtained from Σ by one of these two deformations has the following double lines:

$$\Theta_1 : \begin{cases} V_0^0 \xrightarrow{\xi_0} C_0^- \xrightarrow{\eta} C_1^- \xrightarrow{\xi_1} V_1^1 \\ V_1^0 \xrightarrow{\xi_1} \tilde{C}_1^- \xrightarrow{\eta} C_2^- \xrightarrow{\xi_2} V_2^1 \\ V_2^0 \xrightarrow{\xi_2} \tilde{C}_2^- \xrightarrow{\eta} \tilde{C}_0^- \xrightarrow{\xi_0} V_0^1 \end{cases}, \quad \Theta_2 : \begin{cases} V_0^0 \xrightarrow{\xi_0} C_0^- \xrightarrow{\eta} C_1^- \xrightarrow{\xi_1} V_1^1 \\ V_1^0 \xrightarrow{\xi_1} \tilde{C}_1^- \xrightarrow{\eta} C_2^+ \xrightarrow{\xi_2} V_2^0 \\ V_2^1 \xrightarrow{\xi_2} \tilde{C}_2^+ \xrightarrow{\eta} \tilde{C}_0^- \xrightarrow{\xi_0} V_0^1 \end{cases}.$$

Here we denote close vertices of swallowtails of the initial and deformed fronts by the same letters; an arrow means that a double line goes along a curve written on this arrow; cusps of double lines arising from a point C_k are denoted by $C_k^{\pm}$ and $\tilde{C}_k^{\pm}$, where the sign $+(-)$ means that a given cusp is positive (negative) with respect to the orientation of a double line indicated by the arrows (the orientation of the space R^{3*} is (a, b, c)).

It easy to see that:

1) fronts obtained from Σ by deformations Θ_1, Θ_2 have no closed double lines; this proves Proposition 2.7;

2) the principal flattenings diagrams of curves in R^3 with fronts obtained from Σ by deformations Θ_1 and Θ_2 are diagrams described in claims 1 and 2 of Proposition 3.7 respectively; thus this proposition is proved as well.

6.7. Remark. Admissible homotopies of a curve preserve homotopy type of the dual curve (in the space of closed curves in P^{3*} without self-intersections but with cusps). If a generic curve γ obtained by an admissible homotopy from a curve sufficiently close to Γ has a nonempty principal flattenings diagram, then its dual curve is unknotted. If a curve γ has the empty principal flattenings diagram, then its dual curve is homotopic to the trefoil.

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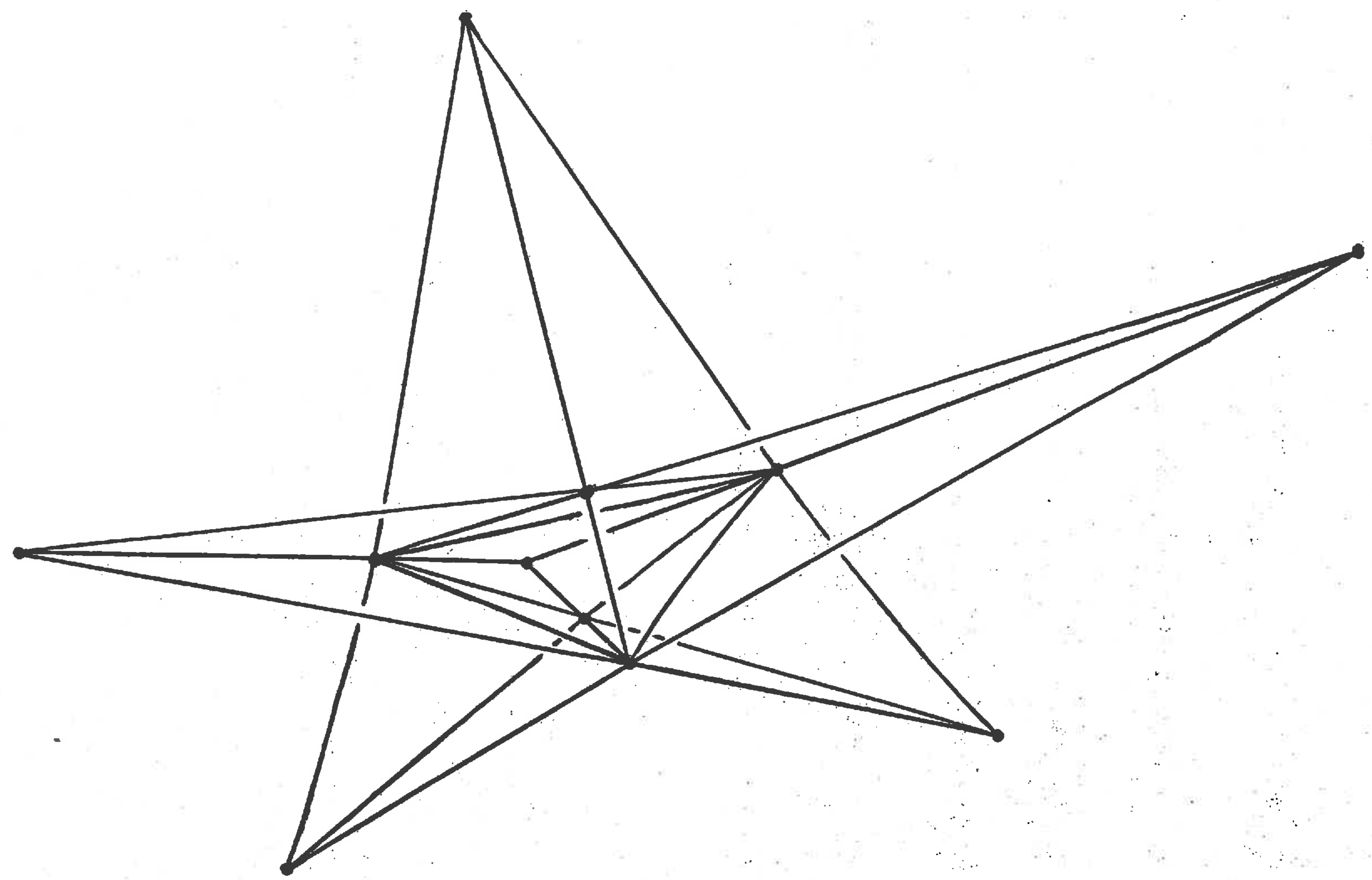

Fig. 1

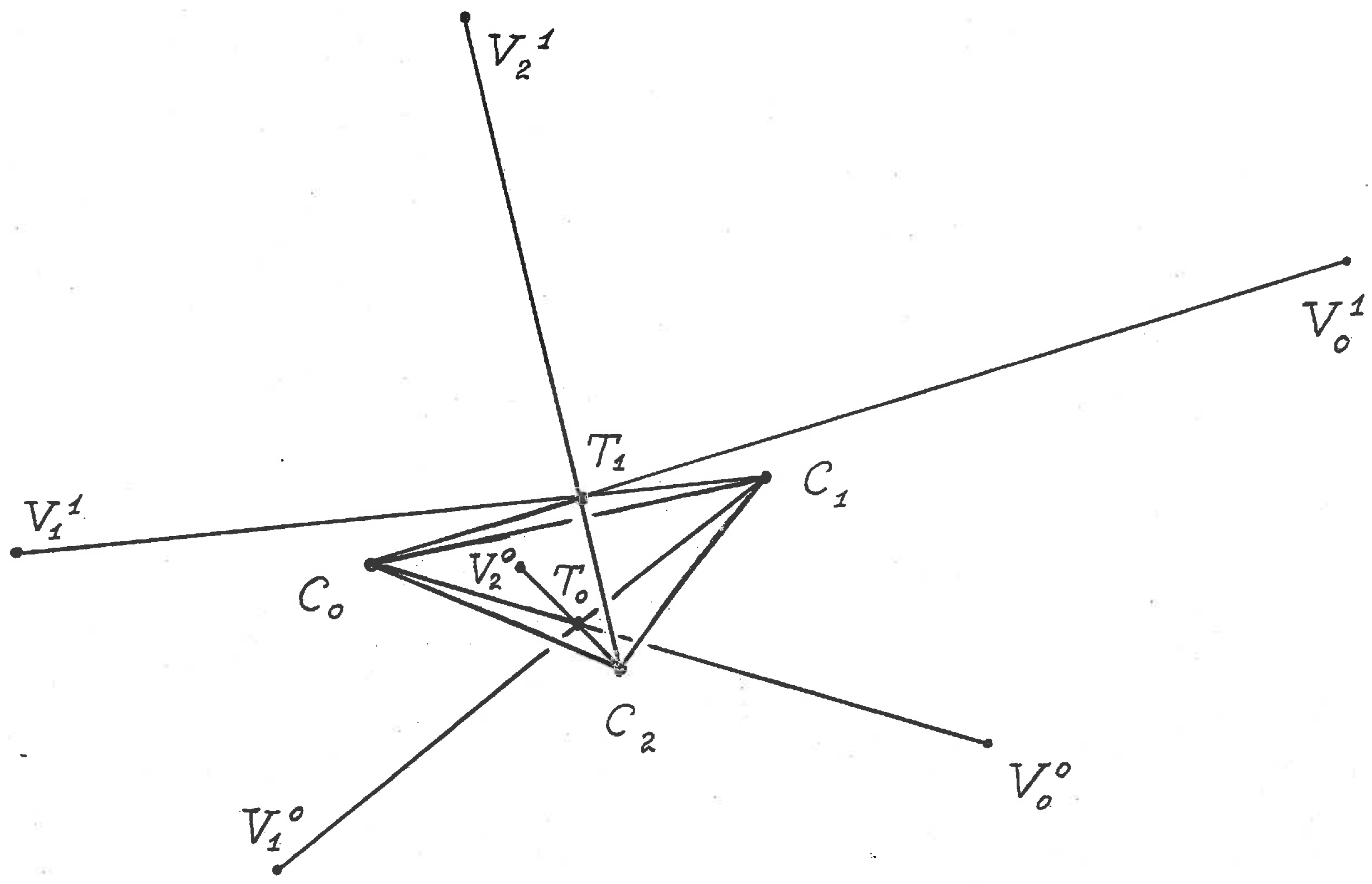

Fig. 2

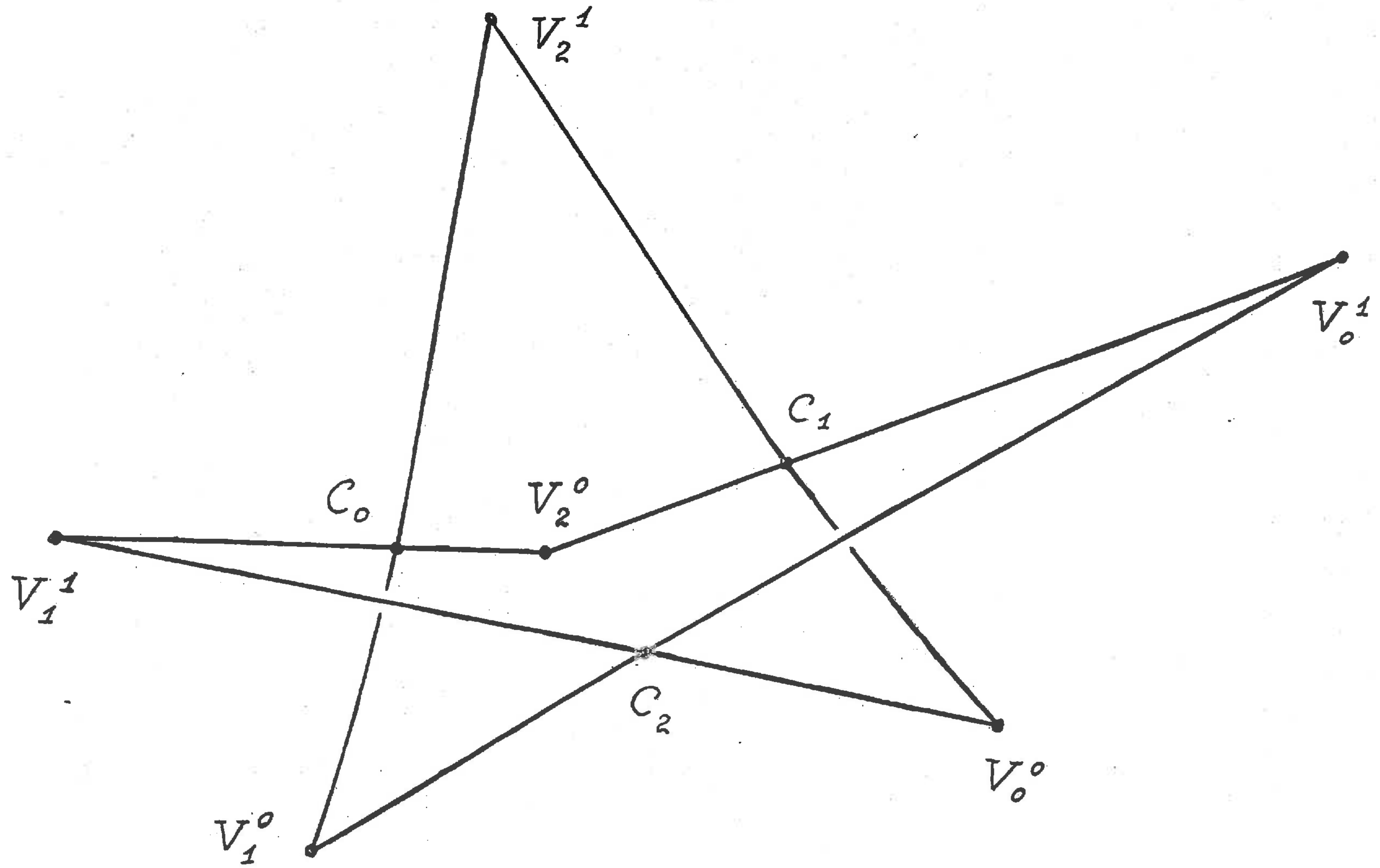

Fig. 3

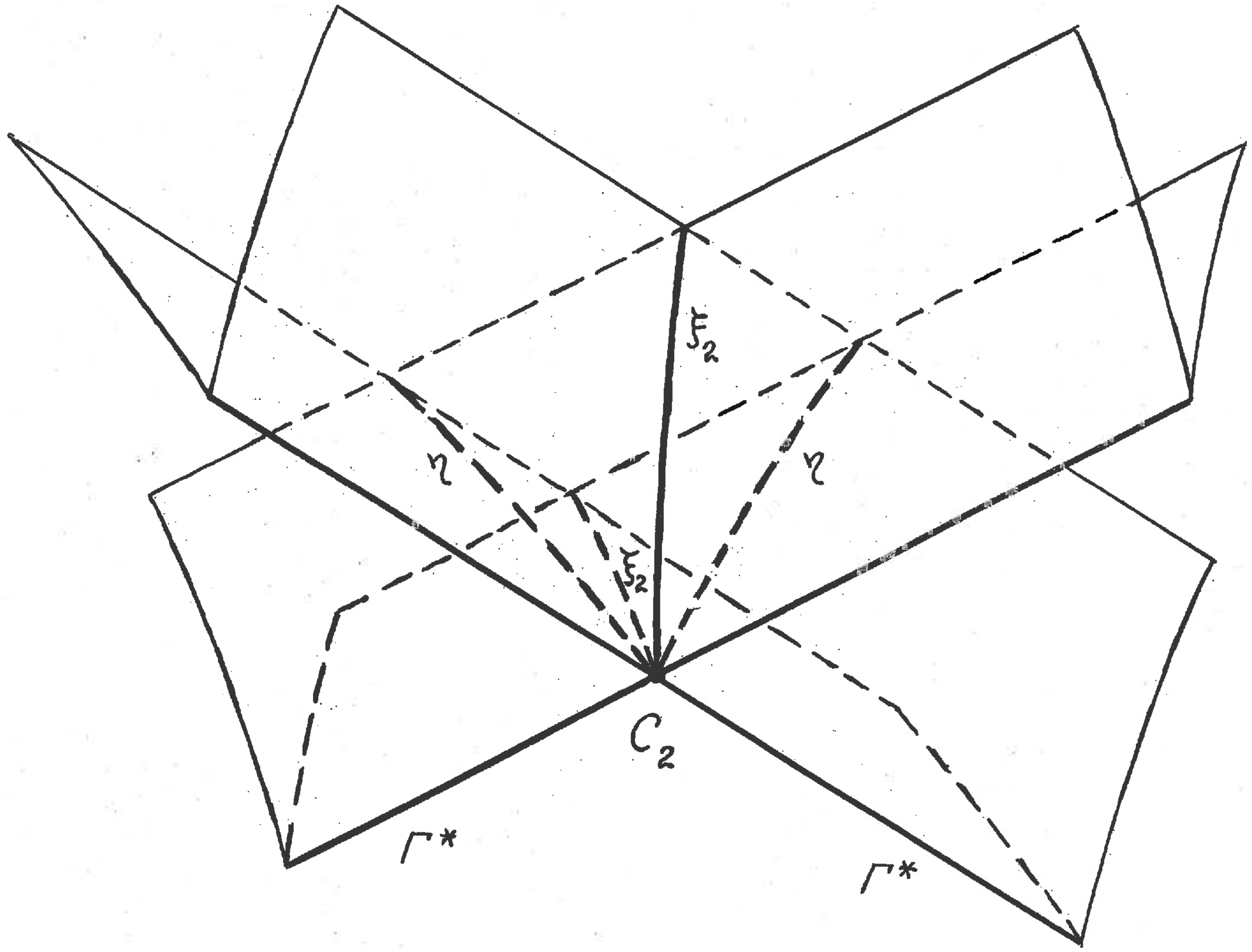

Fig. 4

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