SINGULARITIES OF LINEAR WAVES IN PLANE AND SPACE

I. A. Bogaevsky

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Abstract. The subject of the paper is the propagation of linear waves in 2D and 3D spaces. We describe some typical singularities and metamorphoses of their fronts when the light hypersurface has conical singularities. Such singularities appear if the waves propagate in a nonhomogeneous anisotropic medium and are controlled by a variational principle.

Let us consider the Euler–Lagrange system of linear partial differential equations defined by some variational principle

$$\delta \int L\,dt\,dx_1 \dots dx_n = 0$$

with a Lagrangian $L(t, x, u_t, u_x) = T(t, x, u_t) - V(t, x, u_x)$, where $t, x_1, \dots, x_n$ are the independent variables, $u_1, \dots, u_m$ are the unknown variables, the density of kinetic energy T is a positive definite quadratic form of the first time derivatives of the unknown variables, and the density of potential energy is a nonnegative definite quadratic form of the first space derivatives of the unknown variables. The coefficients of the both quadratic forms are assumed to be smooth functions of t and x. The propagation of perturbations in an elastic medium is a good model example of the above situation, where u is a shift vector of a point of the medium and the quantity of the unknown variables is equal to the space dimension ($m = n$).

The propagation of shock and short waves is described by some hypersurface in the contact space of the projectivised cotangent bundle over space-time. This hypersurface is called *light* and defined as the degeneracy set of the principal matrix symbol of our system of partial differential equations. Let the *big* front of shock wave be the hypersurface in space-time where the solution is discontinuous, a *big* front of short wave approximation be a level hypersurface of its phase. Sections of big fronts with isochrones $t = \text{const}$ are *momentary* fronts propagating with time. The momentary front at the fixed time $t = 0$ is called *initial.* The propagation of a momentary front is defined by the light hypersurface and the initial front. Even if the initial front is smooth the momentary front can become singular after some time.

A momentary front propagating with time can experience metamorphoses. If the light hypersurface is smooth then, in case of a generic smooth initial front, all

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singularities of momentary and big fronts as well as metamorphoses of momentary fronts are known, provided that the dimension of space does not exceed five (see, for example, [A1, Chapter 3] and [B1]). However, if the quantity of the unknown variables is more than one ($m > 1$), there can be singularities on the light hypersurface. When the coefficients of the original variational principle depend generically on a point of space (or space-time), typical singularities of the light hypersurface are classified up to formal contact diffeomorphisms in [A2] (see also [A1, Chapter 8] and [Kh]). These singularities generate *new* singularities of momentary fronts, *new* singularities of big fronts, and *new* metamorphoses of momentary fronts.

In the present paper we classify up to diffeomorphisms some of such new singularities of momentary and big fronts in physically interesting cases $n = 2$ and $n = 3$ provided that the smooth initial front is generic. The nontrivial metamorphoses of momentary fronts defined by our new singularities of big fronts are shown in Fig. 1 for $n = 2$ and in Fig. 2 and Fig. 3 for $n = 3$.

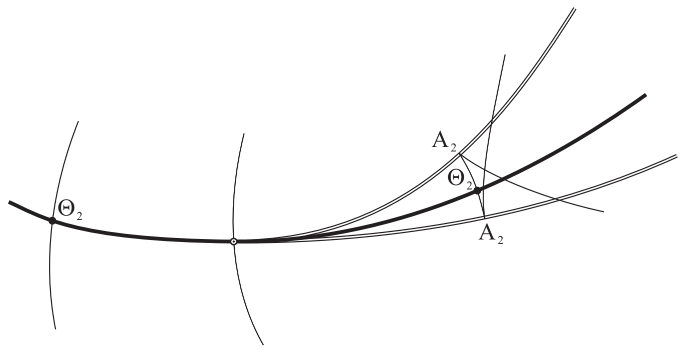

Fig. 1: New metamorphose Θ_3 of momentary fronts in plane.

In mathematical terms the big front is the projection of the Legendre submanifold being the extension of the initial front along characteristics of the light hypersurface. New singularities of momentary and big fronts appear when our Legendre manifold becomes singular. According to [A1, Chapter 8], if $n = 2$ it can have only two different singularities up to contact diffeomorphisms. Their codimensions on the Legendre submanifold are 1 and 2. The corresponding new singularities of big fronts and metamorphoses of momentary fronts are described in [B2] and [B3] only in case of codimension 2 – the simplest but nontrivial case of the singularity of codimension 1 is missed there! The missed metamorphose of a momentary front is shown in Fig. 1.

In Theorem 1 of the present paper we find the stabilizations of the singularities of the Legendre submanifold from [A1, Chapter 8] if $n > 2$. We call these singularities as Λ_1 and Λ_2, on the Legendre submanifold they have the same codimensions 1 and 2 respectively. In case $n = 2$ the singularities Λ_1 and Λ_2 exhaust all possible singularities of the Legendre submanifold up to contact diffeomorphisms provided that the coefficients of the original variational principle depend generically on a point of space and the initial front is generic. In case $n = 3$ there exist other singularities whose classification is an open problem.

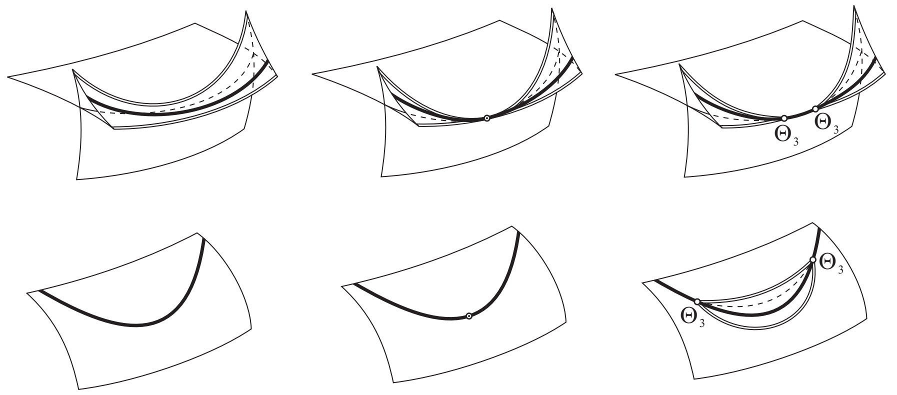

Fig. 2: New metamorphoses Θ_3 of momentary fronts in 3D space.

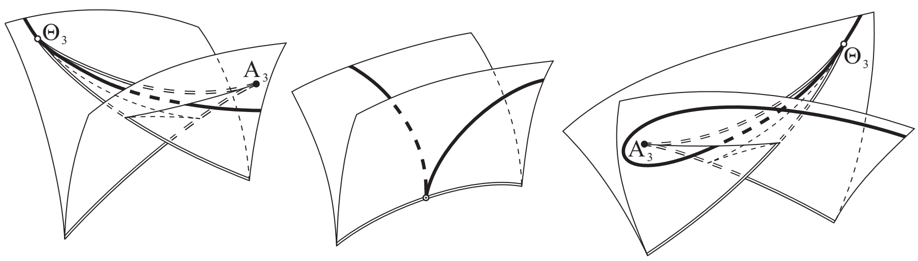

Fig. 3: New metamorphose Ξ_4 of momentary fronts in 3D space.

Main Theorem of the present paper classifies typical fronts of the singularity Λ_1 with respect to diffeomorphisms in physically interesting cases $n \leq 3$. Our classification consists of five new singularities Θ_2, Θ_3, $\Theta_4^{\pm}$, and Ξ_4 of momentary and big fronts. All of them are *stable* in the following sense: after any small perturbation of the initial front and the coefficients of the variational principle the front does not change up to a local diffeomorphism.

Distribution of material. In Section 1 we formulate Main Theorem giving normal forms of our new singularities Θ_2, Θ_3, $\Theta_4^{\pm}$, and Ξ_4 of momentary and big fronts with respect to diffeomorphisms. Main Theorem asserts that in physically interesting cases $n \leq 3$ all generic fronts of singularities Λ_1 of a Legendre submanifold are exhausted by this list up to diffeomorphisms.

Section 2 contains normal forms of typical singularities of the light hypersurface with respect to formal contact diffeomorphisms. These normal forms are found in [A2] provided the light hypersurface satisfies some extra conditions of non-degeneracy. According to the transversality theorem from [Kh], these conditions are realized if the coefficients of the original variational principle depend generically on a point of space-time.

In Section 3 we formulate and prove Theorem 1 describing singularities Λ_1 and Λ_2 of the extension of a generic initial front along characteristics of the light hypersurface which has the singularities from [A2] presented in Section 2. For $n = 2$ the singularity Λ_2 is found in [A1, Chapter 8]. If $n = 2$ the singularities Λ_1 and Λ_2 exhaust all possible singularities of the extension with respect to contact diffeomorphisms preserving the light hypersurface. Even if $n = 3$ there exist other singularities whose classification is an open problem.

Main Theorem is a corollary of Theorem 2 presented in Section 4 and describing normal forms of generic Legendre bundles up to the so-called weak Λ_1-equivalence provided $n \leq 3$. We start to prove Theorem 2 with finding in Section 5 normal forms for separate fibers with respect to contact diffeomorphisms preserving the normal form Λ_1 itself. The corresponding results are formulated in Theorem 3 which is reduced in Section 5 to Lemma 2.

We prove Lemma 2 in Section 6 with the help of the standard homotopy method applied to the group of contact diffeomorphisms preserving the normal form Λ_1. It turns out that the Legendre bundles from Theorem 2 are versal deformations of the separate fibers from Theorem 3 in the class of all smooth Legendre submanifolds. We develop the corresponding versality theory in Section 6. It turns out, for example, that the infinitesimal versality in our situation is nothing but the Givental' criterion of stability of the Legendre mapping of the singularity Λ_1 (see [G, 3.3]).

1. Singularities and Metamorphoses of Fronts

Case $n = 2$. In this case the momentary front at typical time can have cusps A_2 and new stable singularities Θ_2, the latter ones are discontinuities of the third derivative and propagate along rays. At separate times the momentary front can experience the new metamorphose shown in Fig. 1. The singularities Θ_2 of the momentary fronts propagate along a smooth ray and their cusps run through a couple of smooth curves with an infinite order of tangency. Our metamorphose is described by a big front lying in 3D space-time and called Θ_3. The front Θ_3 looks like the usual swallow tail but its cuspidal edge consists of two smooth curves with an infinite order of tangency.

Case $n = 3$. In this case the momentary front at typical time can have cuspidal edges A_2, swallow tails A_3, and new stable singularities Θ_2 and Θ_3. The singularities Θ_2 propagate along rays. New singularities of the big front are Θ_2, Θ_3, $\Theta_4^{\pm}$, and Ξ_4. The two possible metamorphoses Θ_3 are shown in Fig. 2, the metamorphose Ξ_4 – in Fig. 3, the singularity Θ_2 does not give us a new metamorphose of momentary fronts, and the metamorphoses $\Theta_4^{\pm}$ are topologically trivial – before and after their instants the momentary fronts have the singularity Θ_3 and are locally homeomorphic to the momentary fronts at the instants of the metamorphoses. It should be noted that during the metamorphose Ξ_4 the singularities Θ_2 run through the Whitney umbrella.

Normal forms of fronts. Normal forms of the singularities Θ_2, Θ_3, $\Theta_4^{\pm}$, and Ξ_4 are given in local coordinates $(y, z) = (y_1, \dots, y_n, z)$ by the equations

$$z = F(s, y), \quad F_s(s, y) = 0,$$

where $s = (s_1, s_2)$ are parameters and

Θ_2) $F = -s_1^2 \ln (s_1^2/e) + y_1 s_1$;

$\Theta_k^{\pm}$) $F = -s_1^2 \ln (s_1^2/e) + s_1 s_2 \pm s_2^k + y_1 s_2 + \cdots + y_{k-1} s_2^{k-1}$ where $n+1 \geq k \geq 3$;

Ξ_4) $F = -s_1^2 \ln (s_1^2/e) + s_2^3 + y_1 s_1 + y_2 s_2 + y_3 s_1 s_2$.

Remarks. 1) The singularity Θ_2 is given by the equation $z = \varphi(y_1) = y_1^2/\ln y_1^2 + o(y_1^2/\ln y_1^2)$ as $y_1 \to 0$ where φ is just the Legendre transform of the function $-s_1^2 \ln (s_1^2/e)$.

2) The change $s \mapsto -s$ shows that the singularities Θ_k^+ and Θ_k^- are diffeomorphic if k is odd.

3) Removing from F the terms of degree 3 and more we get the following asymptotic normal form for the singularities $\Theta_k^{\pm}$ where $k \geq 3$:

$$F = -s_1^2 \ln (s_1^2/e) + s_1 s_2 + y_1 s_2 + y_2 s_2^2.$$

It shows us that these singularities are homeomorphic to each other.

Definition. A smooth bundle $\pi : E \to B$ is called *Legendre* if its space E is a contact manifold and the fibers are Legendre submanifolds. The image $\pi(\Lambda)$ of a Legendre submanifold $\Lambda \subset E$ is called its *front*.

Remark. In our case $E = PT^*\mathbb{R}^{n+1}$ and $B = \mathbb{R}^{n+1}$.

Main Theorem. *Let $(p, q, u) = (p_1, \dots, p_n, q_1, \dots, q_n, u)$ be coordinates in E such that the contact structure is given by the form*

$$\theta = du - (pdq - qdp)/2,$$
$$\Lambda_1 = \{2p_1 \ln p_1^2 + q_1 = 0,\ p_2 = \cdots = p_n = 0,\ u + p_1^2 = 0\} \subset E$$

be a fixed Legendre submanifold, and $\pi : E \to B$ be a generic Legendre bundle.

Then, provided $n \leq 3$, the fronts of the germs of Λ_1 at its singular points are diffeomorphic to the normal forms Θ_2, Θ_3, $\Theta_4^{\pm}$, and Ξ_4.

2. Singularities of Light Hypersurface

Let $\Sigma^{2n} \subset PT^*\mathbb{R}^{n+1}$ be the light hypersurface of our variational principle. According to [A1, Chapter 8], [A2], and [Kh], if its coefficients depend generically on a point of space-time and $n > 1$, in neighborhoods of typical singular points Σ^{2n} is reduced by formal contact diffeomorphisms to one of the two following normal forms

$$p_1^2 + p_2 q_2 = 0, \quad p_1^2 = p_2^2 + q_2^2$$

which are called *hyperbolic* and *elliptic* respectively. Here (p, q, u) are coordinates in $PT^*\mathbb{R}^{n+1}$ such that the contact structure is given by the form $\theta = du - (pdq - qdp)/2$.

In contrast to the elliptic case, each hyperbolic singularity of Σ^{2n} has two characteristics passing through it. Let $H^{2n-2} \subset \Sigma^{2n}$ be the manifold of all hyperbolic singularities of the light hypersurface and $\widetilde{H}^{2n-1} \subset \Sigma^{2n}$ be the union of all its characteristics passing through H^{2n-2}. In the above local coordinates $\widetilde{H}^{2n-1} = \{p_1 = p_2 = 0\} \cup \{p_1 = q_2 = 0\}$.

3. Singularities of Legendre Submanifolds

An initial front defines a Legendre submanifold $L^n \subset PT^*\mathbb{R}^{n+1}$ consisting of all contact elements which are tangent to the initial front. The transversal intersection $\mathcal{L}^{n-1} = L^n \cap \Sigma^{2n}$ is an integral submanifold of $PT^*\mathbb{R}^{n+1}$ and the union of all characteristics of Σ^{2n} beginning on $\mathcal{L}^{n-1}$ is a Legendre submanifold denoted by $\widetilde{\mathcal{L}}^n \subset PT^*\mathbb{R}^{n+1}$. Its projection is the big front in space-time describing the propagation of momentary fronts defined by the initial one. If some characteristic beginning on L^n ends at a hyperbolic singularity of Σ^{2n}, then after this instant the Legendre submanifold $\widetilde{\mathcal{L}}^n$ acquires singularities which are described by Theorem 1 proved in [A1, Chapter 8] for $n = 2$.

Theorem 1. *Let L^n not pass through singular points of Σ^{2n} and transversally intersect $\widetilde{H}^{2n-1}$ at a point O. Let $P \in H^{2n-1}$ be the endpoint of the characteristic beginning at O. Then the Legendre submanifold $\widetilde{\mathcal{L}}^n$ is reduced in a neighborhood of P to the normal form*

$$\Lambda_2 = \left\{2p_1 \ln p_2 + q_1 = 0,\ p_1^2 + p_2 q_2 = 0,\ p_3 = \cdots = p_n = 0,\ u + p_1^2/2 = 0\right\}$$

by a contact diffeomorphism preserving the light hypersurface $p_1^2 + p_2 q_2 = 0$.

Corollary. *The Legendre submanifold $\widetilde{\mathcal{L}}^n$ described by Theorem 1 has singularities when $p_1 = \cdots = p_n = q_1 = u = 0$ and $q_2 \le 0$. If $q_2 < 0$ then in neighborhoods of these singularities $\widetilde{\mathcal{L}}^n$ is reduced to the normal form*

$$\Lambda_1 = \left\{2p_1 \ln p_1^2 + q_1 = 0,\ p_2 = \cdots = p_n = 0,\ u + p_1^2 = 0\right\}$$

by a local contact diffeomorphism (reducing the light hypersurface to the form $p_2 = 0$).

Proof. Explicitly: $p_2 \mapsto (p_1^2 - p_2)e^{-q_2}$, $q_1 \mapsto q_1 + 2p_1 q_2$, $q_2 \mapsto -e^{q_2}$, and $u \mapsto u + (p_1^2 - p_2 + p_2 q_2)/2$. □

Proof of Theorem 1. Our proof is analogous to the one proposed in [A1, Chapter 8] for $n = 2$. In the above coordinates the characteristics of Σ^{2n} are described by the equations

$$\text{(1)} \qquad \dot{p}_1 = 0,\ \dot{p}_2 = -p_2,\ \dot{q}_1 = 2p_1,\ \dot{q}_2 = q_2,\ \dot{p}_\star = \dot{q}_\star = 0,\ \dot{u} = 0$$

where $p_\star = (p_3, \ldots, p_n)$ and $q_\star = (q_3, \ldots, q_n)$. Let the considered characteristic $\overrightarrow{OP}$ be given by the equations $p_1 = 0$, $p_2 \ge 0$, $q_1 = q_2 = 0$, $p_\star = q_\star = 0$, and $u = 0$ (maybe after an obvious change of the variables). On Σ^{2n} the intersection

$$\widetilde{\mathcal{L}}^{n-1} = \{p_2 = 1\} \cap \widetilde{\mathcal{L}}^n$$

is transversal to $\widetilde{H}^{2n-1}$ whose equation on Σ^{2n} is $p_1 = 0$. So, $(p_1, q_\star)$ will be coordinates on $\widetilde{\mathcal{L}}^{n-1}$ after several contact changes preserving Σ^{2n} and having the form $(p_i, q_i) \mapsto (q_i, -p_i)$ where $i = 3, \ldots, n$. Therefore,

$$\widetilde{\mathcal{L}}^{n-1} = \{p_2 = 1,\ p_\star = f_{q_\star}(p_1, q_\star),\ q_1 = -f_{p_1}(p_1, q_\star),\ q_2 = -p_1^2,\ u = g(p_1, q_\star)\}.$$

Indeed, the function f exists because $d\left((-q_1dp_1 + p_\star dq_\star)|_{\widetilde{\mathcal{L}}^{n-1}}\right) = dp \wedge dq|_{\widetilde{\mathcal{L}}^{n-1}} = -d\theta|_{\widetilde{\mathcal{L}}^{n-1}} = 0$. The symplectic change

$$(p_1, p_2, p_\star, q_1, q_2, q_\star) \mapsto (p_1, p_2, p_\star - f_{q_\star}(p_1, q_\star), q_1 + f_{p_1}(p_1, q_\star), q_2, q_\star)$$

kills f preserving Σ^{2n}. The corresponding contact change reduces g to $-p_1^2/2$ because on $\widetilde{\mathcal{L}}^{n-1}$ we get $du = (p_2dq_2 - q_2dp_2)/2 = -p_1dp_1$ if $f = 0$. Thus,

$$\widetilde{\mathcal{L}}^{n-1} = \{p_2 = 1,\ p_\star = 0,\ q_1 = 0,\ q_2 = -p_1^2,\ u = -p_1^2/2\}.$$

But the functions $2p_1 \ln p_2 + q_1$, $p_1^2 + p_2q_2$, $p_\star$, and $u + p_1^2/2$ are constant along the characteristics (1) and vanish on $\widetilde{\mathcal{L}}^{n-1}$. □

4. Normal Forms of Legendre Bundles

Main Theorem follows from Theorem 2 formulated in this Section and proved in Section 6.

Definition. Let $\Lambda \subset E$ be a Legendre submanifold. Two Legendre bundles $\pi, \pi' : E \to B$ are called *weakly Λ-equivalent* if $\pi' \circ h = f \circ \pi$ where f is a diffeomorphism of B and h is a diffeomorphism of E which preserves Λ.

Remarks. 1) If π and π' are weakly Λ-equivalent then the fronts $\pi(\Lambda)$ and $\pi'(\Lambda)$ are diffeomorphic.

2) The diffeomorphism g is not required to be contact.

Definition. Let $W : \mathbb{R}^n \times \mathbb{R}^{n+1} \to \mathbb{R}$ be a smooth function of p_I, q_J, and $(y_1, \dots, y_n, z) \in \mathbb{R}^{n+1}$, which satisfies the condition of nondegeneracy:

$$\det \left\| \begin{matrix} W_{p_I y} & W_{p_I z} \\ W_{q_J y} & W_{q_J z} \\ W_y & W_z \end{matrix} \right\| \neq 0$$

where $I \cap J = \emptyset$, $I \cup J = \{1, \dots, n\}$. Then W is called a *generating family* of the Legendre bundle $\pi : (p, q, u) \mapsto (y, z)$ whose contact structure and fibers are given by the formulas

$$du - (pdq - qdp)/2 = 0,$$
$$\pi^{-1}(y, z) = \{p_J = W_{q_J}, q_I = -W_{p_I}, u = W - p_I W_{p_I}/2 - q_J W_{q_J}/2\}.$$

Remark. This bundle is correctly defined in consequence of nondegeneracy of W.

Theorem 2. *If $n \leq 3$ then in a neighborhood of a singular point of the Legendre submanifold Λ_1 a generic Legendre bundle is weakly Λ_1-equivalent to one of the normal forms given in a neighborhood of the origin by the following generating families:*

Θ_2 *) $W(p, y, z) = y_1p_1 + \dots + y_np_n - z$;*

$\Theta_k^\pm$*) $W(q_1, p_2, \dots, p_n, y, z) = \pm q_1^k + y_1q_1 + \dots + y_{k-1}q_1^{k-1} + y_2p_2 + \dots + y_np_n - z$, where $n + 1 \geq k \geq 3$;*

Ξ_4) $W(p_1, q_2, p_3 \dots, p_n, y, z) = q_2^3 + y_1 p_1 + y_2 q_2 + y_3 p_1 q_2 + y_3 p_3 + \dots + y_n p_n - z$, *where* $n \geq 3$.

Remarks. 1) The change $(p_1, q_1) \mapsto (-p_1, -q_1)$ shows that the singularities Θ_k^+ and Θ_k^- are reduced to each other if k is odd.

2) One can show that all simple stable mappings of the Legendre submanifold Λ_1 are exhausted by the singularities $\Theta_k^\pm$ and Ξ_4.

Proof of Main Theorem. Explicit check that the fronts of Λ_1 from Theorem 2 give the normal forms from Main Theorem. □

5. Normal Forms of Fibers

We start to prove Theorem 2 with finding normal forms for separate fibers which pass through singular points of the Legendre submanifold Λ_1 with respect to contact diffeomorphisms preserving Λ_1 itself. The corresponding results are formulated in Theorem 3.

Definition. Let $\Lambda \subset E$ be a Legendre submanifold. Two Legendre submanifolds are called Λ-*equivalent* if they are the same with respect to a contact diffeomorphism preserving Λ.

If the contact structure is $du - (pdq - qdp)/2 = 0$ then in the coordinates (p, q, u) any Legendre submanifold L is locally given with the help of at least one of the 2^n *generating functions* $w(p_I, q_J)$ by the formulas

$$p_J = w_{q_J}, \quad q_I = -w_{p_I}, \quad u = w - p_I w_{p_I}/2 - q_J w_{q_J}/2$$

where $I \cap J = \emptyset$ and $I \cup J = \{1, \dots, n\}$. On the other hand, if (p_I, q_J) are local coordinates on L then

$$w(p_I, q_J) = \psi_I|_L, \text{ where } \psi_I(p, q, u) = u - (p_I q_I - p_J q_J)/2,$$

is its generating function. For example, $w(p_1, q_2, \dots, q_n) = p_1^2 \ln(p_1^2/e)$ is a generating function of the Legendre submanifold Λ_1.

Theorem 3. *Let us consider a generic family $L_\star$ of Legendre submanifolds depending on a point $b \in B$ where* $\dim B \leq 4$. *Let L_b be any Legendre submanifold from $L_\star$ which intersects the Legendre submanifold Λ_1 at its singular point. Then for any $n \geq 1$ in a neighborhood of this point L_b is Λ_1-equivalent to one of the normal forms given in a neighborhood of the origin by the following generating functions:*

Θ_2) $w(p) = 0$;

$\Theta_k^\pm$) $w(q_1, p_2, \dots, p_n) = \pm q_1^k$, *where* $\dim B \geq k \geq 3$;

Ξ_4) $w(p_1, q_2, p_3 \dots, p_n) = q_2^3$, *where* $\dim B \geq 4$.

Remark. The fibers of a generic Legendre bundle form a family of Legendre submanifolds depending generically on a point $b \in B$ where $\dim B = n + 1$.

Proof. Let L_b intersect Λ_1 at its singular point $p = 0$, $q = q^0 = (0, q_2^0, \dots, q_n^0)$, $u = 0$. The contact diffeomorphism $p \mapsto p$, $q \mapsto q - q^0$, $u \mapsto u + pq^0/2$ moves this point to 0 and preserves Λ_1. So the Legendre submanifold obtained from L_b can be locally given by a generating function w such that $w(0) = w_{p_I}(0) = w_{q_J}(0) = 0$.

The singularities of Λ_1 form a submanifold of codimension $n + 2$ in E. So the germs of Legendre submanifolds from $L_\star$ at singular points of Λ_1 form a family depending generically on $\dim B + \dim L_b - (n+2) = \dim B - 2 \leq 2$ parameters. Any germ from such a family can be given by a generating function $w(p_I, q_J)$ where $\sharp J = 0$ or 1. Indeed, the condition $\dim TL_b \cap \{dp = du = 0\} > 1$ for the tangent plane TL_b to the germ of L_b requires at least three parameters. It remains to prove the following Lemma 1. $\square$

Notation. For two germs w and w' of generating functions we write $w \sim_\Lambda w'$ if the corresponding germs of Legendre submanifolds are Λ-equivalent.

Lemma 1. *If $w(p_I, q_J)$ is the germ at 0 of a generating function such that $w(0) = w_{p_I}(0) = w_{q_J}(0) = 0$ then*
a) $J = \emptyset \Rightarrow w \sim_{\Lambda_1} 0$;
b) $J = \{1\}$, $\partial^2_{q_1} w(0) = \cdots = \partial^{k-1}_{q_1} w(0) = 0$, $\partial^k_{q_1} w(0) \neq 0$, $k \geq 3 \Rightarrow w \sim_{\Lambda_1} \pm q_1^k$;
c) $J = \{2\}$, $\partial_{p_1}\partial_{q_2} w(0) = \partial^2_{q_2} w(0) = 0$, $\partial^3_{q_2} w(0) \neq 0 \Rightarrow w \sim_{\Lambda_1} q_2^3$.

Proof. It is sufficient to prove the cases a), b) for $n = 1$ and the case c) for $n = 2$. This follows from the equivalence $w \sim_{\Lambda_1} w_0$ where $w_0(p_I, q_J) = w|_{p_{I''}=0}$, $I' = I \cap \{1\}$, and $I'' = I \cap \{2, \ldots, n\}$. The equivalence is performed by the contact diffeomorphism

$$(p_I, p_J, q_I, q_J, u) \mapsto (p_I, p_J - \widehat{w}_{q_J}, q_I + \widehat{w}_{p_I}, q_J, u - \widehat{w} + p_I \widehat{w}_{p_I}/2 + q_J \widehat{w}_{q_J}/2)$$

where $\widehat{w} = w - w_0$. This diffeomorphism preserves Λ_1 because it shifts the plane $p_{I''} = 0$ along only $q_{I''}$ (preserving p_I, p_J, $q_{I'}$, q_J, and u) that follows from the equality $\widehat{w}|_{p_{I''}=0} = 0$.

The infinite chains $a_2 \Rightarrow a_3 \Rightarrow \ldots$, $b^k \Rightarrow b^k_{k+1} \Rightarrow b^k_{k+2} \Rightarrow \ldots$, and $c^6 \Rightarrow c^6_7 \Rightarrow c^6_8 \Rightarrow \ldots$ of propositions of the following Lemma 2 prove the cases a), b), and c) respectively on the level of formal series. To prove Lemma 1 in smooth case it is enough to use the finite-determinacy theorem [AGLV, Chapter 3, § 2] for the nice geometric group of Λ-equivalence [AGLV, Chapter 3, 2.5]. $\square$

Lemma 2. *Let $\alpha_I = \deg p_I$ and $\beta_J = \deg q_J$ be positive integer quasidegrees and $\mathcal{A}_0 \supseteq \mathcal{A}_1 \supseteq \ldots$ be the corresponding quasihomogeneous filtration in the algebra of the germs at 0 of smooth functions of p_I and q_J. Then*
a_d) $n = 1$, $J = \emptyset$, $\alpha_1 = 1$, $w_d \in \mathcal{A}_d$, $d \geq 2 \Rightarrow w_d \sim_{\Lambda_1} 0 \pmod{\mathcal{A}_{d+1}}$;
b^k) $n = 1$, $J = \{1\}$, $\beta_1 = 1$, $w_k \in \mathcal{A}_k$, $\partial^k_{q_1} w_k(0) \neq 0$, $k \geq 3 \Rightarrow w_k \sim_{\Lambda_1} \pm q_1^k \pmod{\mathcal{A}_{k+1}}$;
b^k_d) $n = 1$, $J = \{1\}$, $\beta_1 = 1$, $w_d \in \mathcal{A}_d$ $d > k \geq 3 \Rightarrow \pm q_1^k + w_d \sim_{\Lambda_1} \pm q_1^k \pmod{\mathcal{A}_{d+1}}$;
c^6) $n = 2$, $J = \{2\}$, $\alpha_1 = 3$, $\beta_2 = 2$, $w_6 \in \mathcal{A}_6$, $\partial^3_{q_2} w_6(0) \neq 0 \Rightarrow w_6 \sim_{\Lambda_1} q_2^3 \pmod{\mathcal{A}_7}$;
c^6_d) $n = 2$, $J = \{2\}$, $\alpha_1 = 3$, $\beta_2 = 2$, $w_d \in \mathcal{A}_d$, $d > 6 \Rightarrow q_2^3 + w_d \sim_{\Lambda_1} q_2^3 \pmod{\mathcal{A}_{d+1}}$.

Lemma 2 is proved in Section 6.

6. Contact Vector Fields and Λ-versality

In this Section we prove Theorem 2 and Lemma 2. Theorem 2 follows from Theorem 3 which was reduced to Lemma 2 in Section 5.

A vector field preserving a contact structure on a manifold is called *contact.* It is well known that any contact field is uniquely defined by its *generator.* If the contact structure is given as the null subspaces of a 1-form θ then the generator of a contact vector field v is the function $K = \theta(v)$. In our case $\theta = du - (pdq - qdp)/2$ and v is defined by the formulas

$$\dot{p} = K_q + pK_u/2, \quad \dot{q} = -K_p + qK_u/2, \quad \dot{u} = K - (pK_p + qK_q)/2.$$

Let $L(w)$ be the Legendre submanifold given by a generating function $w(p_I, q_J)$:

$$L(w) = \{p_J = w_{q_J}, q_I = -w_{p_I}, u = w - p_I w_{p_I}/2 - q_J w_{q_J}/2\}$$

and $K(w)$ denote the derivative of the generating function when the Legendre submanifold is acted by the contact vector field v with the generator K.

Lemma 3. $K(w) = K|_{L(w)}$

Proof. Indeed, (p_I, q_J) are local coordinates on $L(w)$ and $w(p_I, q_J) = \psi_I|_{L(w)}$ where $\psi_I(p, q, u) = u - (p_I q_I - p_J q_J)/2$. So, $K(w) = \dot{\psi}_I|_{L(w)} - (\dot{p}_I|_{L(w)} w_{p_I} + \dot{q}_J|_{L(w)} w_{q_J}) = (\dot{\psi}_I + \dot{p}_I q_I - \dot{q}_J p_J)|_{L(w)} = (\dot{u} + \dot{p}q/2 - p\dot{q}/2)|_{L(w)} = K|_{L(w)}$. □

Definition. Let $\Lambda \subset E$ be a Legendre submanifold. Two families $L_\star, L'_\star$ of Legendre submanifolds depending on a point $b \in B$ are called Λ-*equivalent* if $L'_{f(b)} = g_b(L_b)$ where f is a diffeomorphism of B and $g_\star$ is a family of contact diffeomorphisms of E preserving Λ and depending on a point $b \in B$.

Lemma 4. *Two Legendre bundles $\pi, \pi' : E \to B$ are weakly Λ-equivalent if the families of their fibers are Λ-equivalent.*

Proof. Because the families ${\pi'}^{-1}(\star)$, $\pi^{-1}(\star)$ are Λ-equivalent we get ${\pi'}^{-1}(f(b)) = g_b(\pi^{-1}(b)) = h(\pi^{-1}(b))$ where $h(e) = g_{\pi(e)}(e)$, $e \in E$. The mapping h preserves Λ (but not the contact structure in general) and performs diffeomorphisms between the fibers $\pi^{-1}(b)$ and ${\pi'}^{-1}(f(b))$ for any $b \in B$. Therefore, h is a diffeomorphism such that $\pi' \circ h = f \circ \pi$. □

Let $\mathcal{E}$ be the algebra of the smooth functions on E and $\mathcal{I}_\Lambda \subset \mathcal{E}$ be the ideal consisting of all functions vanishing on Λ. Contact vector fields with generators from $\mathcal{I}_\Lambda$ are tangent to Λ. Then, according to Lemma 3, the tangent space to the Λ-equivalence orbit of the Legendre submanifold $L(w)$ is the restriction $\mathcal{I}_\Lambda|_{L(w)}$. Let $W(p_I, q_J, b)$ be a smooth deformation of the generating function $W(p_I, q_J, 0) = w(p_I, q_J)$ and $\dot{W} = \partial_b W|_{b=0} \in \mathcal{E}_{p_I q_J}$ be its initial velocities which are elements of the algebra $\mathcal{E}_{p_I q_J}$ of the smooth functions of p_I and q_J.

Definition. The deformation W of the generating function w is called *infinitesimally Λ-versal* if it is transversal to the Λ-equivalence orbit of $L(w)$:

$$\langle \dot{W} \rangle_{\mathbb{R}} + \mathcal{I}_\Lambda|_{L(w)} = \mathcal{E}_{p_I q_J}.$$

Remarks. 1) If W is a generating family of a Legendre bundle $\pi : E \to B$ then Λ-versality of W is nothing but the Givental' criterion of stability of the Legendre mapping $\Lambda \hookrightarrow E \xrightarrow{\pi} B$ from [G, 3.3].

2) In order to define Λ-versality without coordinates let us consider a deformation $L_\star = g_\star(L)$ of a Legendre submanifold $L_0 = g_0(L)$ where $g_\star$ is a family of contact diffeomorphisms depending smoothly on a point $b \in B$ and L is a fixed Legendre

submanifold. Let $\dot g_0(g_0(e)) = \partial_b g_b(e)|_{b=0}$ be the initial velocities of the deformation $g_\star$ which are contact vector fields on E. In this situation the deformation $L_\star$ is called *infinitesimally* Λ*-versal* if

$$\langle \theta(\dot g_0)\rangle_{\mathbb{R}} + \mathcal{I}_\Lambda + \mathcal{I}_{L_0} = \mathcal{E}$$

where $\mathcal{I}_{L_0} \subset \mathcal{E}$ is the ideal consisting of all functions vanishing on L_0. The point is that contact vector fields with generators from $\mathcal{I}_{L_0}$ move L_0 along itself.

Proof of Theorem 2. The generating families from Theorem 2 are infinitesimally Λ_1-versal deformations of the generating functions from Theorem 3. Indeed, $\mathcal{I}_{\Lambda_1} = (u+p_1^2, p_2, \dots, p_n)$, $b = (y,z)$, and

Θ_2) $w(p) = 0$, $L(w) = \{q = 0, u = 0\}$, $\mathcal{I}_{\Lambda_1}|_{L(w)} = (p_1^2, p_2, \dots, p_n)$, $\langle \dot W\rangle_{\mathbb{R}} = \langle 1, p\rangle_{\mathbb{R}}$;

$\Theta_k^\pm$) $w(q_1, p_2, \dots, p_n) = \pm q_1^k$, $L(w) = \{p_1 = \pm k q_1^{k-1}, q_2 = \dots = q_n = 0, u = \pm(1-k/2)q_1^k\}$ where $k \geq 3$, $\mathcal{I}_{\Lambda_1}|_{L(w)} = (q_1^k, p_2, \dots, p_n)$, $\langle \dot W\rangle_{\mathbb{R}} = \langle 1, q_1, p_2 + q_1^2, \dots, p_{k-1} + q_1^{k-1}, p_k, \dots, p_n\rangle_{\mathbb{R}}$ where $n \geq k-1$;

Ξ_4) $w(p_1, q_2, p_3 \dots, p_n) = q_2^3$, $L(w) = \{p_2 = 3q_2^2, q_1 = q_3 = \dots = q_n = 0, u = -q_2^3/2\}$, $\mathcal{I}_{\Lambda_1}|_{L(w)} = (p_1^2, q_2^2, p_3, \dots, p_n)$, $\langle \dot W\rangle_{\mathbb{R}} = \langle 1, p_1, q_2, p_3 + p_1 q_2, p_4, \dots, p_n\rangle_{\mathbb{R}}$ where $n \geq 3$.

Let $\pi : E \to B$ be a generic Legendre bundle. Then its fibers form a family $L_\star = \pi^{-1}(\star)$ of Legendre submanifolds depending generically on a point $b \in B$. Therefore, provided $n = \dim B - 1 \leq 3$, the germs of the family $L_\star$ at singular points of the Legendre submanifold Λ_1 are infinitesimally Λ_1-versal deformations of Legendre submanifolds described by Theorem 3. So, they are Λ_1-equivalent to deformations described by Theorem 2 in consequence of the general versality theorem [AGLV, Chapter 3, 2.3, 2.5]. Lemma 4 implies the required weak Λ_1-equivalence of the corresponding germs of Legendre bundles. □

Proof of Lemma 2. We use the standard homotopy method. Namely, let $\widehat w_\tau$ be a family of generating functions depending smoothly on a parameter τ and K_τ be a smooth family of generators satisfying the homological equation

$$K_\tau(\widehat w_\tau) + \partial_\tau \widehat w_\tau \equiv 0$$

on a segment $[\tau_0, \tau_1]$. Besides, the corresponding contact vector fields v_{K_τ} are assumed to be tangent to the Legendre submanifold Λ_1 and to preserve 0: $v_{K_\tau}(0) = 0$. For the generators K_τ these conditions mean $K_\tau|_{\Lambda_1} = 0$ and $K_\tau(0) = \partial_p K_\tau(0) = \partial_q K_\tau(0) = 0$ respectively. Now solving the Cauchy problem

$$\dot g_\tau(p,q,u) = v_{K_\tau}(g_\tau(p,q,u)), \quad g_{\tau_0}(p,q,u) = (p,q,u)$$

with respect to a family of diffeomorphisms g_τ on the segment $[\tau_0, \tau_1]$ for small (p,q,u) we get the equivalence $\widehat w_{\tau_0} \sim_{\Lambda_1} \widehat w_{\tau_1}$ performed by the local contact diffeomorphism g_{τ_1} preserving Λ_1 and 0.

a_d) In this case $w_d = a p_1^d$. Let $\widehat w_\tau = \tau p_1^d$ and $[\tau_0, \tau_1] = [0, a]$. Then

$$K_\tau = -(u + p_1^2) p_1^{d-2}$$

is a required solution of the homological equation. Indeed, using Lemma 3 we get

$$K_\tau(\widehat w_\tau) + \partial_\tau \widehat w_\tau = -(\tau(1-d/2)p_1^d + p_1^2)p_1^{d-2} + p_1^d = 0 \pmod{\mathcal{A}_{d+1}}$$

provided $d \geq 2$.

b^k) In this case $w_k = aq_1^k$ where $a \neq 0$. Let $\widehat{w}_\tau = \tau q_1^k$ and $[\tau_0, \tau_1] = [a, \operatorname{sign}(a)]$. Then

$$K_\tau = \frac{u + p_1^2}{\tau(k/2 - 1)}$$

is a required solution of the homological equation. Indeed, using Lemma 3 we get

$$K_\tau(\widehat{w}_\tau) + \partial_\tau \widehat{w}_\tau = \frac{\tau(1 - k/2)q_1^k + \tau^2 k^2 q_1^{2k-2}}{\tau(k/2 - 1)} + q_1^k = 0 \pmod{\mathcal{A}_{k+1}}$$

provided $k \geq 3$.

b_d^k) In this case $w_d = aq_1^d$. Let $\widehat{w}_\tau = \pm q_1^k + \tau q_1^d$ and $[\tau_0, \tau_1] = [0, a]$. Then

$$K_\tau = \pm \frac{u + p_1^2}{k/2 - 1} q_1^{d-k}$$

is a required solution of the homological equation. Indeed, using Lemma 3 we get

$$K_\tau(\widehat{w}_\tau) + \partial_\tau \widehat{w}_\tau =$$
$$\pm \frac{\pm(1 - k/2)q_1^k + \tau(1 - d/2)q_1^d + (\pm k q_1^{k-1} + \tau d q_1^{d-1})^2}{k/2 - 1} q_1^{d-k} + q_1^d = 0 \pmod{\mathcal{A}_{d+1}}$$

provided $d > k \geq 3$.

c^6) In this case $w_6 = ap_1^2 + bq_2^3$ where $b \neq 0$. The contact change $p_2 \mapsto b^{1/3}p_2$, $q_2 \mapsto b^{-1/3}q_2$ preserves Λ_1 and reduces $w_6 \mapsto ap_1^2 + q_2^3$. Let $\widehat{w}_\tau = \tau p_1^2 + q_2^3$ and $[\tau_0, \tau_1] = [0, a]$. Then

$$K_\tau = -(u + p_1^2) - p_2 q_2/6$$

is a required solution of the homological equation. Indeed, using Lemma 3 we get

$$K_\tau(\widehat{w}_\tau) + \partial_\tau \widehat{w}_\tau = -(\tau(1 - 2/2)p_1^2 + (1 - 3/2)q_2^3 + p_1^2) - 3q_2^3/6 + p_1^2 = 0.$$

c_d^6) In this case $w_d = a_{d-6}(p_1, q_2)p_1^2 + b_{d-4}(p_1, q_2)q_2^2$ where $a_{d-6} \in \mathcal{A}_{d-6}$ and $b_{d-4} \in \mathcal{A}_{d-4}$. Let $\widehat{w}_\tau = q_2^3 + \tau w_d$ and $[\tau_0, \tau_1] = [0, 1]$. Then

$$K_\tau = -a_{d-6}(p_1, q_2)(u + p_1^2 + p_2 q_2/6) - b_{d-4}(p_1, q_2)p_2/3$$

is a required solution of the homological equation. Indeed, using Lemma 3 we get

$$K_\tau(\widehat{w}_\tau) + \partial_\tau \widehat{w}_\tau = -a_{d-6}(p_1, q_2)(p_1^2 + \tau(w_d - p_1 \partial_{p_1} w_d/2 - q_2 \partial_{q_2} w_d/3)) -$$
$$- b_{d-4}(p_1, q_2)(q_2^2 + \partial_{q_2} w_d/3) + w_d = 0 \pmod{\mathcal{A}_{d+1}}$$

provided $d > 6$. $\square$

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Independent University of Moscow, Bolshoi Vlas'evskii per. 11, Moscow 121002 Russia

E-mail address: `bogaevsk@mccme.ru`