Special curves and ruled surfaces

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Dedicated to Professor Koichi OGIUE on his sixtieth birthday

Abstract

We study cylindrical helices and Bertrand curves as curves on ruled surfaces. Some results in this paper clarify that the cylindrical helix is related to the Gaussian curvature and the Bertrand curve is related to the mean curvature of the ruled surface. We also study the singularities of the principal normal surfaces of generic curves.

1 Introduction

In [3] we have studied singularities of the rectifying developable (surface) of a space curve. The rectifying developable is defined to be the envelope of the family of rectifying planes along the curve. We observed that the rectifying developable along a curve γ is non-singular if and only if γ is a cylindrical helix. In this case the rectifying developable is a cylindrical surface. The notion of cylindrical helices is a generalization of the notion of circular helices. On the other hand, the notion of Bertrand curves is another generalization of the notion of circular helices. These two curves have been classically studied as special curves in Euclidean space.

In this paper we study these curves from the view point as geometry of curves on ruled surfaces. As we mentioned in the above paragraph, the notion of rectifying developables is deeply related to cylindrical helices. On the other hand, the principal normal surface of a space curve γ is defined to be a ruled surface along γ whose rulings are given by the principal normal of γ. Principal normal surfaces are naturally related to Bertrand curves by definition. We can

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easily show that the principal normal surface of a Bertrand curve has singularities if and only if the curve is a plane curve (cf., Proposition 3.6). For a general space curve, the principal normal surface usually has singularities. We also study singularities of principal normal surfaces of general space curves. We prove that only cross caps appear as generic singularities for principal normal surfaces of curves (cf., Theorem 3.4).

In §2 we review basic notions and properties of space curves and ruled surfaces. We study singularities of rectifying developables and principal normal surfaces of space curves in §3. In §4 we study cylindrical helices and Bertrand curves as curves on ruled surfaces. We prove that a ruled surface is the rectifying developable of γ if and only if γ is a geodesic of the ruled surface which is transversal to rulings and the Gaussian curvature vanishes along γ (Theorem 4.4). As a corollary of Theorem 4.4, we give a characterization of a cylindrical surface as a developable surface by the existence of a geodesic which is a cylindrical helix with non-zero curvature (c.f., Corollary 4.5). We also prove that a ruled surface is the principal normal surface of a space curve γ if and only if γ is an asymptotic curve of the ruled surface which is transversal to rulings and the mean curvature vanishes along γ (Theorem 4.10). As a corollary of Theorem 4.10, we show that if there exist two disjoint asymptotic curves on a ruled surface both of which are transversal to rulings and the mean curvature of the ruled surface vanishes along these curves, then these curves are Bertrand curves (c.f., Proposition 4.12). We also show that if there exist three disjoint Bertrand curves on a ruled surface, then the ruled surface is a helicoid (cf., Proposition 4.11). We give some examples of cylindrical helices and Bertrand curves in §5.

This is the second paper of the authors joint project entitled "Geometry of ruled surfaces and line congruence".

All manifolds and maps considered here are of class C^∞ unless otherwise stated.

2   Basic notions and properties

We now review some basic concepts on classical differential geometry of space curves and ruled surfaces in Euclidean space. For any two vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, we denote $x \cdot y$ as the standard inner product. Let $\gamma : I \longrightarrow \mathbb{R}^3$ be a curve with $\dot{\gamma}(t) \neq 0$, where $\dot{\gamma}(t) = d\gamma/dt(t)$. We always assume that γ is a proper mapping and $\gamma(I)$ is bounded in $\mathbb{R}^3$. We also denote the norm of x by $\|x\|$. The *arc-length* of a curve γ, measured from $\gamma(t_0)$, $t_0 \in I$ is

$$s(t) = \int_{t_0}^{t} \|\dot{\gamma}(t)\| \, dt.$$

Then a parameter s is determined such that $\|\gamma'(s)\| = 1$, where $\gamma'(s) = d\gamma/ds(s)$. So we say that a curve γ is *parameterized by the arc-length* if it satisfies $\|\gamma'(s)\| = 1$. Let us denote $t(s) = \gamma'(s)$ and we call $t(s)$ a *unit tangent vector* of γ at s. We define the *curvature* of γ by $\kappa(s) = \sqrt{\|\gamma''(s)\|}$. If $\kappa(s) \neq 0$, then the *unit principal normal vector* $n(s)$ of the curve γ at s is given by $\gamma''(s) = \kappa(s)n(s)$. The unit vector $b(s) = t(s) \times n(s)$ is called a *unit binormal vector* of the curve γ at s. Then the following Frenet-Serret formula holds:

$$\begin{cases} t'(s) & = & \kappa(s)n(s) \\ n'(s) & = & -\kappa(s)t(s) + \tau(s)b(s) \\ b'(s) & = & -\tau(s)n(s), \end{cases}$$

where $\tau(s)$ is the torsion of the curve γ at s. For any unit speed curve $\gamma : I \longrightarrow \mathbb{R}^3$, we call $D(s) = \tau(s)t(s) + \kappa(s)b(s)$ *the Darboux vector field* of γ (cf., [5], Section 5.2). By using the Darboux vector field, the Frenet-Serret formula is rewritten as follows:

$$\begin{cases} t'(s) &= D(s) \times t(s) \\ n'(s) &= D(s) \times n(s) \\ b'(s) &= D(s) \times b(s) \end{cases}$$

We define a vector field $\widetilde{D}(s) = (\tau/\kappa)(s)t(s) + b(s)$ along γ under the condition that $\kappa(s) \neq 0$ and we call it *the modified Darboux vector field* of γ.

A curve $\gamma : I \longrightarrow \mathbb{R}^3$ with $\kappa(s) \neq 0$ is called a *cylindrical helix* if the tangent lines of γ make a constant angle with a fixed direction. It has been known that the curve $\gamma(s)$ is a cylindrical helix if and only if

$$\frac{\tau}{\kappa}(s) = \text{constant}.$$

If both of $\kappa(s) \neq 0$ and $\tau(s)$ are constant, it is, of course, a cylindrical helix. We call such the curve a *circular helix*. On the other hand, a curve $\gamma : I \longrightarrow \mathbb{R}^3$ with $\kappa(s) \neq 0$ is called a *Bertrand curve* if there exists a curve $\bar{\gamma} : I \longrightarrow \mathbb{R}^3$ such that the principal normal lines of γ and $\bar{\gamma}$ at $s \in I$ are equal. In this case $\bar{\gamma}$ is called a *Bertrand mate* of γ. Any plane curve γ is a Bertrand curve whose Bertrand mates are parallel curves of γ. Bertrand curves have the following fundamental properties.

Proposition 2.1 *Let $\gamma : I \longrightarrow \mathbb{R}^3$ be a space curve.*

(1) Suppose that $\tau(s) \neq 0$. Then γ is a Bertrand curve if and only if there exist nonzero real numbers A, B such that

$$A\kappa(s) + B\tau(s) = 1$$

for any $s \in I$. It follows from this fact that a circular helix is a Bertrand curve.

(2) Suppose that γ is a Bertrand curve. If there exists a point $s_0 \in I$ such that $\tau(s_0) = 0$, then γ is a plane curve.

(3) Let γ be a Bertrand curve with the Bertrand mate $\bar{\gamma}$. Then we have

$$\tau(s)\bar{\tau}(s) = \text{constant} \geq 0,$$

where $\bar{\tau}(s)$ is the torsion of $\bar{\gamma}$.

Proof. Although these assertions are classical results, we give the proof here in order to understand the situation (cf,. [2], pp 26).

Let $\bar{t}(s)$, $\bar{n}(s)$ and $\bar{b}(s)$ be the unit tangent vector, the principal normal vector and the binormal vector of $\bar{\gamma}(s)$ respectively. The curve $\bar{\gamma}$ is the Bertrand mate of γ if and only if $\bar{n}(s) = \varepsilon n(s)$, where $\varepsilon = \pm 1$. By definition, there exists a nonzero smooth function $A(s)$ such that $\bar{\gamma}(s) = \gamma(s) + A(s)n(s)$. By Frenet-Serret formula, we have

$$\frac{d\bar{\gamma}}{ds}(s) = (1 - A(s)\kappa(s))t(s) + A'(s)n(s) + \tau(s)A(s)b(s).$$

Let $\bar{s}$ be the arc-length parameter of $\bar{\gamma}$, then

$$\frac{d\bar{s}}{ds}\bar{t}(s) = (1 - A(s)\kappa(s))t(s) + A'(s)n(s) + \tau(s)A(s)b(s).$$

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Multiplying $\bar{n}(s) = \varepsilon n(s)$ to both sides of this equality, we have $\varepsilon A'(s) \equiv 0$, so that $A(s) \equiv A$ is constant. Since $\bar{n}(s) = \varepsilon n(s)$, we have

$$\langle \bar{t}(s), \bar{b}(s) \rangle_{\mathbb{R}} = \langle t(s), b(s) \rangle_{\mathbb{R}}.$$

It follows from this fact that there exists a smooth function $\theta(s)$ such that

$$\bar{t}(s) = \cos \theta(s) t(s) + \sin \theta(s) b(s).$$

Therefore we have

$$\bar{b}(s) = \bar{t}(s) \times \bar{n}(s) = -\varepsilon \sin \theta(s) t(s) + \varepsilon \cos \theta(s) b(s).$$

We also have

$$\frac{d\bar{\gamma}}{ds}(s) = (\cos \theta)' t(s) + (\kappa(s) \cos \theta(s) - \tau(s) \sin \theta(s)) n(s) + (\sin \theta(s))' b(s).$$

Since $t(s) \cdot \bar{n}(s) = b(s) \cdot \bar{n}(s) = 0$, we have $(\cos \theta(s))' = (\sin \theta(s))' = 0$. This means that $\theta(s) = \theta$ is constant.

Thus we have

$$\frac{d\bar{\gamma}}{ds}(s) = (1 - A\kappa(s)) t(s) + \tau(s) A b(s),$$

so that

$$\cos \theta \frac{d\bar{s}}{ds} t(s) + \sin \theta \frac{d\bar{s}}{ds} b(s) = (1 - A\kappa(s)) t(s) + \tau(s) A b(s).$$

This means that

$$\cos \theta \frac{d\bar{s}}{ds} = (1 - A\kappa(s)) \text{ and } \sin \theta \frac{d\bar{s}}{ds} = \tau(s) A.$$

Suppose that there exists a point $s_0 \in I$ that $\tau(s_0) = 0$, then $\sin \theta = 0$. Therefore $\tau(s) = 0$, so that the curve γ is a plane curve. This complete the proof of the assertion (2).

We now suppose that $\tau(s) \neq 0$. In this case, we now set $B = A \cot \theta$. Then we have $B\tau(s) = 1 - A\kappa(s)$. This complete the proof of the assertion (1). For the proof of the assertion (3), we may assume that $\tau(s) \neq 0$. By the previous arguments, we have

$$\frac{d\bar{b}}{ds}(s) = -\varepsilon(\kappa(s) \sin \theta + \tau(s) \cos \theta) n(s).$$

It also follows from Frenet-Serret formulae for $\bar{\gamma}(\bar{s})$ that

$$\frac{d\bar{s}}{ds} \frac{d\bar{b}}{d\bar{s}}(s) = -\bar{\tau}(s) \frac{d\bar{s}}{ds} \bar{n}(s) = -\bar{\tau}(s) \frac{d\bar{s}}{ds} \varepsilon n(s).$$

Therefore we have

$$\frac{d\bar{s}}{ds} \bar{\tau}(s) = \kappa(s) \sin \theta + \tau(s) \cos \theta.$$

Multiplying $\sin \theta$ to both sides of this equality, we have

$$\bar{\tau}(s) \tau(s) A = \frac{\sin^2 \theta}{A} (A\kappa(s) + A \cot \theta \tau(s)) = \frac{\sin^2 \theta}{A}.$$

Therefore, $\bar{\tau}(s) \tau(s)$ is non-negative constant. $\qquad \square$

We have the following corollary of the proposition.

Corollary 2.2 *Let $\gamma : I \longrightarrow \mathbb{R}^3$ be a space curve with $\kappa(s) \neq 0$ and $\tau(s) \neq 0$. Then γ is a Bertrand curve if and only if there exists a real number $A \neq 0$ such that*

$$A(\tau'(s)\kappa(s) - \kappa'(s)\tau(s)) - \tau'(s) = 0.$$

In this case the Bertrand mate of γ is given by $\bar{\gamma}(s) = \gamma(s) + An(s)$.

Proof. By the proposition, γ is a Bertrand curve if and only if there exist real numbers $A \neq 0$ and B such that $A\kappa(s) + B\tau(s) = 1$. This is equivalent to the condition that there exists a real number $A \neq 0$ such that $\dfrac{1 - A\kappa(s)}{\tau(s)}$ is constant.

Differentiate the both sides of the last equality, we have

$$A(\tau'(s)\kappa(s) - \kappa'(s)\tau(s)) = \tau'(s).$$

The converse assertion is also true. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

On the other hand, *a ruled surface* in $\mathbb{R}^3$ is (locally) the map $F_{(\gamma,\delta)} : I \times \mathbb{R} \to \mathbb{R}^3$ defined by $F_{(\gamma,\delta)}(t, u) = \gamma(t) + u\delta(t)$, where $\gamma : I \to \mathbb{R}^3$, $\delta : I \to \mathbb{R} \setminus \{0\}$ are smooth mappings and I is an open interval or a unit circle S^1. We call γ a *base curve* and δ a *director curve*. The straightlines $u \mapsto \gamma(t) + u\delta(t)$ are called *rulings*. We can calculate that

$$\frac{\partial F_{(\gamma,\delta)}}{\partial t}(t, u) = \gamma'(t) + u\delta'(t), \quad \frac{\partial F_{(\gamma,\delta)}}{\partial u}(t, u) = \delta(t).$$

It follows from the above that

$$\frac{\partial F_{(\gamma,\delta)}}{\partial t}(t, u) \times \frac{\partial F_{(\gamma,\delta)}}{\partial u}(t, u) = \gamma'(t) \times \delta(t) + u\delta'(t) \times \delta(t).$$

Therefore (t_0, u_0) is a singular point of $F_{(\gamma,\delta)}$ if and only if

$$\gamma'(t_0) \times \delta(t_0) + u_0\delta'(t_0) \times \delta(t_0) = \mathbf{0}.$$

In [4] we have shown that only cross caps appear as singularities for generic ruled surfaces. Here, we say that a C^∞-map germ at the origin $g : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^3, 0)$ is *the cross cap* if there are local charts (x_1, x_2) and (y_1, y_2, y_3) around the origins such that $(y_1 \circ g(x_1, x_2), y_2 \circ g(x_1, x_2), y_3 \circ g(x_1, x_2)) = (x_1^2, x_2, x_1 x_2)$. We also say that the origin is the cross cap if g is the cross cap. In Fig.1 we draw the picture of the cross cap by using the normal form.

We say that the ruled surface $F_{(\gamma,\delta)}$ is a *cylindrical surface* if the direction of the director curve is constant. This is equivalent to the condition that $\delta(t) \times \delta'(t) \equiv \mathbf{0}$. We also say that the ruled surface $F_{(\gamma,\delta)}$ is *non-cylindrical* if $\delta(t) \times \delta'(t) \neq \mathbf{0}$. For any ruled surface $F_{(\gamma,\delta)}$, we now denote that $\widetilde{\delta}(t) = \delta(t)/\|\delta(t)\|$. Then the image of $F_{(\gamma,\delta)}$ and $F_{(\gamma,\widetilde{\delta})}$ are equal. It is easy to show that $F_{(\gamma,\delta)}$ is non-cylindrical if and only if $\widetilde{\delta}'(t) \neq \mathbf{0}$. We now consider a curve $\sigma(t)$ on the ruled surface $F_{(\gamma,\delta)}$ with the property that $\sigma'(t) \cdot \widetilde{\delta}'(t) = 0$. We call such a curve *the line of striction*. If $F_{(\gamma,\delta)}$ is non-cylindrical, the line of striction is uniquely determined and given by

$$\sigma(t) = \gamma(t) - \frac{\gamma'(t) \cdot \widetilde{\delta}'(t)}{\widetilde{\delta}'(t) \cdot \widetilde{\delta}'(t)}\widetilde{\delta}(t).$$
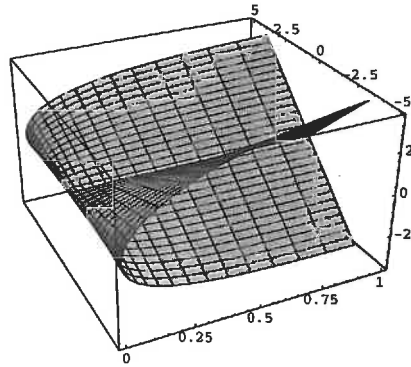
Fig.1. The cross cap

In this paper we consider the following two special ruled surfaces associated to a space curve γ with $\kappa(s) \neq 0$ which are respectively related to cylindrical helices and Bertrand curves. A ruled surface

$$F_{(\gamma,\widetilde{D})}(s,u) = \gamma(s) + u\widetilde{D}(s)$$

is called *the rectifying developable of* γ. We also define a ruled surface

$$F_{(\gamma,n)}(s,u) = \gamma(s) + un(s)$$

which is called *the principal normal surface of* γ.

3   The rectifying developable and the principal normal surface of a space curve

In this section we consider some basic facts on the rectifying developable and the principal normal surfaces of a space curve.

Firstly we consider the rectifying developable of a unit speed space curve $\gamma(s)$ with $\kappa(s) \neq 0$. We can calculate that $\widetilde{D}'(s) = (\tau/\kappa)'(s)t(s)$, so that we have

$$\gamma'(s) \times \widetilde{D}(s) = -n(s), \quad \widetilde{D}'(s) \times \widetilde{D}(s) = -\left(\frac{\tau}{\kappa}\right)'(s)n(s).$$

Therefore (s_0, u_0) is a singular point of $F_{(\gamma,\widetilde{D})}$ if and only if

$$\left(\frac{\tau}{\kappa}\right)'(s_0) \neq 0 \text{ and } u_0 = \frac{-1}{(\tau/\kappa)'(s_0)}.$$

In [3] we have studied singularities of the rectifying developable of γ and given a local classification as follows:

Theorem 3.1 [3] *Let* $\gamma : I \longrightarrow \mathbb{R}^3$ *be a unit speed curve with* $\kappa(s) \neq 0$. *Then we have the following:*
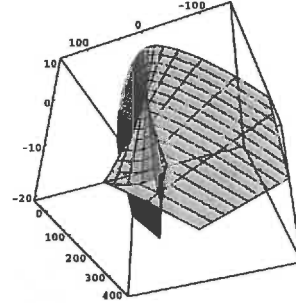
(1) The rectifying developable is locally diffeomorphic to the cuspidal edge $C \times \mathbb{R}$ *at* $\gamma(s_0) + u_0\widetilde{D}(s_0)$ *if and only if* $(\tau/\kappa)'(s_0) \neq 0$, $(\tau/\kappa)''(s_0) \neq 0$ *and* $u_0 = \frac{1}{(\tau/\kappa)'(s_0)}$.

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(2) *The rectifying developable is locally diffeomorphic to the swallow tail SW at $\gamma(s_0) + u_0 + \widetilde{D}(s_0)$ if and only if $(\tau/\kappa)'(s_0) \neq 0$, $(\tau/\kappa)''(s_0) = 0$, $(\tau/\kappa)'''(s_0) \neq 0$ and $u_0 = \frac{1}{(\tau/\kappa)'(s_0)}$.*

Here, $C = \{(x_1, x_2) | x_1{}^2 = x_2{}^3\}$ is the ordinary cusp and $SW = \{(x_1, x_2, x_3) | x_1 = 3u^4 + u^2 v, x_2 = 4u^3 + 2uv, x_3 = v\}$ is the swallow tail.
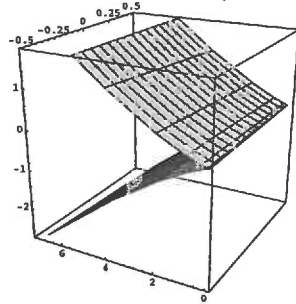
cuspidal edge   swallowtail

Fig.2.

On the other hand, we have the following proposition:

Proposition 3.2 *For a unit speed space curve $\gamma : I \longrightarrow \mathbb{R}^3$ with $\kappa(s) \neq 0$, the following are equivalent.*

    (1) *The rectifying developable $F_{(\gamma, \widetilde{D})} : I \times \mathbb{R} \longrightarrow \mathbb{R}^3$ is a non-singular surface.*

    (2) *γ is a cylindrical helix.*

    (3) *The rectifying developable $F_{(\gamma, \widetilde{D})}$ is a cylindrical surface.*

Proof. By the previous calculation, $F_{(\gamma, \widetilde{D})}$ is non-singular at any point in $I \times \mathbb{R}$ if and only if $(\tau/\kappa)'(s) \equiv 0$. This means that γ is a cylindrical helix.

On the other hand, we have calculated that $\widetilde{D}'(s) = (\tau/\kappa)'(s) t(s)$. The rectifying developable $F_{(\gamma, \widetilde{D})}$ is cylindrical if and only if $\widetilde{D}'(s) \equiv 0$, so that the condition (2) is equivalent to the condition (3). □

Secondary we consider the principal normal surface $F_{(\gamma, n)}(s, u)$ of a unit speed space curve $\gamma(s)$ with $\kappa(s) \neq 0$. We start to consider the singular point of $F_{(\gamma, n)}(s, u)$. By Frenet-Serret formula, we can show that

$$\gamma'(s) \times n(s) + u n'(s) \times n(s) = (1 - u\kappa(s)) b(s) - \tau(s) u t(s).$$

Therefore (s_0, u_0) is a singular point of $F_{(\gamma, n)}$ if and only if $\tau(s_0) = 0$ and $u_0 = \dfrac{1}{\kappa(s_0)}$.

We now consider what kind of singularities appear on the principal normal surface of a generic space curve.

Example 3.3 Consider the space curve defined by $\gamma(t) = (t, t^2, t^4)$. In this case, we can calculate that the principal normal direction is given by

$$\widetilde{n}(t) = (-32t^3, 64t^6 - 2, -32t^4 - 12t^2).$$

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Therefore, the principal normal surface is given by

$$F(t, u) = (t - 32ut^3, t^2 + u(64t^6 - 2), t^4 - u(32t^4 + 12t^2)).$$

We can easily determine that the singular point is $F(0, 1/2) = (0, 1/2, 0)$. We can draw the picture of the surface by using Mathematica in Fig .3. The singular point looks like the cross cap.
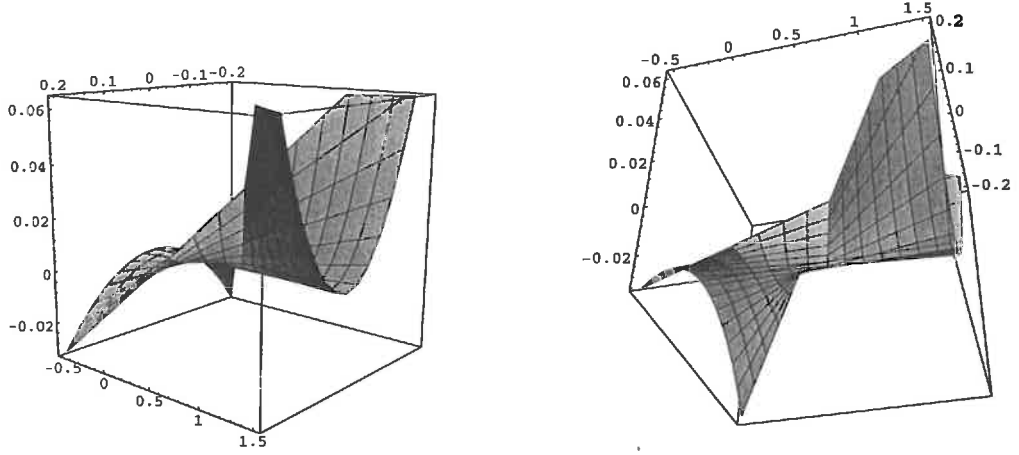

Fig.3. The principal normal surface

It has been known that $g : (\mathbb{R}^2, \mathbf{0}) \longrightarrow (\mathbb{R}^3, \mathbf{0})$ is a cross cap if and only if there exists a local chart (x_1, x_2) around the origin such that the following conditions hold:

$$\frac{\partial g}{\partial x_1}(\mathbf{0}) \neq \mathbf{0}, \quad \frac{\partial g}{\partial x_2}(\mathbf{0}) = \mathbf{0} \text{ and } \det \left(\frac{\partial g}{\partial x_1}(\mathbf{0}), \frac{\partial^2 g}{\partial x_1 \partial x_2}(\mathbf{0}), \frac{\partial^2 g}{\partial x_2^2}(\mathbf{0}) \right) \neq 0.$$

For the principal normal surface, we have the following classification theorem.

Theorem 3.4 *For a unit speed space curve $\gamma : I \longrightarrow \mathbb{R}^3$ with $\kappa(s) \neq 0$, the principal normal surface $F_{(\gamma,n)}(s, u)$ is the cross cap at (s_0, u_0) if and only if*

$$u_0 = \frac{1}{\kappa(s_0)}, \quad \tau(s_0) = 0 \text{ and } \tau'(s_0) \neq 0.$$

Proof. By using Frenet-Serret formula, we can calculate that

$$\frac{\partial F_{(\gamma,n)}}{\partial s}(s, u) = (1 - u\kappa(s))\boldsymbol{t}(s) + \tau(s)u\boldsymbol{b}(s), \quad \frac{\partial F_{(\gamma,n)}}{\partial u}(s, u) = \boldsymbol{n}(s).$$

We also have the second order derivations of $F_{(\gamma,n)}$ as follows:

$$\frac{\partial^2 F_{(\gamma,n)}}{\partial u \partial s}(s, u) = \boldsymbol{n}'(s) = -\kappa(s)\boldsymbol{t}(s) + \tau(s)\boldsymbol{b}(s)$$

$$\frac{\partial^2 F_{(\gamma,n)}}{\partial^2 s}(s, u) = -u\kappa'(s)\boldsymbol{t}(s) + (1 - u\kappa(s)^2 - u\tau(s)^2)\boldsymbol{n}(s) + \tau'(s)\boldsymbol{b}(s).$$

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By the previous argument, (s_0, u_0) is a singular point of $F_{(\gamma,n)}(s, u)$ if and only if

$$u_0 = \frac{1}{\kappa(s_0)}, \quad \tau(s_0) = 0.$$

Therefore we may substitute these relations into the derivatives of $F_{(\gamma,n)}$, so that we have

$$\frac{\partial F_{(\gamma,n)}}{\partial u}(s_0, u_0) = n(s_0)$$

$$\frac{\partial^2 F_{(\gamma,n)}}{\partial u \partial s}(s_0, u_0) = -\kappa(s_0)t(s_0)$$

$$\frac{\partial^2 F_{(\gamma,n)}}{\partial^2 s}(s, u) = -\frac{\kappa'(s_0)}{\kappa(s_0)}t(s_0) + (1 - \kappa(s_0))n(s_0) + \tau'(s)b(s).$$

It follows from these relations that

$$\det\left(\frac{\partial F_{(\gamma,n)}}{\partial u}(s_0, u_0), \frac{\partial^2 F_{(\gamma,n)}}{\partial u \partial s}(s_0, u_0), \frac{\partial^2 F_{(\gamma,n)}}{\partial^2 s}(s, u)\right)$$

$$= \det\left(n(s_0), -\kappa(s_0)t(s_0), -\frac{\kappa'(s_0)}{\kappa(s_0)}t(s_0) + (1 - \kappa(s_0))n(s_0) + \tau'(s)b(s)\right)$$

$$= -\kappa(s_0)\tau'(s_0)\det(n(s_0), t(s_0), b(s_0)) = \kappa(s_0)\tau'(s_0)$$

By the characterization of the cross cap, this completes the proof. $\qquad \square$

We consider the following conditions of a space curve $\gamma : S^1 \longrightarrow \mathbb{R}^3$:
(C_1) There are no points on S^1 with $\tau(s) = \tau'(s) = 0$.
(C_2) The number of the point $s_0 \in S^1$ at where $\tau(s_0) = 0$ and $\tau'(s_0) \neq 0$ is finite.
(C_3) $\kappa(s) \neq 0$ at any point $s \in S^2$.

It has been known [1] that the conditions (C_1, C_2, C_3) are generic conditions on space curves $\gamma : S^1 \longrightarrow \mathbb{R}^3$. Then we have the following corollary.

Corollary 3.5 *For a "generic" space curve $\gamma : S^1 \longrightarrow \mathbb{R}^3$, the number of singular points of $F_{(\gamma,n)}(s, u)$ is finite and each singular point is the cross cap.*

By the previous arguments, the principal normal surface $F_{(\gamma,n)}$ is non-singular under the assumption that $\tau(s) \neq 0$. For example, the principal normal surface of a circular helix is the *helicoid*. For a Bertrand curve, we have the following proposition.

Proposition 3.6 *Let $\gamma : I \longrightarrow \mathbb{R}^3$ be a Bertrand curve. The principal normal surface $F_{(\gamma,n)}$ has a singular point if and only if γ is a plane curve. In this case the image of $F_{(\gamma,n)}$ is a plane in $\mathbb{R}^3$.*

Proof. By the assertion (2) of Proposition 2.1, if there exists a point $s_0 \in I$ such that $\tau(s_0) = 0$, then γ is a plane curve. On the other hand, the singular point of $F_{(\gamma,n)}$ corresponds to the point $s_0 \in I$ with $\tau(s_0) = 0$. The last assertion of the proposition is clear by definition. $\qquad \square$

4 Curves on ruled surfaces

In this section we study cylindrical helices and Bertrand curves from the view point as the theory of curves on ruled surfaces. In the previous sections, we remarked that the rectifying developable of a cylindrical helix is a cylindrical surface and the principal normal surface of a Bertrand curve is non-singular if the Bertrand curve is a space curve. Especially the rectifying developable is a circular cylinder and the principal normal surface is a helicoid if the curve is a circular helix. It has been classically known that the circular cylinder is a non-singular developable surface and the helicoid is the minimal ruled surface. By these facts, we now pay attention to the Gaussian curvature and the mean curvature of ruled surfaces. Let $F_{(\gamma,\delta)}$ be a ruled surface. For convenience, we may assume that $\|\delta(t)\| = 1$. It is easy to show that the Gaussian curvature of $F_{(\gamma,\delta)}$ is

$$K(t, u) = -\frac{(\det(\gamma'(t), \delta(t), \delta'(t)))^2}{(EG - F^2)^2}$$

and the mean curvature of $F_{(\gamma,\delta)}$ is

$$H(t, u) = \frac{-2(\gamma'(t) \cdot \delta(t))\det(\gamma'(t), \delta(t), \delta'(t)) + \det(\gamma''(t) + u\delta''(t), \gamma'(t) + u\delta'(t), \delta(t))}{2(EG - F^2)^{3/2}},$$

where

$$E = E(t, u) = \|\gamma'(t) + u\delta'(t)\|^2, \;\; F = F(t, u) = \gamma'(t) \cdot \delta(t), \;\; G = G(t, u) = 1.$$

Especially the Gaussian curvature of the rectifying developable of a space curve vanishes and the mean curvature of the principal normal surface of a space curve is

$$H(s, u) = \frac{u(\tau'(s) + u(\kappa'(s)\tau(s) - \tau'(s)\kappa(s)))}{(EG - F^2)^{3/2}},$$

where s is the arc-length of γ. It follows from this fact that $H(s, u) = 0$ if and only if $u = 0$ or $\tau'(s) = u(\tau'(s)\kappa(s) - \tau(s)\kappa'(s))$. Thus, the mean curvature of the principal normal surface $F_{(\gamma,n)}$ of γ always vanishes along γ. If there exists a point $s_0 \in I$ such that $\tau'(s_0)\kappa(s_0) - \tau(s_0)\kappa'(s_)) = 0$, then $H(s_0, u_0) = 0$ for some $u_0 \neq 0$ if and only if $\tau'(s_0) = 0$. In this case $\kappa'(s_0) = 0$. Therefore, $H(s_0, u_0) = 0$ for some $u_0 \neq 0$ if and only if $\tau'(s_0) = \kappa'(s_0) = 0$ or

$$u_0 = \frac{\tau'(s_0)}{\tau'(s_0)\kappa(s_0) - \tau(s_0)\kappa'(s_0)}.$$

If $\tau'(s_0) \neq 0$ and $\tau'(s_0)\kappa(s_0) - \tau(s_0)\kappa'(s_0) = 0$, then $H(s_0, u) \neq 0$ for any $u \neq 0$. Moreover, under the assumption that $\tau'(s_0) = \kappa'(s_0) = 0$, $H(s_0, u) = 0$ for any u. Of course, if $\tau'(s)\kappa(s) - \tau(s)\kappa'(s) \neq 0$, the mean curvature vanishes along the curve given by

$$\widetilde{\gamma}(s) = \gamma(s) + \frac{\tau'(s)}{\tau'(s)\kappa(s) - \tau(s)\kappa'(s)}n(s).$$

Let $\gamma : J \longrightarrow F_{(\gamma,\delta)}(I \times \mathbb{R}) \subset \mathbb{R}^3$ be a regular curve. We say that γ is the *minimal locus* of $F_{(\gamma,\delta)}$ if the mean curvature H of $F_{(\gamma,\delta)}$ vanishes on $\gamma(J)$. By the above calculation and Corollary 2.2, we have the following proposition.

Proposition 4.1 *Let γ be a Bertrand curve and $\bar{\gamma}$ be the Bertrand mate of γ. Then $\bar{\gamma}$ is the minimal locus of the principal normal surface of γ.*

Proof. By Corollary 2.2, if $\bar{\gamma}$ is the Bertrand mate of γ, then there exists a real number A such that $A(\tau'(s)\kappa(s) - \tau(s)\kappa'(s)) - \tau'(s) = 0$. and $\bar{\gamma}(s) = \gamma(s) + An(s)$. This means that $H(\bar{\gamma}(s)) = H(s, A) = 0$. This completes the proof. $\qquad\qquad\square$

By definition, γ is a geodesic of the rectifying developable and a asymptotic curve of the principal normal surface of γ itself. The following proposition has been known as the Bonnet's theorem for non-cylindrical ruled surfaces. The assertion, however, holds even for general ruled surfaces.

Proposition 4.2 *Let $F_{(\gamma,\delta)}(s, u) = \gamma(s) + u\delta(s)$ be a ruled surface with $\|\delta(s)\| = 1$. Let $\sigma(s) = \gamma(s) + u(s)\delta(s)$ be a curve on $F_{(\gamma,\delta)}$, where s is the arc-length of $\sigma(s)$. Consider the following three conditions on σ :*
(1) $\sigma(s)$ is a line of striction of $F_{(\gamma,\delta)}$.
(2) $\sigma(s)$ is a geodesic of $F_{(\gamma,\delta)}$.
(3) The angles between $\sigma'(s)$ and $\delta(s)$ are constant.
If we assume that each two of the above three conditions hold, then the another condition holds.

We remark that the above conditions are respectively equivalent to the following conditions:
(1)' $\sigma'(s) \cdot \delta'(s) = 0$.
(2)' $\sigma''(s) \cdot \delta(s) = 0$.
(3)' $\sigma'(s) \cdot \delta(s) = $ constant.
The proof of the above proposition easily follows from the formula that

$$(\sigma'(s) \cdot \delta(s))' = \sigma''(s) \cdot \delta(s) + \sigma'(s) \cdot \delta'(s).$$

We have the following corollary.

Corollary 4.3 *Suppose that there exist two disjoint geodesics $\sigma_i(s)$ ($i = 1, 2$) on a ruled surface $F_{(\gamma,\delta)}(s, u) = \gamma(s) + u\delta(s)$ such that the angles between $\sigma_i'(s)$ and $\delta(s)$ are constant. Then the ruled surface $F_{(\gamma,\delta)}(s, u)$ is a cylindrical surface and both of $\sigma_i(s)$ are cylindrical helices. Moreover, the direction of $\delta(s)$ is equal to the direction of the Darboux vector of $\sigma_i(s)$.*

Proof. By the proposition, $\sigma_i(s)$ are lines of striction of $F_{(\gamma,\delta)}$. If the point $F_{(\gamma,\delta)}(s)$ is a non-cylindrical, then $\sigma_1(s) = \sigma_2(s)$ by the uniqueness of the line of striction, so that the ruled surface is a cylindrical surface. Since $\sigma_i(s)$ are geodesics of $F_{(\gamma,\delta)}$, these are cylindrical helices and the rectifying plane of $\sigma_i(s)$ is the tangent plane of $F_{(\gamma,\delta)}$. This means that $F_{(\gamma,\delta)}$ is the rectifying developable of $\sigma_i(s)$. $\qquad\qquad\square$

Corollary 4.3 gives a characterization of cylindrical surfaces by the existence of geodesics with special properties. Especially, a cylindrical surface is the rectifying developable of a cylindrical helix which is a geodesic of the original surface. We now consider the question when a ruled surface is the rectifying developable of a curve.

Theorem 4.4 *Let $F_{(\gamma,\delta)}(s, u) = \gamma(s) + u\delta(s)$ be a non-singular ruled surface with $\|\delta(s)\| = 1$. Let $\sigma(s) = \gamma(s) + u(s)\delta(s)$ be a curve on $F_{(\gamma,\delta)}$ with $\kappa(s) \neq 0$. Then the following conditions are equivalent :*

(1) $F_{(\gamma,\delta)}$ is the rectifying developable of $\sigma(s)$.

(2) $\sigma(s)$ is a geodesic of $F_{(\gamma,\delta)}$ which is transversal to rulings and $F_{(\gamma,\delta)}$ is a developable surface.

(3) $\sigma(s)$ is a geodesic of $F_{(\gamma,\delta)}$ which is transversal to rulings and the Gaussian curvature of $F_{(\gamma,\delta)}$ vanishes along $\sigma(s)$.

Proof. Since the Darboux vector field always transverse to rulings, the condition (2) holds under the assumption of the condition (1). It is trivial that the condition (3) follows from the condition (2).

We now assume that the condition (3) holds. Since $\sigma(s)$ is transverse to rulings, we may assume that $\sigma(s) = \gamma(s)$. The Gaussian curvature of $F_{(\gamma,\delta)}$ is given by

$$K(s,u) = -\frac{\det(\gamma'(s), \delta(s), \delta'(s))^2}{(EG - F^2)^2},$$

then it vanishes along $\gamma(s)$ if and only if

$$\det(\gamma'(s), \delta(s), \delta'(s)) = 0.$$

Since $\gamma(s)$ is a geodesic of $F_{(\gamma,\delta)}$, $\delta(s)$ is contained in the rectifying plane of γ at $\gamma(s)$. There exists $\lambda(s), \mu(s)$ such that

$$\delta(s) = \lambda(s)t(s) + \mu(s)b(s),$$

where $t(s) = \gamma'(s)$ and $b(s)$ is the binormal vector of γ. By Frenet-Serret formulae, we have

$$\delta'(s) = \lambda'(s)t(s) + \mu'(s)b(s) + (\lambda(s)\kappa(s) - \mu(s)\tau(s))n(s).$$

It follows from this formula that

$$\det(\gamma'(s), \delta(s), \delta'(s)) = (\mu(s)\tau(s) - \lambda(s)\kappa(s))\mu(s).$$

If there exists a point s_0 such that $\mu(s_0) = 0$, then $\delta(s_0) = \lambda(s_0)t(s_0)$. This contradicts to the assumption that γ is transversal to rulings.

Hence, we have $\mu(s)\tau(s) - \lambda(s)\kappa(s) = 0$, so that

$$\tau(s)\delta(s) = \tau(s)\lambda(s)t(s) + \kappa(s)\lambda(s)b(s) = \lambda(s)D(s),$$

where $D(s)$ is the Darboux vector field along γ. □

Since the rectifying developable of a cylindrical helix is a cylindrical surface, we have the following another characterization of cylindrical surfaces as a simple corollary of Theorem 4.4.

Corollary 4.5 *Suppose that $F_{(\gamma,\delta)}$ is a non-singular developable surface. If there exists a cylindrical helix with non-zero curvature on $F_{(\gamma,\delta)}$ which is a geodesics of $F_{(\gamma,\delta)}$, then $F_{(\gamma,\delta)}$ is a cylindrical surface.*

Moreover, we also have another characterization of cylindrical surfaces.

Corollary 4.6 *Let $F_{(\gamma,\delta)}(s,u) = \gamma(s) + u\delta(s)$ be a non-singular ruled surface. If there exists a planer geodesic of $F_{(\gamma,\delta)}$ with non-zero curvature which is perpendicular to rulings at any point, then $F_{(\gamma,\delta)}$ is a cylindrical surface.*

Proof. By Frenet-Serret formulae, a planer geodesic is a line of curvature. Since the tangent vector of such a geodesic is perpendicular to the ruling, the direction of the ruling is also the principal direction. This means that $F_{(\gamma,\delta)}$ is a developable surface. Since any plane curve is a helix, the assertion follows from Corollary 4.5. $\qquad\square$

On the other hand, we now consider asymptotic curves on ruled surfaces. We prepare the following simple lemma on Euclidean plane.

Lemma 4.7 *Let e_1, e_2 be the canonical basis of Euclidean plane $\mathbb{R}^2$. Let v_1, v_2 be unit vectors in $\mathbb{R}^2$. We assume that there exists real numbers $\lambda > 0$ and α such that $v_1 = \lambda(e_1 + \alpha e_2)$. Then $v_2 = \lambda(e_1 - \alpha e_2)$ if and only if*

$$v_2 \cdot e_1 = v_1 \cdot e_1 \text{ and } v_1 \cdot v_2 = \frac{1 - \alpha^2}{1 + \alpha^2}.$$

Let $F_{(\gamma,\delta)}(s, u) = \gamma(s) + u\delta(s)$ be a ruled surface which is non-singular on $\gamma(s)$. In this case, $\gamma(s)$ is transversal to rulings. If the Gaussian curvature is negative along $\gamma(s)$, then there exist two different principal directions $e_1(s), e_2(s)$ along $\gamma(s)$ with principal curvatures $\kappa_1(s), \kappa_2(s)$ respectively. We may assume that $\|e_i(s)\| = 1$. We have the following proposition.

Proposition 4.8 *Under the same situation as the above, $\gamma(s)$ is an asymptotic curve if and only if*

$$\gamma'(s) \cdot e_1(s) = \delta(s) \cdot e_1(s) \text{ and } \gamma(s) \cdot \delta(s) = \frac{\kappa_1(s) + \kappa_2(s)}{\kappa_1(s) - \kappa_2(s)}.$$

Proof. We now consider two tangent vectors at $\gamma(s)$ which is given by

$$v_1 = e_1(s) + \sqrt{-\frac{\kappa_1(s)}{\kappa_2(s)}} e_2(s), \quad v_2 = e_1(s) - \sqrt{-\frac{\kappa_1(s)}{\kappa_2(s)}} e_2(s).$$

Let N be the unit normal of $F_{(\gamma,\delta)}$ at $\gamma(s)$. Since $(-dN)e_i(s) = \kappa_i(s)e_i$ $(i = 1, 2)$, we have

$$(-dN)v_i \cdot v_i = \kappa_1(s) - \frac{\kappa_1(s)}{\kappa_2(s)}\kappa_2(s) = 0.$$

This means that v_1 and v_2 give asymptotic direction at $\gamma(s)$. Since the Gaussian curvature is negative at $\gamma(s)$ and $\delta(s)$ gives an asymptotic direction, we may assume that

$$\delta(s) = \lambda(s)\left(e_1(s) + \sqrt{-\frac{\kappa_1(s)}{\kappa_2(s)}} e_2(s)\right),$$

where $\lambda(s) = 1/\sqrt{1 - \kappa_1(s)/\kappa_2(s)}$. If $\alpha = \sqrt{-\kappa_1(s)/\kappa_2(s)}$, then

$$\frac{1 - \alpha^2}{1 + \alpha^2} = \frac{\kappa_2(s) + \kappa_1(s)}{\kappa_2(s) - \kappa_1(s)}.$$

The assertion follows directly from the above lemma. $\qquad\square$

We have the following corollary which is analogous result to Bonnet's theorem on geodesics of ruled surfaces.

Corollary 4.9 *Let $F_{(\gamma,\delta)}(s,u) = \gamma(s) + u\delta(s)$ be a ruled surface which is nonsingular on $\gamma(s)$. We assume that $\gamma(s)$ is an asymptotic curve of $F_{(\gamma,\delta)}$ and we denote that $\kappa_i(s)$ ($i = 1, 2$) as two different principal curvatures at $\gamma(s)$. Then the following conditions are equivalent:*
(1) The angle between $\gamma'(s)$ and $\delta(s)$ is constant.
(2) $\dfrac{\kappa_1(s)}{\kappa_2(s)}$ is constant.

In the first paragraph of this section we have shown that the mean curvature of the principal normal surface $F_{(\gamma,n)}$ vanishes along γ which is an asymptotic curve of $F_{(\gamma,n)}$. We can show the converse assertion is also true as a corollary of Proposition 4.8. We say that a curve on a surface is a *minimal asymptotic curve* if it is an asymptotic curve and the mean curvature vanishes on the curve.

Theorem 4.10 *Let $F_{(\gamma,\delta)}(s,u) = \gamma(s) + u\delta(s)$ be a ruled surface and $\sigma(s)$ be a curve on $F_{(\gamma,\delta)}$. Then the following conditions are equivalent:*
(1) $F_{(\gamma,\delta)}$ is the principal normal surface of $\sigma(s)$.
(2) The curve $\sigma(s)$ is a minimal asymptotic curve of $F_{(\gamma,\delta)}$ which is transversal to rulings.

Proof. Suppose that the condition (2), then $\kappa_2(s) + \kappa_1(s) = 0$. By Proposition 4.8, $\sigma'(s)$ is perpendicular to $\delta(s)$. Since $\sigma(s)$ is an asymptotic curve, this means that $\delta(s)$ is parallel to the principal normal direction of $\sigma(s)$. The converse assertion has already been proved. $\square$

On the other hand, we have another proof of Theorem 4.10 as follows: Let $\sigma(s) = \gamma(s) + u(s)\delta(s)$ be a curve on $F_{(\gamma,\delta)}$. Suppose that $F_{(\gamma,\delta)}$ is non-singular on $\sigma(s)$. This means that $\sigma(s)$ is transversal to rulings. Since

$$\sigma'(s) = \frac{\partial F_{(\gamma,\delta)}}{\partial s}(s, u(s)) + u'(s)\frac{\partial F_{(\gamma,\delta)}}{\partial u}(s, u(s)),$$

$\sigma(s)$ is an asymptotic curve if and only if

$$\det\left(\gamma''(s) + u(s)\delta''(s), \gamma'(s) + u(s)\delta'(s), \delta(s)\right) + 2\det\left(\delta'(s), \gamma'(s), \delta(s)\right)u'(s) = 0.$$

Under the assumption that $K(s, u(s)) < 0$, $\sigma(s)$ is a minimal asymptotic curve if and only if $u'(s) = -\delta(s) \cdot \gamma'(s)$. We remark that $\sigma'(s) \cdot \delta(s) = 0$ if and only if $u'(s) = -\delta(s) \cdot \gamma'(s)$. This completes the alternate proof of Theorem 4.10.

By using this method, we have the following characterization of helicoids.

Proposition 4.11 *Let $F_{(\gamma,\delta)}(s,u) = \gamma(s) + u\delta(s)$ be a non-singular ruled surface. If there exist three disjoint minimal asymptotic curves on $F_{(\gamma,\delta)}$ which are transversal to rulings, then $F_{(\gamma,\delta)}$ is a helicoid. In this case minimal asymptotic curves which are transversal to rulings are circular helices.*

Proof. By the previous calculation, we remark that the mean curvature of $F_{(\gamma,\delta)}$ is a quadratic functions of the u variable. If there exist three disjoint minimal asymptotic curves on $F_{(\gamma,\delta)}$ which are transversal to rulings, then the mean curvature always vanishes. This means that the surface $F_{(\gamma,\delta)}$ is a minimal surface. It has been classically known that a minimal ruled surface is a helicoid. In this case each minimal asymptotic curves which is transversal to rulings is a circular helix. $\square$

Finally we give a characterization of Bertrand curves as curves on ruled surfaces.

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Proposition 4.12 *Let $F_{(\gamma,\delta)}(s,u) = \gamma(s) + u\delta(s)$ be a non-singular ruled surface. If there exist two disjoint minimal asymptotic curves on $F_{(\gamma,\delta)}$ which are transversal to rulings, then these curves are Bertrand curves and Bertrand mates of each other.*

Proof. Let $\sigma_i(s) = \gamma(s) + u_i(s)\delta(s)$ $(i = 1, 2)$ be minimal asymptotic curves which are transversal to rulings. By Theorem 4.10, $F_{(\gamma,\delta)}$ is the principal normal surface of $\sigma_i(s)$. By the previous argument, we have $u_i'(s) = -\delta(s) \cdot \gamma'(s)$, so that $(u_1 - u_2)'(s) = 0$. Thus there exists a constant A such that $u_1(s) = u_2(s) + A$. It follows from this fact that $\sigma_1(s) = \sigma_2(s) + A\delta(s)$. We may choose s as the arc-length parameter of $\sigma_2(s)$. In this case $\delta(s)$ can be consider as the unite principal normal of $\sigma_2(s)$. By the calculation of the mean curvature of the principal normal surface of $\sigma_2(s)$ and Corollary 2.2, $\sigma_1(s)$ is a Bertrand curve and $\sigma_2(s)$ is a Bertrand mate of $\sigma_1(s)$. $\qquad\square$

5 Examples

In this section we give some examples of cylindrical helices and Bertrand curves and draw their pictures by using Mathematica.

Let $\gamma : I \longrightarrow \mathbb{R}^2$ be a unit speed plane curve with the curvature $\kappa(s)$. We define a space curve

$$\widetilde{\gamma}(s) = \gamma(s) \pm \left(\cot\theta \int \|\gamma'(s)\| ds \right) a,$$

where θ is a constant number and a is a constant vector with $\gamma'(s) \cdot a = 0$ and $\|a\| = 1$. In this case, we can calculate that the curvature of $\widetilde{\gamma}(s)$ is given by $\widetilde{\kappa}(s) = \kappa(s)\sin^2\theta$ and the torsion is $\widetilde{\tau}(s) = \kappa(s)\cot\theta\sin^2\theta$. Therefore, $\widetilde{\gamma}(s)$ is a cylindrical helix. We can also calculate that the Darboux vector of $\widetilde{\gamma}$ is $D(s) = \kappa(s)|\sin\theta|a$, so that the rectifying developable of $\widetilde{\gamma}$ is the cylindrical surface given by $F_{(\widetilde{\gamma},a)}(s,u) = \widetilde{\gamma}(s) + ua$. These arguments mean that we can construct cylindrical helices from plane curves by the above way. Especially, if we choose a plane curve $\gamma(t) = (f(t), g(t), 0)$ and $a = (0, 0, 1)$, we have

$$\widetilde{\gamma}(t) = \left(f(t), g(t), c \int \sqrt{f'(t)^2 + g'(t)^2} \, dt \right),$$

where c is a constant number. Some examples of cylindrical helices and their pictures are given as follows.

Example 5.1 We now consider two plane curves

$$\gamma(\theta) = (2\sin\theta, \cos\theta, 0), \quad \gamma(\theta) = (2\sin\theta, \sin\theta, 0)$$

then we have two cylindrical helices

$$\widetilde{\gamma}(\theta) = (2\sin\theta, \cos\theta, E(\theta, 3/4)) \quad \widetilde{\gamma}(\theta) = (2\sin\theta, \sin\theta, \sqrt{5}|\cos\theta|\tan\theta),$$

where $E(\phi, m) = \int_0^\infty \sqrt{1 - m\sin^2\theta} \, d\theta$. These curves are depicted as in Fig.4 and Fig.5. Pictures in Fig.5. are rectifying developables of curves in Fig.4. We can recognize that these surfaces are cylindrical surfaces.
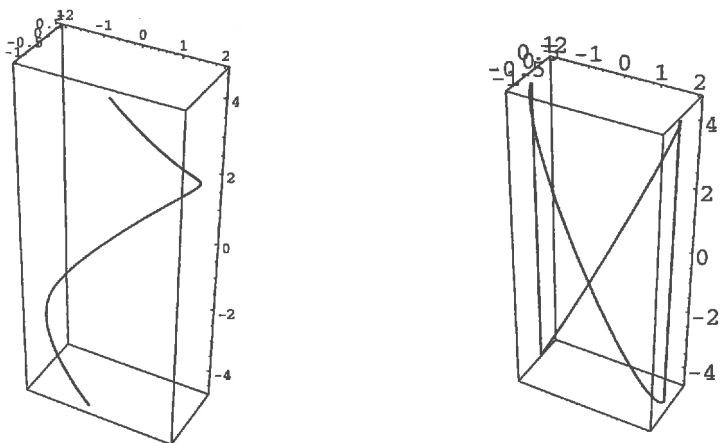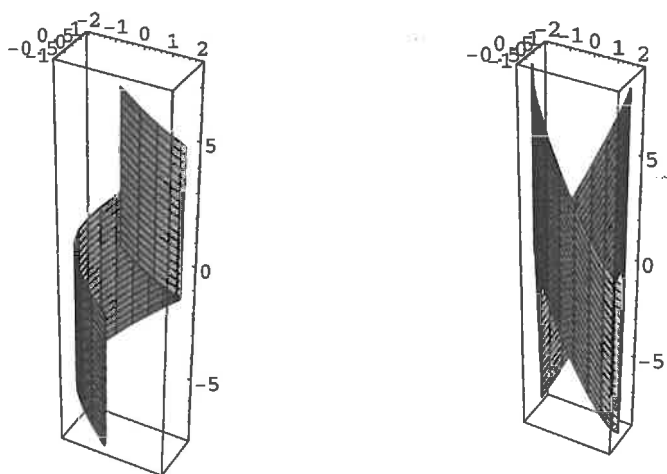
Fig.4.

Fig.5.

Let $\gamma : I \longrightarrow S^2$ be a unit speed spherical curve. We define a space curve

$$\widetilde{\gamma}(s) = a \int \gamma ds + a \cot \theta \int \gamma(s) \times d\gamma(s),$$

where a, θ are constant numbers. In this case we can calculate that the curvature and the torsion of $\widetilde{\gamma}(s)$ is given as follows:

$$\widetilde{\kappa}(s) = \varepsilon \frac{\sin^2 \theta(1 - \kappa_g(s) \cot \theta)}{a} \quad \widetilde{\tau}(s) = \frac{\sin^2 \theta(\kappa_g(s) + \cot \theta)}{a},$$

where $\varepsilon = \pm 1$ and $\kappa_g(s) = |\gamma(s), \gamma'(s), \gamma''(s)|$ is the geodesic curvature of the spherical curve γ. It follows from these formulae that $a(\varepsilon \widetilde{\kappa}(s) + \cot \theta \widetilde{\tau}(s)) = 1$, so that $\widetilde{\gamma}(s)$ is a Bertrand curve.

Example 5.2 We consider a spherical curve $\gamma(\theta) = (\sin \theta, \sin \theta \cos \theta, \cos^2 \theta)$. By using the above method, we have the following Bertrand curve:

$$\widetilde{\gamma}(\theta) = \left(\frac{1}{4}(-\sqrt{2} \cos \theta \sqrt{3 + \cos 2\theta} - 2(\theta + \log(\sqrt{2} \cos \theta + \sqrt{3 + \cos 2\theta}) - \sin 2\theta, \right.$$

$$\frac{1}{12}(3\sqrt{6} - 3\sqrt{6 + 2 \cos 2\theta} - \cos 2\theta \sqrt{6 + 2 \cos 2\theta} + 15 \sin \theta - \sin 3\theta),$$

$$\left. \frac{1}{12}(12\sqrt{2}E(\theta, \frac{1}{2}) - 4\sqrt{2}E(\theta, \frac{1}{2}) - 2 \cos \theta(-5 + \cos 2\theta - \sqrt{6 + 2 \cos 2\theta} \sin \theta) \right).$$

The Bertrand mate of $\widetilde{\gamma}(\theta)$ is given as follows:

$$\bar{\widetilde{\gamma}}(\theta) = \left(\frac{1}{4} \left(-2(\theta + \log(\sqrt{2} \cos \theta + \sqrt{3 + \cos 2\theta}) - \frac{4\sqrt{2} \cos^3 \theta \sin \theta}{(3 + \cos 2\theta)^{3/2}} \right. \right.$$

$$\left. + \cos \theta \left(2 - \sqrt{6 + 2 \cos 2\theta} + \frac{4\sqrt{2} \sin \theta}{\sqrt{3 + \cos 2\theta}} \right) - \sin 2\theta \right),$$

$$\frac{1}{12} \left(3\sqrt{6} - 3\sqrt{6 + 2 \cos 2\theta} - \cos 2\theta \sqrt{6 + 2 \cos 2\theta} + 15 \sin \theta - \sin 3\theta \right.$$

$$\left. + 6 \left(\cos^2 \theta - \frac{\sqrt{2} \cos^2 (-3 + \cos 2\theta) \sin \theta}{(3 + \cos 2\theta)^{3/2}} - \sin^2 \theta + \frac{-5 \sin \theta + 3 \sin 3\theta}{2\sqrt{6 + 2 \cos 2\theta}} \right) \right),$$

$$\frac{1}{12} \left(12\sqrt{2}E(\theta, \frac{1}{2}) + 2 \left(-2\sqrt{2}F(\theta, \frac{1}{2}) + \cos \theta (5 - \cos 2\theta - 6 \sin \theta) \right. \right.$$

$$\left. \left. \left. + \sqrt{6 + 2 \cos 2\theta} \sin \theta - \frac{9\sqrt{2} \sin^2 \theta}{\sqrt{3 + \cos 2\theta}} - \frac{6\sqrt{2} \sin^4 \theta}{(3 + \cos 2\theta)^{3/2}} \right) \right) \right) \right),$$

where $F(\phi m) = \int_0^\phi \frac{d\theta}{1 - m \sin^2 \theta}$. Pictures of both curves and the principal normal surface are given in Fig.6.
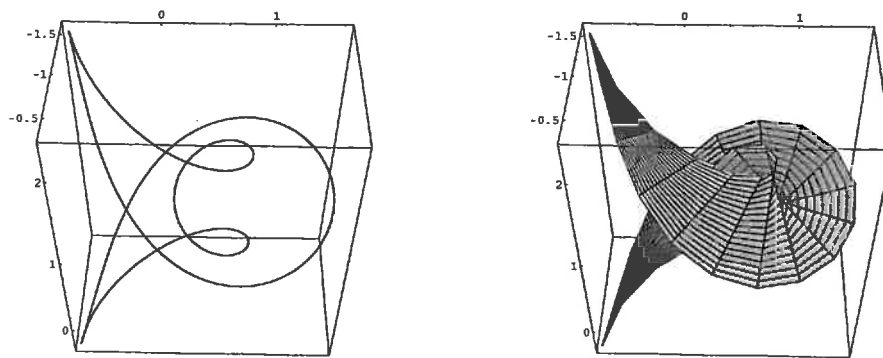
Fig.6.

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