Instability of steady flows of an ideal incompressible fluid

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ABSTRACT

This article is devoted to recent developments and open questions concerning instabilities in ideal fluid flows. It is argued that in some appropriate sense almost all steady flows of an ideal incompressible fluid are unstable. However there are different kinds of instability. Many of the instabilities that are described could be termed "slow" and technically they are associated with the Jordan cell structure of the governing operator as opposed to the "fast" instabilities associated with isolated unstable eigenvalues. Numerous examples are given to stress the importance for the existence of instabilities of the norm in which the growth of disturbance is measured.

1   Introduction

The topic of this article is linear and nonlinear instability of steady (i.e. time independent) flows of an ideal incompressible fluid. Our goal is to discuss our understanding of this subject, rather than to try to survey all works in this domain. We admit that our list of references is quite incomplete and personal. The ideal incompressible fluid is a basic model in fluid dynamics. It is believed to describe correctly the motion of real fluids in conditions such that the effects of viscosity and compressibility are negligible. The stability and instability of steady flows is an old subject with many impressive achievements and many well-developed methods. However there remain significant

unsolved problems, and we may say that the main problems of instability are open.

Let us start from the notion of stability and instability of steady flows. The motion of an inviscid incompressible fluid in a domain $M \subset \mathbf{R}^n$ is described by the Euler equations:

$$\frac{\partial u}{\partial t} + (u, \nabla)u + \nabla p = 0; \tag{1.1}$$

$$\nabla \cdot u = 0. \tag{1.2}$$

Here $u(x,t)$ is the velocity field of the fluid, $p(x,t)$ is the pressure. The boundary conditions are $u_n\big|_{\partial M} = 0$, i.e. the flow is tangent to the boundary; the initial condition is $u(x,0) = u_0(x)$.

Let $U_0(x)$ be the velocity field of a steady flow of an ideal incompressible fluid. U_0 satisfies the steady Euler equation

$$(U_0, \nabla)U_0 + \nabla p_0 = 0; \tag{1.3}$$

$$\nabla \cdot U_0 = 0. \tag{1.4}$$

Loosely speaking, this flow is unstable if a small change of initial velocity field results in a considerable change of the actual flow field $u(x,t)$ at some time T. If, on the other hand, small perturbations result in small changes of the flow during arbitrarily long time, the flow is stable.

Both stability and instability are important features of fluid flows and deserve much attention. It appears that only stable flows may exist for a long time, while the unstable ones break down, or require some external stabilization or some feedback to persist. Note that both breaking down of unstable flows, and their stabilization and control are of great interest for numerous applications in natural sciences and engineering. There exists an inverse relation between the stability and the controllability. The system which is excessively stable is difficult to control, and conversely, unstable system is usually easier to control if one manages to stabilize it. For example, today's jet fighters are intentionally built unstable, so that they cannot fly without the stabilization done by an on-board computer. But this instability makes these planes very maneuverable.

The breakdown of unstable flows itself may be very important; for example, the action of the most of musical instruments is based on some sort

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of instability. The impression that only stable flows persist for a long time without external stabilization may be wrong. There exist different kinds of instability and some of them, which may be called "slow instabilities", are delicate enough and, in their turn, very unstable themselves; they are described in more detail below.

The question of instability falls naturally into two parts, linear and nonlinear instability. The majority of the classical work is connected with the linear problem and studies of properties of the eigenvalues of the operator obtained by linearizing (1.1), (1.2) about a specific steady flow [DR], [Cha]. Much of the work has concentrated on a relatively small number of special fluid configurations, and even in these open questions remain. In section 2 we describe some recent results obtained through a study of the evolution operator for the linearized Euler equations which shed light on the linear instability of rather general Euler flows. We also discuss the existence of unstable discrete eigenvalues in a few specific cases. Linear instability where growth is measured in the energy norm appears to be ubiquitous for 3-dimensional Euler flows.

The problem of nonlinear stability/instability is even more subtle and perplexing than the questions assocated with the linear theory. We address certain aspects of this problem in section 3. We do something that is rather unusual: we work with two definitions of nonlinear stability/instability and discuss ramifications that follow from these two related but nonequivalent definitions. That we proceed in this fashion is at least partly dictated by our lack of full understanding of the nature of fluid instability and partly due to the subtleties of a system which exhibits fascinating degrees of instability. The first definition allows us to discuss a general result of instability in function spaces that are "correct spaces" for the Euler equations, i.e. where local existence and uniqueness of solutions is known. The second definition applies to spaces that are not "correct" in this sense but are natural spaces from a physical point of view. Although we discuss results concerning fairly general flows, much of the section 3 concentrates on one of the most basic classes of flows, namely plane parallel shear flows. The simplest nontrivial example, i.e. a shear flow with a linear profile, illustrates the complexity of fluid stability:

(i) it is linearly (spectrally) stable;
(ii) it is nonlinearly stable in the sense of Arnold in the vorticity norm;
(iii) it is nonlinearly unstable in any norm that includes derivatives of

vorticity;

(iv) it is nonlinearly unstable in the L^2-norm (with no conditions on the derivatives of velocity);

(v) it is linearly and nonlinearly unstable as a 3-dimensional flow in a norm which includes the magnitude of vorticity.

The energy and strength of these instabilities will be different from the "fast" exponential instabilities that exist for shear flows with inflection points in the profile.

Throughout this article we hope to communicate several key observations, namely the crucial importance of the norm in which growth of disturbances is measured and the existence of different types of instability that all influence the evolution of a fluid configuration.

2   Linear instability

We consider the linearized Euler equations for a small perturbation $v(x, t)$ about a steady flow U_0, satisfying (1.3), (1.4):

$$\frac{\partial v}{\partial t} = -(U_0, \nabla)v - (v, \nabla)U_0 - \nabla P; \qquad (2.1)$$

$$\nabla \cdot v = 0, \qquad (2.2)$$

with initial condition

$$v(x, 0) = v_0(x). \qquad (2.3)$$

These equations may be recast in the following more convenient form:

$$\frac{\partial v}{\partial t} = Lv, \qquad (2.4)$$

where

$$Lv = -P\Big((U_0 \cdot \nabla)v + (v \cdot \nabla)U_0\Big); \qquad (2.5)$$

here P is the orthogonal projector in $L^2(M, \mathbf{R}^n)$ onto the space J of incompressible vector fields tangent to ∂M.

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The stability problem for the linearized Euler equations may be posed as follows. Let us choose a function space X of vector fields where the problem (1.1), (1.2) is well posed. We ask if there exists an initial condition $v_0 \in X$ such that $||v(t)||_X$ is *unbounded* on the whole t-axis; in this case the zero solution of linearized equations is *unstable*. Otherwise it is stable. The classical approach to linear stability of fluid motion is based on an investigation of the spectrum of the operator L in a given space X. Much of the discussion in such texts as [Cha] or [DR] concerns properties of eigenvalues of L in the case of of specific, relatively simple flows U_0. There is a more powerful approach in which the spectrum of the evolution operator is studied (see [VF], [V], [FV]).

The spectrum σ of the operator e^{tL} is naturally decomposed into a discrete part consisting of isolated eigenvalues of finite multiplicity and an essential spectrum:

$$\sigma = \sigma_{\text{disc}} \cup \sigma_{\text{ess}}. \tag{2.6}$$

The operator L is a degenerate non-selfadjoint non-elliptic operator. For an arbitrary steady flow $U_0(x)$ the structure of the spectrum σ is remarkably little understood. Although, as we will describe, we have some general results concerning the essential spectrum, the problem of the existence of discrete eigenvalues is at present too difficult for any general results and must be treated on a case by case basis.

2.1   The unstable essential spectrum

Recently Friedlander and Vishik [VF],[FV] developed a useful tool for investigating the unstable essential spectrum of the linearized Euler equation (2.1), (2.2). One of the main ideas in [VF] is to replace the study of the spectrum of L by the study of the spectrum of the evolution operator e^{tL} for $t > 0$. This permits the development of an explicit formula for the growth rate of a small perturbation due to the essential spectrum. The following theorem proved by Vishik [V] gives an expression for the essential spectral radius $r_{\text{ess}}(e^{tL})$ in terms of a geometric quantity that can be considered as a Lyapunov exponent for fluid flow. The results are proved for free space or periodic boundary conditions and are valid in any spatial dimension, although spatial dimensions 2 and 3 are, of course, physically the most interesting. We consider perturbations with $v_0 \in L^2$, $\nabla \cdot v_0 = 0$.

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Theorem 2.1

$$r_{ess}(e^{tL}) = e^{\mu t}, \qquad (2.7)$$

where

$$\mu = \lim_{t \to \infty} \frac{1}{t} \log \sup |b(x_0, \xi_0, b_0; t)|. \qquad (2.8)$$

Here supremum is taken over all triplets (x_0, ξ_0, b_0), such that $|b_0| = 1, |\xi_0| = 1, \xi_0 \cdot b_0 = 1$, and the vector $b = b(x_0, \xi_0, b_0; t)$ is determined by the following system of ODE's which we call the bicharacteristic amplitude equations:

$$\left.\begin{array}{lll}
(a) & \dot{v} = U_0(x) \\[6pt]
(b) & \dot{\xi} = -\left(\dfrac{\partial U_0}{\partial x}\right)^T \xi \\[10pt]
(c) & \dot{b} = -\left(\dfrac{\partial U_0}{\partial x}\right)b + 2\left[\left(\dfrac{\partial U_0}{\partial x}\right)b \cdot \xi\right]\xi/|\xi|^2
\end{array}\right\} \qquad (2.9)$$

with $x(0) = x_0$, $\xi(0) = \xi_0$, $b(0) = b_0$.

This theorem is proved by writing the evolution operator e^{tL} as a product of a pseudo-differential operator and a shift operator along the trajectories of the equilibrium flow U_0. This allows the growth of the evolution operator to be studied to precise exponential asymptotics. A heuristic derivation of this result is obtained by applying a "geometric optics" treatment based on high frequency solutions of equations (2.1), (2.2). In the language of geometric optics, equation (2.9b) is the Eikonal equation and equation (2.9c) is the transport equation. Equation (2.9c) is the evolution equation for the amplitude of a high frequency wavelet initially localized at x_0, with initial wave number vector ξ_0. The quantity on the RHS of (2.8) is the "fluid Lyapunov exponent" corresponding to the maximal exponential growth rate of such an amplitude vector $b(t)$.

The result of Theorem 2.1 gives one piece of information concerning the stability spectrum for inviscid flows, namely the maximum growth rate of instability in the essential spectrum. Moreover it implies that any point z in the spectrum $\sigma(e^{tL})$ such that $|z| > e^{\mu t}$ is necessarily an eigenvalue of finite multiplicity. A positive lower bound for the value of the Lyapunov exponent μ can be explicitly computed in many examples [FV],[FV1]. Furthermore theorem 1 provides an effective sufficient condition for instability of large classes of inviscid fluid flows. Since expression (2.8) involves the supremum

over initial conditions (x_0, ξ_0, b_0), it is only necessary to show there exists at least one initial condition for which the solution to the system (2.9) of ODE gives

$$\lim_{t \to \infty} \frac{1}{t} \log |b| > 0 \qquad (2.10)$$

to conclude that $\mu > 0$, and hence the unstable essential spectrum is nonempty.

We comment briefly on some other results obtained in the context of fluid instability using a "geometric optics" approach. This method has been employed successfully by a number of authors with the earliest work being on the hyperbolic systems for *compressible* fluids by G. Friedlander in 1958 [Fr] and Ludwig in 1960 [L]. Later Eckhoff [E] and Eckhoff and Storesletten [ES] studied the stability of azimuthal shear flows of a compressible fluid and more generally symmetric hyperbolic systems using an approach based on a generalized progressive wave expansion. Eckhoff shows that local instability problems for hyperbolic systems can be essentially reduced to a local analysis involving ODEs. We note that the *incompressible* Euler equations (1.1), (1.2) do not form a strictly hyperbolic system, and the proof of Theorem 1 does not follow directly from Eckhoff. Results using a system of ODEs equivalent to (2.8) to detect instabilities in incompressible flows include those of Bayly et al [BHL], who obtained the growth rate of instabilities for columnar and elliptic vortices, and Lifschitz and Hameiri [LH] who obtained instability conditions for vortex rings.

We will now discuss in a little more detail some of the explicit instability results of Friedlander and Vishik that follow from Theorem 2.1. The idea that exponential stretching of fluid particles could imply instability for the Euler equations is originally due to Arnold [A1]. Friedlander and Vishik [FV] use Theorem 2.1 to prove that every flow with exponential stretching, even at one point, is linearly unstable.

Theorem 2.2 *Consider a steady solution U_0 of 3-dimensional Euler equations. Suppose the flow U_0 has a positive classical Lyapunov exponent at some point x_0. Then $\mu > 0$, and hence the flow U_0 is unstable.*

The proof of this theorem follows from a result obtained from the system (2.9) of ODEs, namely:

$$\frac{d}{dt}(b_1 \times b_2 \cdot \xi) = 0, \qquad (2.11)$$

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where b_1 and b_2 are two linearly independent solutions of (2.9) corresponding to a cotangent vector ξ that satisfies (2.9). Since the flow is volume preserving the existence of a positive Lyapunov exponent (i.e. an exponentially growing tangent vector) implies the existence of an exponentially decaying cotangent vector ξ. From (2.10) we then conclude the existence of at least one exponentially growing amplitude vector b, which implies $\mu > 0$. Therefore we note that in 3 dimensions the classical Lyapunov exponent provides a lower bound on μ.

In 2 dimensions, the system (2.9) of ODEs provides an even stronger constraint on the relation between $b(t)$ and $\xi(t)$, namely

$$\frac{d}{dt}(|b(t)||\xi(t)|) = 0. \tag{2.12}$$

Hence in 2 dimensions the fluid Lyapunov exponent μ and the maximal classical Lyapunov exponent for the dynamical system $\dot{x} = U_0(x)$ are the same.

It follows from Theorem 2 that any flow U_0 (in 2 or 3 dimensions) with a hyperbolic stagnation point x_0 is unstable (i.e. $U(x_0) = 0$ and there exists an eigenvalue of $(\frac{\partial U_0}{\partial x})_{x_0}$ with positive real part). There are large classes of fluid flows U_0 with such stagnation points. For all such flows $r_{ess} > 1$.

A class of flows with presumably chaotic stream lines was identified by Arnold [A2]. An example is the so-called ABC flow $U_0 = (\dot{x}, \dot{y}, \dot{z})$ where

$$\begin{aligned}
\dot{x} &= A \sin z + C \cos y \\
\dot{y} &= B \sin x + A \cos z \\
\dot{z} &= C \sin y + B \cos x.
\end{aligned} \tag{2.13}$$

For general values of the constants A, B and C numerical investigations [OFGHMS] indicate that ABC flows exhibit the phenomenon of Lagrangian chaos which suggests strong exponential stretching. Analytic treatment of ABC flows [FGV], [Chi] proves that for certain ranges of A, B and C there is exponential stretching either at hyperbolic points or associated with hyperbolic closed trajectories. The result of Theorem 2 then proves that these ABC flows are hydrodynamically unstable.

In 3 dimensions the mechanism of vortex tube stretching, which is absent in strictly 2-dimensional flows, can give rise to values of μ which are

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greater than the classical Lyapunov exponent. For example, in [F] analysis of a model equation for 3-dimensional Euler known as the surface quasi-geostrophic equation leads to a quantity analogous to μ that tends to infinity for flows with hyperbolic structures: i.e. there exist perturbations that grow like e^{t^2}. This result suggests that a 3-dimensional flow with hyperbolic structure is strongly unstable. Furthermore in 3 dimensions it is possible to have flows U_0 for which the classical Lyapunov exponents are all zero yet the fluid exponent μ is positive. Such an example is constructed in [FV]. It is proved that the integrable flow $U_0 \times \text{curl} U_0 = -\nabla H$ with $\nabla H \neq 0$ has μ positive provided that a certain geometric condition is satisfied by the stream lines. The following (non sharp) condition ensures $\mu > 0$:

$$\int_0^T \{\kappa \hat{n} \cdot \nabla H - \tau_g U_0 \cdot \text{curl} U_0 / |\nabla H|^2\} dt \geq 0 \qquad (2.14)$$

where, for any stream line of the flow as it wraps around the toroidal surface $H = H_0$, T is the period, κ the curvature, τ_g the geodesic torsion and $\hat{n}$ the principal unit normal to the stream line.

We have described many fluid flows where it can be shown that $\mu > 0$. In a few cases μ can be computed explicitly. For example, the 2-dimensional cellular flow

$$U_0 = (-\sin x \cos y, \cos x \sin y). \qquad (2.15)$$

In this case μ is given by the positive real eigenvalue of the matrix $\left(\frac{\partial U_0}{\partial x}\right)$ at a hyperbolic stagnation point. Thus $\mu = 1$ for this simple cellular flow. There are certain classes of 2-dimensional flows for which it follows from (2.8) that $\mu = 0$ [FSV]. In particular $\mu = 0$ (i.e. there is no unstable essential spectrum) for 2-dimensional flows with no stagnation points or any 2-dimensional plane parallel shear flow.

2.2 Examples of Instability in the Discrete Spectrum

We now turn to the question of existence and distribution of unstable eigenvalues in the discrete spectrum of equation (2.1), (2.2). The linearized Euler operator is degenerate, non-elliptic, and there are no general theorems that may be applied to prove the existence of unstable discrete eigenvalues. However in certain rather special examples it is possible to construct unstable eigenvalues.

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The spectral problem for the linearized Euler operator is considerably simpler in 2 dimensions rather than in 3 dimensions. In particular, in 2 dimensions we can define a scalar stream function to replace the divergence free vector field. We write

$$U_0 = \hat{k} \times \nabla\Psi(x,y), \ v = \hat{k} \times \nabla\phi(x,y,t). \tag{2.16}$$

Hence,

$$\nabla \times U_0 = \hat{k}\nabla^2\Psi(x,y), \ \nabla \times v = \hat{k}\nabla^2\phi(x,y,t). \tag{2.17}$$

Here $\hat{k}$ is the unit vector perpendicular to the 2-dimensional plane with Cartesian coordinates (x,y). The 2-dimensional steady equations (1.3), (1.4) will be satisfied when Ψ satisfies an elliptic equation of the form

$$\nabla^2\Psi = -F(\Psi). \tag{2.18}$$

Taking the curl of equation (2.1) gives the equation for the evolution of the perturbation vorticity $\omega \equiv \nabla \times v$:

$$\frac{\partial\omega}{\partial t} = \{U_0, \omega\} + \{v, \nabla \times U_0\}, \tag{2.19}$$

where $\{,\}$ denotes the Poisson bracket of two vector fields, i.e.

$$\{A, B\} = (B \cdot \nabla)A - (A \cdot \nabla)B. \tag{2.20}$$

In general the second Poisson bracket on the RHS of (2.19) is very difficult to analyse. However in 2 dimensions the problem greatly simplifies because $\hat{k} \cdot \nabla(\cdot) \equiv 0$. The vorticity equation (2.19) reduces to

$$\lambda\omega + (U_0 \cdot \nabla)\omega + (v \cdot \nabla)(\nabla \times U_0) = 0. \tag{2.21}$$

We consider the eigenfunction ϕ and the eigenvalue λ for equation (2.19); after substituting (2.18) and (2.21) into (2.19) we obtain equation

$$\lambda\nabla^2\phi = (\Psi_y \frac{\partial}{\partial x} - \Psi_x \frac{\partial}{\partial y})(\nabla^2\phi + F'(\Psi)\phi). \tag{2.22}$$

We take the boundary conditions to be 2π-periodicity in (x,y).

A simple and very classical example that has received much attention in the literature of the past 100 years is plane parallel shear flow (see, for

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example, [DR], [Cha]). In this case $U_0 = (U(y), 0)$ and (2.22) becomes the so-called Rayleigh equation:

$$(\frac{\lambda}{ik} + U(y))(\frac{d^2}{dy^2} - k^2)\Phi(y) - U''(y)\Phi(y) = 0, \qquad (2.23)$$

where we have written

$$\phi(x, y, t) = \Phi(y)e^{ikx}e^{\lambda t}. \qquad (2.24)$$

The celebrated Rayleigh stability criterion [DR] says that a necessary condition for instability is the presence of an inflection point in the profile $U(y)$. As we remarked in section 2.1, the concept of the "fluid Lyapunov exponent" μ given by expression (2.8) can be used to prove that equation (2.22) with periodic boundary conditions has no unstable essential spectrum for any profile $U(y)$. It remains to discuss the possibility of discrete unstable eigenvalues (i.e. λ such that $\text{Re}\lambda > 0$) associated with equation (2.23) for profiles $U(y)$ that contain at least one inflection point.

Meshalkin and Sinai [MS], followed by Yudovich [Y1] investigated the instability of a *viscous* shear flow $U(y) = \sin my$ using techniques of continued fractions. More recently Friedlander et al [FSV], [BFY], [FH] showed that these techniques could be used for the inviscid equation (2.23) with $U(y) = \sin my$. Eigenfunctions are constructed in terms of Fourier series that converge to C^∞-smooth functions for eigenvalues λ that satisfy the characteristic equation. We write

$$\Phi(y) = \sum_{n=-\infty}^{\infty} a_n e^{iny}. \qquad (2.25)$$

The recurrence relation equivalent to (2.23) yields a tridiagonal infinite algebraic system which is analyzed using continued fractions to yield the characteristic equation relating the eigenvalues to the wavenumbers k and m. The Fourier coefficients a_n decay exponentially with n for each root λ of the characteristic equation. In the example $U(y) = \sin my$ this procedure exhibits the complete unstable spectrum in L^2 of the linearized Euler equation.

The existence of unstable eigenvalues for shear flows with a general rapidly oscillating profile $U(my), m >> 1$, was demonstrated in [BFY] using homogenization techniques to compute the spectral asymptotics. Gordin [G] has solved numerically an interesting problem of finding a "maximally unstable" profile $U(y)$, provided its enstrophy $\int |U'(y)|^2 dy$ is fixed.

On the other hand, if the profile $U(y)$ is close enough to the linear one, i.e. $U(y) = y + \varepsilon f(y)$ for arbitrary smooth function $f(y)$ and sufficiently small $\varepsilon > 0$, then there are no unstable eigenvalues (for a fixed wave number k). This may be deduced from the paper of L. Faddeev [Fa]. We can regard this result as relating to flows on the side surface of a cylinder, in which case the wave number of perturbation cannot be smaller than some constant. We shall discuss these flows in section 3.5.

There are a few results concerning the unstable eigenvalues of (2.22) for somewhat more general flows U_0. A specific "cats-eye" type flow was studied by Friedlander et al [FVY] and the method of averaging was used to construct the formal asymptotic expansion for eigenfunctions of a class of unstable eigenvalues for equation (2.22) with $F(\Psi)$ corresponding to the "cats-eye" flow. We remark that the existence of hyperbolic stagnation points in this flow means that, in contrast with parallel shear flow, μ is positive. Hence both the discrete and the essential unstable spectrum are nonempty.

3 Nonlinear Instability

Problems connected with stability and instability of the full nonlinear Euler equations (2.1)-(2.2) are even more complex than those related to the spectrum of the linearized equation discussed in Section 2. Hence many questions remain open. However some results have been obtained recently. We will describe these "small steps of progress" and indicate some promising paths for future development.

A steady state is called nonlinearly stable if every disturbance that is "small" initially generates a solution to the nonlinear Euler equation which stays "close" to the steady state for all time. There are several natural precise definitions of nonlinear stability and its converse instability. To a certain extent these definitions incorporate a concept of "degrees" of instability. The definitions reflect the crucial dependence of a stable or unstable state on the norm in which growth with time of disturbances is measured. The first definition we give allows us to consider nonlinear stability/instability in function spaces for which it is known that there is local existence and uniqueness. Later in this section we prove a theorem under this definition relating linear instability in L^2 with nonlinear instability in H^s, $s > n/2 + 1$. In the second definition we consider nonlinear stability in L^2 and H^1 which are natural spaces to measure growth of a disturbance but are not "correct" spaces for

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the Euler equation in terms of proven properties of the solutions of the non-linear equation. The elegant nonlinear stability results of Arnold [A3] fall under this second definition, as does the concept of minimal flows introduced by Shnirelman [S1].

3.1 Definitions of Nonlinear Stability/Instability

First definition of nonlinear instability. We define nonlinear stability for a general evolution equation of the form

$$u_t = Lu + N(u), \quad u(0) = u_0, \tag{3.1}$$

where L and N are respectively the linear and nonlinear terms. Let X and Z be a fixed pair of Banach spaces with $X \subset Z$ being a dense embedding. We assume that for any $u_0 \in X$ there exists a $T > 0$ and a unique solution $u(t)$ to (3.1) with

$$u(t) \in L^\infty((0,t); X) \bigcap C([0,T], Z) \tag{3.2}$$

in the sense that for any $\phi \in D(0,T)$

$$\int_0^T \{u(\tau)\phi'(\tau) + (Lu(\tau) + N(w(\tau))\Phi(\tau)\} d\tau = 0. \tag{3.3}$$

The initial condition is assumed in the sense of strong convergence in Z:

$$\lim_{\tau \to 0^+} ||u(\tau) - u_0||_Z = 0. \tag{3.4}$$

Definition 3.1 *The trivial solution $u_0 = 0$ of (3.1) is called* nonlinearly stable *in X (i.e. Lyapunov stable) if for all $\varepsilon > 0$ there exists $\delta > 0$ so that $||u_0||_X < \delta$ implies*
    (a) we can choose T in (3.2) to be $T = \infty$, and
    (b) $||u)t)||_X < \varepsilon$ for a.e. $t \in [0, \infty)$.
The trivial solution is called nonlinearly unstable *in X if it does not satisfy the conditions stated above.*

We remark that by this definition finite time "blow up" (i.e. a maximal *finite* $T > 0$ in (3.1)) is a special case of nonlinear instability. This is valuable in the context of the Euler equations (1.1), (1.2) since in 3 dimensions the

possibility of finite time blow up has not yet been ruled out. In the context of the Euler equations the "natural" choice for the spaces X and Z are H^s with $s > n/2 + 1$ where n is the space dimension and L^2 respectively.

Second definition of nonlinear instability. The second notion of (in)stability is what we call Z-(in)stability. In this definition we do not split the operator into linear and nonlinear parts, because such splitting makes no sense for strong perturbations we are dealing with. Let $X \subset Z$ be a pair of Banach spaces with dense and compact embedding. Consider an operator equation having the form

$$\frac{du}{dt} = A(u). \tag{3.5}$$

Suppose that for every $u_0 \in X$ and every $T > 0$ there exists unique solution $u(t) \in L^\infty((0,T); X) \cap ((0,T); Z)$. Let $U_0 \in X$ be a fixed point, i.e. $A(U_0) = 0$.

Definition 3.2 *The constant solution $u(t) \equiv U_0$ is called stable in Z, or Z-stable, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $v_0 \in X$ and $\|v_0 - U_0\|_Z < \delta$, and if $v(t)$ is a solution of (3.5), satisfying $v(0) = v_0$, then $\|v(t) - U_0\| < \varepsilon$ for all $t \in \mathbf{R}$. Otherwise steady solution U_0 is called Z-unstable.*

3.2 Instability may depend on the functional space

In infinite-dimensional systems like a fluid the choice of functional space may be crucial for the stability/instability of the system, as was emphasized by Yudovich [Y2]. This is both a linear and a nonlinear phenomenon.

To illustrate the dependence of stability on the choice of norm consider the following simple example given in [Y2], namely the Cauchy problem

$$\frac{\partial v}{\partial t} = x \frac{\partial v}{\partial x}, \quad v(0) = \phi(x). \tag{3.6}$$

The unique solution for an arbitrary smooth function $\phi(x)$ is

$$v(x,t) = \phi(xe^t). \tag{3.7}$$

A simple calculation shows that

$$\left|\left|\frac{\partial^k v(\cdot, t)}{\partial x^k}\right|\right|_{L^p(\mathbf{R}} = e^{(k-p^{-1})t}||\phi^{(k)}||_{L^p(\mathbf{R})}. \tag{3.8}$$

Hence this linear equation is asymptotically stable in L^p for any $p, 1 \le p < \infty$ and disturbances decay exponentially. The solution is stable but not asymptotically in $L^\infty(\mathbf{R})$, $C(\mathbf{R})$ and $W^{1,1}(\mathbf{R})$. In any space $W^{k,p}(\mathbf{R})$ with $k > 1$ or $k = 1, p > 1$, the solution is exponentially unstable.

The following simple example of Shnol [Sh] shows that an Euler flow may be linearly stable in one natural function space and unstable in another one. Let $U_0(x)$ be a plane-parallel flow in a strip with a linear profile: $U_0(x, y) = (ay, 0)$, $a \neq 0$. Then its vorticity is constant, and the linearized equation for the perturbation vorticity is

$$\frac{\partial \omega}{\partial t} + ay \frac{\partial \omega}{\partial x} = 0, \tag{3.9}$$

i.e. the vorticity perturbation is transported by the flow. The zero solution of this equation is stable in the space J_1 where $||u||_{J_1} = ||u||_{L^2} + ||\nabla \times u||_{L^2}$. On the other hand, it is unstable in Sobolev spaces H^s for $s > 1$, because the derivatives of vorticity grow in time. Interesting enough, the zero solution is unstable also in $J = L^2$, but the reason for this instability is quite different. Namely, every smooth solution of (3.9) tends weakly to some function depending only on y, as $t \to \infty$; this means that the velocity field tends in L^2 to some field, parallel to the x-axis and depending only on y, and this field is zero, if the mean value of the initial perturbation along the x-axis is identically zero. Thus the subspace of parallel flows in the space J appears to be an attracting set, consisting of fixed points. But (and this is the original reasoning of Shnol) the equation (3.9) is time reversible; this means that there exist arbitrarily small in L^2 perturbations of velocity, which grow arbitrarily big after some time; this means that this simplest nontrivial parallel flow is linearly unstable in L^2. The same is true for arbitrary nontrivial parallel flow. A similar situation may be observed for other classes of steady flows, for example for a potential flow in a multiconnected domain with one or more hyperbolic stagnation points.

As Yudovich [Y3] observes, there exists a class of exact solutions to the full nonlinear Euler equations which imply that 2-dimensional steady flows are, with very few exceptions, unstable with respect to 3-dimensional perturbations in any norm which includes the maximum of vorticity modulus.

In particular, consider the plane parallel shear flow $U_0 = (f(y), 0, 0)$ with $y \in [0, 1]$. This flow is well known to be linearly stable in L^2 if there are no inflection points in the profile of $f(y)$ in $[0, 1]$ (see, for example, [DR]). In an appropriate sense that we discuss in the next section it is also nonlinearly stable to 2-dimensional perturbations. It is easy to check that the following is an exact solution to the full nonlinear Euler equations (1.1)-(1.2) for any smooth functions f and w:

$$u = (f(y), 0, w(x - tf(y))).$$ (3.10)

The corresponding vorticity

$$\nabla \times u = -(tf'(y)w'(x - tf(y)), w'(x - tf(y)), f'(y)).$$ (3.11)

Hence the vorticity of a perturbation $(0, 0, w)$ to the steady shear flow grows linearly with time provided only that f and w are nonconstant functions. Thus a shear flow, even with no inflection points, is nonlinearly unstable to 3-dimensional perturbations in any norm that incorporates the magnitude of vorticity.

This set of exact solutions to the 3-dimensional Euler equation can be generalized to a suitable z-independent perturbation of any 2-dimensional steady flow (see [Y3], [F]), namely

$$u = (-\Psi_x, \Psi_y, w(x, y, t)),$$ (3.12)

where $\Psi(x, y)$ is a stream function for a steady 2-dimensional flow (see Section 2.2) and w satisfies

$$(\frac{\partial}{\partial t} + ((\hat{k} \times \nabla)\Psi) \cdot \nabla)w = 0.$$ (3.13)

The evolution equation for the vorticity gives

$$(\frac{\partial}{\partial t} + ((\hat{k} \times \nabla)\Psi) \cdot \nabla)(\hat{k} \times \nabla w) = (\hat{k} \times \nabla)w \cdot \nabla((\hat{k} \times \nabla)\Psi),$$ (3.14)

i.e. the vorticity component $(\hat{k} \times \nabla)w$ evolves as a tangent vector to the 2-dimensional flow $u_H = (\hat{k} \times \nabla)\Psi$. Hence for almost all choices of Ψ there exists a perturbation $w(x, y, t)$ such that the vorticity of the 3-dimensional Euler flow (3.12) grows with time. This growth can be exponential on a set of measure zero if the flow u_H has a hyperbolic fixed point.

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Note that these instabilities are associated with the essential spectrum and thus have different nature than instabilities connected with discrete eigenvalues. If the perturbation is an eigenfunction of the linearized equations (2.1), (2.2) with an eigenvalue having a positive real part, then, of course, it grows exponentially in any norm.

3.3 A nonlinear instability theorem

In this section we describe a result which applies to nonlinear instability in the sense of our first definition (see section 3.1). In the context of the Euler equation the result relates spectral instability of the evolution operator in L^2 with nonlinear instability in the Sobolev space H^s, $s > n/2 + 1$.

We formulate the relevant theorem in a general setting. We consider the stability of the zero solution of an evolution equation

$$\frac{dv}{dt} = Lv + N(v), \tag{3.15}$$

where L and N are respectively the linearized and nonlinear parts of the governing equation. Once the spectrum of the linear part L is analyzed and shown to have an unstable component (i.e. the zero solution is linearly unstable) then the question arises whether the zero solution is nonlinearly unstable. It is well known (see, for example, Lichtenberg and Lieberman [LL] that the linear instability implies nonlinear instability in the finite-dimensional case (i.e. if (3.15) is an ODE). In the infinite-dimensional case (PDE) such general result is not known, although for many particular types of evolution PDE's it has been shown that linear instability implies nonlinear instability (e.g. such a result for the incompressible Navier-Stokes equations in a bounded domain has been proved by Yudovich [Y2]). Difficulties with deriving the nonlinear instability from the linear one usually appear whenever the essential spectrum of L is non-empty as it generally is for the Euler equation.

In [FSV] Friedlander, Strauss, and Vishik proved the following abstract nonlinear instability theorem under the spectral gap condition.

Theorem 3.1 *Fix a pair of Banach spaces $X \hookrightarrow Z$ with a dense embedding. Let (3.15) admit a local existence theorem in X. Let N and L satisfy the following conditions.*

(1) $\|N(v)\|_Z \le C_0\|v\|_X\|v\|_Z$ for $v \in X$ with $\|v\|_X < \rho$ for some $\rho > 0$. (3.16)

(2) A spectral "gap" condition, i.e. $\sigma(e^{tL}) = \sigma_+ \cup \sigma_-$ with $\sigma_+ \ne \phi$ (3.17)

where

$$\sigma_+ \subset \{z \in C | e^{Mt} < |z| < e^{\Lambda t}\}$$
$$\sigma_- \subset \{z \in C | e^{\lambda t} < |z| e^{\alpha t}\}$$
(3.18)

with

$$-\infty < \lambda < \alpha < M < \Lambda < \infty \text{ and } M > 0.$$
(3.19)

Then the trivial solution $v = 0$ to equation (3.15) is nonlinearly unstable.

The main idea of the proof of this theorem is as follows. We assume the contrary, namely that the trivial solution $v = 0$ is nonlinearly stable. Let $\varepsilon > 0$ sufficiently small be given: it will be specified later. From the definition 1 of nonlinear stability it follows that there exists a global solution $v(t), t \in [0, \infty)$ such that $\|v(t)\|_X < \varepsilon$ provided $\|V(0)\|_X < \delta(\varepsilon)$.

We project $v(t)$ onto two subspaces using the spectral gap condition (3.17), (3.18). We denote by $P_\pm$ the Riesz projection corresponding to the partition of the spectrum created by the gap and introduce a new norm $\||\cdot\||$ on Z. For any $x \in Z$ let

$$\||x\|| = \||P_+x\|| + \||P_-x\||$$
$$= \int_0^\infty \|e^{-\tau L} P_+ x\|_Z e^{\tau M} d\tau + \int_0^\infty \|e^{\tau L} P_- x\|_Z e^{-\tau \alpha} d\tau.$$
(3.20)

The norm $\||\cdot\||$ is equivalent to $\|\cdot\|_Z$, i.e. there exists $C > 0$ such that

$$C^{-1}\|x\|_Z \le \||X\|| \le C\|x\|_Z.$$
(3.21)

Since $v(t)$ is a solution to (3.15) it can be shown that

$$(|||P_+v(t)||| - |||P_-v(t)|||)|_{t_1}^{t_2}$$

$$\geq \int_{t_1}^{t_2} \{M|||P_+v(\tau)||| - \alpha|||P_-v(\tau)||| \tag{3.22}$$

$$+ C^{-1}|||v(\tau)||| - |||N(v(\tau))|||\} d\tau$$

for any interval $0 \leq t_1 \leq t_2$.

We choose the initial condition $v_0 = \delta\bar{v}_0$, where $\bar{v}_0 \in X$ is an arbitrary vector satisfying

$$|||P_+\bar{v}_0||| > |||P_-\bar{v}_0|||, \quad ||\bar{v}_0||_X < 1. \tag{3.23}$$

Since $||v_0||_X < \delta$ our assumption of nonlinear stability implies

$$||v(t)||_X < \varepsilon \quad \text{for a.e. } t \in [0, \infty), \tag{3.24}$$

and from condition (3.23)

$$|||N(v(t))||| \leq Cc_0||v(t)||_Z \leq C^2 c_0 \varepsilon|||v(t)||| \quad \text{for a.e. } t \in [0, \infty). \tag{3.25}$$

Now the inequalities (3.24)-(3.25) plus Gronwall's inequality give

$$|||P_+v(t)||| - |||P_-v(t)|||$$
$$\geq \delta(|||P_+\bar{v}_0||| - P_-\bar{v}_0|||)e^{MT}, \ t \in [0, \infty), \tag{3.26}$$

provided ε is chose so that $\varepsilon < \min(C^{-3}c_0^{-1}, \rho)$. Since $M > 0$, for sufficiently large t the inequality (3.26) contradicts our assumption that $||v(t)||_X < \varepsilon$. Hence the trivial solution to (3.15) is nonlinearly unstable in X.

We now consider Theorem 3.1 in the context of the Euler equations (1.1)-(1.2). We write

$$u = U_0 + v,$$
$$Lv = -((U_0 \cdot \nabla)v - (v \cdot \nabla)U_0 - \nabla p,$$
$$N(v) = -(v \cdot \nabla)v - \nabla q; \tag{3.27}$$

thus the notation of the general theorem applies to instability of the steady flow U_0. The local existence requirement and condition (1) of Theorem 3.1 are easy to satisfy by making the natural choice for the spaces X and Z, namely

$$X = H_s, \quad s > \frac{n}{2} + 1 \text{ and } Z = L^2 \tag{3.28}$$

with the restriction to vector fields that are divergence free and satisfy appropriate boundary conditions. However the spectral gap condition is much more difficult to verify for a given steady solution U_0 because, as we have discussed, the essential spectrum of e^{tL} is non-empty and at least in some examples fills the whole annulus.

One piece of information we have about the structure of the spectrum is the essential spectral radius theorem discussed in section 2.1. In some examples the "fluid Lyapunov exponent" μ can be explicitly calculated. Also the theorem implies, in particular, that any $z \in \sigma(e^{tL})$ with $|z| > e^{\mu t}$ is a point of the discrete spectrum (i.e. an isolated point with finite multiplicity where the range of $(z - e^{tL})$ is closed). Any accumulation point of $\sigma_{\text{disc}}(e^{tL})$ necessarily belongs to $\sigma_{\text{ess}}(e^{tL})$. Thus if

$$\sigma(e^{tL}) \bigcap \{|z| > e^{\mu t}\} \neq \emptyset, \tag{3.29}$$

then there exists a partition

$$\sigma(e^{tL}) = \sigma_+ \bigcup \sigma_- \tag{3.30}$$

satisfying the gap condition (3.17), (3.18).

There are several examples of 2-dimensional flows where μ can be computed and discrete unstable eigenvalues calculated to show that (3.18) holds. These are the examples of discrete unstable eigenvalues discussed in section 2.2. As we remarked in section 2.1, in 2 dimensions the fluid Lyapunov exponent and the classical Lyapunov exponent are equal. Hence $\mu = 0$ for any plane-parallel shear flow. It therefore follows from Theorem 3.1, plus the result of [BFV] that there exist unstable discrete eigenvalues for any shear flow with a rapidly oscillating profile, that all such shear flows are nonlinearly unstable in H^2.

Other recent results concerning nonlinear instability of 2-dimensional shear flows include the work of Grenier [G] who proves nonlinear instability in L^∞ for piecewise linear profiles. Koch [K] proves in 2 dimensions that

nonlinear stability in $C^{1,\alpha}$ requires uniform boundedness of the derivatives of the flow map which implies that all steady shear flows are nonlinearly unstable in $C^{1,\alpha}$.

A more general 2-dimensional flow than parallel shear flow that can be shown to be nonlinearly unstable is the "cats-eye" flow studied in [FVY]. In this case the existence of hyperbolic stagnation points implies that $\mu > 0$. The exact value of μ can be calculated as the positive eigenvalue of the gradient matrix of U_0 at the hyperbolic point. The results of [FV] show that there exist discrete unstable eigenvalues with real part $> \mu$, hence again we can invoke theorem 3.1 to prove that such "cats-eye" flows are nonlinearly unstable.

The problem of verifying the gap condition for the spectrum corresponding to 3-dimensional flows is more difficult and the structure of the spectrum remains an open question. As we discussed in Section 2.1, there is some evidence that in 3 dimensions the combination of vortex tube stretching and hyperbolic stagnation points may provide a situation in which $\mu \to \infty$ and hence it would be very difficult to verify the gap condition. Proving instability is then beyond the tools we have presently available.

3.4   Arnold stable and minimal flows

In this section we discuss some results and open questions concerning non-linear stability in the sense of our second definition.

The most frequently employed method to prove the nonlinear stability of particular flows and classes of flows was developed by Arnold [AK]. He used the *Energy-Casimir method* based on the existence of two different integrals of motion. The simplest example of an application of this method is to the stability of rigid rotation of a fluid in a disk. Here we have two integrals, the energy E and the angular momentum Ω. Consider the space J of all square integrable incompressible vector fields in the disk tangent to its boundary. The fields with given momentum Ω_0 form a hyperplane in this space, and the functional E achieves an absolute minimum E_0 on this hyperplane at the field $U(x)$, which is the velocity field of the rigid rotation. This critical point is nondegenerate (its second variation is positive definite in L^2). Hence, every vector field $u(x)$ of the space J with the energy E and the angular momentum Ω, close resp. to E_0 and Ω_0, is close in the space J to the field $U(x)$ and remains close forever, because the functionals E and Ω do not depend on time.

For more general flows the Energy-Casimir method assumes more sophisticated forms. We use the fact that the vorticity ω is transported by the fluid. In other words, the fluid moves in such a manner that its vorticity field at every moment is obtained from the vorticity at the initial moment by some volume-preserving diffeomorphism depending on t: $\omega(x, t) = \omega(g_t^{-1}(x), 0)$ (in the terminology of Arnold, the velocity fields of the flow at any two moments are *isovortical*). The relation of equivorticity defines partition of the space of velocity fields into equivalence classes which may be regarded as a sort of generalized Casimir. For a given field $U_0(x)$ the class of isovortical fields is an infinite-dimensional manifold V, which is the orbit of the group $\mathcal{D}$ of volume-preserving diffeomorphisms in the space X of incompressible vector fields. Arnold has proved [AK] that the steady solutions of the Euler equations are exactly the fields of V which are critical points of the energy functional E, restricted on V. If the critical point is a point of a strict local maximum or minimum of E, then the flow is nonlinearly stable in the space J_1, whose elements are incompressible vector fields $u(x)$ in the flow domain, tangent to the boundary and having a finite norm $\|u\|_{J_1} = \|u\|_{L^2} + \|\nabla \times u\|_{L^2}$. The development of this idea gives rise to the celebrated results of Arnold concerning the nonlinear stability of certain classes of steady flows. In particular, Arnold's methods show that for plane-parallel shear flow in 2 dimensions the Rayleigh criterion (i.e. no inflection points in the profile) guarantees not only spectral stability but also nonlinear stability in J_1 (see [AK] for more details).

This theory has several weak points. Firstly, on the manifold V of isovortical fields there may be no critical points at all. At least, the functional E usually does not assume its maximum and minimum on the surface V. Here is the typical example of situation, where the minimum and/or maximum of E on V cannot be achieved. Consider some velocity field u_0 in a strip $0 < y < 1$ with the period L along the x-axis. Let $\omega_0(x, y)$ be its vorticity, and suppose that $\int\int \omega \, dx \, dy = 0$. We are looking for a flow $u(x, y)$ with vorticity $\omega(x, y)$ such that ω is obtained from ω_0 by a volume preserving permutation of points, i.e. element of the group $\mathcal{D}$; in other words, u and u_0 are on the same manifold V and the energy $E(u)$ should be minimal (maximal). But the minimum cannot be achieved, because it is zero. In fact, we can construct a sequence $g_1, g_2, \cdots$ of diffeomorphisms, becoming more and more "mixing", which transform ω_0 into $\omega_1, \omega_2, \cdots$, and this sequence of vorticities tends weakly to 0. Corresponding velocity fields $u_1, u_2, \cdots$ tend to 0 strongly in L^2; thus, $E(u_i) \to 0$. The maximum value of E is also not always achiev-

able. In fact, the supremum and infimum of energy on V depend only on the value distribution of ω_0, and not on the topology of its level lines. Thus, if for some u_0 the maximum is achieved (say, at the same u_0), then it cannot be achieved for any other function u_0' with the same value distribution of vorticity, but with different topology. For example, if the flow domain is a disk, and ω_0 is positive, depends only on the radius, decreases when the radius is growing, and is concentrated in a small neighborhood of the center of the disk, and ω_0' is another vorticity, having the form of *two* such spots whose size is $\sqrt{2}$ times smaller, then for the corresponding flow u_0' the maximum of E on its surface V' cannot be achieved. In all cases we see no evidence of the existence of other local maxima and minima, different from the global ones.

Secondly, as Sadun and Vishik [SV] observe, there is a serious drawback in applying the Arnold method in 3 or more dimensions. A natural way to prove that a critical point is a strict local maximum or minimum of E is to show that the second variation of the energy (the Hessian) defined on the tangent space to V is negative or positive definite at a critical point u_0. However Sadun and Vishik show in [SV] that in 3 or more dimensions the spectrum of the Hessian is not only *never* definite, but is generally unbounded from below as well as from above. The only exception are harmonic flows (i.e. both divergence and curl are zero) in which case the Hessian is identically zero. This result is suggestive, but does not prove that most flows in 3 dimensions are likely to be nonlinearly *unstable* in the Arnold's sense.

Another approach to nonlinear stability in 2 dimensions introduced by A. Shnirelman [S1] is based on the following ideas. Consider the space J_1 of incompressible vector fields in the flow domain M, which are tangent to the boundary and have a finite norm $||u||^2_{J_1} = ||u||^2_{L^2} + ||\nabla \times u||^2_{L^2}$. Consider the group $\mathcal{D}$ of volume preserving diffeomorphisms of M; it acts in $L^2(M)$ by the formula $g \cdot f(x) = f(g^{-1}(x))$. This is a unitary operator in $L^2(M)$, and we shall identify the group $\mathcal{D}$ with the group of these unitary transformations. Now, for every $g \in \mathcal{D}$ and every $u \in J_1$ we define $g \cdot u$ as a unique field $v \in J_1$, such that $\nabla \times v(x) = \nabla \times u(g^{-1}(x))$ (assume for simplicity that the domain M is simply connected). Thus, we have defined an action of the group $\mathcal{D}$ in the space J_1. Hence we see that the manifolds of isovortical fields are just the orbits of this action. Consider now an extension of the group $\mathcal{D}$. It consists of linear operators in L^2, having the form $Kf(x) = \int K(x,y)f(y)dy$, where the kernel $K(x,y)$ satisfies the following conditions:

$$\text{(1)} \qquad K(x,y) \geq 0 \quad \text{(i.e. } K(x,y)dxdy \text{ is a positive measure) ;}$$

$$\text{(2)} \qquad \int_M K(x,y)dx \equiv 1; \qquad\qquad\qquad\qquad\qquad\qquad \text{(3.31)}$$

$$\text{(3)} \qquad \int_M K(x,y)dy \equiv 1.$$

Such operators, usually called bistochastic, or, in the terminology of A. Vershik, polimorphisms, form a semigroup $\mathcal{P}$; it consists of contracting operators in L^2, and the group $\mathcal{D}$ is dense in $\mathcal{P}$ in a weak operator topology.

Let us define the action of the semigroup $\mathcal{P}$ in the space J_1. For every $u \in J_1$ and $K \in \mathcal{P}$ we define $K \cdot u$ as a unique field $v \in J_1$, such that $\nabla \times v = K(\nabla \times u)$. We can define a partial order relation in the space J_1: suppose $u, v \in J_1$; we say that $u \prec v$, if $u = K \cdot v$ for some $K \in \mathcal{P}$.

For any $u_0 \in J_1$ let us consider the set S_{u_0} of vector fields $u \in J_1$, such that $u \prec u_0$, and $||u||_{L^2} = ||u_0||_{L^2}$. This set is a lattice with respect to the binary relation $\prec$; using Zorn's lemma, we prove that there exists a minimal element $v \in S_{u_0}$ (not necessarily unique). It turns out that this minimal element, which is a vector field, is a velocity field of a steady flow in M. We call such flows *minimal flows*.

Minimal flows have a clear physical meaning. The flow of an ideal incompressible fluid transports its own vorticity. It is natural to assume that the vorticity is permanently distorted by the flow, and effectively mixed. But the mixing operators are just polimorphisms. So, at any remote time moment $t >> 1$ the vorticity is (presumably) close in a weak sense to the result of the action of some operator $K_t \in \mathcal{P}$ on the initial vorticity $\omega_0 = \nabla \times u_0$, which means that the velocity field $u(x,t)$ is L^2-close to $K_t \cdot u_0(x)$.

The mixing of the vorticity field by the flow is practically irreversible, and we may assume that it proceeds until some constraint makes further mixing impossible. These constraints may be any integral of the motion, the primary one being the energy $E(u)$; further mixing is impossible, if every operator $K \in \mathcal{P}$ changes the kinetic energy of the flow. Thus minimal flows are the most degenerate states of fluid motion.

Our conjecture is that all (generic) 2-dimensional flows of an ideal incompressible fluid have a similar asymptotic behavior as $t \to \infty$: every such flow tends to some minimal flow. This hypothesis appears difficult to prove. It sounds close to (but in fact is very far from) "statistical hydrodynamics" in the sense of J. Miller and R. Robert [M], [R].

We do not expect the regularity of minimal flows to be very high; even if $u_0 \in C^\infty$, the corresponding minimal flow $v \in S_{u_0}$ has *a priori* only bounded vorticity, and it is unclear whether its vorticity is at least continuous.

There exist three classes of minimal flows. If a minimal flow belongs to the first class, then every mixing operator applied to its vorticity can only increase the kinetic energy. For a flow from the second class, every mixing of its vorticity decreases its energy. The third class contains only one flow (up to a multiplicative constant), namely the flow with constant vorticity.

Every minimal flow realizes a *global* minimum or maximum (for minimal flows respectively of the first and the second class) of the energy E on its orbit V; for the flow of the third class its orbit consists of one point. But we cannot assert (and it may be false) that this critical point is nondegenerate. In some examples the maximum of energy is attained on a compact set $H \subset V$, containing v (the convexity considerations show that the global minimum of energy is always assumed at a single point). In fact, consider the following example. Suppose that the flow domain M is a circular disk $|x| < 1$, and the velocity field $u_0(x)$ has the form of two small spots of vorticity of opposite sign, such that the total vorticity is zero. If we decrease the size of these spots, keeping the total vorticity (i.e. the circulation along the contour encircling each of the spots) constant, we can make the energy E arbitrarily big, because it grows as logarithm of the spot diameter. We assume that the size of both spots is small enough. The set S_{u_0}, defined above, contains a minimal flow v; if the size of vortices is small enough, this flow looks like two standing vortices of opposite signs (this configuration realizes the global maximum of the energy E on its isovortical manifold V). But this flow is not axisymmetric, and therefore the same minimal value of E is assumed on the whole circle in V, consisting of the flows, obtained from v by rotations. If we add to v the velocity field of a slow rotation of the disk, we obtain a nonsteady solution of the Euler equations in the disk, having the form of two small vortices of opposite signs, slowly rotating in the disk on the backdrop of a small uniform vorticity. (This type of flows is analogous to the precession of the axisymmetric rotating body; see [S1] for more details).

Thus minimal flows are not generally stable. What may be asserted is that they are "compactly unstable". This means that if v is a minimal flow, then there exists a compact $H \subset J_1$, containing v, such that for all initial velocities $w(x, 0)$, close to v, the flow $w(x, t)$ is close to H for all $t > 0$.

3.5 Spectrally stable and Arnold stable flows

We call a steady flow U_0 *spectrally stable* if the linearized equations (2.1), (2.2) have no exponentially growing solutions, i.e. the linearized operator L has no eigenvalues with a positive real part. What is the relation between the classes of spectrally stable and Arnold stable (or minimal) flows? In the case of a channel flow the essential spectrum of the linearized evolution operator e^{tL} lies on the unit circle; applying the theorem 3.1 we conclude that if the flow is spectrally unstable, it is nonlinearly unstable; thus, it is not Arnold stable. For generic steady flows the question is less clear, because unstable eigenvalues of the operator e^{tL} may occur on its essential spectrum. This may happen, if, for example, the flow U_0 contains a thin jet in one part of the flow domain M, and a hyperbolic stagnation point elsewhere in M. In this case the essential spectrum of e^{tL} contains an annulus $e^{-\Lambda t} \le \|z\| \le e^{\Lambda t}$, and the eigenvalue $e^{\lambda t}$ satisfies $1 < e^{\lambda t} < e^{\Lambda t}$. Theorem 3.1 is no more applicable, and it is now unclear whether the flow U_0 is stable (while it is unlikely that a distant saddle point could stabilize the jet).

In the other direction the answer is clearer: there exist steady flows which are spectrally stable, and are not Arnold stable. Our examples are plane-parallel flows in a channel $-1 < y < 1$, periodic in the x-direction with the period L (so that the flow domain may be regarded as a side surface of a cylinder). In the first example the velocity profile $U(y)$ has the form $U(y) = y + \varepsilon f(y)$ for an arbitrary smooth function $f(y)$ and small $\varepsilon > 0$; in the second example $U(y)$ may be an arbitrary smooth function, but the period L in the x-direction is small (depending on U). In both examples, if ε or respectively L are small enough, there are no unstable eigenvalues. This is proved in the paper of Faddeev [Fa] which is excellent for its clarity.

However these flows, for some profiles U, are not Arnold stable. For example, consider $U(y) = \sin(m\pi y)$ for integer $m > 1$ and sufficiently small period L so that the flow is spectrally stable. Let $U_0 = (U(y), 0)$ denote the flow field and let $\omega_0 = \nabla \times U_0$ be its vorticity. Let us show that this field is neither the point of a local minimum nor the point of a local maximum of the energy E among the fields u isovortical with U_0. To do this, it is sufficient to show that for every $\varepsilon > 0$ there exist two volume preserving diffeomorphisms η and ζ such that $\|\omega_0 \circ \eta^{-1} - \omega_0\|_{L^2} < \varepsilon$, $\|\omega_0 \circ \zeta^{-1} - \omega_0\|_{L^2} < \varepsilon$, and $E(\omega_0 \circ \eta^{-1}) < E(\omega_0)$, $E(\omega_0 \circ \zeta^{-1}) > E(\omega_0)$, where $E(\omega)$ denotes the kinetic energy of incompressible flow with vorticity ω. We divide the flow domain $M = \{(x, y) | 0 \le x < L, \ -1 \le y \le 1\}$ into small equal cells M_k;

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for example, they may be equal squares of size δ. Consider permutations of these cells, and for every permutation τ consider the function $\tau \cdot \omega_0 = \omega_0 \circ \tau^{-1}$ (here τ is regarded as a measurable transformation of the flow domain M, preserving the Lebesgue measure). As it is proven in [S2], for every permutation τ there exists a smooth volume preserving diffeomorphism τ', such that $\tau' = \tau$ outside an arbitrarily small neighborhood of $\bigcup_k \partial M_k$. Thus, $\|\omega_0 \circ \tau' - \omega_0 \circ \tau\|_{L^2}$ may be made arbitrarily small, and we shall consider now only the action of permutations of cells on the vorticity and velocity fields. We define two permutations, τ_1 and τ_2, such that the action of τ_1 on ω_0 increases the energy of the flow, while the action of τ_2 decreases the energy. Note that if ω_0 and ω_0' are two vorticity fields, which are equimeasurable, i.e. $\mathrm{mes}\{\omega_0 < c\} = \mathrm{mes}\{\omega_0' < c\}$ for every c, then there exist a partition of M into sufficiently small equal squares M_k and a permutation τ of these cells such that $\|\omega_0' - \omega_0 \circ \tau\|_{L^2}$ is arbitrarily small (see [S2]). Furthermore, there exist functions ω_0' which are equimeasurable with ω, $E(\omega_0') > E(\omega_0)$, and $\|\omega_0' - \omega_0\|_{L^2}$ is as small as we wish. For example, we can inflate a little one period of $\omega_0(x,y) = m \sin my$ in the y-direction while shrinking other periods. Now we can find a permutation τ_1, which approximately transforms ω_0 into ω_0', and then approximate this permutation by a smooth volume preserving diffeomorphism η. Our construction shows that $\|\omega_0 \circ \eta - \omega_0\|_{L^2}$ may be made arbitrarily small and that $E(\omega_0 \circ \eta) > E(\omega_0)$. Thus ω_0 is not a point of local minimum of E on V.

To show that ω_0 is not a point of local maximum of E on V, observe first that the mean value of ω_0 is zero. This makes the following construction possible. Let us divide M into N equal cells and pick $n << N$ cells by random. Let τ_2 be a random permutation of the chosen n cells. If $N \to \infty$, $n \to \infty$, $n/N \to \delta$, then, with probability 1, $\omega_0 \circ \tau_2$ tends weakly in $L^2(M)$ to $(1 - \varepsilon)\omega_0$; on the other hand, $\|\omega_0 \circ \tau_2 - \omega_0\|_{L^2} < c\delta$; hence, we can find a diffeomorphism ζ, such that $\|\omega_0 \circ \zeta - \omega_0\|_{L^2} < c\varepsilon$, while $E(\omega_0 \circ \zeta) < E(\omega_0) - \varepsilon^2$. This shows that ω_0 is not a point of local minimum of E on V. Hence we have the following

Problem. We have shown that there exist steady, plane-parallel flows such that the linearized problem has no unstable eigenvalues, but these flows don't satisfy the conditions of the Arnold stability (or minimality, which is essentially the same). Are they stable in J_1? On one hand, there is no "fast" exponential instability. On the other hand, the vorticity integrals are not constraints that prevent a flow from going far away from U_0, if initially it was close to U_0 in J_1. So, either there exist other constraints of unknown

nature, or the above flows are unstable in J_1. However such an instability is quite different from instabilities with which we are familiar.

3.6 Problems and conjectures on the Arnold stable and minimal flows.

We have four remarkable classes of steady flows: Lyapunov stable flows, Arnold stable, minimal flows, and spectrally stable ones (in the sense that the linearized equation has no unstable eigenvalues). What are relations between these classes?

Suppose that the steady flow $U_0(x)$ does not satisfy Arnold's condition of stability. This means that U_0 is a critical point of the energy E restricted on the surface V of isovortical vector fields in the space J_1, but this flow is not the local minimum or maximum of the functional E on V. Then we may anticipate that this flow is unstable, because there is nothing to hold the perturbed flow $u(x,t)$ close to $U_0(x)$; the Energy-Casimir method breaks down. If the flow U_0 is spectrally unstable, then Theorem 3.1 shows that it is nonlinearly unstable. But there is a wide gap between spectrally unstable flows and those which do not satisfy the Arnold's conditions. We conjecture that those flows which are neither Arnold stable nor spectrally unstable are nonlinearly unstable in the space J_1, but the nature of their instability is different from that of linearly unstable flows.

3.7 L^2-instability

In the above theories of nonlinear stability we considered stability of smooth flows with respect to small perturbations, which are small in the J_1 (i.e. in H^1) sense: the vorticity of perturbations should be small in L^2. The theory breaks down if we drop the condition on the vorticity perturbation and consider all (smooth) velocity fields $u(x,0)$ which are close to $U_0(x)$ in L^2 without any conditions on derivatives. Note that this class of perturbations is no less physically significant than the previous one, because it describes perturbations with small energy. Such perturbations may be easily created, for example, by inserting small obstacles in the flow. In this case, the vorticity integrals are completely destroyed and it appears that nothing prevents the flow from going far away from U_0. Hence, the natural conjecture is that every nontrivial flow (i.e. flow having a nonconstant velocity) is unstable with respect to small in L^2 perturbations.

Consider the simplest basic steady flow, namely a parallel flow. Let M be a strip $0 \leq x_2 \leq 1$ in the (x_1, x_2)-plane. We restrict ourselves to the flows having period L along the x_2-axis; this period is the same for all flows that are considered below. Suppose that the velocity field $U_0(x)$ has the form $(U(x_2), 0)$, where U is a given smooth function (the velocity profile). The original question asked, for which profiles U is the flow U_0 is stable. Our first result is the following

Theorem 3.2 *For every nontrivial (i.e. different from constant) velocity profile U the flow U_0 is L^2-unstable. This means that for every function $U(x_2) \neq$ const there exists $C > 0$, such that for every $\varepsilon > 0$ the following is true. There exist $T > 0$ and a smooth force $f(x,t)$, defined in $M \times [0,T]$, such that $\int_0^T \| f(\cdot,t) \|_{L^2} \, dt < \varepsilon$, and f transfers the flow U_0 during the time interval $[0,T]$ into a steady flow u_1, such that $\| U_0 - u_1 \|_{L^2} > C$.*

This theorem is proved in [S3] by an explicit construction, based on the variational method. The next result is much stronger, but here we use a weaker notion of instability (see [S4]). Let X be a Banach space of incompressible vector fields in M, tangent to the boundary. Consider the Euler equations with a nonzero right hand side (i.e. external force):

$$\frac{\partial u}{\partial t} + (u, \nabla)u + \nabla p = f; \qquad (3.32)$$

$$\nabla \cdot u = 0. \qquad (3.33)$$

Here $f = f(x,t)$ is a smooth in x vector field such that $\nabla \cdot f = 0$ and $f(x,t)|_{\partial M}$ is parallel to ∂M. Consider the behavior of $u(x,t)$ when f is small in the following sense: $\int_0^T \| f(\cdot,t) \|_{L^2} \, dt$ is small, where $[0,T]$ is the time interval (assumed to be long) where the flow is considered. For example, if f has the form $f(x,t) = F(x)\delta(t)$ we return to the initial stability problem.

Definition 3.1. *Suppose that $u(x_1, x_2)$ and $v(x_1, x_2)$ are two steady flows. We say that the force f transfers the flow u into the flow v during the time interval $[0,T]$, if the following is true: if $w(x,t)$ is the solution of the nonhomogeneous Euler equations (3), (4) with the initial condition $w(x,0) = u(x)$, then $w(x,T) = v(x)$.*

We note that if the force f satisfies a stronger condition $\int_0^T \| \omega(\cdot,t) \|_{L^2} \, dt < \varepsilon$, where $\omega = \nabla \times f$ is the vorticity, then for every Arnold stable flow

U_0, the resulting perturbation at time t will be small, too. But the following theorem shows that situation in L^2 is quite different.

Theorem 3.3 *Suppose that $U(x_2)$ and $V(x_2)$ are two velocity profiles, such that $\int_0^1 U(x_2)dx_2 = \int_0^1 V(x_2)dx_2$, and $\int_0^1 \frac{1}{2}|U(x_2)|^2 dx_2 = \int_0^1 \frac{1}{2}|V(x_2)|^2 dx_2$; let $U_0(x_1, x_2) = (U(x_2), 0)$, $v_0(x_1, x_2) = (V(x_2), 0)$ be corresponding steady parallel flows (having equal momenta and energies). Then for every $\varepsilon > 0$ there exist $T > 0$ and a smooth force $f(x,t)$, such that $\int_0^T \parallel f(\cdot, t) \parallel_{L^2} < \varepsilon$, and f transfers u into v during the time interval $[0, T]$.*

In other words, the flow may be considerably changed by arbitrarily small in L^2 force, provided the time interval is sufficiently long.

This means that the flow of an ideal incompressible fluid is perfectly controllable by arbitrarily small force.

Theorem 3.2 is proven by an explicit construction of the flow. Note first, that if $U_1, U_2, \cdots, U_N$ are velocity profiles, and Theorem 2 is true for every pair (U_i, U_{i+1}) of velocity profiles, then we can pass from U_1 to U_N by simply concatenating the flows connecting U_i and U_{i+1}; thus Theorem 2 is true for the pair (U_1, U_N). Therefore it is enough to construct the sequence of steady flows with profiles $U_1, \cdots, U_N$, and the intermediate nonsteady flows connecting every two successive steady ones. We also note that it is enough to construct a sequence of piecewise-smooth flows, because it is not difficult to smooth them, so that the necessary force will have arbitrarily small norm in $L^1(0, T; L^2(M))$.

As a first step, we change the flow with the profile $U = U_1$ by a piecewise-constant profile U_2 with sufficiently small steps; this may be done by a force with arbitrarily small norm. Thus, $U_2(x_2)$ is a step function, $U_2(x_2) = U_2^{(k)}$ for $x_2^{(k-1)} < x_2 < x_2^{(k)}$, $k = 1, \cdots, K$. Every successive profile U_i is also a step-wise function. We are free to subdivide the steps and change a little the values of velocity, if these changes are small enough.

Every flow u_k is obtained from the previous one u_{k-1} by one of two operations, described in the following theorems.

Theorem 3.4 *Let $U(x_2)$ be a step function, $U(x_2) = U^{(k)}$ for $x_2^{(k-1)} < x_2 < x_2^{(k)}$; let $V(x_2)$ be another step function, obtained by transposition of two adjacent segments $[x_2^{(k-1)}, x_2^{(k)}]$ and $[x_2^{(k)}, x_2^{(k+1)}]$. Let $u(x_1, x_2)$, $v(x_1, x_2)$ be parallel flows with velocity profile $U(x_2), V(x_2)$. Then for every $\varepsilon > 0$ there exist $T > 0$ and a piecewise-smooth force $f(x,t)$, such that $\int_0^T \parallel f(\cdot, t) \parallel_{L^2} <$*

ε, and the force f transfers the flow u into the flow v during the time interval $[0, T]$.

To formulate the next theorem, we recall the law of an elastic collision of two bodies. Suppose that two point masses m_1 and m_2, having velocities u_1 and u_2, collide elastically. Then their velocities after collision will be $v_1 = 2u_0 - u_1$, $v_2 = 2u_0 - u_2$, where $u_0 = (m_1 u_1 + m_2 u_2)/(m_1 + m_2)$ is the velocity of the center of masses. The transformation $(u_1, u_2) \to (v_1, v_2)$ is called a *transformation of elastic collision*.

Theorem 3.5 *Assume that the profile $U(x_2)$ is like the profile in Theorem 3, and the profile $V(x_2)$ is equal to $U(x_2)$ outside the segment $x_2^{(k-1)} < x_2 < x_2^{(k+1)}$; on the last segment, $V(x_2) = v^{(k)}$, if $x_2^{(k-1)} < x_2 < x_2^{(k)}$, and $V(x_2) = v^{(k+1)}$, if $x_2^{(k)}) < x_2 < x_2^{(k+1)}$, where $(v^{(k)}, v^{(k+1)})$ is obtained from $(u^{(k)}, u^{(k+1)})$ by the transformation of elastic collision, the lengths $x_2^{(k)} - x_2^{(k-1)}$, $x_2^{(k+1)} - x_2^{(k)}$ playing the role of masses m_1, m_1. Let $u(x_1, x_2), v(x_1, x_2)$ be parallel flows with profiles $U(x_2), V(x_2)$. Then for every $\varepsilon > 0$ there exist $T > 0$ and a force $f(x, t)$, such that $\int_0^T \| f(\cdot, t) \|_{L^2} < \varepsilon$, and the force f transfers the flow u into flow v.*

Assume now that $U(x_2)$ and $V(x_2)$ are two velocity profiles, having equal momenta and energies. Then it is not difficult to construct a sequence of step functions $U_2(x_2), U_3(x_2), \cdots, U_N(x_2)$, so that U_2 is L^2-close to $U_1 = U$, U_N is L^2-close to V, and every profile U_k is obtained from U_{k-1} by one of two operations, described in Theorems 3 and 4. Using these theorems and the discussion above, we construct a piecewise-smooth force $f(x, t)$ such that $\int_0^T \| f(\cdot, t) \|_{L^2} dt < \varepsilon$ and f transfers U into V during the time interval $[0, T]$.

Theorems 3.4 and 3.5 are proved by explicit construction of the flows. They are true also for circular flows in a disk, with the angular momentum taking the place of momentum in Theorem 3.5. But for generic 2-dimensional domains the situation is not so clear. We don't know whether there is an integral of motion, similar to the angular momentum, in any domain different from the disk. If such integral does not exist, which is most likely, then the natural conjecture is that for any two flows with equal energies the conclusion of Theorem 3.3 is true. But this behavior is paradoxical: just imagine a nearly circular flow in a nearly circular domain (e.g. ellipse), which after some long

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time changes the sign of the angular velocity. This question requires more study.

3.8 L^2-instability and scattering for the Euler equations

The L^2-instability may be regarded as another side of of the hypothetical picture of an asymptotic behavior of generic flow as $t \to \infty$, developed in section 3.4. According to this hypothesis, the vorticity carried by the flow is mixed more and more until its further mixing becomes impossible because of the energy conservation. Thus the flow $u(x,t)$ tends to some minimal flow $u_+(x)$. This passage from an initial (arbitrary) flow $u_0(x) = u(x,0)$ to the final state (minimal flow) $u_+(x)$ is analogous to the scattering of linear waves on an obstacle (or potential); to make this analogy closer, we can continue the flow back in the direction of negative t; as $t \to \infty$, the flow $u(x,t)$ tends to some minimal flow $u_-(x)$. Consider the passage from u_- to u_+; this is the exact analog of the scattering operator. The vorticity field for $|t|$ very big is a highly oscillating function in the flow domain, which approaches the vorticity of the final flow only in a weak sense. Thus the velocity field $u(x,t)$ for $t < 0$ and $|t|$ very big may be arbitrary close in L^2 to $u_-(x)$. If we take it as the initial condition for the Euler equation, we obtain an example of a small in L^2 perturbation of a minimal flow $u_-(x)$, which grows considerably on a large time interval.

The scattering property was proved by Caglioti and Maffei [CM1] for 1-dimensional Vlasov-Poisson equation, which has some features similar to the 2-dimensional Euler equation but is much simpler. As for the Euler equations themselves, they attempted in [CM2] to construct an asymptotic decomposition of their hypothetical scattering solution as $|t| \to \infty$, having the form of a collection of long and narrow vorticity filaments (this form would be assumed by the vorticity field if it is transported as a passive scalar by a smooth and steady field). However they managed to construct only the first term of this asymptotics. The difficulty lies in the fact that the interaction between the oscillating part of the vorticity field and the mean field decreases very slowly (as $|t|^{-1}$) when $|t| \to \infty$; the vorticity perturbation remains "active" for all t, and in no way can be regarded as a passive scalar. The opposite difficulty was pointed out by Isichenko in his paper [I]; in this smooth picture the transverse motions of fluid parcels decay too rapidly (as

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$|t|^{-5/2})$ when $|t| \to \infty$, and therefore the state of the minimal flow is out of reach.

Our hypothesis is that the approach to the final state is not that smooth. The flow picture at small scales is being transformed all the time, infinitely many times. Thus, the asymptotic solution in the form assumed in [CM2] simply does not exist. The true picture is much more violent (including infinite series of refolding of vorticity filaments and appearance of new, secondary filaments, which complicates the picture even more). This problem deserves extensive study.

4 Conclusion and further questions

In this paper we did not try to cover all the vast field of the fluid instability; rather we have concentrated on some particular aspects of it. We tried to show that there exist different kinds of instability. In fact the differences are so big that they deserve to be regarded as different phenomena, and not as different kinds of one phenomenon. The difference may be illustrated by the following simple example. Consider a pendulum balanced upside down in the top position. It is certainly unstable; almost any small disturbance will grow in time. Consider, on other hand, a particle moving freely in Euclidean space which is at rest in some point. This equilibrium is also unstable, because every small impulse will result in steady motion, which after a long time will move the particle far away. These two instabilities appear quite different. Technically, the second one is associated with the Jordan cell structure of the governing operator, while in the first case it is an unstable eigenvalue that determines the instability.

In the context of fluid motion, we see the same two sorts of instability but in a much stronger form. Most of the linear and nonlinear instabilities considered in this work belong to the second class; they may be called "slow" instabilities, as opposed to "fast" instabilities, associated with isolated unstable eigenvalues. The reason for the ubiquity of slow instabilities is the fact that steady flows in 2 dimensions themselves have a Jordanian structure at almost every point in the sense that the differential of the flow map (i.e. diffeomorphism produced by the flow during some time) is a Jordan matrix. This fact alone is enough to explain linear growth of perturbations in the smooth norms. On the other hand, instability in the energy (L^2) norm is associated directly with the simple picture of a freely moving particle described

above. And finally, the nature of a hypothetical instability in a vorticity norm beyond "Arnold" stability and spectral instability is quite unclear.

If there are unstable discrete eigenvalues in the spectrum of linearized operator, then the flow is definitely unstable. This is "fast" instability with exponential growth of disturbances. Less clear is the situation when the unstable continuous spectrum of the evolution operator is nonempty (e.g., when the basic steady flow has a hyperbolic stagnation point). In this case the spectrum fills an annulus, and for each point of the spectrum we can construct a solution of the linearized equation which grows in time, but not monotonically; rather it has "outbreaks" in some rare time moments, being small most of the time. It is unclear whether we can construct a growing solution to the full nonlinear Euler equations showing similar behavior. May be we have here one more kind of instability, unaccounted for at present in the traditional scheme of instability..

The case of an unstable eigenvalue embedded in the essential spectrum is also unclear. Does a growing solution appear, or will the continuous spectrum "damp" it? What will happen if we change a little the basic steady flow? Does the eigenvalue "dissolve" in the continuous spectrum? Is there something analogous to the Fermi rule?

So we can be optimistic: there is a lot of work ahead of us.

4.1 Acknowledgements

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