On Lagrange and symmetric degeneracy loci

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The study of topology of the real Lagrange Grassmannians has various important applications in the theory of Hamiltonian systems, symplectic and contact geometry, and others fields of mathematics. The notion of the Lagrange Grassmannian can be complexified in several ways (cf. [A, MSS]) and the corresponding objects of complex geometry also have many important applications. In this paper we study the most straightforward complex version of the isotrope and the Lagrange Grassmannians, namely, the manifold of all complex subspaces of a fixed dimension in $\mathbb{C}^{2n}$, which are isotrope with respect to a fixed non-degenerate bilinear skew-symmetric form.

The main result of the paper are various formulas for cohomology classes dual to different kinds of Lagrange and orthogonal degeneracy loci: Schubert cells in isotrope Grassmann and flag varieties, intersections of isotrope subbundles of symplectic and orthogonal bundles, degenerations of symmetric and skew-symmetric maps of bundles etc. This subject was intensively studied last years, here is a non-complete list of references: [HT, JLP, F2, P1, P2, P3, PR, FP, LP1, LP2]. A part of results presented here can be found in these papers. Nevertheless, I decided to write this note essentially for two reasons. First, to the best of my knowledge, the main results of this paper, namely the formulas of Theorems 1.1 and 2.1 are not known. Their particular case solves J. Harris' problem. I used these formulas in the computations of the characteristic classes dual to Lagrange, Legendre, and critical point function singularities [K3]. Two different proofs are presented here. The analogues of these formulas for orthogonal case are formulated in Appendix D. The second, and perhaps, more important reason was an attempt to present a brief review of basic ideas and methods of the theory which would be self-contained and clear for non-specialists in intersection theory. I dare hope that the respect to the theory will not be lost if it would be shown to be much simpler then it is sometimes presented. I am certainly sure that the intension 'to make things clear' will always be a strong tradition of *Moscow Mathematical Journal*.

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1 Main results

Consider vector bundles $L \subset E$ over some manifold M and another flag F of subbundles

$$F_{k_1} \subset F_{k_2} \subset \ldots \subset E, \quad \operatorname{rk} F_k = n+1-k, \qquad \text{where} \quad n = \operatorname{rk} E - \operatorname{rk} L.$$

Different *degeneracy loci* are defined as the subsets of the base M with the prescribed dimensions for the intersections of the corresponding fibers of the bundles F_i and L. Namely, for a given decreasing sequence of integers $\lambda_1 > \lambda_2 > \ldots > \lambda_r > 0$ we consider the *degeneracy locus* $S_{\lambda_1,\ldots,\lambda_r} \subset M$ as the set of points $x \in M$ such that

$$\dim F_{\lambda_i x} \cap L_x \geq i \tag{1}$$

for all $i = 1, \ldots, r$ (assuming that the bundles F_{λ_i} are present in the flag F.)

Assume that the fibers of E are equipped with a linear symplectic structure, i.e. we are given a nowhere degenerating section of the bundle $\Lambda^2 E^*$. Assume also that the subbundles F_i and L are *isotrope* i.e. the fibers of F_{k_i} and L are isotrope subspaces in the fibers of E. This assumption implies additional restrictions on the degeneracy loci which change, in particular, their expected codimensions. We present a formula for the cohomology classes dual to these loci in terms of the Chern classes of the bundles E, V, F_i.

Remark. Denote by $\rho(k)$ the largest number $k+j$ such that $\lambda_k = \lambda_{k+1}+1 = \ldots = \lambda_{k+j}+j$. The index k is called *redundant* if $\rho(k) \neq k$. Otherwise it is called *essential.* The conditions (1) for redundant indices follow from those for the essential ones. They can be dropped in the definition of the degeneracy locus $S_{\lambda_1,\ldots,\lambda_r}$.

For any collection of formal series $c^{(i)} = 1 + c_1^{(i)} + c_2^{(i)} \ldots$, $i = 1, \ldots, r$, and any sequence (not necessary decreasing) of integers $\lambda_1, \ldots, \lambda_r$ we define the *generalized Schur Q-polynomials* $Q_{\lambda_1,\ldots,\lambda_r}(c^{(1)}, \ldots, c^{(r)})$ as follows:

- for $r = 1$ we set $Q_k(c) = c_k$;
- for $r = 2$ we set
$$Q_{k,l}(c^{(1)}, c^{(2)}) = c_k^{(1)} c_l^{(2)} - 2c_{k+1}^{(1)} c_{l-1}^{(2)} + 2c_{k+2}^{(1)} c_{l-2}^{(2)} - 2c_{k+3}^{(1)} c_{l-3}^{(2)} + \ldots;$$
- for any even $r \geq 4$ we set
$$Q_{\lambda_1,\ldots,\lambda_r}(c^{(1)}, \ldots, c^{(r)}) \quad = \quad \operatorname{Pf} |Q_{\lambda_i,\lambda_j}(c^{(i)}, c^{(j)})|_{1 \leq i,j \leq r}; \tag{2}$$
- for any odd $r \geq 3$ we set
$$Q_{\lambda_1,\ldots,\lambda_r}(c^{(1)}, ..., c^{(r)}) = \sum_{k=1}^{r} (-1)^{k-1} c_{\lambda_k}^{(k)} \, Q_{\lambda_1,\ldots,\widehat{\lambda_k},\ldots\lambda_r}(c^{(1)}, ..., \widehat{c^{(k)}}, ..., c^{(r)}). \tag{3}$$

Here Pf is the Pfaffian of a skew-symmetric matrix (see Appendix C). This definition makes sense only if

$$Q_{\lambda_i,\lambda_j}(c^{(i)}, c^{(j)}) + Q_{\lambda_j,\lambda_i}(c^{(j)}, c^{(i)}) = 0 \quad \text{for all} \quad 1 \leq i,j \leq r. \tag{4}$$

If this condition holds, then the polynomial $Q_{\lambda_1,\ldots,\lambda_r}(c^{(1)}, ..., c^{(r)})$ depends skew-symmetrically with respect to the permutations of indices λ_i and simultaneous permutations of $c^{(i)}$,

$$Q_{\lambda_{s(1)},\ldots,\lambda_{s(r)}}(c^{(s(1))}, ..., c^{(s(r))}) = (-1)^{|s|} Q_{\lambda_1,\ldots,\lambda_r}(c^{(1)}, ..., c^{(r)}),$$

where s is a permutation and $|s|$ is its sign. This follows from the fact that the Pfaffian is skew-symmetric with respect to simultaneous permutations of rows and columns of the matrix. In particular, $Q_{\lambda_1,\dots,\lambda_r}(c^{(1)},\dots,c^{(r)})$ vanishes if for some $i \neq j$ one has $\lambda_i = \lambda_j$ and $c^{(i)} = c^{(j)}$.

Remark. The distinction between the cases of even and odd r is apparent. For instance, the following reduction formula holds for any $r > 1$ with positive $\lambda_1, \dots, \lambda_r$

$$Q_{\lambda_1,\dots,\lambda_r,0}(c^{(1)},\dots,c^{(r+1)}) = Q_{\lambda_1,\dots,\lambda_r}(c^{(1)},\dots,c^{(r)})$$

(whenever these classes are defined). For $r = 2$ this evidently follows from the definition. For greater r it can be easily derived from (2) and (3) by induction in r.

Let $\lambda_1 > \dots > \lambda_r > 0$. If the flag F contains the bundles F_{λ_i} for all essential indices i then the degeneracy locus $S_{\lambda_1,\dots,\lambda_r} \subset M$ is well defined.

Theorem 1.1. *Generically the cohomology class dual to $S_{\lambda_1,\dots,\lambda_r}$ is given by*

$$[S_{\lambda_1,\dots,\lambda_r}] = Q_{\lambda_1,\dots,\lambda_r},$$

where

$$Q_{\lambda_1,\dots,\lambda_r} = Q_{\lambda_1,\dots,\lambda_r}(E{-}L{-}F_{\lambda_{\rho(1)}},\dots,E{-}L{-}F_{\lambda_{\rho(r)}}).$$

If the flag F contains the bundles F_{λ_i} for all redundant indices i then we have also

$$Q_{\lambda_1,\dots,\lambda_r} = Q_{\lambda_1,\dots,\lambda_r}(E{-}L{-}F_{\lambda_1},\dots,E{-}L{-}F_{\lambda_r}).$$

First verify that *the classes entering these formulas are defined*, that is the condition (4) is satisfied. Indeed, denoting $X_k = E - L - F_k$, we get

$$\begin{aligned} Q_{k,l}(c(X_{k'}),c(X_{l'})) + Q_{l,k}(c(X_{l'}),c(X_{k'})) &= 2\sum_{i=-\infty}^{\infty}(-1)^i c_{k+i}(X_{k'})\, c_{l-i}(X_{l'}) \\ &= \pm 2\, c_{k+l}(X^*_{k'} - X_{l'}) = \pm 2\, c_{k+l}((L^\perp/L)^* \oplus F^\perp_{k'}/F_{l'}). \end{aligned}$$

Here we used the isomorphisms $E/L^\perp \cong L^*$, $E/F^\perp_{k'} \cong F^*_{k'}$ provided by the symplectic structure (the orthogonal complement is considered with respect to the symplectic form). The bundle in the brackets has the rank

$$(\operatorname{rk} E - 2\operatorname{rk} L) + (\operatorname{rk} E - \operatorname{rk} F_{k'} - \operatorname{rk} F_{l'}) = k' + l' - 2.$$

The $(k+l)$th Chern class of this bundle vanishes if, for example, $k' \le k$, $l' \le l$. Therefore all classes of the form $Q_{\lambda_1,\dots,\lambda_r}(c(E{-}L{-}F_{\lambda'_1}),\dots,c(E{-}L{-}F_{\lambda'_r}))$ with $\lambda'_i \le \lambda_i$ are always well defined. □

Remark. The *genericity condition* of Theorem is formulated as follows. Consider the locally trivial bundle $Y \to M$ of 'geometrical configurations' whose fibers are formed by products $\mathbf{F}_x \times \Lambda_x$ where $\mathbf{F}_x$ and Λ_x are manifolds of isotrope flags and isotrope planes in E_x of the dimensions corresponding to the ranks of the bundles F_k, L. The canonical bundles over Y define degeneracy loci on Y and the genericity condition in this case is, by definition, satisfied. The given flag of bundles F and the bundle L define a section $s : M \to Y$. The genericity condition means that this section is transversal to every singularity locus on Y. In this case the equality for M is induced by s^* from the corresponding equality for Y. Remark that the class $s^*[S_{\lambda_1,\dots,\lambda_r}]$ is well defined on M and the equality of Theorem holds for this class even if the section s is not transversal. Similar trick can be applied for other situations in order to avoid problems with non-transversality. In particular it is implied in the definition of the manifolds Z_k in the next section. □

Remark. Similar trick is used also to show that *it is sufficient to prove Theorem* 1.1 *only for the case when the flag* F *contains subbundles* F_{λ_i} *for all redundant indices* i. Indeed, consider the flag bundle $Y \to M$ the fibers of which are formed by complete isotrope flags $F'_{nx} \subset F'_{(n-1)x} \subset \ldots \subset E_x$ such that $F'_{\lambda_i x} = F_{\lambda_i x}$ for all essential indices i. Then the validity of the assertion of Theorem 1.1 for Y implies its validity for M since the induced homomorphism of the cohomology $H^*(M) \to H^*(Y)$ is injective, see Appendix A. □

Proof of the second equality of Theorem 1.1,

$$Q_{\lambda_1,\ldots,\lambda_r}(c(X_1),\ldots,c(X_r)) = Q_{\lambda_1,\ldots,\lambda_r}(c(X_{\rho(1)}),\ldots,c(X_{\rho(r)})), \quad X_k = E{-}L{-}F_{\lambda_k}. \tag{5}$$

By definition, the classes $Q_{\lambda_1,\ldots,\lambda_r}(c(X_1),\ldots,c(X_r))$ are polilinear with respect to the total Chern classes $c(X_k)$. Suppose that the index k is redundant. Then from the equality

$$c_*(X_k) = c_*(F_{\lambda_{\rho(k)}}/F_{\lambda_k} + X_{\rho(k)}) = \sum_{j=0}^{\rho(k)-k} c_j(F_{\lambda_{\rho(k)}}/F_{\lambda_k})\; c_{*-j}(X_{\rho(k)})$$

we get

$$Q_{\ldots,\lambda_k,\ldots}(\ldots,X_k,\ldots) = \sum_{j=0}^{\rho(k)-k} c_j(F_{\lambda_{\rho(k)}}/F_{\lambda_k})\; Q_{\ldots,\lambda_k-j,\ldots}(\ldots,X_{\rho(k)},\ldots).$$

When we apply similar expansion to other redundant indices we obtain a linear combination of different classes of the form $Q_{\lambda'}(c(X_{\rho(1)}),\ldots,c(X_{\rho(r)}))$. In this combination all terms except one with $\lambda' = \lambda$ will have repeating indices and so they vanish. □

Two different proofs of the first equality of Theorem 1.1 are presented in two subsequent sections. In this Section we discuss some applications of Theorem.

In particular case when the flag F consists of only one plane $L' = F_k$ the formula of Theorem 1.1 answers the problem of J. Harris about the class dual to the locus given by the intersection ranks for the fibers of two isotrope bundles L and L' (see [PR], where the answer for the case when both L and L' are Lagrangian is given in much more complicated form).

Proposition. *Let* L, L' *be isotrope subbundles in a symplectic bundle* E. *Then the cohomology class dual to the locus* $\Omega_r = \{x \in M \mid \dim L_x \cap \dim L'_x \geq r\}$ *is given by*

$$[\Omega_r] = Q_{m+r,m+r-1,\ldots,m+1}(c(E{-}L{-}L'),\ldots,c(E{-}L{-}L')),$$

where $m = \operatorname{rk} E - \operatorname{rk} L - \operatorname{rk} L'$. □

In particular, the last formula for the case $r = \operatorname{rk} L = \operatorname{rk} L'$ describes the cohomology class dual to the diagonal in the product of two isotrope Grassmannians (or, more general, to the diagonal bundle in the fiber product of two isotrope Grassmannian bundles associated with the given symplectic vector bundle $E \to M$ (cf. [P2] where this class for the case of Lagrange Grassmannians is presented in much more complicated form),

$$[\Delta] = Q_{m+r,m+r-1,\ldots,m+1}(c(E{-}L{-}L'),\ldots,c(E{-}L{-}L')), \tag{6}$$

where $r = \operatorname{rk} L = \operatorname{rk} L'$, $m = \operatorname{rk} E - 2r$, and L_1, L_2 are the tautological bundles over the Grassmannians.

The simplest examples of degeneracy loci are provided by Schubert varieties on Grassmannians and flag manifolds of isotrope subspaces in $\mathbb{C}^{2n}$. In particular, this gives the following formula (for the case of Lagrange Grassmannians it was proved in [P2]).

Proposition. *On any Grassmannian of isotrope subspaces in symplectic vector space the cohomology class dual to the Schubert cycle S_λ defined by dimensions of intersections with a fixed isotrope flag is given by*

$$[S_{\lambda_1,\ldots,\lambda_r}] \quad = \quad Q_{\lambda_1,\ldots,\lambda_r}(c(-L),\ldots,c(-L)). \tag{7}$$

More examples are provided by symmetric degeneracy loci. Consider some vector bundle $V \to M$ and a symmetric bundle map $f : V \to V^*$ (it can be thought as the section of the bundle $\mathrm{Sym}^2 V^*$). Denote by $\Omega_r \subset M$ the locus of points $x \in M$ for which the kernel $\ker f_x$ has dimension at least r. Then we have (cf. [HT, JLP, P1])

$$[\Omega_r] = Q_{r,r-1,\ldots,1}(c(V^*-V),\ldots,c(V^*-V)).$$

Indeed, the sum $V \oplus V^*$ carries the natural symplectic structure (due to the natural isomorphism $V_x \oplus V_x^* \cong T^*V_x$). The condition that the map f_x is symmetric is equivalent to the condition that the graph $L_x \subset V_x \oplus V_x^*$ of this map is Lagrangian (in fact, the graphs of symmetric maps $V_x \to V_x^*$ form an open cell in the Grassmannian of Lagrange subspaces in $V_x \oplus V_x^*$). Hence the locus Ω_r is the degeneracy locus $S_{r,r-1,\ldots,1}$ defined with respect to the Lagrange subbundles L and $V \oplus 0$ in $V \oplus V^*$ and we can apply the formula of Theorem 1.1. □

In fact, all assertions about Lagrange degeneracy loci can be reformulated in terms of the corresponding symmetric degeneracies. (The inverse is also true, see Remark at the end of Appendix A.) The following assertions are the direct reformulations of Theorem 1.1 for the case of symmetric degeneracy loci in the same way as explained above.

Proposition. *Consider a flag of vector bundles $F_n \subset \ldots \subset F_1 = V \to M$, $\mathrm{rk}\, F_k = n+1-k$, and a symmetric bundle map $f : V \to V^*$. Denote by $\Omega_{\lambda_1,\ldots,\lambda_r} \subset M$ the locus of points $x \in M$ such that $\dim F_{\lambda_i x} \cap \ker f_x \geq i$. Then generically*

$$[\Omega_{\lambda_1,\ldots,\lambda_r}] = Q_{\lambda_1,\ldots,\lambda_r}(c(V^*-F_{\lambda_{\rho(1)}}),\ldots,c(V^*-F_{\lambda_{\rho(r)}})).$$ □

A particular case of the last Proposition is the following simplification of the formulas from [LP2] for degeneracies of symmetric maps. Let $F \subset E$ be vector bundles over some base M. The bundle map $f : F \to E^*$ is called *symmetric* if the bilinear form $(u,v) \mapsto \langle f_x(u), v\rangle$ on $F_x \times E_x$ is symmetric when restricted to $F_x \times F_x$ for all $x \in M$. One can easily see that the linear map $f_x : F_x \to E_x^*$ is symmetric if and only if its graph $L(f_x) = \{y \oplus (f_x(y)) \in E_x \otimes E_x^* \mid y \in F_x\}$ is an isotrope subspace in $E_x \oplus E_x^*$. The kernel of f_x is identified with the intersection of the isotrope subspaces $L(f_x)$ and $E \oplus 0$. Hence,

Proposition. *Let $F \to E^*$ be a symmetric map. Then the cohomology class Poincaré dual to the locus $\Omega_r \subset M$ of points $x \in M$ such that $\dim\ker f_x \geq r$ is given by*

$$[\Omega_r] = Q_{n+r,n+r-1,\ldots,n+1}(c(E^*-F),\ldots,c(E^*-F)), \qquad n = \mathrm{rk}\, E - \mathrm{rk}\, F.$$ □

The following interpretation of Theorem 2.1 of the next Section was used in [K3]. For a given vector bundle $V \to M$ denote by $\mathbf{D}_r(V) \to M$ the associated locally trivial flag bundle whose fibers are formed by all flags $D^1,\ldots,D^r \subset V_x$, $\dim D^i = i$. If $f : V \to V^*$ is a generic symmetric bundle map define the submanifold $Z_r \subset \mathbf{D}_r(V)$ by the condition $D^r \subset \ker f_x$. Denote $t_i = -c_1(D^i/D^{i-1})$. Let $p : Z_r \to M$ be the natural projection.

Theorem. *For any monomial* $t_1^{s_1}\cdots t_r^{s_r}\in H^*(Z_r)$, *we have*

$$p_*(t_1^{s_1}\cdots t_r^{s_r}) = Q_{s_1+1,\dots,s_r+1}(c(V^*-V),\dots,c(V^*-V)). \qquad \square$$

The analogues for the orthogonal case of the formulas of this Section are formulated in Appendix D.

2 The push-forward formula

In this section we prove Theorem 1.1 by means of resolutions of degeneracy loci. Let $L\subset E$ and $F_{k_1}\subset F_{k_2}\subset\cdots\subset E$, $\operatorname{rk}F_k = n+1-k$, $n=\operatorname{rk}E-\operatorname{rk}L$, be as in previous Section. For a given decreasing (not necessary strictly) sequence of integers $\lambda_1\ge\ldots\ge\lambda_r$ such that the planes F_{λ_i} are present in the flag F, we consider the flag bundle space $\mathbf{D}_r\to M$, whose fibers are formed by all isotrope flags

$$D_x^1\subset D_x^2\subset\ldots D_x^r\subset E_x, \qquad \dim D^i = i,$$

such that $D_x^i\subset F_{\lambda_i x}$ for $i=1,\dots,r$. Denote by $Z_r\subset\mathbf{D}_r$ the submanifold given by the condition $D_x^r\subset L_x$, where D_x^r is the largest plane of the flag. Generically Z_r is a smooth manifold of (complex) dimension $\dim Z_r=\dim M-\sum\lambda_i$. We study the push-forward homomorphism

$$p_*: H^*(Z_r)\to H^*(M)$$

corresponding to the natural projection $p: Z_r\to M$. We shall use the same notations D^i, L for the corresponding tautological bundles on $\mathbf{D}_r$, and Z_r. Denote $t_i=-c_1(D^i/D^{i-1}) = c_1((D^i/D^{i-1})^*)$.

Theorem 2.1. *For any monomial* $t_1^{s_1}\cdots t_r^{s_r}\in H^*(Z_r)$ *we have*

$$p_*(t_1^{s_1}\cdots t_r^{s_r}) = Q_{s_1+\lambda_1,\dots,s_r+\lambda_r}(c(E-L-F_{\lambda_1}),\dots,c(E-L-F_{\lambda_r}))\in H^*(M).$$

Theorem 1.1 is a corollary of Theorem 2.1. Indeed, suppose that $\lambda_1>\lambda_2>\ldots>\lambda_r>0$. Then the image $p(Z_r)$ coincides with $S_{\lambda_1,\dots,\lambda_r}$. Moreover, the restriction of p to Z_r is one-to-one over an open dense set in $S_{\lambda_1,\dots,\lambda_r}$. Therefore, applying Theorem 2.1 to the case $s_1=\ldots=s_r=0$ we get the formula of Theorem 1.1:

$$[S_{\lambda_1,\dots,\lambda_r}] = p_*(1) = Q_{\lambda_1,\dots,\lambda_r}(c(E-L-F_{\lambda_1}),\dots,c(E-L-F_{\lambda_r})). \qquad \square$$

To prove Theorem 2.1 we represent the map p as the composition of the following chain of bundles and embeddings

$$\begin{array}{ccccccccccc} & & P(E_{r-1}) & & & & & P(E_1) & & P(E) & \\ & \nearrow & \downarrow & & & & \nearrow & \downarrow & \nearrow & \downarrow & \\ Z_r & \xrightarrow{p_r} & Z_{r-1} & \xrightarrow{p_{r-1}} & \cdots & \xrightarrow{p_3} & Z_2 & \xrightarrow{p_2} Z_1 & \xrightarrow{p_1} & M & \end{array} \qquad (8)$$

where E_k is the restriction to Z_k of the bundle F_{λ_k}/D^k. At the points $y\in Z_k$ we should have $D_y^k\subset L_y\subset D_y^{k\perp}$. The projective planes of the bundle $P(E_k)$ are formed by all possible positions of the line D_y^{k+1}/D_y^k in $F_{\lambda_k y}/D_y^k$. The submanifold $Z_{k+1}\subset P(E_k)$ is given by condition: the plane $L_y/D_y^k\subset D_y^{k\perp}/D_y^k$ contains the line D_y^{k+1}/D_y^k. This explains the diagram (8).

For the computation of the homomorphism $p_{r*}: H^*(Z_r) \to H^*(Z_{r-1})$ we remark that the classes $t_1, \dots, t_{r-1}$ come from Z_{r-1} and so the multiplication by these classes commutes with p_{r*}. Hence, it is sufficient to compute $p_{r*}(t_r^s)$.

Lemma. *The homomorphism $p_{r*}: H^*(Z_r) \to H^*(Z_{r-1})$ is given by*

$$p_{r*}(t_r^s) = R_0^{(r)} a_{s+\lambda_r}^{(r)} + R_1^{(r)} a_{s+\lambda_r-1}^{(r)} + R_2^{(r)} a_{s+\lambda_r-2}^{(r)} + \dots,$$

where $a_i^{(r)} = c_i(E - L - F_{\lambda_r})$ and $R_i^{(r)} = c_i(D^{r-1} - D^{(r-1)*})$ are polynomials of degree i in $t_1, \dots, t_{r-1}$ that are independent of s, λ and given by the expansion

$$R_0^{(r)} + R_1^{(r)} + R_2^{(r)} + \dots = \prod_{i=1}^{r-1} \frac{1 - t_i}{1 + t_i} = \prod_{i=1}^{r-1} \left(1 - 2t_i + 2t_i^2 - 2t_i^3 + \dots\right).$$

Proof of Theorem 2.1. Iterating this Lemma we can compute the direct image $p_* = p_{1*} \dots p_{r*}$ of any particular monomial. This solves, in principle, the problem of finding the cohomology class dual to any degeneracy locus. It follows without further computations that $p_*(t_1^{s_1} \cdots t_r^{s_r})$ is expressed as a universal polynomial (depending only on λ_i, s_i, $i = 1, \dots, r$) in classes $a_i^{(k)} = c_i(E - L - F_{\lambda_k})$. It is a matter of algebra to show that the result has the nice form of Theorem 2.1. This algebraic proof is given in Appendix C. □

Example. For $r = 2$ applying Lemma twice we get

$$\begin{aligned} [S_{k,l}] &= p_{1*}p_{2*}(1) = p_{1*}\left(a_l^{(2)} - 2t_1 a_{l-1}^{(2)} + 2t_1^2 a_{l-2}^{(2)} - \dots \pm 2t_1^l\right) \\ &= p_{1*}(1)\, a_l^{(2)} - 2p_{1*}(t_1)\, a_{l-1}^{(2)} + 2p_{1*}(t_1^2)\, a_{l-2}^{(2)} - \dots \pm 2p_{1*}(t_1^l) \\ &= a_k^{(1)} a_l^{(2)} - 2a_{k+1}^{(1)} a_{l-1}^{(2)} + 2a_{k+2}^{(1)} a_{l-2}^{(2)} - \dots \pm 2a_{k+l}^{(1)} = Q_{k,l}(a^{(1)}, a^{(2)}), \end{aligned}$$

where $a^{(1)} = c(E - L - F_k)$, $a^{(2)} = c(E - L - F_l)$. □

Proof of Lemma. Represent p_r as the composition $Z_r \xrightarrow{i} P(E_{r-1}) \xrightarrow{q} Z_{r-1}$ according to the diagram (8). The homomorphism i_* is given by the multiplication by the fundamental cycle of $Z_r \subset P(E_{r-1})$. The submanifold $Z_r \subset P(E_{r-1})$ is given, as explained above, by the condition that the line $D_y^r / D_y^{r-1} \subset D_y^{(r-1)\perp} / D_y^{r-1}$ is contained in the plane L_y / D_y^{r-1}. This may be reformulated as vanishing of the section for the bundle $\mathrm{Hom}(D^r / D^{r-1}, D^{(r-1)\perp} / L)$. Therefore

$$\begin{aligned} i_*(t_r^s) &= t_r^s\, c_{n-r+1}(\mathrm{Hom}(D^r / D^{r-1}, D^{(r-1)\perp} / L)) \\ &= t_r^s \sum_{i+j=n-r+1} c_i(D^{(r-1)\perp}/L)\, t_r^j \\ &= \sum_{i+j=n+s-r+1} c_i(D^{(r-1)\perp}/L)\, t_r^j \\ &= c_{n+s-r+1}(D^{(r-1)\perp}/L - D^r/D^{r-1}) \\ &= c_{n+s-r+1}(D^{r-1} - D^{(r-1)*} + E - L - F_{\lambda_r} + F_{\lambda_r}/D^r) \\ &= \sum_{i+j=n+s-r+1} c_j(D^{r-1} - D^{(r-1)*} + E - L - F_{\lambda_r})\, c_i(F_{\lambda_r}/D^r) \end{aligned}$$

Now we apply q_*. The classes $c_j(D^{r-1} - D^{(r-1)*} + E - L - F_{\lambda_r})$ commute with this homomorphism as they come from Z_{k-1}. The bundle F_{λ_r}/D^r is the canonical quotient bundle over $P(E_{r-1})$. Therefore $q_* c_d(F_{\lambda_r}/D^r) = 1$, where $d = \mathrm{rk}(F_{\lambda_r}/D^r) = n+1-r-\lambda_r$ and $q_* c_i(F_{\lambda_r}/D^r) = 0$ for $i \neq d$. Therefore,

$$
\begin{aligned}
p_{r*}(t_r^s) = q_* i_*(t^s) &= c_{s+\lambda_r}(D^{r-1} - D^{(r-1)*} + E - L - F_{\lambda_r}) \\
&= \sum_{i=0}^{\infty} c_i(D^{r-1} - D^{(r-1)*})\, a^{(r)}_{s+\lambda_r - i}. \qquad \square
\end{aligned}
$$

3 Inverse induction

In this section we present an alternative proof of Theorem 1.1. Namely, we first establish the equality of Theorem in one particular case when $M = \mathbf{F}_N$ is the flag manifold of complete isotrope subspaces in $\mathbb{C}^{2N}$, the bundle $E = \mathbb{C}^{2N}$ is trivial, F is the tautological flag of bundles and $L \subset \mathbb{C}^{2N}$ is a fixed Lagrange subspace. This space $\mathbf{F}_N$ for sufficiently large N together with the induced Schubert partition on it is considered as the classifying space for our problem. It means that under condition of Theorem 1.1 there is a classifying map $\kappa : M \to \mathbf{F}_N$ which induces both the partition of M into the degeneracy loci and the characteristic classes. So the equality of Theorem 1.1 for M is induced from the corresponding equality for $\mathbf{F}_N$. Remark that in general we can not avoid the consideration of C^∞-manifolds and maps. The reader who does not like this kind of arguments may consider the proof as a motivation for finding new formulas (which can be applied as well in other problems).

Let $\mathbf{F} = \mathbf{F}_n$ be the space of complete isotrope flags $F_n \subset \ldots \subset F_1 \subset \mathbb{C}^{2n}$, $\dim F_k = n+1-k$. For a strictly decreasing sequence of integers $n \geq \lambda_1 > \ldots > \lambda_r > 0$ we define the Schubert cycle $S_{\lambda_1,\ldots,\lambda_r} \subset \mathbf{F}$ by conditions $\dim(F_{\lambda_1} \cap L) \geq i$, $i = 1, \ldots, r$, where $L \subset \mathbb{C}^{2n}$ is a fixed Lagrange subspace.

Theorem 3.1. *The cohomology class dual to the Schubert cycle in* $\mathbf{F}$ *is given by*

$$[S_{\lambda_1,\ldots,\lambda_r}] = Q_{\lambda_1,\ldots,\lambda_r}(c(-F_{\lambda_1}), \ldots, c(-F_{\lambda_r})).$$

In the proof we use the method of 'divided differences', see [BGG, D, F1, F2, P3]. Namely, we introduce operations $\partial_k : H^*(\mathbf{F}_n) \to H^{*-2}(\mathbf{F}_n)$, $k = 1, \ldots, n$, that decrease by 1 the complex codimension of the cycles representing the cohomology classes. Then we compute these operations in terms of degeneracy loci and in terms of the Chern classes. Then from the validity of Theorem for the 'deepest' locus we can establish the validity of Theorem for other loci by reverse induction over the indices λ.

The operations ∂_k are defined as follows. Denote by $\mathbf{F}^{(k)}$ the manifold of isotrope flags $F_n \subset \cdots F_{k+1} \subset F_{k-1} \subset \cdots \subset F_1$ (with the kth subspace of rank $n+1-k$ omitted). The natural projection

$$p_k : \mathbf{F} \to \mathbf{F}^{(k)}$$

is a smooth locally trivial bundle with the fibers isomorphic to $\mathbb{C}P^1$. The total space $\mathbf{F}$ of this bundle can be identified with the projectivization on the bundle F_{k-1}/F_{k+1} of rank 2 over $\mathbf{F}^{(k)}$. This description is valid for all $k = 1, 2, \ldots, n$ if we set $F_{n+1} = 0$, $F_0 = F_2^\perp$. Indeed, since the plane F_1 is Lagrangian, the condition $F_2 \subset F_1$ implies $F_1 \subset F_2^\perp$.

Definition. $\partial_k = p_k^* p_{k*} : H^*(\mathbf{F}) \to H^{*-2}(\mathbf{F})$.

Lemma S. *Let* $k \geq 2$. *Then*

$$\partial_k [S_{\lambda_1,\ldots,k,\ldots,\lambda_r}] = [S_{\lambda_1,\ldots,k-1,\ldots,\lambda_r}]$$

if $k \in \lambda$ *and* $k-1 \notin \lambda$*; otherwise* $\partial_k[S_{\lambda_1,\dots,\lambda_r}] = 0$.

For $k=1$ *we have*

$$\partial_1[S_{\lambda_1,\dots,\lambda_{r-1},1}] = [S_{\lambda_1,\dots,\lambda_{r-1}}]$$

and $\partial_1[S_{\lambda_1,\dots,\lambda_r}] = 0$ *if* $\lambda_r > 1$.

Lemma C. *In order to obtain the action of* ∂_k *in terms of Chern classes one should to replace the classes of* $[S_{\lambda_1,\dots}]$ *in the statement of previous Lemma by the corresponding polynomials* $Q_{\lambda_1,\dots}(c(-F_{\lambda_1}),\dots)$.

Proof of Theorem 3.1. Every Schubert cycle in $\mathbf{F}$ can be obtained from the cycle $S_{n,n-1,\dots,1}$ by a sequence of operations ∂_k for different k. If we would prove the equality of Theorem 3.1 for this cycle $S_{n,n-1,\dots,1}$ then by Lemmas S and C we get that similar equality holds for the other Schubert cycles. In the multi-index $(n, n-1, \dots, 1)$ all entries except the last one are redundant. Therefore, $Q_{n,n-1,\dots,1}(c(-F_n),\dots,c(-F_1)) = Q_{n,n-1,\dots,1}(c(-F_1),\dots,c(-F_1))$. It follows that it is sufficient to verify the required equality on the Lagrange Grassmannian Λ_n. But in this case the degree of the class $Q_{n,n-1,\dots,1}(c(-F_1),\dots,c(-F_1))$ is equal to the dimension of Λ_n and $S_{n,n-1,\dots,1}$ is a point. Therefore,

$$Q_{n,n-1,\dots,1}(c(-F_1),\dots,c(-F_1)) = b\,[\mathrm{pt}] = b\,[S_{n,n-1,\dots,1}] \in H^{2\frac{n(n+1)}{2}}(\Lambda_n) \cong H_0(\Lambda_n),$$

where the constant $b \in \mathbb{Z}$ is equal to the value of the characteristic class $Q_{n,n-1,\dots,1}(c(-F_1),\dots,c(-F_1))$ on the fundamental cycle of Λ_n. Assume that $b \neq 1$. Then again by inverse induction we get that for all Schubert cycles on $\mathbf{F}_n$ we should have $Q_\lambda(c(-F_{\lambda_1}),\dots,c(-F_{\lambda_r})) = b\,[S_\lambda]$ and to find the constant b it is sufficient to compute $[S_\lambda]$ for any particular λ.

Set $\lambda = (n)$. Then $S_n \subset \mathbf{F}_n$ is given by the condition $F_1 \subset L$. This is equivalent to the vanishing of the section of the bundle $\mathrm{Hom}(F_1, \mathbb{C}^{2n}/L)$. Therefore,

$$[S_n] = c_n(\mathrm{Hom}(F_1, \mathbb{C}^{2n}/L)) = c_n(-F_1) = Q_n(c(-F_1)).$$

This proves that, in fact, $b=1$. Theorem is proved. $\square$

Proof of Lemma S. If $k \notin \lambda$ then the planes of the bundle F_k are not used in the definition of the cycle S_λ. Therefore, this cycle is the inverse image of the corresponding cycle $S'_\lambda \subset \mathbf{F}^{(k)}$. Hence

$$p_{k*}[S_\lambda] = p_{k*}p^{k*}[S'_\lambda] = [S'_\lambda]\, p_{k*}p^{k*}(1) = 0,$$

and so $\partial_k[S_\lambda] = 0$. If $k = \lambda_i$ for some number i but $\lambda_{i+1} = k-1$ then the index i is redundant and we can apply the same arguments.

Assume now that $k = \lambda_i$ and that the index i is not redundant. Geometrically the homomorphism ∂_k can be described as follows. Suppose that a cohomology class in the total space of the bundle p_k is represented by a cycle C which meets every fiber at at most one point. Then the class $\partial_k[C]$ is represented by the union of all fibers through the points of C. It follows from this description that the cycle representing the class $\partial_k[S_\lambda]$ is defined by conditions $\dim F_{\lambda_j} \cap L \geq j$ for $j \neq i$ and the corresponding condition for the index i is replaced by the following one: *there exist a plane* F'_k *of dimension* $n+1-k$ *such that* $F_{(k+1)} \subset F'_k \subset F_{(k-1)}$ *and* $\dim F'_k \cap L \geq i$. Clearly, this condition is equivalent to the condition $\dim F_{(k-1)} \cap L \geq i$, that is $\partial_k[S_\lambda]$ is represented by the cycle $S_{\lambda_1,\dots,\lambda_{i-1},k-1,\lambda_{i+1},\dots,\lambda_r}$. $\square$

P r o o f o f L e m m a C. Observe that ∂_k commutes with the multiplication by characteristic classes of bundles that are defined on $\mathbf{F}^{(k)}$. In particular, if $k \notin \lambda$ then the class Q_λ is the pull back of the corresponding class on $\mathbf{F}^{(k)}$. Therefore,

$$\partial_k Q_\lambda = Q_\lambda \partial_k(1) = 0.$$

Assume now that $k = \lambda_i > 1$ for some (unique) i. Then the equality

$$c(-F_k) = c(F_{k-1}/F_k - F_{k-1}) = (1+t)\, c(-F_{k-1}),$$

where $t = c_1(F_{k-1}/F_k)$ implies

$$Q_{\dots,k,\dots}(\dots, c(-F_k), \dots) = A + t\, B,$$

where $A = Q_{\dots,k,\dots}(\dots, c(-F_{k-1}), \dots)$ and $B = Q_{\dots,k-1,\dots}(\dots, c(-F_{k-1}), \dots)$ are classes that are defined on $\mathbf{F}^{(k)}$. Therefore,

$$\partial_k Q_\lambda(\dots) = \partial_k(A + t\, B) = A\partial_k(1) + B\partial_k(t).$$

$\partial_k(1)$ vanishes by dimensional reason and the class $t = c_1(F_{k-1}/F_k)$ is the top Chern class of the canonical quotient bundle for the fiber bundle $P(F_{k-1}/F_{k+1})$ over $\mathbf{F}^{(k)}$. Therefore, $\partial_k(t) = p^{*k}p_{*k}(t) = 1$. We get finally

$$\partial_k Q_{\dots,k,\dots}(\dots, c(-F_k), \dots) = B = Q_{\dots,k-1,\dots}(\dots, c(-F_{k-1}), \dots).$$

Observe that if $k-1 \in \lambda = (\dots, k, \dots)$ then $\lambda' = (\dots, k-1, \dots)$ has repeating indices and so $B = 0$ in this case. This proves Lemma in case when $k > 1$.

In the case $k = 1 = \lambda_r$ the computation above can be applied as well if we denote $F_0 = F_2^\perp$ and gives

$$\partial_1 Q_{\dots,\lambda_{r-1},1}(\dots, c(-F_{\lambda_{r-1}}), c(-F_1)) = Q_{\dots,\lambda_{r-1},0}(\dots, c(-F_{\lambda_{r-1}}), c(-F_0)).$$

By Remark before Theorem 1.1, the class on the right hand side is equal to $Q_{\dots,\lambda_{r-1}}(\dots, c(-F_{\lambda_{r-1}}))$. Lemma is proved. □

P r o o f o f T h e o r e m 2.1. Assume we are given vector bundles $V \subset E \to M$, $F_{k_1} \subset \dots \subset E \to M$, E is symplectic, L, F_i are isotrope. Without loss of generality we can assume that E splits into the sum $E = L \oplus L^* \oplus K$, where K is symplectic and the symplectic structure ω_E on E is the sum of the corresponding symplectic structures $\omega_{L\oplus L^*}$ on $L \oplus L^*$ and ω_K on K. Indeed, the fiber L'_x of the bundle $L' \cong L^*$ is chosen among the isotrope subspaces in E_x transversal to $L_x^\perp$ and having the complementary dimension. The space of possible choices is contractible (it is homeomorphic to a cell). Hence such subbundle L' exists (in general, it is a complex C^∞-bundle). Then we set $K_x = L_x^\perp \cap L_x'^\perp$.

Now choose the bundle $U \to M$ such that $L \oplus K \oplus U = \mathbb{C}^N$ is a trivial bundle (again, in general, U is a complex C^∞-bundle). Define the bundles $\widetilde{E}, \widetilde{L}, \widetilde{F}_i$ and the symplectic structure $\omega_{\widetilde{E}}$ on $\widetilde{E}$ according to the following table

$$\begin{array}{lcccccccccc}
\widetilde{E} & = L & \oplus & L^* & \oplus & K & \oplus & K & \oplus U & \oplus & U^* \\
\omega_{\widetilde{E}} & = & & \omega_{L\oplus L^*} & \oplus & \omega_K & \oplus & -\omega_K & \oplus & & \omega_{U\oplus U^*} \\
\widetilde{L} & = L & \oplus & 0 & \oplus & & \Delta & & \oplus U & \oplus & 0 \\
\widetilde{F}_k & = & & F_k & & & \oplus & 0 & \oplus 0 & \oplus & U^*
\end{array}$$

Here $\Delta \in K \oplus K$ is the diagonal bundle. Recall that the natural symplectic structure on the sum $K_x \oplus K_x$ of two symplectic spaces is defined as the *difference* of the symplectic structures induced from the two summands. If it is defined this way then the diagonal $\Delta_x = \{z \oplus z \mid z \in K_x\}$ as well as the anti-diagonal $\Delta'_x \cong \Delta^*_x = \{z \oplus (-z) \mid z \in K_x\}$ are Lagrange subspaces. In particular, they define the canonical isomorphism $K \oplus K = \Delta \oplus \Delta^*$.

By construction, the degeneracy loci for this new problem are the same. But now the bundles $\widetilde{L} \cong L \oplus K \oplus U \cong \mathbb{C}^N$ and $\widetilde{E} = \widetilde{L} \oplus \widetilde{L}^* \cong \mathbb{C}^{2N}$ are trivial and the fibers of the bundles $\widetilde{F}_k$ belong to the same symplectic space $\mathbb{C}^{2N}$. Therefore, they define the map

$$\kappa : M \to \mathbf{F}'_N,$$

where $\mathbf{F}'_N$ is the the manifold of (incomplete) flags of isotrope subspaces in $\mathbb{C}^{2N}$ of dimensions $N-n+1, N-n+2, \ldots, N$.

The formula of Theorem 3.1 can be applied to the classes of Schubert cycles on $\mathbf{F}'_N$ since the projection $\mathbf{F}_N \to \mathbf{F}'_N$ induces an injective homomorphism of cohomology, see Appendix A. The degeneracy locus $S_\lambda(M) \subset M$ is the inverse image of the Schubert cycle $S_\lambda(\mathbf{F}'_N) \subset \mathbf{F}'_N$. The characteristic classes induced by κ are $\kappa^* c(\widetilde{F}_k) = c(F_k + U^*) = c(F_k + L - E)$. Therefore,

$$\begin{aligned} [S_\lambda(M)] &= \kappa^*[S_\lambda(\mathbf{F}'_N)] = \kappa^* Q_\lambda(c(-\widetilde{F}_{\lambda_1}), \ldots, c(-\widetilde{F}_{\lambda_r}) \\ &= Q_\lambda(c(E - L - F_{\lambda_1}), \ldots, c(E - L - F_{\lambda_r}) \end{aligned}$$

Theorem 1.1 is proved. □

Remark. The map κ is holomorphic in case when the degeneracy locus is the diagonal in the product of two isotrope Grassmannians or it is a Schubert cell on the Grassmannian of isotrope subspaces in $\mathbb{C}^{2N}$. Therefore the proof of formulas (6) (for the case $E = \mathbb{C}^{2N}$) and (7) of Section 1 by the method of this Section does not require considerations of C^∞-maps and bundles.

Remark. Lemmas C and S remain valid if $\mathbf{F}_N$ is replaced by the bundle of isotrope flags associated with some symplectic vector bundle, sf. [F2]. It follows that the formula of Theorem 1.1 can be derived from the formula (8) of Section 1 for the class of the diagonal in the fiber product of two Lagrange Grassmann bundles. The direct proof of (8) would imply the proof of Theorem 1.1 by divided differences method without using C^∞-maps and bundles.

Appendix A. Complex Lagrange Grassmannian

It was shown in previous Section that the Lagrange Grassmannian Λ_N, $N \to \infty$ plays the role of a classifying space for many geometrical problems. The limit cohomology group $\lim_{N\to\infty} H^*(\Lambda_N)$ is called *the ring of Lagrange characteristic classes.* In this section we describe the topology of Λ_N. All results of this section are proved in [P2] but our presentation is more elementary.

Consider an even-dimensional vector space $\mathbb{C}^{2n}$ and a fixed skew-symmetric bilinear form (the *symplectic form*) $\sum dp_i \wedge dq_i$ on it, where dp_i, dq_i are elements of some fixed basis on the dual space $\mathbb{C}^{*2n}$.

Definition. An n-dimensional subspace is called *Lagrangian* if the restriction of the symplectic form to it vanishes. The *complex Lagrange Grassmannian* $\Lambda_n^{\mathbb{C}}$ is the manifold of all Lagrange subspaces in $\mathbb{C}^{2n}$.

The topology of the real Lagrange Grassmannian $\Lambda_n^{\mathbb{R}} \cong \mathrm{U}(n)/\mathrm{O}(\mathrm{n})$ is well known. Its $\mathbb{Z}_2$-cohomology is $H^*(\Lambda^{\mathbb{R}}, \mathbb{Z}_2) \cong \Lambda_{\mathbb{Z}_2}(\alpha_1, \ldots, \alpha_n)$. The (integer) cohomology of complex Lagrange Grassmannian $\Lambda_n^{\mathbb{C}} \cong \mathrm{Sp}(2n)/\mathrm{U}(n)$ has a similar description.

Theorem ([P2]). *The ring $H^*(\Lambda_n^{\mathbb{C}})$ is isomorphic to the quotient of the polynomial ring in variables $a_1, a_2, \ldots, a_n$ of degrees $2, 4, \ldots, 2n$ over the ideal generated by elements*

$$a_i^2 - 2a_{i+1}a_{i-1} + 2a_{i+2}a_{i-2} - 2a_{i+3}a_{i-3} + \ldots. \tag{9}$$

The group $H^*(\Lambda_n^{\mathbb{C}})$ is torsion free and the monomials $a_1^{i_1} \ldots a_n^{i_n}$, $i_k \in \{0, 1\}$ form a free additive basis.

In the relations above we assume $a_0 = 1$ and $a_i = 0$ for $i < 0$ or for $i > n$. The classes $a_i \in H^{2i}(\Lambda_n^{\mathbb{C}})$ are Chern classes of the tautological bundle $L \to \Lambda_n^{\mathbb{C}}$ (or inverse images of Chern classes of the usual Grassmannian $G_{n,2n}^{\mathbb{C}}$ under the embedding $\Lambda_n^{\mathbb{C}} \subset G_{n,2n}^{\mathbb{C}}$.

The second assertion of Theorem follows from the first one. To express an element of this ring in terms of this basis one should apply repeatedly relation of Theorem to every monomial which contains squares of generators. This will require finite number of steps since every newly appeared monomial has degree strictly less than the original one if one uses the 'strange' filtration with the degree of a_i equal i^2.

The relations (9) can be rewritten in the form

$$(1 + a_1 + a_2 + \ldots + a_n)(1 - a_1 + a_2 - \ldots \pm a_n) = 1.$$

In this form they follow from the fact that the symplectic form induces an isomorphism $\mathbb{C}^{2n}/L \cong L^*$.

In the proof of Theorem we use the following well-known lemma. Let $E \to M$ be a complex vector bundle of rank d over some manifold M. Let $P = P(E)$ be the projectivization of the bundle E, i.e. the bundle space over M whose fibers are projective spaces formed by lines in fibers of E. Consider $H^*(P)$ as a module over the ring $A = H^*(M)$. Let $L \to P$ be the tautological line bundle and $t = c_1(L^*) = -c_1(L)$.

Lemma. *Additively the group $H^*(P)$ is isomorphic to $H^*(M \times \mathbb{C}P^{d-1})$ independently of the bundle E. Moreover, it is freely generated by elements $1, t, \ldots, t^{d-1}$ as an A-module. As a ring $H^*(P)$ is isomorphic to the quotient ring of $A[t]$ over the ideal generated by*

$$t^d + c_1(E)t^{d-1} + c_2(E)t^{d-2} + \ldots + c_d(E) \in A[t].$$

The Gysin homomorphism $\pi_* : H^*(P) \to H^*(M)$ maps the element $u_0 + u_1 t + \ldots + u_{d-1}t^{d-1} \in H^*(P)$ to $u_{d-1} \in H^*(M)$.

Recall that the *Gysin* or *push-forward* homomorphism or *transfer* π_* associated with a proper map $\pi : P \to M$ of smooth manifolds is the composition of Poincaré duality in P, usual homomorphism of homology, and Poincaré duality in M. The main property of Gysin homomorphism is the identity $\pi_*(\pi^* a\, b) = a\pi_*(b)$ for any elements $a \in H^*(M)$, $b \in H^*(P)$. In other words, π_* is the homomorphism of $H^*(M)$-modules.

The proof of Lemma is simple. First note that $\pi_* t^i = 0$ for $i < d-1$ (by dimensional reason) and $\pi_* t^{d-1} = 1 \in H^0(M)$ (this means that $d-1$ hyperplanes in $\mathbb{C}P^{d-1}$ intersect at one point). Therefore,

$$\pi_*(u_0 + u_1 t + \ldots + u_{d-1}t^{d-1}) = u_0\pi_*(1) + u_1\pi_*(t) + \ldots + u_{d-1}\pi_*(t^{d-1}) = u_{d-1}.$$

Therefore the elements $1, t, \ldots, t^{d-1}$ are independent over A in $H^*(P)$ since any relation of the form $u_0 + \ldots + u_k t^k = 0$ with $u_k \neq 0$ would imply $0 = \pi_*(t^{d-k-1}(u_0 + \ldots + u_k t^k)) = u_k$. It follows from the spectral sequence of the bundle that the elements of the form $u_0 + \ldots + u_{d-1}t^{d-1}$ exhaust all cohomology of P. This proves Lemma. Note that $t^d + c_1(E)t^{d-1} + c_2(E)t^{d-2} + \ldots + c_d(E)$ is equal to zero since it is the dth Chern class of the $(d-1)$-dimensional quotient bundle π^*E/L. □

The lemma is applied as follows. Denote by $\mathbf{F}_{i_1,\ldots,i_r}$ the space of flags consisting of *isotrope* subspaces $U_{i_1} \subset \ldots \subset U_{i_r}$, $\dim_{\mathbb{C}} U_l = l$. Consider the following diagram of projections

$$\mathbb{C}P^{2n-1} = \mathbf{F}_1 \leftarrow \mathbf{F}_{1,2} \leftarrow \ldots \leftarrow \mathbf{F}_{1,2,\ldots,n} \rightarrow \mathbf{F}_{2,\ldots,n} \rightarrow \cdots \rightarrow \mathbf{F}_n = \Lambda_n^{\mathbb{C}}.$$

All arrows in this diagram are projectivizations of certain vector bundles. This allows us to compute inductively the cohomology groups of all spaces in this diagram. It follows without any calculation that all these cohomology groups are torsion free. Minimal calculations give the total rank of $H^*(\Lambda_n^{\mathbb{C}})$. Comparing this diagram with a similar diagram for usual flags and Grassmannian we see that the Chern classes $a_1, \ldots, a_n$ generate all cohomology ring of $\Lambda_n^{\mathbb{C}}$. We know already some set of relations, and, comparing dimensions we see that there are no other relations. □

Exercise. Find generators and relations for the cohomology ring of any flag manifold of isotrope subspaces on $\mathbb{C}^{2n}$. Present a free additive basis for each case.

The correspondence between symmetric and Lagrange degeneracy loci is formalized as follows. Consider the homomorphism

$$\psi : \mathbb{Z}[a_1, a_2, \ldots] \rightarrow \mathbb{Z}[c_1, c_2 \ldots], \qquad 1 + a_1 + a_2 + \ldots \mapsto \frac{1 + c_1 + c_2 + \ldots}{1 - c_1 + c_2 - \ldots}.$$

Proposition. *The homomorphism ψ induces an embedding of the ring of Lagrange characteristic classes to the polynomial ring $\mathbb{Z}[c_1, c_2, \ldots]$.*

P r o o f. Since both $\mathbb{Z}[c_1, c_2, \ldots]$ and the ring of Lagrange characteristic classes are torsion free it is sufficient to prove this assertion over $\mathbb{Q}$. Introduce a new system of generators $\tilde{a}_i$, $\tilde{c}_i$ in the polynomial rings by setting

$$\tilde{c}_1 + \tilde{c}_2 + \ldots = \log(1 + c_1 + c_2 + \ldots)$$

and similarly for $\tilde{a}_i$ (the classes $\tilde{c}_k$, $\tilde{a}_k$ coincide up to $\pm 1/(k-1)!$ with the homogeneous components of the corresponding Chern characters). In terms of these generators the homomorphism

$$\psi : \mathbb{Q}[\tilde{a}_1, \tilde{a}_2, \ldots] \longrightarrow \mathbb{Q}[\tilde{c}_1, \tilde{c}_2, \ldots]$$

is given by $\tilde{a}_{2k-1} \mapsto 2\tilde{c}_{2k-1}$, $\tilde{a}_{2k} \mapsto 0$. The ideal (9) of defining relations in the ring of Lagrange characteristic classes is generated by $2\tilde{a}_2, 2\tilde{a}_4, \ldots$. Proposition follows. □

Remark. This proposition implies that all identities valid in the case of symmetric degeneracies are satisfied also in the ring of Lagrange characteristic classes. Therefore, Theorems 1.1 and 2.1 are equivalent to the corresponding statements about symmetric degeneracies formulated at the end of Section 1. Remark that in particular, the solution to J. Harris' problem (see Section 1) is contained implicitly already in [HT]! (That is the polynomial $Q_{m+r,\ldots,m+1}(c(E-L-L'), \ldots, c(E-L-L'))$ in classes $a_i = c_i(E-L-L')$ is uniquely determined by the condition that the image of this polynomial under ψ coincides with the class found in [HT].) I would claim even more, that some papers on symmetric (or skew-symmetric) degeneracy loci and those on isotrope degeneracy loci are in much extent duplicates of each others.

Appendix B. Twisted degeneracy loci

Assume that the symplectic form on the fibers of vector bundle $E \to M$ takes values not in $\mathbb{C}$ but in fibers of some line bundle $I \to M$. In other words, this twisted symplectic structure on the fibers is given by nowhere degenerating section of $\Lambda^2 E^* \otimes I$. The degeneracy loci for isotrope subbundles $L \subset E$ and $F_{k_1} \subset F_{k_2} \subset \ldots \subset E$, $k_i = \operatorname{rk} E - \operatorname{rk} L - \operatorname{rk} F_{k_i} + 1$, are defined in the same way as in Section 1. The cohomology classes dual to these loci can be obtained in the following way.

Assume first that the bundle I is a tensor square of another line bundle: $I = J^{\otimes 2}$. Then $\Lambda^2 E^* \otimes I = \Lambda^2 (E \otimes J^*)^*$ and we can apply Theorem 1.1 to the bundles $\widetilde{E} = E \otimes J^*$, $\widetilde{L} = L \otimes J^* \subset \widetilde{E}$, $\widetilde{F}_k = F_k \otimes J^* \subset \widetilde{E}$:

$$[S_{\lambda_1,\ldots,\lambda_r}] = Q_{\lambda_1,\ldots,\lambda_r}(c(\widetilde{E}-\widetilde{L}-\widetilde{F}_{\lambda_1}),\ldots,c(\widetilde{E}-\widetilde{L}-\widetilde{F}_{\lambda_1})). \tag{10}$$

Then we substitute

$$c(\widetilde{E}-\widetilde{L}-\widetilde{F}_k) = (1-u/2)^{k-1}\left(1 + \frac{c_1(E-L-F_k)}{1-u/2} + \frac{c_2(E-L-F_k)}{(1-u/2)^2} + \ldots\right), \tag{11}$$

where $u = c_1(I) = 2c_1(J)$. The expression for $[S_\lambda]$ in terms of $c_i(E-L-F_k)$ and $u = c_1(I)$ obtained in this way can be applied for general case since it is universal (the existence of universal polynomial expressing $[S_\lambda]$ in terms of $c_1(I), c_i(E-L-F_k)$ follows from either of the two methods of the proof of Theorem 1.1 presented in this paper, cf. also [HT]). Moreover, though we used division by 2 to obtain it, *this polynomial expression for $[S_\lambda]$ has integer coefficients and so it can be applied for any coefficient ring.* This (algebraic) assertion has the following topological proof.

Fix some integer $N \gg 0$ and define the manifold $Y = Y_N$ as the total space of the bundle over $\mathbb{C}P^N$ with the fibers formed by all isotrope flags in the twisted symplectic spaces $\mathbb{C}^N \oplus (\mathbb{C}^N \otimes I_x)$, $x \in \mathbb{C}P^N$, where $I \to \mathbb{C}P^N$ is the tautological line bundle (the twisted symplectic form on $\mathbb{C}^N \oplus (\mathbb{C}^N \otimes I_x) = (T^*\mathbb{C}^N) \otimes I_x$ with values in I_x is the obvious generalization of the standard symplectic form on $\mathbb{C}^{2N} = T^*\mathbb{C}^N$). It is not difficult to compute explicitly the cohomology ring of Y (using the same arguments as in Appendix A). This ring is generated by classes $u = c_1(I)$ and by the Chern (or Segre) classes $a_i^{(k)} = c_i(-F'_k)$ of the tautological bundles $F'_k \to Y$.

Now, we *define* the polynomial $Q_{\lambda_1,\ldots,\lambda_r} \in H^*(Y)$ as the cohomology class (expressed in terms of the generators $u, a_i^{(k)}$) dual to the twisted Schubert cycle $S_{\lambda_1,\ldots,\lambda_r} \subset Y$ defined with respect to the Lagrange subbundle $\mathbb{C}^N \oplus 0 \subset \mathbb{C}^N \oplus (\mathbb{C}^N \otimes I)$. By definition, this is a polynomial with integer coefficients. Since $H^*(Y)$ is torsion free, this polynomial coincides with that defined by (10)–(11) (with $a_i^{(k)}$ in place of $c_i(E-L-F_k)$).

For general case it is not difficult to verify that the classes $c_i(E-L-F_k)$ satisfy all relations for the classes $a_i^{(k)}$, provided that N is sufficiently large. In other words the characteristic homomorphism

$$\kappa^* : H^*(Y) \to H^*(M), \quad a_i^{(k)} \mapsto c_i(E-L-F_k)$$

is well defined though it is not necessary induced by a map $\kappa : M \to Y$. This proves that the class defined by (10)–(11) is integer. □

Appendix C. Identities in Pfaffians

Recall that the Pfaffian of a skew-symmetric matrix $\omega = \|\omega_{i,j}\|$ of even order $2n$ is, by definition,

$$\operatorname{Pf} \|\omega_{i,j}\| = \sum \pm \omega_{i_1,i_2} \cdots \omega_{i_{2n-1},i_{2n}},$$

where the sum is over all $(2n-1)!!$ ways to represent $\{1,2,\dots,2n\}$ as a union of n pairs $\{i_1,i_2\}\cup\dots\cup\{i_{2n-1},i_{2n}\}$ and $\pm$ is the sign of the permutation $(1,2,\dots,2n)\mapsto(i_1,i_2,\dots,i_{2n})$.

Consider a skew-symmetric function $f[k,l]=-f[l,k]$ with values in some commutative ring and defined on some discrete set I. For any ordered collection $(\lambda_1,\dots,\lambda_r)$, $\lambda_i\in I$, we set

- $f[\lambda_1,\dots,\lambda_r]=\operatorname{Pf}\|f[\lambda_i,\lambda_j]\|$, r even;
- $f[\lambda_1,\dots,\lambda_r]=f[\lambda_1,\dots,\lambda_r,0]=\sum_{i=1}^r(-1)^{i-1}f[\lambda_1,\dots,\widehat{\lambda_i},\dots,\lambda_r]$, r odd, where for $0\notin I$ we put formally $f[k,0]=-f[0,k]=1$.

Proposition. *The function f satisfies the identity*

$$f[\lambda_1,\dots,\lambda_r]=\prod_{1\le i<j\le r}f[\lambda_i,\lambda_j]$$

if and only if it satisfies this identity for $r=3$.

Proof. This proposition generalizes similar statement from [Kn], where the identity for even r was proved under assumption that it is satisfied for $r=4$. Observe that adding 0 does not spoil the identity for $r=3$. Then, it is easily verified for $r=4$, and so the result follows form Knuth's theorem. Remark that the direct proof of Proposition is a little simpler than the original proof of Knuth's theorem. □

Example. The identity is satisfied for the function $g[x,y]=\dfrac{y-x}{x+y}$.

Example (V. Kryukov). Denote by $\mathcal{L}$ the completion of the ring of Laurent polynomials $\mathbb{Z}[t_1,t_1^{-1},\dots,t_r,t_r^{-1}]$ with help of infinite formal series in $t_1/t_2,t_2/t_3,\dots,t_{r-1}/t_r$ (the variables t_i can enter in monomials with arbitrary large positive or negative exponents but every monomial of the product R_1R_2 of two such series is determined by only finite number of monomials in R_1 and R_2). For $i<j$ set

$$f[i,j]=-f[j,i]=\frac{1-t_i/t_j}{1+t_i/t_j}=1-2\frac{t_i}{t_j}+2\frac{t_i^2}{t_j^2}-2\frac{t_i^3}{t_j^3}+\ldots\in\mathcal{L}$$

This function also satisfies the condition of Proposition. Indeed, if for $i<j<k$ we set $a=t_i/t_j$, $b=t_j/t_k$, then the identity we should verify reduces to the correct identity

$$\frac{1-a}{1+a}-\frac{1-ab}{1+ab}+\frac{1-b}{1+b}=\frac{(1-a)(1-ab)(1-b)}{(1+a)(1+ab)(1+b)}.$$

□

Completion of the proof of Theorem 2.1. Denote by A the polynomial ring in variables $a_k^{(i)}$. Consider the homomorphism

$$p_{k*}:A[t_1,\dots,t_k]\to A[t_1,\dots,t_{k-1}],$$

that commutes with $t_1,\dots,t_{k-1}$ and

$$p_{k*}t_k^s=\sum_{i=0}^{\infty}R_i^{(k)}a_{s+\lambda_r-i}^{(k)},\quad\text{where}\quad R_0^{(k)}+R_1^{(k)}+R_2^{(k)}+\ldots=\prod_{i=1}^{k-1}\frac{1-t_i}{1+t_i}.$$

Our goal is to compute $p_{1*}\dots p_{r*}t_1^{s_1}\cdots t_r^{s_r}$. Let $\mathcal{L}_k\subset\mathcal{L}$ denote the subring of series that do not contain negative exponents of variables $t_1,\dots,t_k$. Consider the $\mathbb{Z}[t_1,\dots,t_k]$-linear homomorphism

$$\mathcal{L}_k\xrightarrow{\varphi}A[t_1,\dots,t_k],\qquad t_1^{m_1}\cdots t_r^{m_r}\mapsto t_1^{m_1}\cdots t_k^{m_k}\,a_{m_{k+1}}^{(k+1)}\cdots a_{m_r}^{(r)}.$$

In terms of this homomorphism the image $p_{k*}t_k^s$ is given by

$$p_{k*}t_k^s = \varphi \sum_{i=0}^{\infty} R_i^{(k)} t_k^{s+\lambda_k-i} = \varphi\, t_k^{\lambda_k+s} \sum_{i=0}^{\infty} R_i^{(k)} t_k^{-s} = \varphi\, t_k^{\lambda_k+s} \prod_{i=1}^{k-1} \frac{1-t_i/t_k}{1+t_i/t_k}.$$

In other words, in terms of φ the homomorphism p_{k*} is just the multiplication by $t_k^{\lambda_k} \prod_{i=1}^{k-1} \frac{1-t_i/t_k}{1+t_i/t_k}$. After r steps we obtain

$$p_{1*}\cdots p_{r*}t_1^{s_1}\cdots t_r^{s_r} = \varphi\, t_1^{s_1+\lambda_1}\cdots t_r^{s_r+\lambda_r} \prod_{1\le i<j\le r} \frac{1-t_i/t_j}{1+t_i/t_j}.$$

Now we use the identity for the function f of Example above. Assume r is even. Denoting $\tilde f[i,j] = t_i^{s_i+\lambda_i} t_j^{s_j+\lambda_j} f[i,j]$, we have

$$\begin{aligned}
p_{1*}\ldots p_{r*}t_1^{s_1}\ldots t_r^{s_r} &= \varphi\, t_1^{s_1+\lambda_1}\cdots t_r^{s_r+\lambda_r} f[1,2,\ldots,r] \\
&= \varphi \sum \pm \tilde f[i_1,i_2]\, \tilde f[i_3,i_4]\cdots \\
&= \sum \pm \varphi(\tilde f[i_1,i_2])\, \varphi(\tilde f[i_3,i_4])\cdots \\
&= \sum \pm Q_{\lambda_{i_1}+s_{i_1},\lambda_{i_2}+s_{i_2}}(a^{(i_1)},a^{(i_2)})\, Q_{\lambda_{i_3}+s_{i_3},\lambda_{i_4}+s_{i_4}}(a^{(i_3)},a^{(i_4)})\cdots \\
&= Q_{\lambda_1+s_1,\ldots,\lambda_n+s_n}(a^{(1)},\ldots,a^{(r)}).
\end{aligned}$$

The case of odd r is considered in a similar way or we can simply reduce the problem to the previous case by setting $\lambda_{r+1}=0$. □

Appendix D. Orthogonal degeneracy loci

In this section we state the analogue of Theorem 1.1 for the case when the bundle E is *orthogonal* i.e. it is equipped with a nondegenerate *symmetric* bilinear form given as a nowhere degenerating section of the bundle $\mathrm{Sym}^2 E^*$. The proofs will appear elsewhere.

The main difficulty arising in the orthogonal case is that the isotrope bundles have characteristic classes that are not expressed in terms of their Chern classes. In other words, the cohomology rings of the isotrope Grassmann and flag manifolds are not generated by the Chern classes of the tautological bundles. Almost all such classes still can be expressed via the Chern classes if the division by 2 is allowed. Therefore in this Appendix we assume that the coefficient ring of all cohomology groups contains $1/2$.

There are still extra characteristic classes in the case when $\mathrm{rk}\, E = 2n$ is even. For instance the Grassmannian of *maximal* (i.e. n-dimensional) isotrope subspaces in $\mathbb{C}^{2n}$ has two components. Namely, the dimension of the intersection $\dim(L_1\cap L_2)$ of two maximal isotrope subspaces $L_1, L_2 \subset \mathbb{C}^{2n}$ may jump only by even numbers when L_1 and L_2 are changing continuously. These two planes belong to the same component iff $\dim(L_1\cap L_2) \equiv n \pmod 2$. For an isotrope subbundle $L\subset E$ the missed characteristic class is defined as follows. Assume that the bundle L is a subbundle of some maximal isotrope bundle $\widehat L$. Then *the top Chern class* $c_m(\widehat L/L)$, $m = n - \mathrm{rk}\, L$, *is independent on* $\widehat L$ *up to a sign.* More precisely,

Lemma. *If* $\widehat L, \widehat L' \subset E$, $\mathrm{rk}\, E = 2n$, *are two maximal isotrope subbundles containing* L, *then* $c_m(\widehat L/L) = \pm c_m(\widehat L'/L)$, *where the sign* $\pm$ *is positive (negative) if* $\dim(\widehat L_x \cap \widehat L'_x) \equiv n \pmod 2$ *(resp.* $\dim(\widehat L_x\cap \widehat L'_x) \equiv n-1 \pmod 2$*) for any point* $x\in M$.

Definition. Let $F, L \subset E$ be two isotrope subbundles. If the rank $\operatorname{rk} E = 2n$ is even we set

$$e(L,F) = (-1)^{\dim(\widehat{L}_x \cap \widehat{F}_x)} c_k(\widehat{L}/L \oplus \widehat{F}/F), \quad k = 2n - \operatorname{rk} L - \operatorname{rk} F,$$

where $\widehat{L}, \widehat{F}$ are some maximal isotrope subbundles containing L and F respectively.

If $\operatorname{rk} E = 2n+1$ is odd we set $e(L,F) = 0$.

Remark. The Lemma implies that the class $e(L,F)$ depends neither on the bundles $\widehat{L}, \widehat{F}$ nor on the choice of the point $x \in M$. Moreover this class is well defined even if the maximal isotrope bundles $\widehat{L}, \widehat{F}$ do not exist. Indeed, we can pass from M to the total space of the bundle $G \to M$ whose fibers are formed by pairs of maximal isotrope subspaces in E_x containing L_x and F_x respectively, $x \in M$; it is sufficient to define the corresponding characteristic class on G since the induced homomorphism of the cohomology $H^*(M) \to H^*(G)$ is injective (over $\mathbb{Z}[\frac{1}{2}]$).

Consider an isotrope subbundle $L \subset E$ and a flag of isotrope subbundles

$$F_{k_1} \subset F_{k_2} \subset \ldots \subset E, \quad \operatorname{rk} F_k = \operatorname{rk} E - \operatorname{rk} V - k.$$

(Remark the shift by 1 in the numeration of the planes F_k comparing with the symplectic case.) The degeneracy loci $S_{\lambda_1,\ldots,\lambda_r} \subset M$, $\lambda_1 > \ldots > \lambda_r$ are defined similarly to the symplectic case by conditions

$$\dim F_{\lambda_i x} \cap L_x \geq i, \quad i = 1, \ldots, r.$$

We may always assume that $\lambda_r > 0$. Indeed, if $\operatorname{rk} E$ is even and both F_0 and L are maximal then the fact that $\dim(F_{0x} \cap L_x) = \text{const} \pmod 2$ implies that the condition on the dimension of the intersection $F_{0x} \cap L_x$ is equivalent to the corresponding condition on the dimension of the intersection $F_{1x} \cap L_x$.

Theorem. *Generically the cohomology class dual to the locus $S_{\lambda_1,\ldots,\lambda_r}$ is given by*

$$[S_{\lambda_1,\ldots,\lambda_r}] = P_{\lambda_1,\ldots,\lambda_r},$$

where the characteristic class $P_{\lambda_1,\ldots,\lambda_r} \in H^{2\sum \lambda_i}(M)$ is defined as follows:

- *if $r = 1$ then $2P_k = a_k^{(k)} - e(L,F_k)$, where $a_i^{(k)} = c_i(E-L-F_k)$;*
- *if $r = 2$ then $4P_{k,l} = \left(a_k^{(k)} - e(L,F_k)\right)\left(a_l^{(l)} + e(L,F_l)\right) + 2\sum_{i=1}^{l}(-1)^i a_{k+i}^{(k)} a_{l-i}^{(l)}$;*
- *if $r > 3$ is even then $P_{\lambda_1,\ldots,\lambda_r} = \operatorname{Pf}|P_{\lambda_i,\lambda_j}|_{1\leq i,j\leq r}$;*
- *if $r > 2$ is odd then $P_{\lambda_1,\ldots,\lambda_r} = \sum_{i=1}^{r}(-1)^{i-1} P_{\lambda_i} P_{\lambda_1,\ldots,\widehat{\lambda_i},\ldots,\lambda_r}$.*

The genericity condition is formulated in the same way as in the remark after Theorem 1.1. All corollaries of Theorem 1.1 listed in Section 1 and Appendix B have the corresponding reformulations for the case of orthogonal degeneracy loci (getting rid of redundant indicies, Schubert classes on isotrope flag and Grassmann manifolds, degeneracy of skew-symmetric maps of bundles, classes of twisted degeneracy loci etc.). We present just two examples; the interested reader can easily formulate the others. The formulas of the statements below simplify the corresponding formulas from [PR, LP2].

Let $L, L' \subset E$ be two isotrope subbundles in an orthogonal bundle E. For any integer $r > 0$ consider the locus $\Omega_r = \{x \in M \mid \dim(L_x \cap L'_x) \geq r\}$. Denote $a = c(E - L - L')$,

$m = \operatorname{rk} E - \operatorname{rk} L - \operatorname{rk} L'$. Then m is strictly positive unless $\operatorname{rk} E$ is even and both L, L' are maximal.

Theorem (cf. [PR]). *Assume that $m > 0$. Then generically*

$$[\Omega_r] = \frac{1}{2^r}\left(Q_{m+r-1,\ldots,m+1,m}(a,\ldots,a) + (-1)^r e(L,L')\, Q_{m+r-1,\ldots,m+1}(a,\ldots,a)\right).$$

If $m = 0$ (i.e. $\operatorname{rk} E = 2n$ is even and the isotrope sbbundles L, L' are maximal) we assume that $\dim(L_x \cap L'_x) \equiv r \pmod 2$. *Then generically*

$$[\Omega_r] = \frac{1}{2^{r-1}} Q_{r-1,\ldots,1}(a,\ldots,a).$$

Let $F \subset E$ be vector bundles over some base M. The bundle map $f : F \to E^*$ is called *skew-symmetric* if the bilinear form $\langle f_x(u), v\rangle$ on $F_x \times E_x$ is skew-symmetric when restricted to $F_x \times F_x$ for all $x \in M$. Denote $m = \operatorname{rk} E - \operatorname{rk} F$, $a = c(E^* - F)$.

Corollary (cf. [LP2]). *Let $F \to E^*$ be a skew-symmetric map of vector bundles $F \subset E$. If $F \neq E$ (i.e. $m > 0$) then the Poincaré dual to the locus $\Omega_r \subset M$ of points $x \in M$ such that* $\dim\ker f_x \geq r$ *is given by*

$$[\Omega_r] = \frac{1}{2^r}\left(Q_{m+r-1,\ldots,m}(a,\ldots,a) + (-1)^r c_m(E/F)\, Q_{m+r-1,\ldots,m+1}(a,\ldots,a)\right).$$

If $F = E$ then generically

$$[\Omega_r] = \frac{1}{2^{r-1}} Q_{r-1,\ldots,1}(a,\ldots,a).$$

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