ON COMBINATORIAL FORMULAS FOR COHOMOLOGY OF SPACES OF KNOTS

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Abstract. We develop homological techniques for finding explicit combinatorial expressions of finite-type cohomology classes of spaces of knots in $\mathbb{R}^n, n \geq 3$, generalizing Polyak–Viro formulas [10] for invariants (i.e. 0-dimensional cohomology classes) of knots in $\mathbb{R}^3$.

As the first applications we give such formulas for the (reduced mod 2) *generalized Teiblum–Turchin cocycle* of order 3 (which is the simplest cohomology class of *long knots* $\mathbb{R}^1 \hookrightarrow \mathbb{R}^n$ not reducible to knot invariants or their natural stabilizations), and for all integral cohomology classes of orders 1 and 2 of spaces of *compact knots* $S^1 \hookrightarrow \mathbb{R}^n$. As a corollary, we prove the nontriviality of all these cohomology classes in spaces of knots in $\mathbb{R}^3$.

1. Introduction

There is a wide family of cohomology classes of spaces of knots $S^1 \hookrightarrow \mathbb{R}^n$ $(n \geq 3)$, called *finite-type cohomology classes*; see [14], [16], [18]. For $n > 3$ they cover all of the cohomology group of the space of knots in $\mathbb{R}^n$, for $n = 3$ their 0-dimensional part are the finite-type knot invariants.

These classes are defined as linking numbers (in the space of all smooth maps $S^1 \to \mathbb{R}^n$) with appropriate cycles (of infinite dimension but finite codimension) in the *discriminant space* Σ (cf. [1]); in our case this space consists of maps which are not smooth embeddings. The group of all such classes is filtered by their *orders* induced by some filtration of (some resolution of) the discriminant: roughly speaking, the order of a cohomology class indicates how much complicated strata of Σ participate in the definition of its dual variety.

In [10], M. Polyak and O. Viro have proposed some combinatorial formulas for the finite-type invariants of knots in $\mathbb{R}^3$. Later, M. Goussarov has proved that any finite-type invariant can be represented by a formula of this type, see [6].

We describe some calculus for finding (and proving) combinatorial formulas for arbitrary finite type cohomology classes, in particular show what the answers can look like. *Any such combinatorial formula is nothing else than some semialgebraic chain in the space of maps $S^1 \to \mathbb{R}^n$, such that its boundary lies in Σ and our*

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cohomology class is equal to the linking number with this boundary. We introduce several natural families of semialgebraic subvarieties of the space of such maps, of which the desired chains are built. These varieties are defined by easy differential geometrical conditions; they arise naturally in the direct calculation of the main spectral sequence converging to the (finite type) cohomology group of the space of knots. It is not surprising that some elements of this calculus repeat pictures from [10], [6], and also from the A. B. Merkov's works on invariants of plane curves [9], [8].

We accomplish these calculations explicitly for several cohomology classes of low orders. Before describing them three remarks more.

1. Long and compact knots. We shall distinguish two kinds of knot spaces. The *compact knots* in $\mathbb{R}^n$ are any smooth embeddings $S^1 \to \mathbb{R}^n$, while the *long knots* are the smooth embeddings $\mathbb{R}^1 \to \mathbb{R}^n$ coinciding with a standard linear embedding outside some compact subset in $\mathbb{R}^1$. The invariants of knots of both types in $\mathbb{R}^3$ naturally coincide, but generally the cohomology ring of the space of compact knots is more complicated: it is built of the similar ring for long knots (playing the role of a "coefficient" ring) and homology groups of the space S^1 and certain its configuration spaces.

2. Stabilization. If numbers n and m are of the same parity, then the theories of (finite type) cohomology groups of spaces of knots in $\mathbb{R}^n$ and $\mathbb{R}^m$ are very similar. Namely, the first terms of spectral sequences calculating both groups and generated by the natural filtration of resolved discriminants coincide up to shifts of indices:

$$E_1^{p,q-pn}(\mathbb{R}^n) \simeq E_1^{p,q-pm}(\mathbb{R}^m). \tag{1}$$

Moreover, for spectral sequences calculating $\mathbb{Z}_2$-cohomology groups this identity is true also if n and m are of different parities. M. Kontsevich has proved (but not published) that in the case of complex coefficients our spectral sequence degenerates at the first term: $E_\infty^{p,q} \equiv E_1^{p,q}$, therefore also the limit groups of finite type cohomology classes are very similar. (I conjecture that in the case of long knots the similar degeneration holds also for any coefficients.)

3. This paper is very much a work in the differential geometry of spatial curves and their projections to different subspaces, although almost all results of this kind are hidden in the formulas for boundary operators in our homological calculations.

1.1. **Results for long knots.**

Accordingly to [14], [11], [18], [16], all cohomology classes of orders ≤ 3 of the space of long knots in $\mathbb{R}^n$, $n \geq 3$, are as follows.

Proposition 1. *There are no cohomology classes of order 1. The classes of order 2 are only in dimension* $2n-6$ *and form a group isomorphic to* $\mathbb{Z}$ (*for* $n=3$ *it is generated by the simplest knot invariant*). *In order 3 additional classes can be in exactly two dimensions more:* $3n-9$ *and* $3n-8$. *In dimension* $3n-9$ *they form a group isomorphic to* $\mathbb{Z}$ (*for* $n=3$ *it is generated by the next simple knot invariant*). *In* *dimension* $3n-8$ *the same is true if* $n > 3$, *and for* $n = 3$ *the similar* (*1-dimensional*) *cohomology group is cyclic* (*maybe of order 1 or* ∞).

It was conjectured in [16], [18] that the latter group for $n = 3$ also is isomorphic to $\mathbb{Z}$; we shall prove it in the present work.

For any n we call the generator of this $(3n - 8)$-dimensional cohomology group the *Turchin–Teiblum cocycle.* In the case of odd n its existence was discovered by D. M. Teiblum and V. E. Turchin about 1995 ([11]). Its (quite different) superanalog for even n was found in [16], [18]. However, all these works contain only an implicit proof of the existence of such a class: namely, the calculation of the third column of our spectral sequence (which is responsible for the third order cohomology classes and is isomorphic to $\mathbb{Z}$ for exactly two values of q), and the remark that all further differentials acting from or to this column are trivial by some dimensional reasons.

In §3 we prove the following combinatorial expression for this class reduced mod 2.

Let us choose a direction "up" in $\mathbb{R}^n$, and say that a point $x \in \mathbb{R}^n$ is *above* the point y if the vector $\overrightarrow{(yx)}$ has the chosen direction. Let $\mathbb{R}^{n-1}$ be the quotient space of $\mathbb{R}^n$ by this direction, and $\mathbf{p} : \mathbb{R}^n \to \mathbb{R}^{n-1}$ the corresponding projection. We choose a direction "to the right" in $\mathbb{R}^{n-1}$ and say that the point $x \in \mathbb{R}^n$ *is to the right* of the point y if the vector $\overrightarrow{(\mathbf{p}(y), \mathbf{p}(x))} \in \mathbb{R}^{n-1}$ has this chosen direction.

Theorem 1. *For any* $n \geq 3$, *the value of the reduced mod 2 Teiblum–Turchin class on any generic* $(3n - 8)$*-dimensional singular cycle in the space of long knots in* $\mathbb{R}^n$ *is equal to the parity of the number of points of this cycle corresponding to such knots* $f : \mathbb{R}^1 \to \mathbb{R}^n$ *that one of three holds:*

a) there are five points $a < b < c < d < e$ *in* $\mathbb{R}^1$ *such that* $f(a)$ *is above* $f(d)$, *and* $f(e)$ *is above* $f(c)$ *and* $f(b)$;

b) there are four points $a < b < c < d$ *in* $\mathbb{R}^1$ *such that* $f(a)$ *is above* $f(c)$, $f(b)$ *is below* $f(d)$, *and the projection of the derivative* $f'(b)$ *to* $\mathbb{R}^{n-1}$ *is directed to the right;*

c) there are three points $a < b < c$ *in* $\mathbb{R}^1$ *such that* $f(a)$ *is above* $f(b)$ *but below* $f(c)$, *and the "exterior" angle in* $\mathbb{R}^{n-1}$ *formed by projections of* $f'(a)$ *and* $f'(b)$ *contains the direction "to the right"* (*i.e. this direction is equal to a linear combination of these projections, and at least one of coefficients in this combination is nonpositive*).

These intersection points should be counted with multiplicities equal to the number of different point configurations for which the corresponding condition is satisfied (*note however that for* $n > 3$ *a generic* $(3n-8)$*-dimensional cycle cannot have points for which this multiplicity is greater than 1*).

We prove this theorem in §3. In the next works I am planning to accomplish all the same calculations taking respect on the orientations, and thus to obtain similar results with integer coefficients.

Corollary 1. *The group of order 3 one-dimensional cohomology classes of the space of long knots in $\mathbb{R}^3$ is free cyclic and generated by the* (*integral*) *Teiblum–Turchin class.*

More precisely, let us consider the connected sum of two equal (long) trefoil knots in $\mathbb{R}^3$ and a path in the space of knots connecting this knot with itself as in the proof of the commutativity of the knot semigroup: we shrink the first summand, move it "through" the second, and then blow up again.

Proposition 2. *This closed path in the space of long knots has an odd intersection number with the union of three varieties indicated in items a, b and c of Theorem 1.*

The proof will be given in § 3.7.

On the other hand, for any n the Teiblum–Turchin cocycle is a well-defined integral cohomology class. By the previous proposition it takes a nonzero value on a well-defined integral cycle, hence is not a torsion element, and Corollary 1 is proved.

1.2. **Answers for compact knots.** Nontrivial cohomology classes in the space of *compact knots* $S^1 \hookrightarrow \mathbb{R}^n$ appear already in filtrations 1 and 2. We assume that a cyclic coordinate in S^1, i.e. an identification $S^1 \simeq \mathbb{R}^1/2\pi\mathbb{Z}$, is fixed.

Proposition 3 (see [17], [18]). *For any $n \geq 3$ the group of $\mathbb{Z}_2$-cohomology classes of order 1 of the space of compact knots in $\mathbb{R}^n$ is nontrivial only in dimensions $n-2$ and $n-1$, and is isomorphic to $\mathbb{Z}_2$ in these dimensions. Moreover, for* (*only*) *even n similar integral cohomology groups in these dimensions are isomorphic to $\mathbb{Z}$. The generator of the $(n-2)$-dimensional group is Alexander dual to the set $\mathcal{L}$ of discriminant maps $S^1 \to \mathbb{R}^n$ gluing together some two* opposite *points of S^1, and the generator of the $(n-1)$-dimensional group is dual to the set of maps gluing some* chosen *opposite points, say* 0 *and* π.

Proposition 4 (see [16], [18]). *Additional classes of order 2 exist in exactly two dimensions: $2n-6$ and $2n-3$. In dimension $2n-6$ they for any n form a group isomorphic to $\mathbb{Z}$* (*for $n = 3$ it is generated by the simplest knot invariant*). *The group in dimension $2n-3$ is isomorphic to $\mathbb{Z}$ for $n > 3$ and cyclic for $n = 3$; its generator is Alexander dual to the cycle in the discriminant, whose* principal part (*see Definition 1 in §2 below*) *in the double selfintersection of Σ is swept out by such maps $f : S^1 \to \mathbb{R}^n$ that for some $\alpha \in S^1$ we have $f(\alpha) = f(\alpha + \pi)$, $f(\alpha + \pi/2) = f(\alpha + 3\pi/2)$.*

Below we prove in particular that for $n = 3$ the last group also is free cyclic, see Corollary 3. Now we give explicit combinatorial formulas for all classes mentioned in Propositions 3 and 4.

Theorem 2. *For any $n \geq 3$, the values of any of these four basic cohomology classes on any generic cycle of corresponding dimension in the space $\mathcal{K}_n \setminus \Sigma$ of compact knots $S^1 \hookrightarrow \mathbb{R}^n$ is equal to the number of points of this cycle, corresponding to knots satisfying the following conditions* (*and in the case of integer coefficients taken with appropriate signs*).

A. For the $(n-1)$*-dimensional class of order* 1*: projections of* $f(0)$ *and* $f(\pi)$ *into the plane* $\mathbb{R}^{n-1}$ *coincide, and* $f(0)$ *is above* $f(\pi)$*.*

B. For the $(n-2)$*-dimensional class of order 1, one of the following two conditions:*

a) there is a point $\alpha \in [0, \pi)$ *such that the projections of* $f(\alpha)$ *and* $f(\alpha+\pi)$ *to* $\mathbb{R}^{n-1}$ *coincide, and moreover* $f(\alpha)$ *is above* $f(\alpha+\pi)$*;*

b) the projection of the point $f(0)$ *to* $\mathbb{R}^{n-1}$ *lies "to the right" from the projection of* $f(\pi)$*.*

C. For the $(2n-3)$*-dimensional class of order* 2*, one of following two conditions:*

a) there is a point $\alpha \in [0, \pi/2)$ *such that projections of* $f(\alpha)$ *and* $f(\alpha+\pi)$ *to* $\mathbb{R}^{n-1}$ *coincide, projections of* $f(\alpha+\pi/2)$ *and* $f(\alpha+3\pi/2)$ *to* $\mathbb{R}^{n-1}$ *coincide, and additionally* $f(\alpha+\pi)$ *is above* $f(\alpha)$ *and* $f(\alpha+\pi/2)$ *is above* $f(\alpha+3\pi/2)$*;*

b) projections of $f(0)$ *and* $f(\pi)$ *to* $\mathbb{R}^{n-1}$ *coincide,* $f(\pi)$ *is above* $f(0)$*, and the projection of* $f(\pi/2)$ *to* $\mathbb{R}^{n-1}$ *lies "to the right" from the projection of* $f(3\pi/2)$*.*

D. For the $(2n-6)$*-dimensional class of order* 2*, one of two conditions:*

a) there are four distinct points $\alpha, \beta, \gamma, \delta \in S^1$ *(whose cyclic coordinates satisfy* $0 \leq \alpha < \beta < \gamma < \delta < 2\pi$*) such that projections of* $f(\alpha)$ *and* $f(\gamma)$ *to* $\mathbb{R}^{n-1}$ *coincide, projections of* $f(\beta)$ *and* $f(\delta)$ *to* $\mathbb{R}^{n-1}$ *coincide, and additionally* $f(\gamma)$ *is above* $f(\alpha)$ *and* $f(\beta)$ *is above* $f(\delta)$*.*

b) If $n = 3$ *then second condition is void (and we have only the first one coinciding with the combinatorial formula from [10]), but for* $n > 3$ *we have additional condition: there are three distinct points* β, γ, δ *(whose cyclic coordinates satisfy* $0 < \beta < \gamma < \delta < 2\pi$*) such that projections of* $f(\gamma)$ *and* $f(0)$ *to* $\mathbb{R}^{n-1}$ *coincide,* $f(\gamma)$ *is above* $f(0)$*, and the projection of* $f(\delta)$ *to* $\mathbb{R}^{n-1}$ *lies "to the right" of the projection of* $f(\beta)$*.*

Proofs see in § 4.

Corollary 2. *For any* $n \geq 3$*, the basic class of order* 2 *and dimension* $2n-3$ *takes value* ± 1 *on the submanifold of the space of knots, consisting of all naturally parametrized great circles of the unit sphere in* $\mathbb{R}^n$*.*

Indeed, the variety a) of statement C does not intersect this submanifold, and variety b) has with it exactly one intersection point. □

In the case of even n the fact that this variety in the space of knots is not homologous to zero was proved in [5] by very different methods.

Corollary 3. *The group of* $(2n-3)$*-dimensional cohomology classes of order 2 is free cyclic for* $n = 3$ *as well.* □

I am indebted to A. B. Merkov very much for many interesting conversations. I appreciate the hospitality of the Isaac Newton Institute, Cambridge, where this work was finished.

2. Methodology and nature of combinatorial expressions.

In fact, our main purpose is to develop a general method of finding combinatorial formulas of this type.

Any such formula is just a relative cycle in the space of knots (modulo the discriminant Σ) whose boundary in Σ is Alexander dual to our cohomology class. The problem is to construct such a variety explicitly and as simply as possible.

Our method of doing it consists in the conscientious calculation of our spectral sequence. In this subsection we outline the definition of this sequence and this calculation. This spectral sequence for spaces of knots is very analogous to that calculating the cohomology of complements of plane arrangements (see [15]); let us demonstrate their main common features on the latter more simple example.

2.1. **Simplicial resolutions for plane arrangements.** Let $L \subset \mathbb{R}^N$ be an *affine plane arrangement,* i.e. the union of finitely many affine planes L_i of any dimensions, $L = \bigcup_{i=1}^k L_i$. The cohomology group of its complement is Alexander dual to the homology group of L: $H^j(\mathbb{R}^N \setminus L) \simeq \bar{H}_{N-j-1}(L)$; here $\bar{H}_*$ denotes the *Borel–Moore homology group*, i.e. the homology group of the one-point compactification reduced modulo the added point. To calculate the latter group it is convenient to use the *simplicial resolution* of L (which is a continuous version of the combinatorial formula of inclusions and exclusions).

For some three line arrangements in $\mathbb{R}^2$ (shown in the lower part of Fig. 1) the corresponding simplicial resolutions are given above them in the same picture. These resolutions are constructed as follows.

First, we embed the set of indices $\{1, \ldots, k\}$ into a space $\mathbb{R}^T$ of dimension $T \geq k-1$ in such a way that their convex hull is a $(k-1)$-dimensional simplex. The resolution will be constructed as a subset in $\mathbb{R}^T \times \mathbb{R}^N$. For any point $x \in L$ denote by $\tilde{\Delta}(x)$ the convex hull in $\mathbb{R}^T$ of images of such indices i that $L_i \ni x$, i.e. the simplex with vertices at images of all these indices. Denote by $\Delta(x)$ the simplex $\tilde{\Delta}(x) \times \{x\} \subset \mathbb{R}^T \times \mathbb{R}^N$. Denote by L' the union of all simplices $\Delta(x)$, $x \in L$. The obvious projection $L' \to L$ (sending any $\Delta(x)$ to x) is a homotopy equivalence, as well as its extension to the map of one-point compactifications $\bar{L}' \to \bar{L}$. In particular $\bar{H}_*(L') \equiv \bar{H}_*(L)$.

On the other hand, L' has a very useful filtration. For any set of indices $I \subset \{1, \ldots, k\}$, denote by L_I the plane $\cap_{i \in I} L_i$. Let $L'_I \subset L'$ be the *proper preimage* of L_I, i.e. the closure of the union of complete preimages of all *generic* points of L_I (i.e. of points not from even smaller strata $L_J \subset L_I$, $L_J \neq L_I$). There is obvious homeomorphism $L'_I \simeq \tilde{\Delta}(I) \times L_I$, where $\tilde{\Delta}(I) \subset \mathbb{R}^T$ is the simplex whose vertices correspond to all indices i such that $L_i \supset L_I$. (It is equal to $\tilde{\Delta}(x)$ where x is any generic point of L_I.)

By definition, $L' = \bigcup L'_I$, where the union is taken over all *geometrically different* planes L_I. We define the term F_p of the desired filtration of L' as the similar union

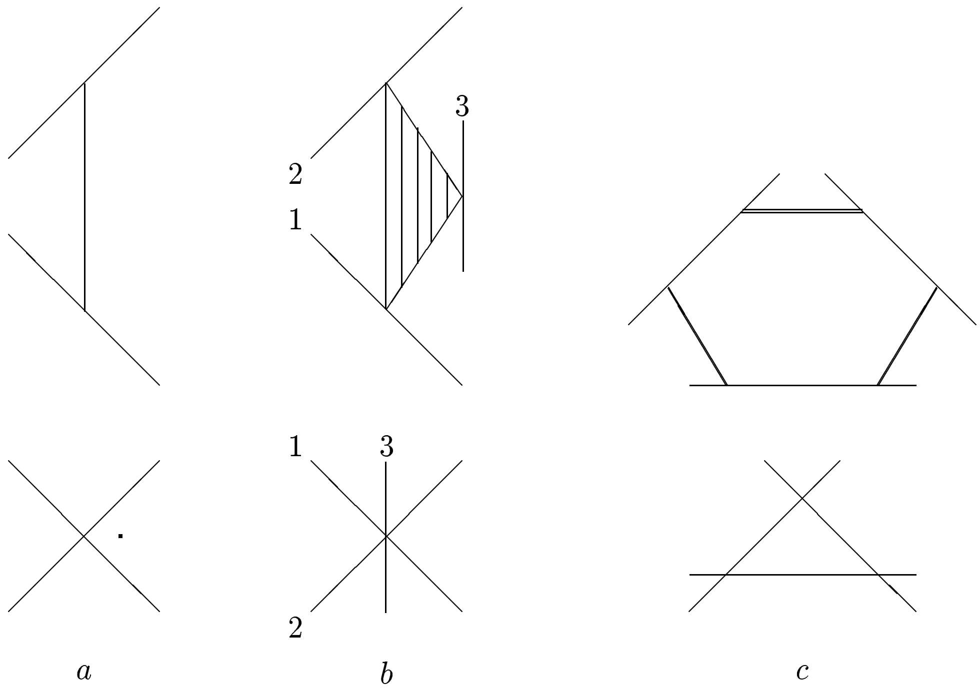


FIGURE 1. Examples of line arrangements

of prisms L'_I over all planes L_I *of codimension* $\leq p$. Then we extend it to a filtration on the one-point compactification $\overline{L'}$ of L' setting $F_0 = \{\text{the added point}\}$.

This filtration defines a spectral sequence calculating the group $\bar{H}_*(L') \simeq \bar{H}_*(L)$: by definition its term $E^1_{p,q}$ is equal to $\bar{H}_{p+q}(F_p \setminus F_{p-1}) \equiv H_{p+q}(\overline{F_p}/\overline{F_{p-1}})$. This space $F_p \setminus F_{p-1}$ splits into a disjoint union (over all spaces L_I of codimension exactly p) of spaces $\check{L}'_I \stackrel{def}{=} \check{\Delta}(I) \times L'_I$, where $\check{\Delta}(I)$ is the simplex $\tilde{\Delta}(I)$ from which several faces are removed: namely such faces $\tilde{\Delta}(J)$, $J \subset I$, that the plane L_J is strictly greater than L_I. For instance, for the configurations shown in pictures a), b), c) of Fig. 1 the planes L_I of codimension 2 are: the point $(1,2)$, the point $(1,2,3)$, and three points $(1,2)$, $(1,3)$, $(2,3)$ respectively. The proper preimages of them are: one segment, one triangle (shadowed vertically in the picture), and three segments. In all these cases the corresponding spaces $\check{L}_I$ coincide with $\check{\Delta}_I$, namely they are: an open interval, a triangle without vertices, and three open intervals, respectively.

In general, any face of the simplex $\tilde{\Delta}(I)$ is characterized by its vertices, i.e. some indices $i \in \{1, \dots, k\}$. A face of $\tilde{\Delta}(I)$ is called *marginal* if the intersection of planes L_i labeled by its vertices is strictly greater than L_I. $\check{\Delta}(I)$ is equal to $\tilde{\Delta}(I)$ with all marginal faces removed. By the Künneth formula, $E^1_{p,q} = \bigoplus \bar{H}_{p+q-(N-p)}(\check{\Delta}(I))$, summation over all planes L_I of codimension p.

The geometrical sense of the corresponding filtration in the Alexander dual group $H^*(\mathbb{R}^N \setminus L)$ is as follows: any element of this group has filtration p if and only if it is equal to a linear combination of finitely many elements γ_j, any of which can be represented by the intersection index with some semilinear[1] subvariety $V_j \subset \mathbb{R}^N$, $\partial V_j \subset L$, invariant under the group $\mathbb{R}^{N-p_j}$ of translations in all directions parallel to some $(N-p_j)$-dimensional plane L_{I_j} with $p_j \leq p$.

Proposition 5 (see [15])**.** *Our filtration of the space $\overline{L'}$ always homotopically splits, i.e. we have the homotopy equivalence*

$$\bar{L}' \sim \bar{F}_1 \vee (\bar{F}_2/\bar{F}_1) \vee \ldots \vee (\bar{F}_N/\bar{F}_{N-1}). \tag{2}$$

In particular, the spectral sequence degenerates in the first term: $E^\infty \equiv E^1$, and we have $\bar{H}_{p+q}(\bar{L}') = \oplus_{p=1}^N E^1_{p,q}$. □

An equivalent statement was proved in [20].

This theorem reduces the structure of cohomology groups of $\mathbb{R}^N \setminus L$ to dimensions of all spaces L_I. However, it does not allow us to calculate the value of an arbitrary element of the group $E^1_{p,q}$ on any cycle in $\mathbb{R}^N \setminus L$. For instance, in the case of the arrangement shown in Fig. 1a, the entire group $E^1_{2,*}$ appears from the unique crossing point $L_{\{1,2\}}$. This group is nontrivial only for $* = -1$, is isomorphic to $\mathbb{Z}$ and is generated by the homology class of the segment $\Delta(1,2)$ modulo its endpoints (lying in F_1). The splitting formula (2) means that we *can* extend this relative cycle of $\bar{F}_2$ (mod $\bar{F}_1$) (or, equivalently, a closed locally finite cycle in $F_2 \setminus F_1$) to a cycle in entire $\bar{L}'$ (respectively, in entire L'). However, to be able to define the value of this point or of this segment on any 0-dimensional cycle (i.e. on a point) in $\mathbb{R}^2 \setminus L$ we need to *choose* such an extension explicitly. Then we project it to L and get a cycle there. Finally, we need to choose a relative cycle in $\mathbb{R}^2$ (mod L) whose boundary coincides with this cycle. Then we call this relative cycle "a combinatorial formula": its value on a point in $\mathbb{R}^2 \setminus L$ is equal to the multiplicity of this cycle in the neighborhood of this point.

If we have a more complicated plane arrangement, then the most convenient way to extend an element of $E^1_{p,q}$ to a closed cycle in L' is to do it step by step over our filtration. Our starting element $\gamma \in E^1_{p,q}$ is represented by a cycle with closed supports in $F_p \setminus F_{p-1}$. We take its *first boundary* $d_1(\gamma)$, which is a cycle in F_{p-1} (mod F_{p-2}). Then we *span* it, i.e. construct a closed chain $\tilde{\gamma}_1$ in $F_{p-1} \setminus F_{p-2}$ such that $\partial\tilde{\gamma}_1 = d_1\gamma$ there. Then we take the boundary of $\gamma + \tilde{\gamma}_1$ in the space $F_{p-2} \setminus F_{p-3}$ and span it there by a chain $\tilde{\gamma}_2$, etc. The degeneration formula (2) ensures that all this sequence of choices can be accomplished. See [20], [9] for some precise algorithms of doing it in the case of plane arrangements.

[1]= semialgebraic distinguished by only linear equations and inequalities

Space $\mathbb{R}^N$	Space $\mathcal{K}_n$ of smooth maps $\mathbb{R}^1 \to \mathbb{R}^n$ with a fixed behavior at ∞
Union of planes $L = \cup L_i \subset \mathbb{R}^N$	Discriminant subset $\Sigma \subset \mathcal{K}_n$
Set of indices $\{1, \dots, k\}$	Chord space $\overline{B(\mathbb{R}^1, 2)}$
A plane L_i	A subspace $L(a, b)$, $a, b \in \mathbb{R}^1$
Disjoint union of hyperplanes L_i	Tautological resolution $F_1\sigma$ of Σ
Simplicial resolution L' of L	Simplicial resolution σ of Σ
Subsets $I \subset \{1, \dots, k\}$ with $\text{codim} L_I = p$	Combinatorial types of chord configurations J with $\text{codim} L(J) = pn$
A prism L'_I	A J-block in σ
Künneth isomorphism for homology of $\check{L}'_I = \check{\Delta}(I) \times L_I$	Thom isomorphism for the fibration of pure J-blocks by spaces $L(J')$
Homotopy splitting (2)	Kontsevich's degeneration theorem

Table 1

2.2. **All the same for knots.** The case of knots (say, of long knots) is very similar to that of plane arrangements. A list of parallel notions is given in Table 1 (whose left part was explained in the previous subsection, and the right-hand part will be explained in the present one).

So, instead of $\mathbb{R}^N$ we consider the affine space $\mathcal{K}_n$ of all smooth maps $\mathbb{R}^1 \to \mathbb{R}^n$ coinciding with a fixed linear embedding "at infinity", and instead of L the discriminant space $\Sigma \subset \mathcal{K}_n$ of all such maps which are not smooth embeddings.

Of course, the space $\mathcal{K}_n$ is infinitedimensional, and formally we cannot use the Alexander duality in it: the (finitedimensional) cohomology classes of the space of knots $\mathcal{K}_n \setminus \Sigma$ should correspond to "infinitedimensional cycles" in Σ, whose definition requires some effort. The strict definition of such cycles corresponding to finite-type cohomology classes was proposed in [14] and is as follows. We consider a sequence of finitedimensional approximating subspaces $\mathcal{K}_n^j$ in $\mathcal{K}_n$, calculate (some) cohomology classes of $\mathcal{K}_n^j \setminus \Sigma$ dual to certain cycles in Σ, and then prove a stabilization theorem when $j \to \infty$. It follows from the Weierstrass approximation theorem that these stable cocycles are well-defined cohomology classes in $\mathcal{K}_n \setminus \Sigma$. A rigorous reader can either read [14] or [16] for all justifications or to think of the spaces $\mathcal{K}_n$ as of such approximating spaces of very high but finite dimension. Let us denote this virtual dimension of $\mathcal{K}_n$ by ω.

Again, Σ is the union of a family of subspaces of very simple nature. For any pair of points (a, b) in $\mathbb{R}^1$, denote by $L(a, b)$ the space of all maps $f \in \mathcal{K}_n$ such that

$$f(a) = f(b) \text{ (if } a \neq b \text{) or } f'(a) = 0 \text{ (if } a = b). \tag{3}$$

Such spaces form a 2-parametric family parametrized by all points (a, b) of the space $\overline{B(\mathbb{R}^1, 2)}$ of all unordered collections of two points in $\mathbb{R}^1$. Since [14] such pairs are depicted by arcs connecting the points a, b (called *chords* in [2]), so the space $\overline{B(\mathbb{R}^1, 2)}$ will be called here the *chord space.* Its degenerated points (corresponding to pairs $a = b$) are depicted by an asterisk at the point a.

The *tautological resolution* $F_1\sigma$ of Σ is constructed as a subspace of the direct product $\overline{B(\mathbb{R}^1, 2)} \times \mathcal{K}_n$: this is the space of pairs $((a, b), f)$ satisfying (3). It is the space of an $(\omega - n)$-dimensional vector bundle over $\overline{B(\mathbb{R}^1, 2)}$. Therefore by the Thom isomorphism we have $\bar{H}_*(F_1\sigma) \simeq \bar{H}_{*-(\omega-n)}(\overline{B(\mathbb{R}^1, 2)}) \equiv 0$: indeed, $\overline{B(\mathbb{R}^1, 2)}$ is homeomorphic to the closed halfplane. There is obvious projection $F_1\sigma \to \Sigma$; it is a map onto, and close to generic points of Σ is a homeomorphism.

Further, we insert simplices spanning preimages of nongeneric points of Σ. As previously, we embed the space $\overline{B(\mathbb{R}^1, 2)}$ generically and algebraically into a space $\mathbb{R}^T$ of a huge dimension ($T \approx \omega^3$). Then for any point $f \in \Sigma$ we mark all the points $(a, b) \in \overline{B(\mathbb{R}^1, 2)}$ such that $L(a, b) \ni f$, and denote by $\tilde{\Delta}(f)$ the convex hull of images of all these points in $\mathbb{R}^T$.

Of course, there exist points $f \in \Sigma$ having infinitely many preimages. However they form a subset of infinite codimension in $\mathcal{K}_n$, and we can ignore them by considering only finitedimensional approximations $\mathcal{K}_n^j$ in general position with the stratification of Σ. Then all the sets $\tilde{\Delta}(f)$, $f \in \mathcal{K}_n^j$, still will be the simplices with vertices at the images of all corresponding points (a, b) of the chord space. The simplicial resolution $\sigma \subset \mathbb{R}^T \times \mathcal{K}_n$ is defined as the union of all simplices $\Delta(f) \equiv \tilde{\Delta}(f) \times \{f\}$.

Again, σ has a useful increasing filtration. Let $I \subset \overline{B(\mathbb{R}^1, 2)}$ be a finite set of chords (a, b). The intersection of corresponding planes $L(a, b)$ is a subspace $L(I) \subset \mathcal{K}_n$, whose codimension is a multiple of n. The proportionality coefficient $\text{codim} L(I)/n$ is called the *complexity* of I. Consider all the points $(a, b) \in \overline{B(\mathbb{R}^1, 2)}$ such that $L(a, b) \supset L(I)$, and denote by $\tilde{\Delta}(I) \subset \mathbb{R}^T$ the convex hull of images of all these points. (It is equal to the space $\tilde{\Delta}(f)$ where f is a *generic* point of the space $L(I)$.) Set $L'(I) = \tilde{\Delta}(I) \times L(I) \subset \mathbb{R}^T \times \mathcal{K}_n$. Finally, define the term $F_p(\sigma)$ of the filtration as the union of all simplices $\Delta(I)$ over all I of complexity $\leq p$.

Definition 1. A cohomology class of the space of knots $\mathcal{K}_n \setminus \Sigma$ is a *finite type class of order* p if it can be defined as the linking number with the direct image in Σ of a cycle (with closed support) lying in the term F_p of the standard filtration of σ. For any such class of order p and dimension d, its *principal part* is the class of the corresponding cycle in the group $\bar{H}_{\omega-d-1}(F_p \setminus F_{p-1})$.

The important property of this filtration is as follows: any its term $F_p \setminus F_{p-1}$ is the space of an $(\omega - pn)$-dimensional affine bundle over some finitedimensional semialgebraic base: the projection of this bundle is induced by the obvious projection $\mathbb{R}^T \times \mathcal{K}_n \to \mathbb{R}^T$. In particular, the Thom isomorphism reduces the Borel–Moore homology group of this term to the homology group of locally finite chains of this base (in the case of odd n with coefficients in some system of groups locally isomorphic to $\mathbb{Z}$, which is constant only for $p = 1$).

These finitedimensional bases, and hence also entire spaces $F_p \setminus F_{p-1}$ of our filtration, admit an easy description, in particular their one-point compactifications have a natural structure of CW-complexes. First let us describe all the spaces $L(I)$ of complexity exactly p.

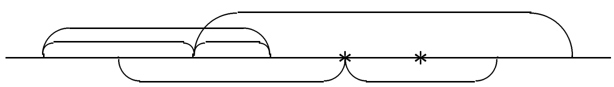
Definition 2 (see [14])**.** Let A is a unordered finite collection of naturals $A = (a_1, \dots, a_{\#A})$, $a_j \geq 2$, and b any nonnegative integer. Then an (A, b)-*configuration* in $\mathbb{R}^1$ is any collection of distinct $a_1 + \cdots + a_{\#A}$ points in $\mathbb{R}^1$ separated into groups of cardinalities $a_1, \dots, a_{\#A}$, plus a collection of b distinct points in $\mathbb{R}^1$ (some of which can coincide with the points of the A-part). A map $f : \mathbb{R}^1 \to \mathbb{R}^n$ *respects* an (A, b)-configuration J if it maps all points of any of groups of cardinality a_j, $j = 1, \dots, \#A$, into one point (these points for different groups may coincide), and $f' = 0$ at all points of the b-part of the configuration. The space of all maps f respecting a fixed (A, b)-configuration J is denoted by $L(J)$. Two (A, b)-configurations are *equivalent* if they can be transformed one into the other by an orientation-preserving homeomorphism of $\mathbb{R}^1$. The *complexity* of an (A, b)-configuration is the number $\sum_{j=1}^{\#A}(a_j - 1) + b$.

Obviously the codimension in $\mathcal{K}_n$ of any space $L(J)$ is equal to n times the complexity of J. The space of all (A, b)-configurations of complexity 1 is the chord space $\overline{B(\mathbb{R}^1, 2)}$. Any intersection of finitely many planes $L(a, b)$, $(a, b) \in \overline{B(\mathbb{R}^1, 2)}$, is a plane of form $L(J)$ for some (A, b)-configuration J. The corresponding simplex $\tilde{\Delta}(J)$ in $\mathbb{R}^T$ has exactly $\sum_{i=1}^{\#A} \binom{a_i}{2} + b$ vertices. The J-*block* $\Box(J)$ in σ is the union of all pairs $(x, f) \subset \mathbb{R}^T \times \mathcal{K}_n$, such that x belongs to the simplex $\tilde{\Delta}(J')$ for some (A, b)-configuration J' equivalent to J, and f respects this configuration J'. It belongs to the term F_p of our filtration, where p is the complexity of J.

The *pure* J-*block* $\check{\Box}(J)$ is equal to $\Box(J) \setminus F_{p-1}$. It is fibered over the space of (A, b)-configurations J' equivalent to J, with fiber equal to $\check{\Delta}(J) \times L(J)$, where $\check{\Delta}(J)$ is the union of several (nonmarginal in some sense) faces of $\tilde{\Delta}(J)$. The base of this fiber bundle is an open cell, thus the bundle is trivial, and we have a canonical decomposition of $\check{\Box}(J)$ into open cells corresponding to all such nonmarginal faces.

The canonical notation of any such cell is a *generalized chord diagram*, i.e. a finite collection of arcs connecting some points of $\mathbb{R}^1$ and of asterisks marking some points, say as in the picture

(4)

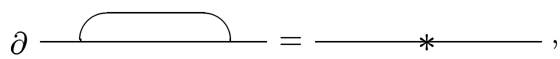


presenting one of cells of a certain equivalence class of $((4, 3), 2)$-configurations. Namely, for any such cell related with a class of equivalent (A, b)-configurations, we fix some configuration $J \subset \mathbb{R}^1$ of this class, mark by asterisks all points of its

b-part ("singular points") and draw a chord between any its two points a, b such that the point $(a, b) \in \overline{B(\mathbb{R}^1, 2)}$ is a vertex of the face of $\tilde{\Delta}(J)$ corresponding to this cell.

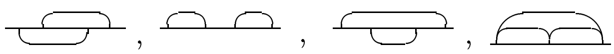
The space $F_p \setminus F_{p-1}$ is the union of such pure blocks $\check{\Box}(J)$ over (finitely many) equivalence classes of all (A, b)-configurations of complexity exactly p. So we get also the decomposition of this space into finitely many open cells. This decomposition can be extended to the structure of a CW-complex on the one-point compactification of $F_p \setminus F_{p-1}$. Its structure and incidence coefficients are explicitly described in [14], [16], which gives also an algorithm for calculating the term E^1 of the spectral sequence generated by this filtration and converging to the group of all finite-type cohomology classes of the space of knots. In particular, if $n = 3$ then all J-blocks of complexity p which (by dimensional reasons) can be valuable for the calculation of knot invariants, are only the blocks with (A, b) equal to $((2, \ldots, 2), 0)$ (chord diagrams), $((2, \ldots, 2), 1)$ (*one-term relations*, see [3], [2]), and $((3, 2, \ldots, 2), 0)$ (any such block corresponds to the totality of *4-term relations* arising from the neighborhood of a triple point: there are three such relations, any two of which are independent).

Example 1. The term F_1 consists of exactly two cells, one of which is the boundary of the other:

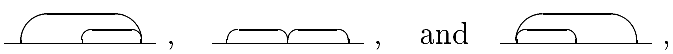
(5) 

thus there are no cohomology classes of order 1 of the space of long knots. (Here and in the next example we assume some natural orientations of such cells, see [14], [16].)

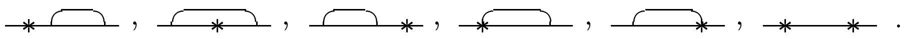
Example 2. The term $F_2 \setminus F_1$ consists of the following cells: four cells of maximal dimension

(6) 

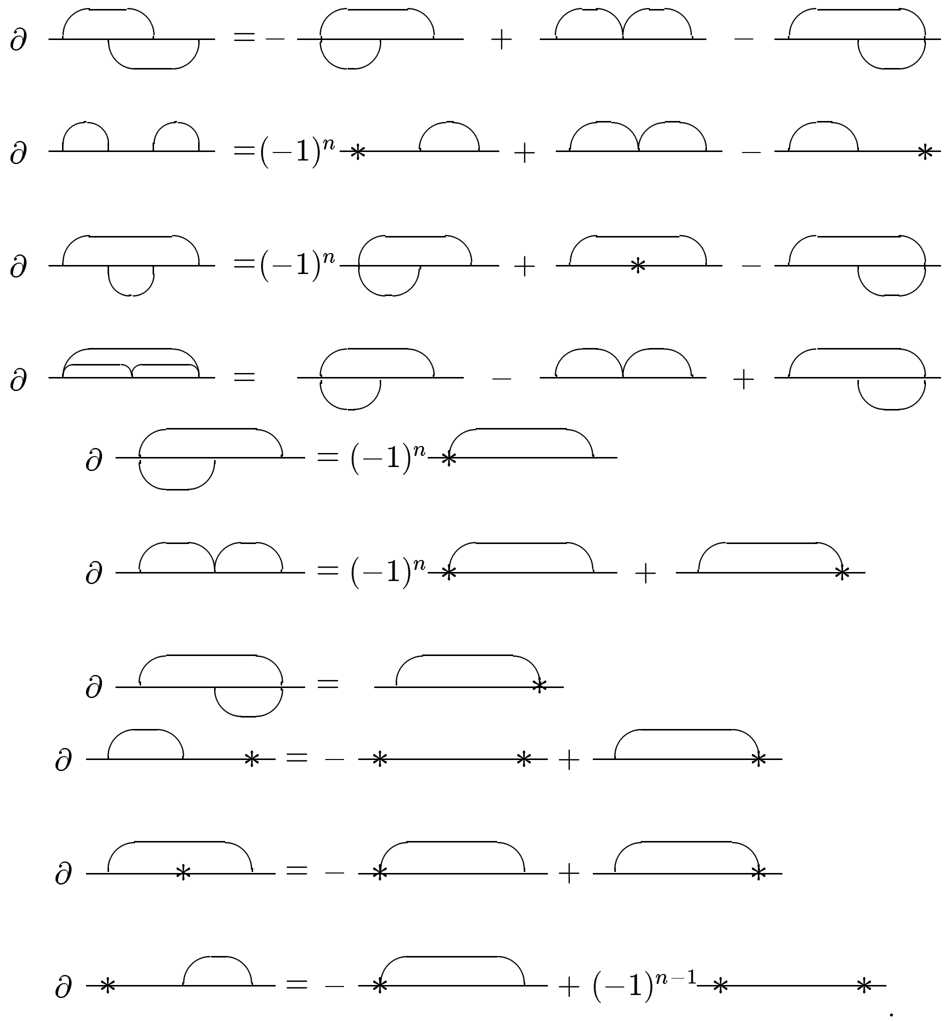
(only the first and the last of which will be interesting for us), three cells forming the boundary of any of these two cells,

(7) 

and also six cells defined by maps with singular points (i.e. in whose notation the asterisks $*$ participate):

(8) 

The boundary operator in this term $F_2 \setminus F_1$ (i.e. the vertical differential d^0 of the spectral sequence) acts as follows:



In particular there is exactly one nontrivial group $\bar{H}_i(F_2 \setminus F_1) \equiv E^1_{2,i-2}$, namely such group with $i = \omega - (2n-5)$ is isomorphic to $\mathbb{Z}$ and is generated by the sum of the first and the last cells in (6).

Thus we obtain a proof of the first statement of Proposition 1. The group $\bar{H}_*(F_3 \setminus F_2)$ of (possible) principal parts of third order cohomology classes was calculated in [11] for odd n (another proof, not relying on the computer's honesty, see in [18]) and in [18] for even n. In both cases, there are exactly two nontrivial groups $\bar{H}_i(F_3 \setminus F_2) \simeq E^1_{3,i-3}$, namely with $i = \omega - (3n-8)$ and $\omega - (3n-7)$; they both are isomorphic to $\mathbb{Z}$. By the dimensional reasons both these groups for any $n \geq 3$ coincide with corresponding groups $E^\infty_{3,i-3}$, and their generators extend to well-defined cohomology classes of spaces of long knots in $\mathbb{R}^n$. By similar considerations (see e.g. [18]) for

$n \geq 4$ these generators are nontrivial and free. If $n = 3$ then for the first of these classes the same follows from the fact that it coincides with the second simple knot invariant (calculated in [14]) whose nontriviality is well known. The other class is exactly the Teiblum–Turchin class studied below; the fact that it also is nontrivial and free for $n = 3$ will follow from the proof of Corollary 1.

Remark 1. It is often convenient to replace formally our homological spectral sequence calculating $\bar{H}_*(\sigma)$ by the "Alexander dual" cohomological spectral sequence

$$E_r^{p,q} \equiv E^r_{-p,\omega-q-1}.$$

It lies in the second quadrant in the wedge $\{(p,q)|p \leq 0, q + pn \geq 0\}$ and converges to some subgroup of the group $H^*(\mathcal{K}_n \setminus \Sigma)$ (if $n > 3$ then to entire this group).

Remark 2. There are beautiful algebraic structures on the above-described spectral sequence, and hence on its limit filtered group $H^*(\mathcal{K}_n \setminus \Sigma)$ and the corresponding adjoint graded group, see [12].

All of this theory can be extended almost literally to the cohomology of the space of compact knots $S^1 \hookrightarrow \mathbb{R}^n$.[2] However, in this case the chord space $\overline{B(S^1, 2)}$ is not topologically trivial (it is a closed Möbius band); also the spaces of equivalent (A, b)-configurations are not the cells. To get the cell decomposition of all spaces $F_p \setminus F_{p-1}$ we need to mark one point in S^1 ("the origin") and call two configurations equivalent if they are transformed one into the other by a homeomorphism of S^1 preserving the origin and the orientation.

The direct calculation of the spectral sequence and obtaining the *combinatorial formulas* for the finite-type cohomology classes of spaces of knots is formally the same process as in the case of plane arrangements. However, the exact choice of the spanning chains in all the consecutive terms of the filtration and in entire $\mathcal{K}_n$ depends very much of the features of the knot space.

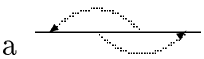
Remark 3. There is another, sometimes more convenient construction of the resolution of discriminant sets, namely the *conical resolutions* based on the notion of a continuous order complex of a topologized partially ordered set, see e.g. [19]. In particular it allows us to resolve the points of Σ with infinitely many preimages in the tautological resolution space. However, for the calculations in the present work it will be enough to use the "naive" simplicial resolution described above.

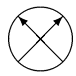
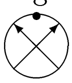
2.3. **Finite type knot invariants and Polyak-Viro combinatorial formulas.** Suppose that $n = 3$ and we are interested in the knot invariants, i.e. the 0-dimensional cohomology classes of $\mathcal{K}_n \setminus \Sigma$. For any such class of finite filtration p, its principal part in $F_p \setminus F_{p-1}$ is a linear combination of cells depicted by p-chord diagrams (i.e. collections of p chords with distinct endpoints) and *$\tilde{p}$-configurations*, i.e. collections

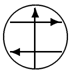
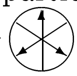
[2] As well as of compact links, i.e. of embeddings of a disjoint union of several circles into $\mathbb{R}^n$; we shall not discuss here the latter theory

of $p-2$ chords with different endpoints and one triple of points joined by three chords. E.g., among all diagrams in (4)–(8) only the left picture in (5) and three left pictures in (6) are chord diagrams, and only the last picture in (6) is a $\tilde{2}$-configuration. The coefficients with which all these cells can enter the linear combination satisfy the homological condition. In particular the coefficients at $\tilde{p}$-configurations are determined by these at p-chord diagrams, and any admissible linear combination is characterized uniquely only by the collection of latter coefficients, which is called a *weight system.*

The elementary characterization of these invariants is as follows (see e.g. §0.2 in [14]). Let us consider any immersion $\mathbb{R}^1 \to \mathbb{R}^3$ with exactly k transverse selfintersection points. We can resolve any of these points in two locally distinct ways to get a knot without intersections. One of these two local resolutions can be invariantly defined as a positive, and the other as the negative one. The k-th index of a knot invariant at our singular immersion is equal to the alternated sum of its values at all 2^k knots obtained by all different possible resolutions of double points: the value at a knot is counted with sign 1 or -1 depending on the parity of the number of positive local resolutions. A knot invariant is of filtration p if and only if all its indices at all immersions with $k > p$ selfintersections are equal to 0. The same definition can be applied to define the filtration of invariants of compact knots $S^1 \to \mathbb{R}^3$. On the other hand, it is easy to see that there is a natural one-to-one correspondence between connected components of spaces of long and compact knots, in particular the theories of their invariants naturally coincide.

Some combinatorial formulas for the simplest finite-type knot invariants — of orders 2 and 3 — were found in [7]. Another, more convenient formulas were introduced by M. Polyak and O. Viro in [10]. These formulas for long knots look as the linear combinations of chord diagrams with oriented chords. E.g. the formula  should be read as follows. Consider a generic long knot $f : \mathbb{R}^1 \to \mathbb{R}^3$. A *representation* of the above picture in this knot is any collection of points $a < b < c < d \subset \mathbb{R}^1$ such that $f(a)$ lies below $f(c)$ and $f(d)$ lies below $f(b)$. The value of this picture on our knot is equal to the number of its representations (counted with appropriate signs). An immediate calculation shows that this number is a knot invariant of order 2.

In the case of compact knots, there are Polyak-Viro formulas of two types: absolute and punctured ones. They also look as oriented chord diagrams, but with endpoints in the oriented circle S^1 instead of $\mathbb{R}^1$; moreover, a punctured Polyak-Viro diagram contains a point in S^1 not coinciding with the endpoints of chords. E.g. a representation of the diagram  in a compact knot is any collection of four points in S^1 with cyclic order $a < b < c < d < a$, satisfying the same conditions as previously. A representation of the punctured diagram  is such a collection of points in the parametrized circle, whose cyclic coordinates satisfy a more strong condition $0 < a < b < c < d < 2\pi$; the origin $0 = 2\pi \in S^1$ corresponds to the marked point in the diagram. It is easy to see that the number of representations of the last punctured diagram (counted with natural signs) is a knot invariant, and the similar number for the absolute diagram is not.

However, some of finite type knot invariants can be realized by absolute diagrams: in particular the unique invariant of order 3 can be given by the diagram $\frac{1}{2}$  $+\frac{1}{3}$ , see [10].

M. Goussarov has proved that any finite-type knot invariant of long knots can be represented by a formula of Polyak–Viro type, see [6].

Similar (but more complicated) formulas appear naturally in the calculation of higherdimensional cohomology classes of spaces of knots, see the next sections.

Remark 4. The homological calculations discussed below provide numerous possibilities to make a mistake: to miss some component of the boundary, to calculate wrongly some orientation, etc. Fortunately, we always can check our calculations. If we have calculated some boundary operator and suspect that it is not correct, just calculate the boundary of this boundary, and try to understand why it is not equal to zero! My experience says that no mistakes survive this examination.

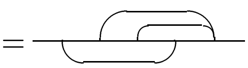
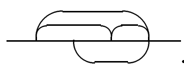
3. Proof of Theorem 1

3.1. **Principal part of the cocycle.** In the original calculation [11], the principal part of the Teiblum-Turchin cocycle in the term $F_3 \setminus F_2$ of the natural filtration of the resolved discriminant was found as a linear combination of some 8 cells of the canonical cell decomposition, see e.g. [18], [16].

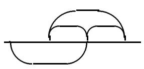
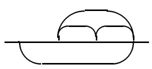
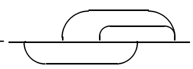
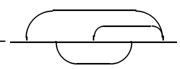
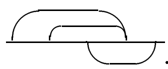
This expression can be simplified, especially if n is even.

Proposition 6. *For any $n \geq 3$, the group of order 3 cohomology classes of dimension $3n-8$ of the space of long knots $\mathbb{R}^1 \to \mathbb{R}^n$ is cyclic; for $n \geq 3$ it is free Abelian.*

If n is even, then this group is generated by the sum of only two cells:

(9) $TT =$  $+$ .

For odd n it is generated by the linear combination

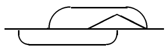
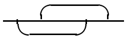
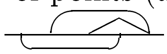
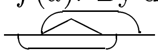
(10)  $+$  $+$  $+$  $-$ .

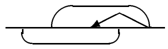
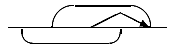
The first statement of this proposition for odd n was essentially proved by Teiblum and Turchin [11]; the justification of entire statement see in §6 of [18] or §V.8.8 of [16].

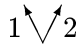
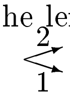
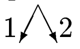
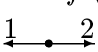
Since we consider our class mod 2, the stabilization formula (1) allows us to use any of expressions (9), (10) in the case of any n. We shall use the shorter "even" version (9).

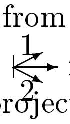
All further calculations in this section are mod 2 only.

3.2. **On the pictures.** The system of notation in this work is an extension of that used in § 2 for the cells of the natural simplicial resolution of the discriminant. Any of our pictures consists of a horizontal segment (the *Wilson loop* symbolyzing the line $\mathbb{R}^1$), several asterisks placed on it, and several arcs ("chords") connecting some its points (these data determine such a cell), plus some additional furniture consisting of broken lines (*zigzags*) and subscripts, which distinguish certain subvarieties in these cells.

For instance, the picture  means, first of all, that we are in the cell  of the term $F_2 \setminus F_1$. This cell can be considered as the space of all triples (α, t, f) where α is some quadruple of points $a < b < c < d$ in $\mathbb{R}^1$, f is a smooth map $\mathbb{R}^1 \to \mathbb{R}^n$ such that $f(a) = f(c)$, $f(b) = f(d)$, and t is a point of the segment $\tilde{\Delta}(J)$, $J = ((a, c); (b, d))$, participating in the construction of the resolution: its endpoints correspond formally to the pairs of points (a, c) and (b, d) glued together by f. The additional zigzag in the picture  distinguishes the subvariety in this cell, consisting of such triples (α, t, f) that there exists one point $\lambda \in \mathbb{R}^1$ more, $b < \lambda < c$, such that $f(\lambda) = f(d)$. By definition, this subvariety is identical with the one encoded by the picture .

The picture  (respectively, ) will denote almost the same, but with the condition $f(\lambda) = f(d)$ replaced by the condition that $f(\lambda)$ has the same projection to $\mathbb{R}^{n-1}$ as $f(d) = f(b)$ and lies *below* (respectively, *above*) $f(d)$ in the line of all points with the same projection.

The subscript of type  under a picture denotes the condition that the "vertical" direction in $\mathbb{R}^n$ lies in the angle between the tangent directions $f'(a_1)$ and $f'(a_2)$, where a_1 and a_2 are the first and the second from the left points of $\mathbb{R}^1$ participating actively in the picture. Similarly, the subscript  (respectively, ) says that the vertical direction lies in the angle between the vectors $-f'(a_1)$ and $f'(a_2)$ (respectively, between the vectors $-f'(a_1)$ and $-f'(a_2)$). The subscript  means that the tangent directions $f'(a_1)$ and $f'(a_2)$ are opposite in $\mathbb{R}^n$. The notation of all these types appears only if the condition $f(a_1) = f(a_2)$ is satisfied (and can be seen from the chords and zigzags on the picture).

The subscript  means that the distinguished direction "to the right" in $\mathbf{R}^{n-1}$ lies between the projections of such tangents $f'(a_1)$, $f'(a_2)$ to $\mathbb{R}^{n-1}$ (i.e. this direction is the linear combination of these projections with nonnegative coefficients). The subscript $1 \leftrightarrow\!\!\!\!\leftrightarrow 2$ means that the projections of tangents $f'(a_1), f'(a_2)$ to $\mathbb{R}^{n-1}$ have opposite directions. The notation of last two types can appear only if the projections of corresponding points $f(a_1), f(a_2)$ coincide in $\mathbb{R}^{n-1}$.

The sum of varieties distinguished by conditions of types $1 \nwarrow\!\!\vee\!\!\nearrow 2$ and $1 \swarrow\!\!\wedge\!\!\searrow 2$ in one and the same cell is equal to the variety of type $1 \leftarrow\!\!\!+\!\!\!\rightarrow 2$; "of type" here means that some other two numbers instead of 1 and 2 can stay in all three pictures. This identity is not symmetric: indeed, the variety of type $1 \leftarrow\!\!\!+\!\!\!\rightarrow 2$ can be well defined even when the former two varieties have no sense.

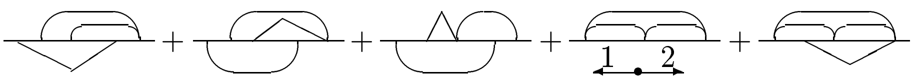
The subscript $2 \updownarrow$ (respectively, $2 \uparrow$, respectively, $2 \downarrow$) means that the tangent vector $f'(a_2)$ is vertical (respectively, vertical directed up, respectively, vertical directed down). The subscript of type $2 \mapsto$ means that the projection of the tangent $f'(a_2)$ to $\mathbb{R}^{n-1}$ is directed "to the right".

Abbreviation f_1 replaces the composition $\mathbf{p} \circ f : \mathbb{R}^1 \to \mathbb{R}^{n-1}$. Finally, a collection of vectors in a framebox means that these vectors are linearly dependent. For instance the subscript $\boxed{f_1'(1), f_1''(2), \mapsto}$ means that some three vectors in $\mathbb{R}^{n-1}$, namely the projection of $f'(a_1)$, the projection of $f''(a_2)$, and the direction "to the right", span a subspace of dimension ≤ 2. Several more specific abbreviations will be explained later, close to their first use.

The boundary of the variety distinguished in any cell by the condition $1 \nwarrow\!\!\vee\!\!\nearrow 2$ (respectively, $1 \swarrow\!\!\wedge\!\!\searrow 2$) is equal to the sum of varieties distinguished by conditions $\underset{\leftarrow\bullet\rightarrow}{1\quad 2}$, $1 \uparrow$ and $2 \uparrow$ (respectively, $\underset{\leftarrow\bullet\rightarrow}{1\quad 2}$, $1 \downarrow$ and $2 \downarrow$) plus maybe something in the boundary of the cell. Similarly, the boundary of the variety distinguished by the condition $\overset{1}{\underset{2}{\models\!\!\rightarrow}}$ is equal (modulo the boundary of the cell) to the sum of varieties distinguished in the same cell by conditions $1 \leftarrow\!\!\!+\!\!\!\rightarrow 2$, $1 \mapsto$, and $2 \mapsto$. The boundary of the condition $2 \mapsto$ is equal to $2 \updownarrow$ plus something in smaller cells.

3.3. The first differential. Formula (9) defines a relative cycle in the term F_3 of our filtration modulo F_2. In this subsection we calculate its boundary in the term $F_2 \setminus F_1$, and span it by some chain with closed supports in this term (i.e. we represent it as the boundary of such a chain).

Proposition 7. *The boundary of the cycle* (*9*) *in* $F_2 \setminus F_1$ *is equal to the chain*



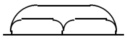
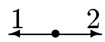
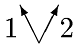
(11) $\qquad A \qquad B \qquad C \qquad D \qquad E$.

(*Namely, the boundary of the first term of* (*9*) *consists of four chains A, B, C and D in* (*11*), *and the boundary of the second is equal to the fifth chain E.*)

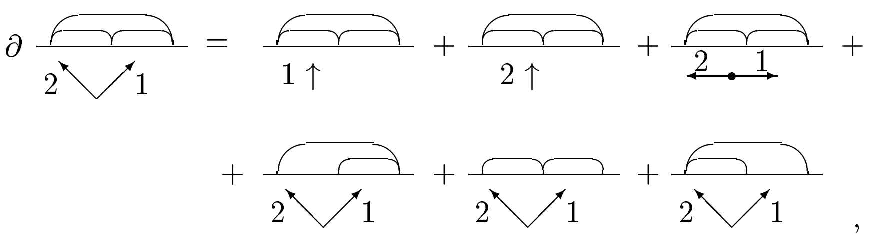
The unique nontrivial term of this formula is the 4th one: it appears when the first point of the 5-configuration participating in the first term of (9) tends to the second, and simultaneously the fourth point tends to the third. $\square$

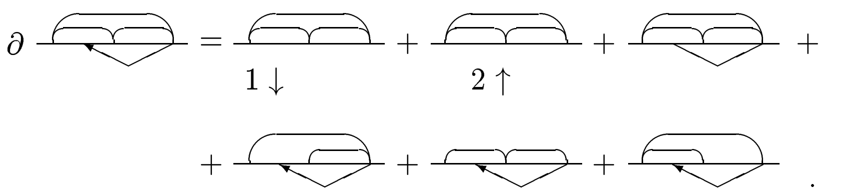
Exercise: to check that the chain (11) actually is a cycle in $F_2 \setminus F_1$.

Now, let us span this cycle by a chain in the term $F_2 \setminus F_1$. The cellular structure of this term was described in Example 2 of section 2.

First we span the components D and E of (11) inside the cell  , i.e. we construct the homology between their sum and some chain in the boundary of this cell. It is natural to span a chain with condition of type  by a similar chain with condition of type , and a chain having zigzag without arrows by a similar chain with an arrow added at one of endpoints of the zigzag. The chains obtained in this way from the ones encoded by parts D and E of (11) are indicated in the left parts of the next two equations (12) and (13) respectively.

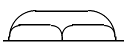
In the right-hand parts of these formulas, as well as in all forthcoming expressions for boundary operators in this work, we first count the components of the boundary defined by the degenerations of the subvarieties in the corresponding cells, distinguished by arrowed zigzags and subscripts. Then we count the components defined by the limit positions of these varieties when the cell itself degenerates because of the collision of some points forming its underlying J-configuration in $\mathbb{R}^1$. The latter degenerations appear in the lexicographic order: first by the number of colliding pairs of points in $\mathbb{R}^1$, and then by their positions in $\mathbb{R}^1$.

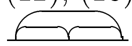
(12) 

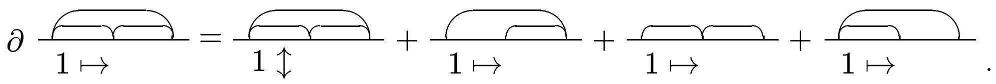
(13) 

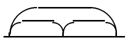
Proposition 8. *The equalities (12), (13) are correct, i.e. the algebraic boundaries (mod 2) in $F_2 \setminus F_1$ of the varieties indicated in their left parts are equal to the sums of varieties indicated in their right-hand parts.* □

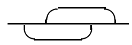
In (13) first two summands are degenerations of the variety defined by the zigzag when its arrowed endpoint tends to one of boundaries of the corresponding segment, and the third summand belongs to its boundary as the equality of type $\phi(x) = \phi(y)$ defines a component of the boundary of the set defined by the inequality $\phi(x) \geq \phi(y)$.

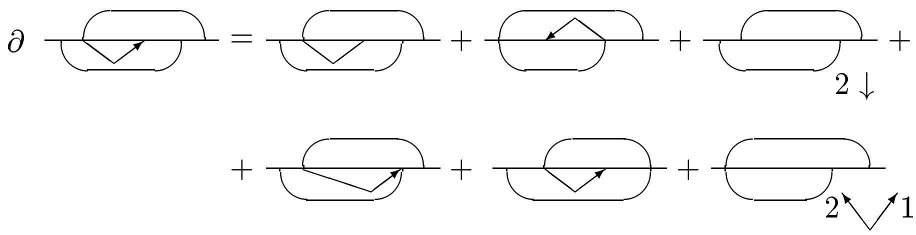
The last three summands in both (12) and (13) belong to the boundary (7) of the cell  .

The sum of all varieties indicated in right-hand parts of (12), (13) consists of part D + E of (11), some chain in the boundary of the cell  , and the first chain in the right-hand part of the equation

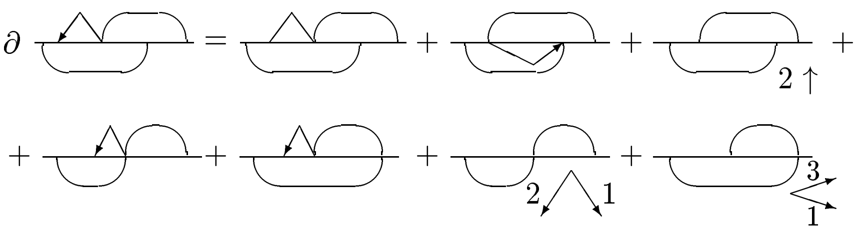
(14) 

In other words, the sum of three chains in the left parts of equations (12), (13) and (14) realizes homology between the sum of chains D and E and some chain in the boundary of the cell  .

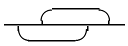
Now we span the summands B and C of (11) inside the open cell  . We need to find varieties in this cell, whose boundaries include these summands. The obvious candidates for this are the chains shown in the left parts of equations (15) and (16) respectively.



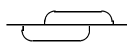
(15)

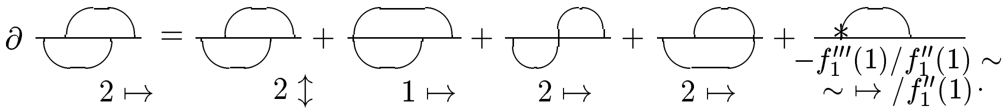


(16)

Again, all summands in lower rows of these equalities belong to the boundary of the cell  .

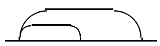
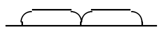
Proposition 9. *The equalities* (*15*), (*16*) *are correct, i.e. the algebraic* (*mod 2*) *boundaries in* $F_2 \setminus F_1$ *of the varieties indicated in their left parts are equal to the sums of varieties indicated in their right-hand parts.* $\square$

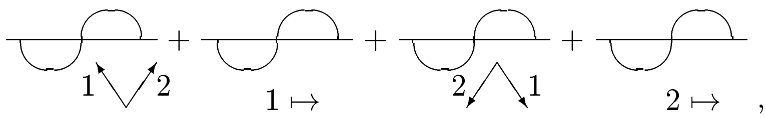
In particular we get that the boundary of the sum of these two left-side varieties is equal to the sum of varieties denoted in (11) by B and C, plus some chain in the boundary of the cell  , plus the variety distinguished in this cell by the additional condition 2 $\updownarrow$. The last variety is a part of the boundary of the similar set distinguished by the condition 2 $\mapsto$. Entire boundary of this set in $F_2 \setminus F_1$ is expressed by the formula

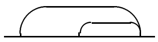
(17) 

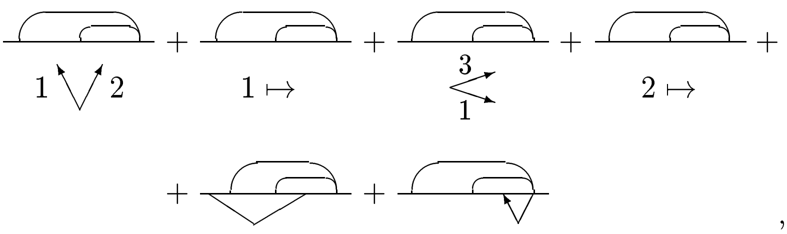
The subscript under the last term of (17) means, that the projections of second and third derivatives of f at the point a_1 into $\mathbb{R}^{n-1}$ lie in the same 2-plane as the direction "to the right", and two frames in this 2-plane obtained by adding to the projection of $f''(a_1)$ either the projection of $f'''(a_1)$ or the direction "to the right" have *opposite* orientations. This term occurs when both endpoints a_1, a_3 of the "lower" arc in the left picture of (17) tend from different sides to the first endpoint a_2 of the "upper" arc.

Finally we get that the cycle $d^1(TT)$ shown in (11) is homologous in $F_2 \setminus F_1$ to a chain lying in the union of cells of nonmaximal dimensions listed in (7), (8); this homology is provided by the sum of six varieties indicated in the left parts of equalities (12), (13), (14), (15), (16), and (17).

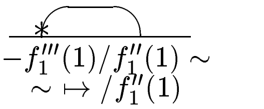
Namely, this cycle homologous to $d^1(TT)$ is as follows. In the cell  it is zero, in the cell  it is equal to the chain

(18) 

in the cell  it is equal to the chain

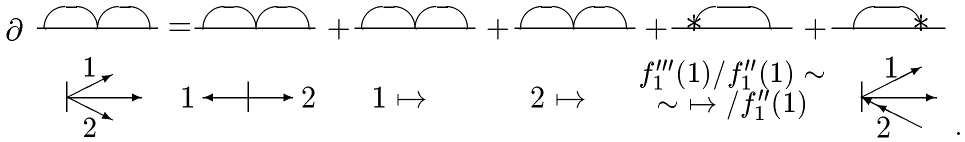
(19) 

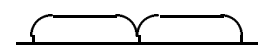
in the 4th cell of (8) it is equal to

(20) 

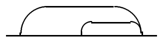
and its intersections with all other cells are empty.

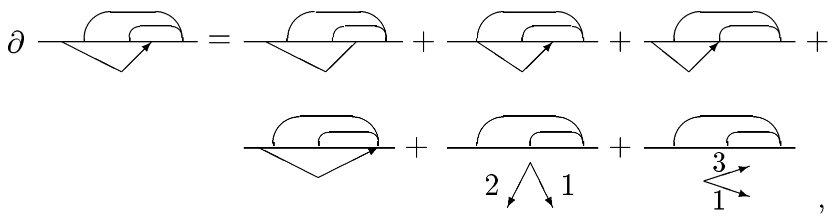
The sum of the first and the third terms in (18) is equal to the variety denoted by the subscript $1 \leftrightarrow 2$. To kill it (and something else) we consider the equality

(21) 

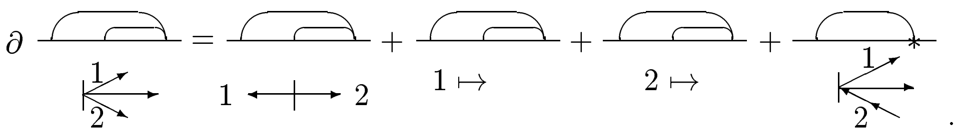
The variety in its left part consists of such points of the cell  that the direction "to the right" in $\mathbb{R}^{n-1}$ lies between the projections of $f'(a_1)$ and $f'(a_2)$ to $\mathbb{R}^{n-1}$. The sum of three first terms in the right-hand part of (21) is equal to entire (18). The subscript under the fourth term in (21) means almost the same as in (17) or (20), but now the two frames compared there should define *equal* orientations.

Finally, the last term in (21) belongs to the 5th cell in (8). This cell can be considered as the space of triples (α, t, f) where α is a pair of points $(a < b)$ in $\mathbb{R}^1$, f a map $\mathbb{R}^1 \to \mathbb{R}^n$ such that $f(a) = f(b), f'(b) = 0$, and t is a point of a segment participating in the construction of the simplicial resolution (its endpoints formally correspond to the above two linear conditions). The subscript under the picture of this cell in (21) denotes a subvariety in the space of such triples, defined by the following additional condition: the direction "to the right" in $\mathbb{R}^{n-1}$ belongs to the angle between projections of vectors $f'(a)$ and $-f''(b)$. Here the number of arrows labeled by 2 shows us the order of the derivative at the second point b participating in this condition, and the reversed direction of these arrows indicates that we need to take this derivative with the opposite sign.

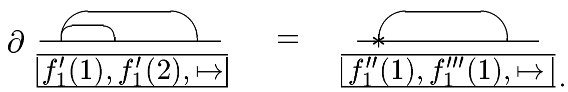
Now we span the chain (19) inside the cell . First of all we kill the 5th picture in (19) by the variety shown in the left part of the next equality:

(22) 

thus reducing it to the sum of other five pictures in the right part of this equality. The last picture in the upper row of (22) and the first picture in the lower row denote one and the same set and annihilate. The first term in (19) together with the second from the end term in (22) form a subvariety in the same cell defined by the condition of the type $1 \leftarrow\!\!\!+\!\!\!\rightarrow 2$. It is natural to kill it by the left part of the following equation:

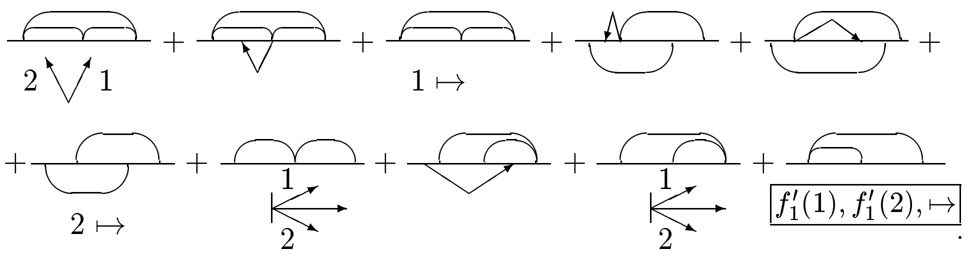
(23) 

Summing up all terms in right-hand parts of equations (21)–(23) and subtracting the chains (18), (19), (20), we annihilate almost all of their summands except for the term (20) and the second from the right term of (21). The sum of these two terms is equal to the right-hand part of the identity

(24) 

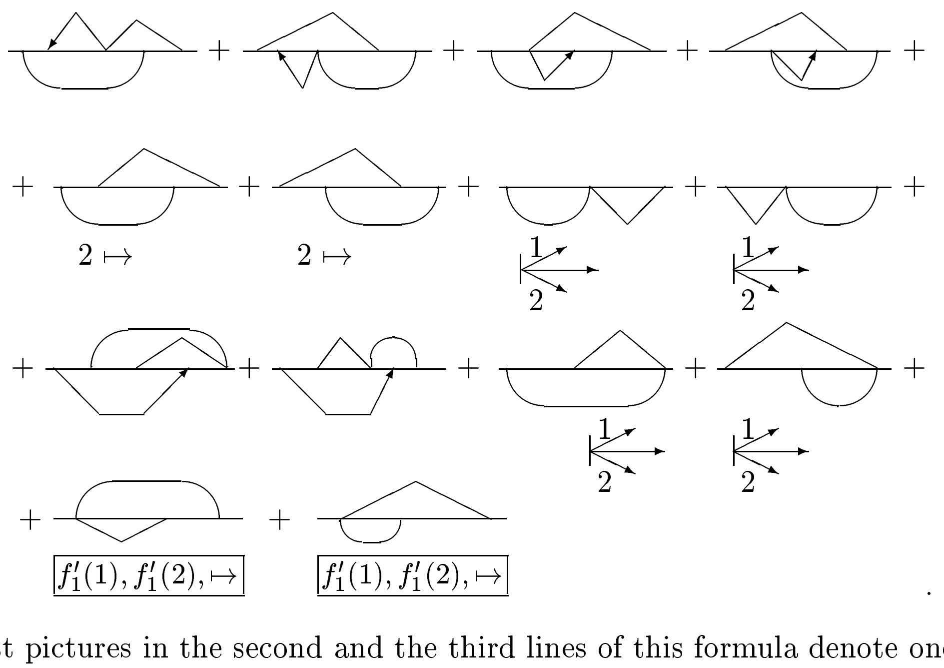
The subscript under this right-hand part means that the projections of $f''(a_1)$ and $f'''(a_1)$ to $\mathbb{R}^{n-1}$ and the direction "to the right" should be linearly dependent; the subscript in the left part says the same about projections of vectors $f'(a_1)$ and $f'(a_2)$. If $n = 3$ then both these subscripts mean nothing.

Summarizing, we get that for the desired chain spanning (11) in $F_2 \setminus F_1$ we can take the sum of varieties shown in left parts of equalities (12)–(17) and (21)–(24), i.e. the chain



(25)

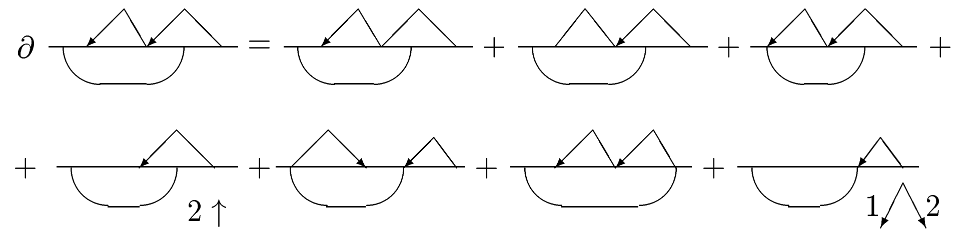
3.4. **The second differential and its homology to zero.** Now let us consider the boundary of the chain (25) in the term F_1 of the filtration. This term consists of two cells, one of which is characterized by a single chord and the second by one asterisk; see (5). It is easy to see that the first three summands in (25) do not have any homological boundary in these cells, and the next seven have two components of the boundary each, and these pairs of components are shown consecutively in the next formula (26):



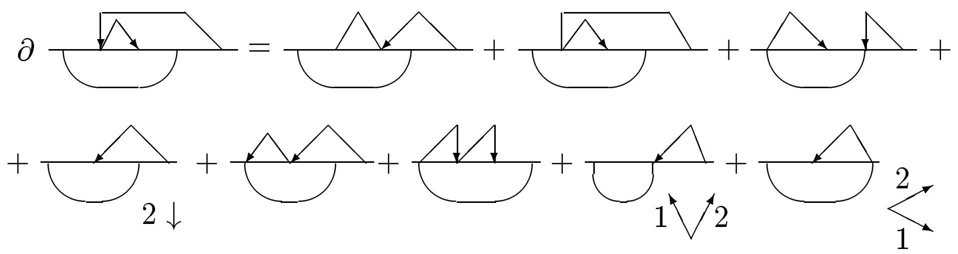
(26)

The last pictures in the second and the third lines of this formula denote one and the same variety and annihilate, so we get only the sum of remaining twelve varieties.

Now let us span this sum by a chain in F_1. As usual, any time as we have a variety characterized by a picture with a zigzag (without arrows) we represent it as a component of the boundary of a variety, whose picture is obtained from this one by adding an arrow at one of the ends of the zigzag. Performing this systematically, we find some ten varieties of codimension 1 in the greater cell of F_1. These varieties are encoded in the left parts of the following equations (27)–(36), whose right-hand parts express the boundaries of these varieties.



(27)



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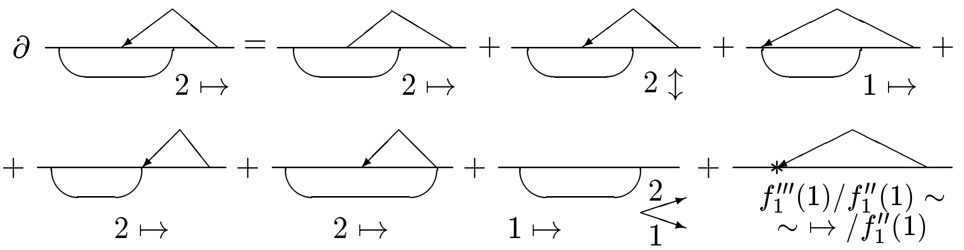

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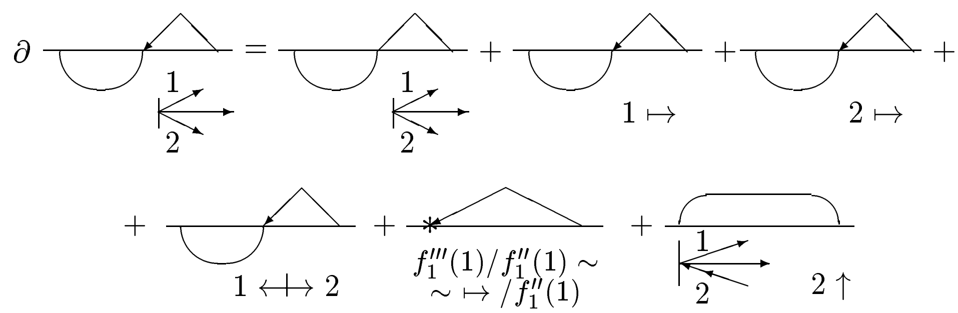

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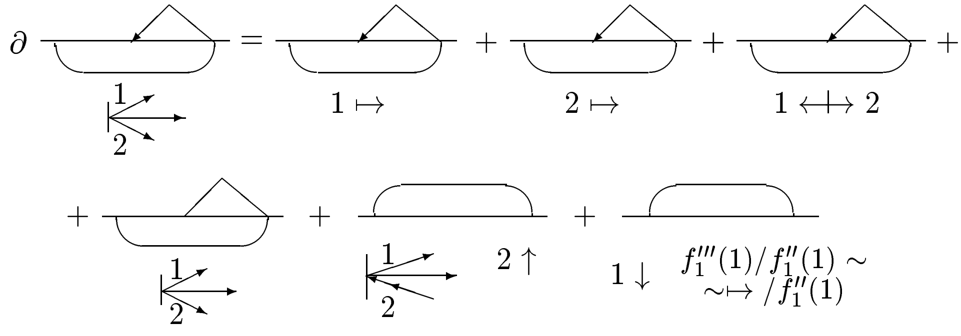

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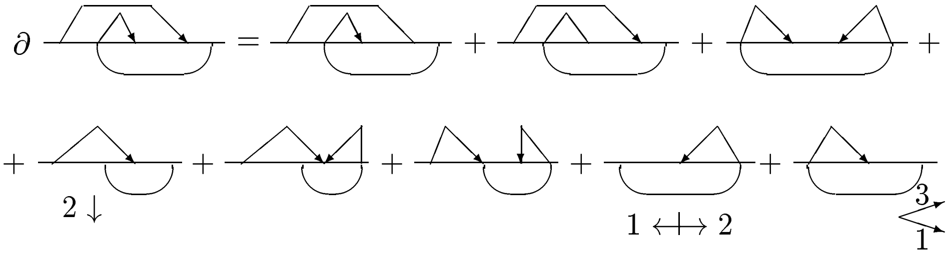

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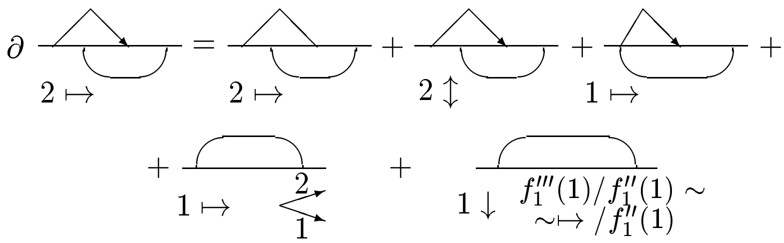

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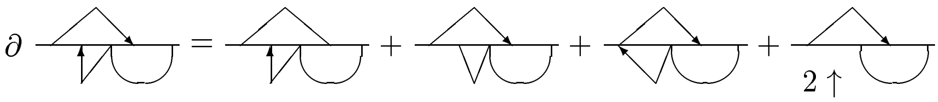

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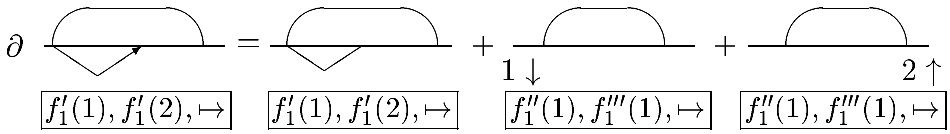

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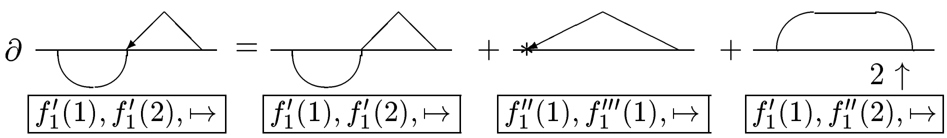


(36)

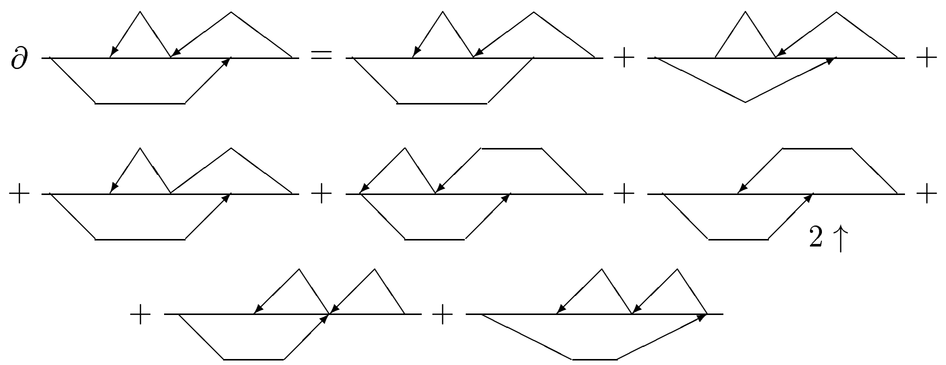
Summing up the right-hand parts of these equations, we get the following statement.

Proposition 10. *The cycle $d^2(TT) \subset F_1$ presented by the linear combination* (*26*) *is equal to the boundary* (*mod 2*) *of the sum of ten varieties shown in the left parts of equations* (*27*)*–*(*36*).

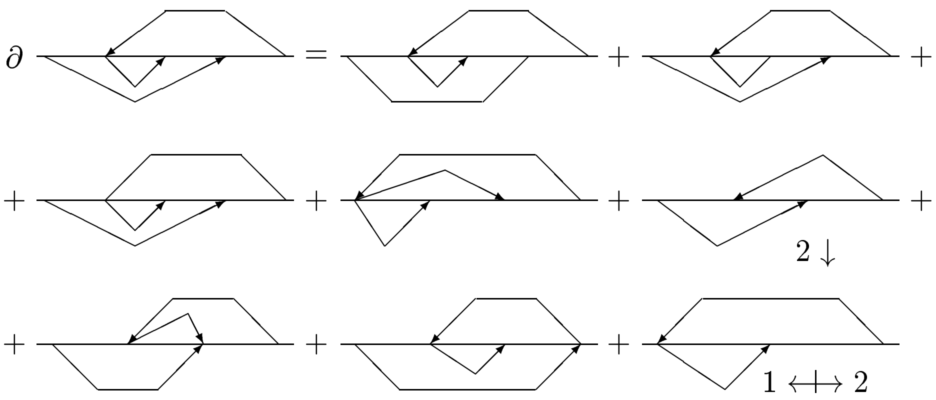
In this summation we use the following relations. Let us denote by (a;b) the b-th summand in the right-hand part of the equation (a). Then (27;6) + (28;5) = (29;2); (27;7) + (28;7) = (30;4); (27;5) + (28;6) = (32;3); (34;3) + (32;6) = (32;5); (34;4) + (32;4) = (33;2).

3.5. **The third differential.** Ten varieties described by left parts of equations (27)–(36) form a chain in the term F_1 of the resolved discriminant, i.e. in the *tautological resolution* of this discriminant, see §2.

Finally, we consider the image of this chain in the discriminant itself. The image of any of ten components of this image is a subvariety in the space $\mathcal{K}_n$ of maps $\mathbb{R}^1 \to \mathbb{R}^n$, distinguished by conditions, whose notation is obtained from the notation of the corresponding variety in F_1 by replacing its unique chord by a zigzag with the same endpoints. It remains to span the sum of these varieties by a chain in the space $\mathcal{K}_n$. Proceeding as before, we find five varieties indicated in the left parts of the following identities (37)–(41).



(37)



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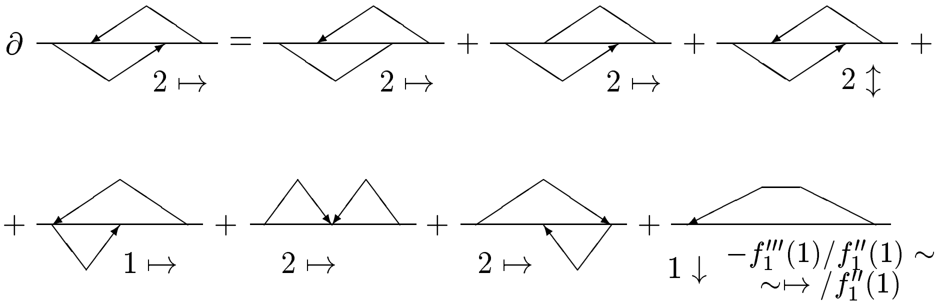

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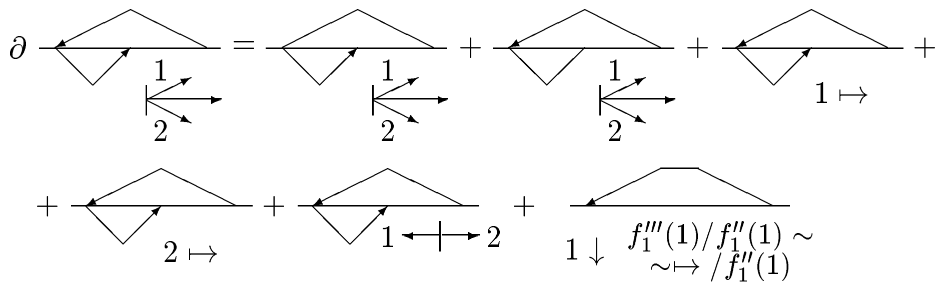

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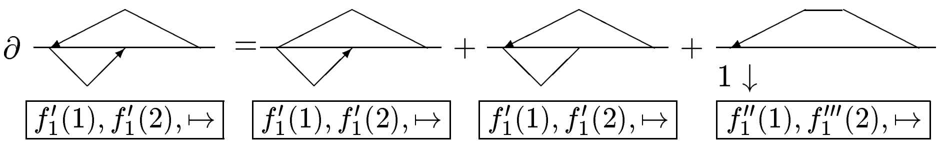


(41)

It is easy to check that the homological sum of the right-hand parts of these identities is equal to the sum of our ten discriminant varieties obtained from left parts of equalities (27)–(36). (We use the following relations: (37;5) + (38;5) = (39;3); (39;6) + (40;4) = (39;5); (39;4) + (40;6) = (41,3); (37;4) + (37;6) + (37;7) + (38;4) + (38;6) + (38;7) = 0.)

Therefore the sum of the five varieties indicated in left parts of equalities (37)–(41) is the desired relative cycle in $\mathcal{K}_n$ (mod Σ).

The sum of the first and the second of these five varieties (respectively, the third variety, respectively, the difference of the fifth and the fourth varieties) is exactly the variety indicated in item a) (respectively, b), respectively, c)) of Theorem 1, which is thus completely proved.

3.6. **Problems. 1. Algorithmization.** To write a computer algorithm doing all the same for any other finite-type cohomology class of the space of knots. Let us consider any homology class γ of the discriminant of the space of knots, having some finite filtration ("order") p and presented by its "principal part", i.e. by the corresponding homology class in the term $F_p \setminus F_{p-1}$ of the resolved discriminant. This class always is described by some linear combination of pictures (generalized chord diagrams) as in (9), (10), see § 2. To get the combinatorial description of a cohomology class with this principal part, we need to calculate all the steps of the spectral sequence starting from this part. To do it, on any step we need to find the chains spanning the consecutive boundaries $d^r(\gamma) \subset F_{p-r} \setminus F_{p-r-1}$. Above we have used some obvious rules: if a piece of our cycle $d^r(\gamma)$ is described by a picture like in (11)–(40), then it is natural to kill it by a piece of the spanning chain, described by almost the same picture, only replacing some one zigzag without arrows by the same zigzag with arrow at one its end, or replacing some condition of type $\underleftrightarrow{1 \;\bullet\; 2}$ by the condition of type $1 \nwarrow\!\!\nearrow 2$, or replacing some condition $1 \leftrightarrow\!\!\!\!\!+\!\!\!\!\!\rightarrow 2$ by the condition $\overset{1}{\underset{2}{\vdash\!\!\leftleftarrows\!\!\to}}$, etc. But how to decide, which of these fragments (and for which piece of the cycle) to replace first? At which endpoint to put the arrow? Is it possible to do it always in such a way that all the other components of the boundary of this spanning variety would be in some sense "of lesser complexity" than the killed one, so that our algorithm converges inductively? Which other subvarieties in the cells of $F_p \setminus F_{p-1}$ can occur in the process of performing this algorithm ? What are the formal rules for calculating their boundaries ?

I presume that the main filtering degree should be the number of points in $\mathbb{R}^1$ participating in the definition of the subvariety, and the orientation of arrows is not important: say, the algorithm will work if we orient all of them from the right to the left (although, of course, other choice can provide somewhat easier formulas).

2. Orientable case.

To do all the same for homology with integer coefficients, i.e. taking into account orientations of our varieties. In this problem, the answers for odd and even n will be different: already the chain (9) is a $\mathbb{Z}$-cycle in $F_3 \setminus F_2$ only for even n. If n is even, is it correct, that all the calculations of § 3 remain valid after imposing appropriate signs before the pictures ?

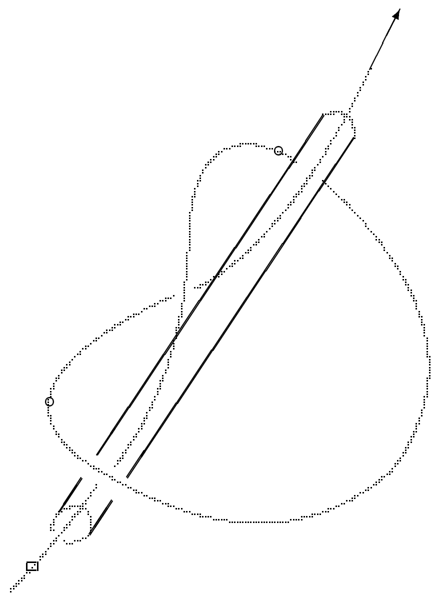


FIGURE 2

3.7. **Proof of Proposition 2.** First we specify a loop in the space of long knots as in this Proposition. We can assume that the standard embedding $\mathbb{R}^1 \to \mathbb{R}^3$ (with which all long knots should coincide close to the infinity) lies in the plane $\mathbb{R}^2$ and has angle $\pi/4$ with the chosen direction "to the right". Let us consider the standard long trefoil as shown in Fig. 2.

Namely, we assume that close to all crossing points the projections of tangent directions to $\mathbb{R}^2$ are separated from the direction "to the right" or "to the left": for certainty, let us make the angles between the direction "to the right" and these projected tangent directions at consecutive 6 points to be equal to $\pi/4$, $3\pi/10$, $-\pi/4$, $3\pi/4$, $2\pi/10$, and $\pi/4$ respectively.

We call this knot a "large" one, and tie a very small homothetic knot on its initial segment indicated by a tiny square in Fig. 2. Then we shrink very much the large knot in the "vertical" direction (orthogonal to the plane of our picture) so that it becomes almost flat and its derivative almost horizontal, not changing the small knot. Then we move this small knot along the large one in such a way as if it would be frozen in a small hard bead put on this large knot. (On the same Fig. 2 we show by the thick lines the channel of the bead; in this case all the picture should be considered as that of the small homothetic knot. In particular all the points of this knot where the direction of its derivative is sufficiently far from the standard one, are inside the bead.)

More precisely, we associate with this bead a orthonormal frame in $\mathbb{R}^3$ whose first vector in the initial instant is vertical (i.e. orthogonal to the plane of the picture) and the second vector is directed along the channel.

Lemma 1. *Suppose that a) the ratio of the diameter of the channel to its length is equal to a sufficiently small number ε, b) the coefficient of the flattening of the large knot in the vertical direction is of order ε^2, so that the absolute value of the "vertical" part of the derivative of the large knot shown in Fig. 2 is nowhere greater than ε^2 times the length of its "horizontal" part, and c) the size of the bead (i.e. the homothety coefficient of two knots) is equal to ε^3.*

Then we can move our bead along the entire large knot in such a way that

A) the first vector of its associated frame remains vertical all the time, and

B) there is a smooth one-parametric family of long knots in $\mathbb{R}^3$ such that at any instant

i) they coincide with the large knot everywhere outside the convex hull of the bead,

ii) their intersection with the bead itself remains fixed and is as shown in Fig. 2,

iii) in all the points of the knot inside the channel of the bead, the angle between the derivative of the knot and the direction of the channel is less than $\pi/4$. □

The first and last instants of this one-parameter family of knots obviously can be joined by a homotopy not changing the topology of the knot diagram, and we get a closed loop in the space of knots. Now let us calculate the intersection number of this loop with the chain described in Theorem 1.

This loop can intersect the varieties indicated in statements a) and c) of this Theorem only when triple intersections of the projection occur. This can happen only if one of crossing points of the smaller knot moving along some branch of the large knot passes above or below the other its branch: in total 18 suspicious instants. These instants should be counted with multiplicities. In the case of variety described in statement a) the multiplicity is equal (mod 2) to the number of other crossing points of the composite knot forming together with this triple point a configuration satisfying all other conditions of this statement; for variety described in c) the multiplicity is equal to 0 or 1 depending on the condition on the tangent frame.

It is easy to calculate that the desired configurations for the variety a) exist only when our small knot passes the first time (i.e. along the lower branch) the third crossing point of the large knot: moreover, all three instants when one of crossing points of the small knot pass this point have multiplicity 1. Therefore the total number of intersections of our path with variety a) is equal to 3. Similarly, we meet the variety c) only once, when our small knot (more precisely, its second crossing point) passes the first time the first crossing point of the large knot. So, the intersection number with variety c) is equal to 1.

The configurations of type b) can appear by two reasons. First, when the small knot passes a crossing point of the large one (and namely an undercrossing) then all its points go under the other branch of the large knot; at some instant this happens with the point with the distinguished tangent direction. Again, any such instant should be counted with multiplicities depending on the order of other crossing points of the composite knot. It is easy to calculate that only once this multiplicity can be not equal to zero. Namely, when our small knot undercrosses the third crossing point, then at some instant this situation appears with multiplicity 2. Further, when our small knot moves and rotates together with the derivative of the large one, some of tangent lines at its own crossing points can instantly become directed "to the right". (Namely, only the tangent line at the undercrossing branch of the first or third crossing point of the small knot is interesting for us.) There are exactly two points of the large knot at which it happens: in Fig. 2 they are indicated by small circles. The multiplicity of the "lower" (in this picture) point is equal to 1, and the multiplicity of the "upper" one is equal to 0.

Finally, the total number of intersection points of our path with the variety indicated in Theorem 1 is equal to $3 + 1 + 2 + 1 = 7$, and proposition 2 is completely proved.

4. Comments on and proof of Theorem 2

Four statements A, B, C, D of this theorem are discussed in corresponding parts of this section.

A. The variety in $\mathcal{K}_n$ given by the condition $f(0) = f(\pi)$ is a vector subspace of codimension n. It is equal to the boundary of the variety $\mathcal{A}$ described in statement A of Theorem 2. The map $\mathcal{K}_n \to \mathbb{R}^{n-1}$, sending any curve f to the vector $f_1(\pi) - f_1(0)$, defines an isomorphism between $\mathbb{R}^{n-1}$ and the normal bundle of $\mathcal{A}$, in particular induces a coorientation of $\mathcal{A}$ from any orientation of $\mathbb{R}^{n-1}$. Thus for any integral $(n-1)$-dimensional cycle in $\mathcal{K}_n \setminus \Sigma$ its intersection index with $\mathcal{A}$ is well defined and is equal to its linking number with the subspace $\{f | f(0) = f(\pi)\}$. It follows from calculations in [17], [16], [18] that such integral cycles exist only if n is even. For instance, let S^{2k-1} be the unit sphere, and consider all the fibers of the Hopf bundle $S^{2k-1} \to \mathbb{CP}^{k-1}$ supplied with natural parametrizations respecting the natural orientations of these fibers. The set of all these parametrized fibers is obviously homeomorphic to S^{2k-1} (to any parametrized fiber there corresponds the zero of the parameter) and has exactly one intersection point with the variety $\mathcal{A}$.

B. The algorithm of finding the spanning chain is as follows. The variety $\mathcal{L}$ described in Proposition 3 is swept out by 1-parametric family of subspaces $L(\alpha) \subset \mathcal{K}_n$ of codimension n: they are parametrized by points α of the halfcircle $S^1/\pm = \mathbb{R}^1/\pi\mathbb{Z}$ and defined by conditions $f(\alpha) = f(\alpha + \pi)$. Let us try to span all these spaces separately. Consider the trivial bundle $\mathcal{K}_n \times [0, \pi] \to [0, \pi]$ and subset in it consisting of pairs (α, f) such that $f(\alpha)$ is above $f(\alpha+\pi)$ in $\mathbb{R}^n$. This subset is a smooth submanifold with boundary, and its projection to $[0, \pi]$ is a smooth fiber bundle. Forgetting the second coordinate α defines the projection of this manifold to $\mathcal{K}_n$. Its image is exactly the variety $\mathcal{B}a$ described in statement Ba of Theorem 2. Its boundary consists of the variety $\mathcal{L}$ and images of fibers of the above-described fiber bundle over the points 0 and π. The union of these two fibers is equal to the subspace distinguished by the condition $f_1(0) = f_1(\pi)$, and is equal to the boundary of the halfspace $\mathcal{B}b$ described in statement Bb of Theorem 2.

Now we choose coorientations of these varieties. The variety $\mathcal{B}a$ is singular. Any its regular point f satisfies the condition $f_1(\alpha) = f_1(\alpha+\pi)$ for exactly one $\alpha \in [0,\pi)$ and has transverse selfintersection of the curve $f_1(S^1)$ at this point. Close to such a point f the coorientation of $\mathcal{B}a$ is defined as follows. Fix our point α and define the map $(\mathcal{K}_n, f) \to TS^{n-2}$ associating to any parametrized curve $g \approx f$ the point of S^{n-2} equal to the direction of the vector $g_1'(\alpha+\pi) - g_1'(\alpha)$, and the tangent vector at this point in S^{n-2} equal to the projection of the vector $g(\alpha+\pi) - g(\alpha)$ to the plane orthogonal to this vector $g_1'(\alpha+\pi) - g_1'(\alpha)$. The preimage of the zero section of TS^{n-2} under this map is tangent in $\mathcal{K}_n$ to the variety $\mathcal{B}a$, in particular if we have a generic germ of a $(n-2)$-dimensional subvariety (simplex) in $\mathcal{K}_n$ at the point g then it is transversal to both varieties and we can induce its desired orientation from (any fixed) orientation of the bundle TS^{n-2}.

The coorientation of the variety $\mathcal{B}b$ is induced from a chosen orientation of $\mathbb{R}^{n-2}$ by the map $\mathcal{K}_n \to \mathbb{R}^{n-2}$ by a map sending any f to the direction of the vector $f_2(\pi) - f_2(0) \in \mathbb{R}^{n-2} \equiv \mathbb{R}^n/\{\uparrow, \mapsto\}$.

C. Recall that the term F_1 of the simplicial resolution of Σ is the space of pairs

$$((\alpha,\beta), f) \in \overline{B(S^1,2)} \times \mathcal{K}_n \tag{42}$$

such that $f(\alpha) = f(\beta)$. In particular it is a vector bundle over $\overline{B(S^1,2)}$. Let $\mathcal{M} \subset F_2 \setminus F_1$ be the principal part of the $(2n-3)$-dimensional class of order 2 described in Proposition 4. Its first differential $d^1(\mathcal{M})$ is realized by the subvariety in F_1 consisting of such pairs (42) that $\beta = \alpha+\pi$ and f satisfies not only the condition $f(\alpha) = f(\alpha+\pi)$ but also the condition $f(\alpha+\pi/2) = f(\alpha-\pi/2)$. The set of such pairs (α,β) is the circle $\mathbb{R}^1/\pi\mathbb{Z}$, so our cycle $d^1(\mathcal{M})$ is the space of a vector bundle over the circle. To span it in F_1 consider the subvariety $\mathcal{M}_1' \subset F_1$ consisting of such pairs (42) that again $\beta = \alpha+\pi$, $f(\alpha) = f(\beta)$, but the image of one of points $f(\alpha \pm \pi/2)$ is *above* the other: namely, the image of those of these two points which is separated from $0 \in S^1$ by the points $\alpha, \alpha+\pi$ is above the image of its antipode. This subvariety also forms a fiber bundle over the circle $\mathbb{R}^1/\pi\mathbb{Z}$ of all such pairs $(\alpha, \alpha+\pi)$. There is exactly one position of α over which this fiber bundle fails to be locally trivial, namely $\alpha = 0 (\mathrm{mod}\ \pi)$. The boundary of this subvariety is equal to the sum of the cycle $d^1(\mathcal{M})$ and the space of points (42) where $\alpha = 0 (\mathrm{mod}\ \pi)$, $f(0) = f(\pi)$ and $f_1(\pi/2) = f_1(-\pi/2)$. We span the latter space by the similar *half*space $\mathcal{M}_1''$, defined by the condition that $f(0) = f(\pi)$ and $f_1(\pi/2)$ lies to the right of $f_1(-\pi/2)$. The sum $\mathcal{M}_1' + \mathcal{M}_1''$ is the desired chain in F_1 whose boundary is equal to $d^1(\mathcal{M})$. Now we consider the image $d_2(\mathcal{M})$ of this chain in Σ and try to represent it as a boundary of some relative cycle in $\mathcal{K}_n(\text{mod } \Sigma)$. The image of $\mathcal{M}_2''$ is obvious, and the image of $\mathcal{M}_1'$ consists of maps f such that there exists $\alpha \in [0, \pi]$ such that $f(\alpha) = f(\alpha + \pi)$, and the image of one of points $\alpha \pm \pi/2$ (namely, the one separated from 0 by α and $\alpha + \pi$) is above the other.

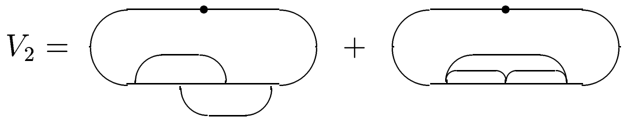
It is natural to kill this variety by the space of all maps f described in statement Ca of Theorem 2. Its boundary consists of this image of $\mathcal{M}_1'$ and the space of such maps f that $f(\pi)$ is above $f(0)$ and $f_1(\pi/2) = f_1(-\pi/2)$. The boundary of the variety described in statement Cb of Theorem 2 is equal to the sum of the latter space and the image of $\mathcal{M}_1''$.

Statement C of Theorem 2 is thus proved for $\mathbb{Z}_2$-homology; the proof of its integer version requires additionally only an accounting of orientations.

D. We shall use the pictures like in § 3, only the Wilson loop will be shown not by a segment but by an oval with marked "zero" point on its top. This point is referred to as 0 in subscripts, and all the other points participating in the definition of cells and their subvarieties are numbered accordingly to the (counterclockwise) orientation of the Wilson loop. All the calculus remains the same as in §3, only the boundary operators will include the limit positions of our cells and their subvarieties when some of defining them points tend to 0.

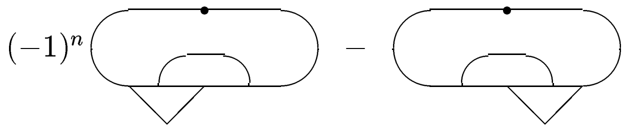
As we are interested in integral homology classes, we shall take care of orientations of all our varieties in the cells of the standard cell decompositions of terms $F_i \setminus F_{i-1}$. This orientation consists of the orientation of the cell and the (co)orientation of the subvariety in it. The choice of these orientations will follow the guidelines indicated in §3 of [14] or §V.3.3 of [16]. Namely, they consist of the following orientations (taken in that order): a) the orientation of the simplex participating in the construction of the simplicial resolution (i.e. the simplex $\tilde{\Delta}(J)$ or some its nonmarginal face); b) the coorientation of the subspace $L(J)$ of the space $\mathcal{K}_n$; c) the orientation of the space of equivalent point configurations $J \subset S^1$; d) the (co)orientation of the subvariety in the cell. The first three orientations are specified exactly as in §3 of [14] (but now in c) we can move only *nonzero* points). Often the subvariety in the cell is given by several conditions of the form: "there are additional points in $\mathbb{R}^1$ whose images $f(\cdot) \in \mathbb{R}^n$ (or their projections to some fixed subspace) coincide with one another or with images of some points participating in the definition of the cell", or at least our subvariety forms an open subset in a subvariety defined in such a way. In this case the orientation d) also is defined by the sequence consisting of α) the (co)orientation of the vector subspace defined by these conditions in the vector spaces counted in the step b) above, and β) the orientation of the space of configurations of additional points. These orientations also are specified as in [14], [16]; to define the coorientations of subspaces we assume that the direction "up" in $\mathbb{R}^n$ is the first vector of the canonical frame, and the direction "to the right" is the second. All the forthcoming calculations refer to exactly this choice of orientations.

The principal part of the considered class in $F_2 \setminus F_1$ is as follows:



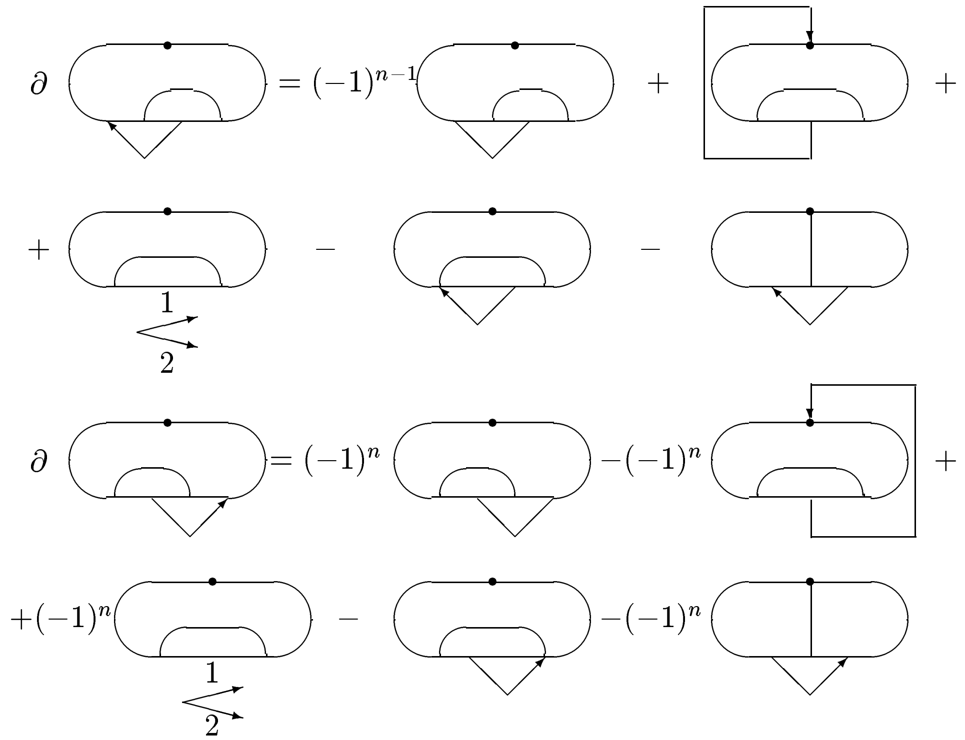
(43) ,

see [14]. The second summand has no boundary in F_1, the boundary of the first is as follows:



(44) .

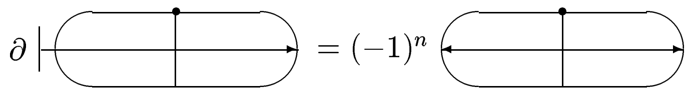
Arguing as previously, we try to span the two terms of this chain by the varieties encoded in left parts of the next two equations, respectively:



(45)

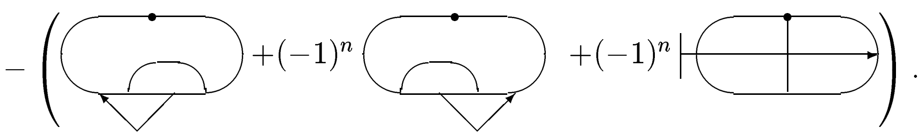
The varieties shown by the second from the end pictures in both these equations coincide geometrically, and their canonical orientations differ by the factor $(-1)^{n-1}$. Therefore the linear combination of left parts of these equations taken with coefficients -1 and $(-1)^{n-1}$ respectively is equal in F_1 to the sum of the expression (44) and two last varieties in these equations. If $n = 3$ then the sum of last two varieties is equal to zero. Indeed, any of these varieties consists of pairs (42) with $\alpha = 0$, $f(\alpha) = f(\beta)$, taken with some multiplicities. These multiplicities always are opposite, because they are equal (up to signs) to different combinatorial expressions for the linking numbers of two "smoothened" loops into which the point $f(0) = f(\beta)$ breaks the curve $f(S^1)$.

However, if $n > 3$ then the sum of these two varieties is only homologous to zero, but not equal to it. We shall encode this sum by the picture in the right-hand part of the following equation:

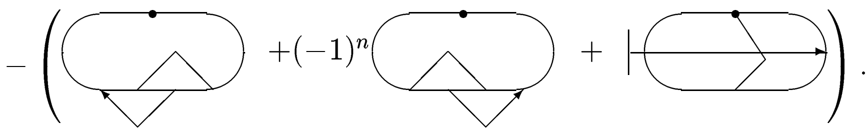
(46) 

The variety assumed in the left part of this equation consists of all pairs (42) in F_1 such that $\alpha = 0$ and additionally there are points $\gamma \in (0, \beta)$ and $\delta \in (\beta, 2\pi)$ such that the projection of $f(\delta)$ to $\mathbb{R}^{n-1}$ lies "to the right" from that of $f(\beta)$.

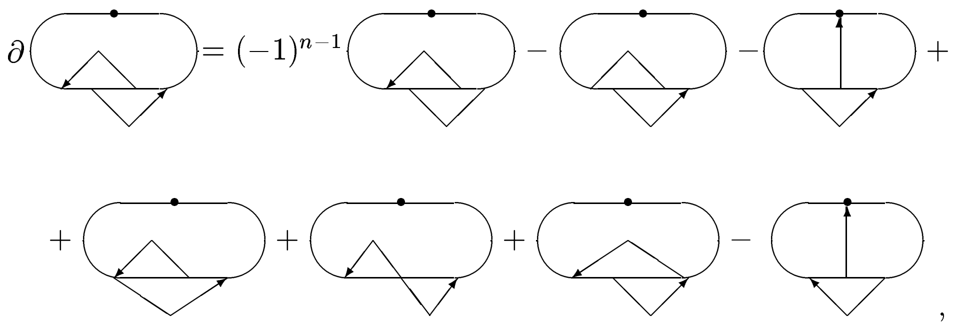
So, the desired chain in F_1 spanning the cycle (44) is equal to

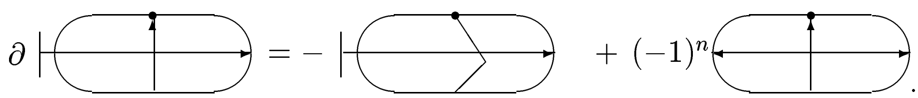
(47) 

Its image in Σ is expressed by the formula

(48) 

We need to span this chain by a relative cycle in $\mathcal{K}_n$ (mod Σ). For this we have

(49) 

(50) 

The sum of the fourth, fifth and sixth terms in the right-hand part of (49) is equal to zero. The sum of varieties encoded by the pictures in the third and seventh terms is equal to the variety shown by the last picture in (50). Therefore the desired relative cycle is equal to the linear combination of left parts of (49) and (50) taken with coefficients 1 and $(-1)^n$ respectively. Theorem 2 is completely proved.

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