STOCHASTIC DIFFERENTIAL GEOMETRY AND THE RANDOM FLOWS OF VISCOUS AND MAGNETIZED FLUIDS IN SMOOTH MANIFOLDS AND EUCLIDEAN SPACE

by

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Abstract: In this article we integrate in closed implicit form the Navier-Stoker equations for an incompressible fluid in a smooth compact manifold without boundary, and in particular, in a compact manifold which is isometrically embedded in Euclidean space, and finally in Euclidean space itself. We further integrate the kinematic dynamo problem of magnetohydrodynamics, i.e. the equations of passive transport of a magnetic field on a fluid. We carry out these integrations through the application of the methods of Stochastic Differential Geometry, i.e. the gauge theoretical formulation of diffusion processes on smooth manifolds. Thus we start by defining the invariant infinitesimal generators of diffusion processes of differential forms on smooth compact manifolds, in terms of the laplacians (on differential forms) associated with the Riemann-Cartan-Weyl (RCW) metric compatible connections. These geometries have a torsion tensor which reduces to a trace 1-form, whose conjugate vector field is the drift of the diffusion of scalar fields. We construct the diffusion processes of differential forms associated with these laplacians by using the property that the solution flow of the stochastic differential equation corresponding to the scalar diffusion is -assuming Hoelder or Sobolev regularity conditions- a random diffeomorphism of the manifold. We apply these constructions to give a new characterization of the Navier-Stokes equation for the velocity one-form of an incompressible fluid as a non-linear diffusion process determined by a RCW connection. We prove the Navier-Stokes equations to be equivalent to a linear diffusion equation for the vorticity and the Poisson-de Rham equation for the velocity with the vorticity as a source, extending thus to manifolds, the approach in practice in the random vortex method in Computational Fluid Dynamics. We give the invariant random stochastic differential equations for the position (as a Lagrangian representation) of the fluid particles and thus obtain a random diffeomorphism which is a solution of the Navier-Stokes equation. We solve the Cauchy problem for the heat equation for the vorticity two-form and the Dirichlet problem for the Poisson-de Rham equation for the velocity one-form in two different instances. Firstly, using

the gradient flows obtained by a Nash isometric embedding of the manifold in Euclidean space, and secondly, for the more general case of an arbitrary manifold, by running curvature and fluid-deformation tensor dependant random flows on the tangent bundle, projecting on the random fluid' particles flow. We implement our general construction on Euclidean space by simply taking the former construction under the identity embedding. We extend these methods to the integration of the kinematic dynamo problem. From the fact that any metric compatible connection can be constructed in terms of the push-forward connection and using simple stochastic analysis considerations, we prove that any diffusion process generated by a RCW connection admits a random flow representable through a purely diffusive process (i.e. zero drift), in any dimension other than 1. We apply this to prove that the random flow of a viscous fluid obeying the classical Navier-Stokes equations on a smooth manifold of dimension other than 1, can be represented as a purely diffusive process, where the new diffusion tensor can be constructed in terms of the velocity and the original diffusion tensor associated to the metric and the kinematical viscosity. A similar construction we prove to be valid for the fluid flow of a passively magnetized viscous or inviscid fluid.

1 Introduction.

The purpose of this article is two-fold: Firstly to give implicit random representations for the solutions of the Navier-Stokes equation for an incompressible fluid and for the kinematic dynamo problem of magnetohydrodynamics, in several instances; firstly, on an arbitrary compact orientable smooth manifold (without boundary) -following our presentation in [58,59]- and further in the case in which the manifold is isometrically embedded in Euclidean space, to finally give the expressions for Euclidean space itself. Secondly, to present as a basis for such an integration, the gauge-theoretical structures of Brownian motion theory and the stochastic analysis rules associated to them. Thus, the method of integration we shall apply for our objectivs stems from stochastic differential geometry, i.e. the gauge theory of Brownian processes in smooth manifolds and Euclidean space developed in the pioneering works by Ito [15], Elworthy and Eells [13], P. Malliavin [11] and further elaborated by Elworthy [12], Ikeda and Watanabe [14], P. Meyer [40], and Rogers and Williams [37]. Associated to these geometrical structures which can be written in terms of the Cartan calculus on manifolds of classical differential geometry, we shall present the rules of stochastic analysis which describe the transformation of differential forms along the paths of generalized Brownian motion generated by these geometries, setting thus the method for the integration of linear evolution equations for differential forms; this is the well known martingale problem approach to the solution of partial differential equations on manifolds [30].

While classical Hamiltonean systems with finite degrees of freedom may appear to have a random behavior, in fluid dynamics it is known that the Euler equation for an inviscid fluid is a hamiltonean system with infinite degrees of freedom supporting as well infinite conserved quantities; such a system appears to be non-random [56]. The situation is radically different in the case of a viscous fluid described by the Navier-Stokes equations. In this case, there is a second-order partial derivative associated to the kinematical viscosity, which points out to the fact that there is a diffusion term, which can be thougth as related to a Brownian motion. Thus in the viscous case, there is from the very beginning a random element. While in the Euler case the group of interest is the (Riemannian) volume preserving of diffeomorphims, it will turn out in the course of these studies, that there is an active group of **random** diffeomorphims which represent the Lagrangian random flow of the viscous fluidparticles. In this case, when there is a non-constant diffusion tensor describing the local amplification of noise, these diffeomorphisms do not preserve the volume measure, contributing at a dynamical level -as it will turn out- to the complicated topology of turbulent and magnetized flows [56].

The essential role of randomness in Fluid Dynamics appears already at an experimental level. The analysis of the velocimetry signal of a turbulent fluid shows that its velocity is a random variable, even though that the dynamics is ruled by the deterministic Navier-Stokes equation [8]. The concept of a turbulent fluid as a stochastic process was first proposed by Reynolds [16], who decomposed the velocity into mean velocity plus fluctuations. The Reynolds approach is currently used in most numerical simulations of turbulent fluids in spite of the fact that it leads to unsurmountable non-closure problems of the transport equations.; see Lumley [18], Mollo and Christiansen [38]. Furthermore, the Reynolds decomposition is non invariant alike the usual decomposition into drift and white noise perturbation in the non-invariant theory of diffusion processes. Other treatments of stochasticity in turbulence were advanced from the point of view of Feynman path integrals, as initiated by Monin and Yaglom [17]. From the point of view of diffusion processes, invariant measures for stochastic modifications of the Navier-Stokes equations on euclidean domains, have been constructed by Vishik and Fursikov [36] and Cruzeiro and Albeverio [42]. (It is important to remark that the existance of an invariant measure for NS as a *classical* dynamical system is the starting point of the classical dynamical systems approach to turbulence; see Ruelle [48].) Contemporary investigations develop the relations between randomness and the many-scale structure of turbulence which stems from the Kolmogorov theory as presented by Fritsch[8] and Lesieur [3], and apply the renormalization group method; see Orzsag [19].

A completely new line of research followed from the understanding of the fundamental importance of the vorticity (already stressed by Leonardo da Vinci) in the self-organization of turbulent fluids, which was assessed by numerical simulations by Lesieur [3,4], and theoretically by Majda [39] and Chorin [1]. It was found that the Navier-Stokes equations for an incompressible fluids on Euclidean domains yields a linear diffusion equation for the vorticity which becomes a source for the velocity through the Poisson equation: Solving the latter equation, we can obtain an expression for the fluids velocity in 2D. This observation was the starting point for the random vortex method in Computational Fluid Mechanics largely due to Chorin [1,2,6]. This conception lead to apply methods of statistical mechanics (as originally proposed by Onsager [20]) to study the complex topology of vortex dynamics and to relate this to polymer dynamics [1]. In the random vortex method a random lagrangian representation for the position of the incompressible fluid particles was proposed. Consequently, the Navier-Stokes linear (heat) equation for the vorticity was integrated only for two dimensional fluids (implicit to this is the martingale problem approach quoted above), while the general case was numerically integrated by discretization of the this heat equation; see Chorin [1,2]. The difficulty for the exact integration in the general case apparently stems from the fact that while in dimension 2 the vorticity 2-form can be identified with a density and then the integration of the Navier-Stokes equation for the vorticity follows from the application of the well known Ito formula for scalar fields, in the case of higher dimension this identification is no longer valid and an Ito formula for 2-forms is required to carry out the integration. This formula became only recently available in the works by Elworthy [27] and Kunita [24], in the context of the theory of random flows on smooth manifolds.

The importance of a Stochastic Differential Geometry treatment of the Navier-Stokes equation on a smooth n-manifold M stems from several fundamental facts which are keenly interwoven. For a start, it provides an intrinsic geometrical characterization of diffusion processes of differential forms which follows from the characterizations of the laplacians associated to non-Riemannian geometries with torsion of the trace type, as the infinitesimal generators of the diffusions. In particular, this will allow to obtain a new way of writing the Navier-Stokes equation for an incompressible fluid in terms of these laplacians acting on differential one-forms (velocities) and two-forms (vorticities). Furthermore, these diffusion processes of differential forms, are constructed starting from the scalar diffusion process which under Hoelder continuity or Sobolev regularity conditions, yields a time-dependant random diffeomorphism of M which will represent the Lagrangian trajectories for the fluid particles position. This diffeomorphic property will allow us to use the Ito formula for differential forms (following the presentation due to Elworthy) as the key instrument for the integration of the Navier-Stokes equation (NS for short, in the following) for an incompressible fluid. Thus, it is the knowledge of the rules of stochastic analysis what sets the martingale problem approach to the solution of NS when he have transformed it to an equivalent system which is essentially linear. This transformation of non-linear equations to linear ones appears as well in an identical gauge-theoretical approach to quantum mechanics as a theory of Brownian motion. In this case, the massive Dirac-Hestenes equation for a Dirac-Hestenes operator field (conceived as a geometrical matter field) can be transformed in a Clifford bundle approach to the sourceless Maxwell equation [26]. We must remark that the geometrical structures on which the gauge-theoretical foundations of Brownian motion are introduced, belong in fact originally to the gauge theory of gravitation, including not only translational degrees of freedom, but additionally spinor fields [46,57].

For the benefit of completeness, we have included in this article an Appendix which reviews briefly the main probabilistic and analytical notions appearing in this article.

2 Riemann-Cartan-Weyl Geometry of Diffusions

The objective of this section is to show that the invariant definition of a "heat" (Fokker-Planck) operator requires the introduction of linear connections of a certain type.

Let us consider for a start, a smooth n-dimensional compact orientable manifold M (without boundary), on which we shall consider a second-order smooth differential operator L. On a local coordinate system, $(x^{\alpha}), \alpha = 1, \ldots, n$, L is written as

$$L = \frac{1}{2} g^{\alpha\beta}(x)\partial_{\alpha}\partial_{\beta} + B^{\alpha}(x)\partial_{\alpha} + c(x). \tag{1}$$

From now on, we shall fix this coordinate system, and all local expressions shall be written in it.

Although formally, there is no restriction as to the nature of M, we are really thinking on a n-dimensional space (or space-time) manifold, and **not** in a phase-space manifold of a dynamical system.

The *principal symbol* σ of L, is the section of the bundle of real bilinear symmetric maps on T^*M, defined as follows: for $x \in M$, $p_i \in T^*_xM$, take C^2 functions, $f_i : M \to R$ with $f_i(x) = 0$ and $df_i(x) = p_i$, $i = 1, 2$; then,

$$\sigma(x)(p_1, p_2) = L(f_1 f_2)(x). \tag{2}$$

Note that σ is well defined, i.e., it is independent of the choice of the functions $f_i, i = 1, 2$.

If L is locally as in (1), then σ is locally represented by the matrix $(g^{\alpha\beta})$. We can also view σ as a section of the bundle of linear maps $L(T^*M, TM)$, or as a section of the bundle $TM \otimes TM$, or still as a bundle morphism from T^*M to TM. If σ is a bundle isomorphism, it induces a Riemannian structure g on M, $g : M \to L(TM, TM)$:

$$g(x)(v_1, v_2) := < \sigma(x)^{-1} v_1, v_2 >_x,$$

for $x \in M$, $v_1, v_2 \in T^*_xM$. Here, $< .,. >_x$ denotes the natural duality between T^*_xM and T_xM. Locally, $g(x)$ is represented by the matrix $\frac{1}{2}(g^{\alpha\beta}(x))^{-1}$. Consider the quadratic forms over M associated to L, defined as

$$Q_x(p_x) = \frac{1}{2} < p_x, \sigma_x(p_x) >_x,$$

for $x \in M$, $p_x \in T^*_xM$. Then, with the local representation (1) for L, Q_x is represented as $\frac{1}{2}(g^{\alpha\beta}(x))$. Then, L is an elliptic (semi-elliptic) operator whenever for all $x \in M$, Q_x is positive-definite (non-negative definite). We shall assume in the following that L is an elliptic operator. In this case, σ is a bundle isomorphism and the metric g is actually a Riemannian metric. Notice, as well, that $\sigma(df) = grad\ f$, for any $f : M \to R$ of class C^2, where grad denotes the Riemannian gradient.

We wish to give an intrinsic description of L, i.e. a description independent of the local coordinate system. This is the essential prerequisite of covariance.

For this, we shall introduce for the general setting of the discussion, an arbitrary connection on M, whose covariant derivative we shall denote as $\tilde{\nabla}$. We remark here that $\tilde{\nabla}$ will **not** be the Levi-Civita connection associated to g; we shall precise this below. Let $\sigma(\tilde{\nabla})$ denote the second-order part of L, and let us denote by $X_0(\tilde{\nabla})$ the vector field on M given by the first-order part of L. Finally, the zero-th order part of L is given by $L(1)$, where 1 denotes the constant function on M equal to 1. We shall assume in the following, that $L(1)$ vanishes identically.

Then, for $f : M \to R$ of class C^2, we have

$$\sigma(\tilde{\nabla})(x) = \frac{1}{2}trace(\tilde{\nabla}^2 f)(x) = \frac{1}{2}(\tilde{\nabla}df)(x)), \tag{3}$$

where the trace is taken in terms of g, and $\tilde{\nabla}df$ is thought as a section of $L(T^*M, T^*M)$. Also, $X_0(\tilde{\nabla}) = L - \sigma(\tilde{\nabla})$. If $\tilde{\Gamma}^{\alpha}_{\beta\gamma}$ is the local representation for the Christoffel symbols of the connection, i.e. $\tilde{\nabla}_{\frac{\partial}{\partial x^\alpha}}\frac{\partial}{\partial x^\beta} = \tilde{\Gamma}^{\nu}_{\alpha\beta}\frac{\partial}{\partial x^\nu}$, then the local representation of $\sigma(\tilde{\nabla})$ is:

$$\sigma(\tilde{\nabla})(x) = \frac{1}{2}g^{\alpha\beta}(x)(\partial_\alpha\partial_\beta + \tilde{\Gamma}^{\gamma}_{\alpha\beta}(x)\partial_\gamma), \tag{4}$$

and

$$X_0(\tilde{\nabla})(x) = B^\alpha(x)\partial_\alpha - \frac{1}{2}g^{\alpha\beta}(x)\tilde{\Gamma}^{\gamma}_{\alpha\beta}\partial_\gamma. \tag{5}$$

If in particular, we take $\tilde{\nabla}$ the Levi-Civita connection associated to g, which we shall denote as ∇^g, then for any $f : M \to R$ of class C^2:

$$\sigma(\nabla^g)(df) = \frac{1}{2}\mathrm{trace}((\nabla^g)^2 f) = \frac{1}{2}\mathrm{trace}(\nabla^g df) = -1/2\mathrm{div_g}\ \mathrm{grad}\ f = 1/2\triangle_g f. \tag{6}$$

Here, $\triangle_g$ is the Levi-Civita laplacian operator on functions; locally it is written as

$$\triangle_g = g^{-1/2}\partial_\alpha((g^{1/2}g^{\alpha\beta}\partial_\beta); \quad g = det(g_{\alpha\beta}), \tag{7}$$

and div_g is the Riemannian divergence operator on vector fields $X = X^\alpha(x)\partial_\alpha$:

$$div_g(X) = -g^{-1/2}\partial_\alpha(g^{1/2}X^\alpha). \tag{8}$$

Note the relation we already have used in eqt. (6) and will be used repeatedly; namely:

$$div_g(X) = -\delta\tilde{X}, \tag{9}$$

where δ is the co-differential operator (see (23) below), and $\tilde{X}$ is the one-form conjugate to the vector field X, i.e. $\tilde{X}_\alpha = g_{\alpha\beta}X^\beta$.

We now take $\tilde{\nabla}$ to be a Cartan connection with torsion [10,46], which we additionally assume to be compatible with g, i.e. $\tilde{\nabla}g = 0$. Then $\sigma(\tilde{\nabla}) = \frac{1}{2}\text{trace}(\tilde{\nabla}^2)$. Let us compute this. Denote the Christoffel coefficients of $\tilde{\nabla}$ as $\tilde{\Gamma}^\alpha_{\beta\gamma}$; then,

$$\tilde{\Gamma}^\alpha_{\beta\gamma} = \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} + 1/2K^\alpha_{\beta\gamma}, \tag{10}$$

where the first term in (10) stands for the Christoffel Levi-Civita coefficients of the metric g, and

$$K^\alpha_{\beta\gamma} = T^\alpha_{\beta\gamma} + S^\alpha_{\beta\gamma} + S^\alpha_{\gamma\beta}, \tag{11}$$

is the cotorsion tensor, with $S^\alpha_{\beta\gamma} = g^{\alpha\nu}g_{\beta\kappa}T^\kappa_{\nu\gamma}$, and $T^\alpha_{\beta\gamma} = \tilde{\Gamma}^\alpha_{\beta\gamma} - \tilde{\Gamma}^\alpha_{\gamma\beta}$ the skew-symmetric torsion tensor.

Let us consider the Laplacian operator associated with this Cartan connection, defined -in extending the usual definition- by

$$H(\tilde{\nabla}) = 1/2\text{trace}\tilde{\nabla}^2 = 1/2g^{\alpha\beta}\tilde{\nabla}_\alpha\nabla_\beta; \tag{12}$$

then, $\sigma(\tilde{\nabla}) = H(\tilde{\nabla})$. A straightforward computation shows that that $H(\tilde{\nabla})$ only depends in the trace of the torsion tensor and g:

$$H(\tilde{\nabla}) = 1/2\Delta_g + g^{\alpha\beta}Q_\beta\partial_\alpha \equiv H_0(g, Q), \tag{13}$$

with $Q = \tilde{T}^\nu_{\nu\beta}dx^\beta$, the trace-torsion one-form.

Therefore, for the Cartan connection $\tilde{\nabla}$ defined in (10), we have that

$$\sigma(\tilde{\nabla}) = \frac{1}{2}\text{trace}(\tilde{\nabla}^2) = \frac{1}{2}\Delta_g + \hat{Q}, \tag{14}$$

with $\hat{Q}$ the vector-field conjugate to the 1-form Q: $\hat{Q}(f) =< Q, \text{grad } f >$, $f : M \to R$. In local coordinates,

$$\hat{Q}^\alpha = g^{\alpha\beta}Q_\beta.$$

We further have:

$$X_0(\tilde{\nabla}) = B - \frac{1}{2}g^{\alpha\beta}\begin{Bmatrix}\gamma\\ \alpha\beta\end{Bmatrix}\partial_\gamma - \hat{Q}, \tag{15}$$

Therefore, the invariant decomposition of L is

$$\frac{1}{2}\mathrm{trace}(\tilde{\nabla}^2) + X_0(\tilde{\nabla}) = \frac{1}{2}\triangle_g + b, \tag{16}$$

with

$$b = B - \frac{1}{2}g^{\alpha\beta}\begin{Bmatrix}\gamma\\ \alpha\beta\end{Bmatrix}\partial_\gamma. \tag{17}$$

Notice that (15) can be thought as arising from a gauge transformation: $\tilde{b} \to \tilde{b} - Q$, with $\tilde{b}$ the 1-form conjugate to b.

If we take for a start $\tilde{\nabla}$ with Christoffel symbols of the form

$$\Gamma^\alpha_{\beta\gamma} = \begin{Bmatrix}\alpha\\ \beta\gamma\end{Bmatrix} + \frac{2}{(n-1)}\left\{\delta^\alpha_\beta\, Q_\gamma \; - \; g_{\beta\gamma}\, Q^\alpha\right\} \tag{18}$$

with

$$Q = \tilde{b}, \quad \text{i.e.} \quad \hat{Q} = b, \tag{19}$$

we have in writing now ∇ for the covariant derivative of (18)

$$X_0(\nabla) = 0,$$

and

$$H_0(g, Q) = \sigma(\nabla) = \frac{1}{2}\mathrm{trace}(\nabla^2) = \frac{1}{2}\mathrm{trace}((\nabla^g)^2) + \hat{Q} = \frac{1}{2}\triangle_g + b. \tag{20}$$

Therefore, for ∇ as in (18) we obtain a gauge theoretical invariant representation for L given by

$$L = H(\nabla) = \frac{1}{2}\nabla^2 = \sigma(\nabla) = \frac{1}{2}\mathrm{trace}((\nabla^g)^2) + \hat{Q} = H_0(g, Q). \tag{21}$$

The restriction we have placed in the metric-compatible $\tilde{\nabla}$ to be as in (18), i.e. only the trace component of the irreducible decomposition of the torsion tensor is taken, is due to the fact that all other components of this tensor do not appear at all in the laplacian of (the otherwise too general) $\tilde{\nabla}$; in other words,

$H_0(g,Q) = \frac{1}{2}(\nabla)^2 = \frac{1}{2}(\tilde{\nabla})^2 = \frac{1}{2}\triangle_g + \hat{Q}$, with $\tilde{\nabla}$ given by $(10-11)$ and ∇ given by (18). In the particular case of dimension 2, this is automatically satisfied. In the case we actually have assumed, g is Riemannian, the expression (21) is the most general invariant laplacian (with zero potential term) acting on functions defined on a smooth manifold. This restriction, will allow us to establish a one-to-one correspondance between Riemann-Cartan connections of the form (18) with (generalized Brownian) diffusion processes. These metric compatible connections we shall call RCW geometries (short for Riemann-Cartan-Weyl), since the trace-torsion is a Weyl 1-form [10]. Thus, these geometries do not have the historicity problem which lead to Einstein's rejection of the first gauge theory ever proposed by Weyl. We would like to remark that we first encountered these connections on developing a pre-symplectic structure for the derivation of the dynamics of relativistic massive spinning systems subjected to exterior gravitational fields [57].

To obtain the most general form of the RCW laplacian, we only need to apply to the trace-torsion one-form the most general decomposition of one-forms on a smooth compact manifold. This amounts to give the constitutive equations of the particular theory of fluctuations under consideration on the manifold M; see [22,26,49]. The answer to this problem, is given by the well known de Rham-Kodaira-Hodge theorem, which we present now.

We consider the Hilbert space of square summable ω of smooth differential forms of degree k on M, with respect to vol_g. We shall denote this space as $L^{2,k}$. The inner product is

$$<< \omega, \phi >>:= \int_M < \omega(x), \phi(x) > vol_g, \tag{22}$$

where the integrand is given by the multiplication between the components $\omega_{\alpha_1 \ldots \alpha_k}$ of ω and the conjugate tensor:$g^{\alpha_1\beta_1} \ldots g^{\alpha_k\beta_k}\phi_{\beta_1\ldots\beta_k}$; alternatively, we can write in a coordinate independent way: $< \omega(x), \phi(x) > vol_g = \omega(x) \wedge *\phi(x)$, with $*$ the Hodge star operator, for any $\omega, \phi \in L^{2,k}$.

The de Rham-Kodaira-Hodge operator on $L^{2,k}$ is defined as

$$\triangle_k = -(d+\delta)^2 = -(d\delta + \delta d), \tag{23}$$

where δ is the formal adjoint defined on $L^{2,k+1}$ of the exterior differential operator d defined on $L^{2,k}$:

$$<< \delta\phi, \omega >>=<< \phi, d\omega >>,$$

for $\phi \in L^{2,k+1}$ and $\omega \in L^{2,k}$. Then, $\delta^2 = 0$.

Let $R : (TM \oplus TM) \oplus TM \to TM$ be the (metric) curvature tensor defined by: $(\nabla^g)^2 Y(v_1, v_2) = (\nabla^g)^2 Y(v_2, v_1) + R(v_1, v_2)Y(x)$. From the Weitzenbock formula [14] we have

$$\triangle_1 \phi(v) = \text{trace } (\nabla^g)^2 \phi(-,-) - Ric_x(v, \hat{\phi}_x),$$

for $v \in T_x M$ and $Ric_x(v_1, v_2) = \text{trace } < R(-, v_1)v_2, - >_x$. Then, $\triangle_0 = (\nabla^g)^2 = \triangle_g$ so that in the case of $k = 0$, the de Rham-Kodaira operator coincides with the Laplace-Beltrami operator on functions.

The de Rham-Kodaira-Hodge theorem states that $L^{2,1}$ admits the following invariant decomposition. Let $\omega \in L^{2,1}$; then,

$$\omega = d\, f + A_1 + A_2, \tag{24}$$

where $f : M \to R$ is a smooth function on M, A_1 is a co-closed smooth 1-form: $\delta A_1 = -div_g \hat{A}_1 = 0$, and A_2 is a co-closed and closed smooth 1-form:

$$\delta A_2 = 0, dA_2 = 0. \tag{25}$$

Otherwise stated, A_2 is an harmonic one-form, i.e.

$$\triangle_1 A_2 = 0. \tag{26}$$

Furthermore, this decomposition is orthogonal in $L^{2,1}$, i.e.:

$$<< df, A_1 >>=<< df, A_2 >>=<< A_1, A_2 >>= 0. \tag{27}$$

Remark 1. Note that $A_1 + A_2$ is itself a co-closed one-form. If we consider an augmented configuration space $R \times M$ for an incompressible fluid, this last decomposition will be the fluid's velocity. If we consider instead a four-dimensional Lorentzian manifold provided with a Dirac-Hestenes spinor operator field (DHSOF), one needs the whole decomposition (24) associated to an invariant density ρ of the diffusion (i.e. a solution of the equation $H_0(g, Q)^\dagger \rho = 0$) to describe two electromagnetic potentials such that when restricted to the spin-plane of the DHSOF, they enforce the equivalence between the Dirac-Hestenes equation for the DHSOF on a manifold provided with a RCW connection, and the free Maxwell equation on the Lorentzian manifold; see [26, 64].

3 Generalized Laplacians on Differential Forms

Consider the family of zero-th order differential operators acting on smooth k-forms, i.e. differential forms of degree k ($k = 0, \ldots, n$) defined on M:

$$H_k(g, Q) := 1/2 \triangle_k + L_{\hat{Q}}, \tag{28}$$

The second term in (28) denotes the Lie-derivative with respect to the vectorfield $\hat{Q}$. Recall that the Lie-derivative is independant of the metric:for any smooth vectorfield X on M

$$L_X = i_X d + d i_X,, \tag{29}$$

where i_X is the interior product with respect to X: for arbitrary vectorfields $X_1, \ldots, X_{k-1}$ and ϕ a k-form defined on M, we have $(i_X\phi)(X_1, \ldots, X_{k-1}) = \phi(X, X_1, \ldots, X_{k-1})$. Then, for f a scalar field, $i_X f = 0$ and

$$L_X f = (i_X d + d i_X) f = i_X df = g(\tilde{X}, df) = X(f). \tag{30}$$

where $\tilde{X}$ denotes the 1-form associated to a vectorfield X on M via g. We shall need later the following identities between operators acting on smooth k-forms, which follow easily from algebraic manipulation of the definitions:

$$d\triangle_k = \triangle_{k+1} d, \ k = 0, \ldots, n, \tag{31}$$

and

$$\delta\triangle_k = \triangle_{k-1}\delta, \ k = 1, \ldots, n, \tag{32}$$

and finally, for any vectorfield X on M we have that $dL_X = L_X d$ and therefore

$$dH_k(g, Q) = H_{k+1}(g, Q) d, \ k = 0, \ldots, n. \tag{33}$$

In (28) we retrieve for scalar fields ($k = 0$) the operator $H(g, Q)$ defined in (21).

Proposition 1: Assume that g is non-degenerate. There is a one-to-one mapping

$$\nabla \rightsquigarrow H_k(g, Q) = 1/2\triangle_k + L_{\hat{Q}}$$

between the space of g-compatible affine connections ∇ with Christoffel coefficients of the form

$$\Gamma^{\alpha}_{\beta\gamma} = \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} + \frac{2}{(n-1)} \left\{ \delta^{\alpha}_{\beta}\, Q_\gamma - g_{\beta\gamma}\, Q^{\alpha} \right\} \tag{34}$$

and the space of elliptic second order differential operators on k-forms ($k = 0, \ldots, n$) with zero potential term.

4 Riemann-Cartan-Weyl Connections and the Laplacians for Differential Forms

In this section we shall construct the diffusion processes for scalar fields.

In the following we shall further assume that $Q = Q(\tau, x)$ is a time-dependant 1-form, so that we have a time-dependant RCW connection on M, which we think of as a space manifold. The stochastic flow associated to the diffusion generated by $H_0(g, Q)$ has for sample paths the continuous curves $\tau \mapsto x_\tau \in M$ satisfying the Ito invariant non-degenerate s.d.e. (stochastic differential equation)

$$dx(\tau) = X(x(\tau))dW(\tau) + \hat{Q}(\tau, x(\tau))d\tau. \tag{35}$$

In this expression, the diffusion tensor $X = (X^\alpha_\beta(x))$ is a linear surjection $X(x) : R^m \rightarrow T_xM$ satisfying $X^\alpha_\nu X^\beta_\nu = g^{\alpha\beta}$, and $\{W(\tau), \tau \geq 0\}$ is a standard Wiener process on R^n. Thus $< W_\tau >= 0$ and $< W^i_\tau W^j_\tau >= \delta_{ij}\tau$, where $< - >$ denotes expectation with respect to the zero-mean standard Gaussian function on R^m $(m \geq n)$. Here τ denotes the time-evolution parameter of the diffusion (in a relativistic setting it should not be confused with the time variable), and for simplicity we shall assume always that $\tau \geq 0$. Consider the canonical Wiener space Ω of continuous maps $\omega : R \rightarrow R^n, \omega(0) = 0$, with the canonical realization of the Wiener process $W(\tau)(\omega) = \omega(\tau)$. The (stochastic) flow of the s.d.e. (35) is a mapping

$$F_\tau : M \times \Omega \rightarrow M, \quad \tau \geq 0, \tag{36}$$

such that for each $\omega \in \Omega$, the mapping $F_.(. \,, \omega) : [0, \infty) \times M \rightarrow M$, is continuous and such that $\{F_\tau(x) : \tau \geq 0\}$ is a solution of equation (35) with $F_0(x) = x$, for any $x \in M$.

Let us assume in the following that the components X^α_β, $\hat{Q}^\alpha$, $\alpha, \beta = 1, \ldots, n$ of the vectorfields X and $\hat{Q}$ on M in (35) are predictable functions which further belong to $C^{m,\epsilon}_b$ $(0 \leq \epsilon \leq 1$, m a non-negative integer), the space of Hoelder bounded continuous functions of degree $m \geq 1$ and exponent ϵ, and also that $\hat{Q}^\alpha(\tau) \in L^1(R)$, for any $\alpha = 1, \ldots, n$. With these regularity conditions, if we further assume that $\{x(\tau) : \tau \geq 0\}$ is a semimartingale (see Appendix) on a probability space $(\Omega, \mathcal{F}, P)$, then it follows from Kunita [24] that (35) has a modification (which with abuse of notation we denote as)

$$F_\tau(\omega) : M \rightarrow M, \quad F_\tau(\omega)(x) = F_\tau(x, \omega), \tag{37}$$

which is a diffeomorphism of class C^m, almost surely for $\tau \geq 0$ and $\omega \in \Omega$. We can obtain an identical result if we assume instead Sobolev regularity conditions.

Indeed, assume that the components of σ and $\hat{Q}$, $\sigma_i^\beta \in H^{s+2}(T^*M)$ and $\hat{Q}^\beta \in H^{s+1}(T^*M)$, $1 \leq i \leq m$, $1 \leq \beta \leq n$, where the Sobolev space $H^s(T^*M) = W^{2,s}(T^*M)$ with $s > \frac{n}{2} + m$ [52]. Then, the flow of (35) for fixed ω defines a diffeomorphism in $H^s(M, M)$ (see [53]), and hence by the Sobolev embedding theorem, a diffeomorphism in $C^m(M, M)$ (i.e. a mapping from M to M which is m-times continuously differentiable as well as its inverse.) In any case, for $1 \leq m$ we can consider the Jacobian (velocity) flow of $\{x_\tau : \tau \geq 0\}$. It is a random diffusion process on TM, the tangent bundle of M.

Remarks 2: In the differential geometric approach -pioneered by V. Arnold- for integrating NS on a smooth manifold as a perturbation (due to the diffusion term we shall present below) of the geodesic flow in the group of volume preserving diffeomorphisms of M (as the solution of the Euler equation), it was proved that under the above regularity conditions on the initial velocity, the solution flow of NS defines a diffeomorphism in M of class C^m; see Ebin and Marsden [9]. The difference of this classical approach with the one presented here, is to integrate NS through a time-dependant **random** diffeomorphism associated with a RCW connection. As wellknown, these regularity conditions are basic in the usual functional analytical treatment of NS pioneered by Leray [45] (see also Temam [7]), and they are further related to the multifractal structure of turbulence [41]. This diffeomorphism property of random flows is fundamental for the construction of their ergodic theory (provided an invariant measure for the processes exists), and in particular, of quantum mechanics amd non-linear non-equilibrium thermodynamics [10,21,22,26,49].

Let us describe now the Jacobian flow. We can describe it as the stochastic process on the tangent bundle, TM, given by $\{v(\tau) := T_{x_0}F_\tau(v(0)) \in T_{F_\tau(x_0)}M$, $v(0) \in T_{x_0}M\}$; here T_zM denotes the tangent space to M at z and $T_{x_0}F_\tau$ is the linear derivative of F_τ at x_0. The process $\{v_\tau, \tau \geq 0\}$ can be described (see [27]) as the solution of the invariant Ito s.d.e. on TM:

$$dv(\tau) = \nabla^g \hat{Q}(\tau, v(\tau))d\tau + \nabla^g X(v(\tau))dW(\tau) \tag{38}$$

If we take U to be an open neighborhood in R^n so that $TU = U \times R^n$, then $v(\tau) = (x(\tau), \tilde{v}(\tau))$ is described by the system given by integrating (35) and the covariant Ito s.d.e.

$$d\tilde{v}(\tau)(x(\tau)) = \nabla^g X(x(\tau))(\tilde{v}(\tau))dW(\tau) + \nabla^g \hat{Q}(\tau, x(\tau))(\tilde{v}(\tau))d\tau, \tag{39}$$

with initial condition $\tilde{v}(0) = v_0 \in T_{x(0)}$. Thus, $\{v(\tau) = (x(\tau), \tilde{v}(\tau)), \tau \geq 0\}$ defines a random flow on TM.

Theorem 2 : For any differential 1-form ϕ of class $C^{1,2}(R \times M)$ (i.e. in a local coordinate system $\phi = a_\alpha(\tau)dx^\alpha$, with $a_\alpha(\tau, .) \in C^2(M)$ and $a_\alpha(., x) \in C^1(R)$) we have the Ito formula (Corollary 3E1 in [27]):

$$\begin{aligned}
\phi(v_\tau) &= \phi(v_0) + \int_0^\tau \phi(\nabla^g X(v_s) dW_s + \int_0^\tau [\frac{\partial}{\partial s} + H_1(g,Q)]\phi(v_s)ds \\
&+ \int_0^\tau \nabla^g \phi(X(x)dW_s)(v_s) \\
&+ \int_0^\tau \text{trace } d\phi(X(x_s)-, \nabla^g X(v_s))(-)ds \qquad (40)
\end{aligned}$$

In the last term in (40) the trace is taken in the argument $-$ of the bilinear form and further we have the mappings

$$\nabla^g Y : TM \to TM; \nabla^g \phi : TM \to T^*M.$$

Remarks 3 : From (40) we conclude that the infinitesimal generators (i.g., for short in the following) of the derived stochastic process is not $\partial_\tau + H_1(g,Q)$, due to the last term in (40). This term vanishes identically in the case we shall present in the following section, that of gradient diffusions. An alternative method which bypasses the velocity process is the construction of the generalized Hessian flow further below. Both methods will provide for the setting for the integration of the Navier-Stokes equations.

5 Riemann-Cartan-Weyl Gradient Diffusions

Suppose that there is an isometric embedding of an n-dimensional compact orientable manifold M into a Euclidean space R^m:$f : M \to R^m, f(x) = (f^1(x),\ldots,f^m(x))$. Suppose further that $X(x) : R^m \to T_xM$, is the orthogonal projection of R^m onto T_xM the tangent space at x to M, considered as a subset of R^m. Then, if $e_1,\ldots,e_m$ denotes the standard basis of R^m, we have

$$X = X^i e_i, \text{ with } X^i = \text{grad } f^i, i = 1,\ldots,m. \qquad (41)$$

The second fundamental form [25] is a bilinear symmetric map

$$\alpha_x : T_xM \times T_xM \to \nu_xM, x \in M, \qquad (42)$$

with $\nu_xM = (T_xM)^\perp$ the space of normal vectors at x to M. We then have the associated mapping

$$A_x : T_xM \times \nu_xM \to T_xM, < A_x(u,\zeta), v >_{R^m} = < \alpha_x(u,v), \zeta >_{R^m}, \qquad (43)$$

for all $\zeta \in \nu_xM$, $u, v \in T_xM$. Let $Y(x)$ be the orthogonal projection onto ν_xM

$$Y(x) = e - X(x)(e), x \in M, e \in R^m. \qquad (44)$$

Then:

$$\nabla^g X(v)(e) = A_x(v, Y(x)e), v \in T_xM, x \in M. \tag{45}$$

For any $x \in M$, if we take $e_1, \dots, e_m$ to be an orthonormal base for R^m such that $e_1, \dots, e_m \in T_xM$, then for any $v \in T_xM$,we have

$$\text{either } \nabla^g X(v)e_i = 0, \text{ or } X(x)e_i = 0. \tag{46}$$

We shall consider next the RCW gradient diffusion processes, i.e. for which in equation (35) we have specialized taking $X = \text{grad}f$. Let $\{v_\tau : \tau \geq 0\}$ be the associated derived velocity process. We shall now give the Ito-Elworthy formula for 1-forms.

Theorem 3. Let $f : M \to R^m$ be an isometric embedding. For any differential form ϕ of degree 1 in $C^{1,2}(R \times M)$, the Ito formula is

$$\begin{aligned} \phi(v_\tau) &= \phi(v_0) + \int_0^\tau \nabla^g \phi(X(x_s)dW_s)v_s + \int_0^\tau \phi(A_x(v_s, Y(x_s)dW_s) \\ &+ \int_0^\tau [\frac{\partial}{\partial s} + H_1(g, Q)]\phi(v_s)ds, \end{aligned} \tag{47}$$

i.e. $\partial_\tau + H_1(g, Q)$, is the i.g. (with domain the differential 1-forms belonging to $C^{1,2}(R \times M)$) of $\{v_\tau : \tau \geq 0\}$.

Proof: It follows immediately from the facts that the last term in the r.h.s. of (40) vanishes due to (46), while the second term in the r.h.s. of (40) coincides with the third term in (47) due to (45).

Consider the value Φ_x of a k-form at $x \in M$ as a linear map: $\Phi_x : \Lambda^k T_xM \to R$. In general, if E is a vector space and $A : E \to E$ is a linear map, we have the induced maps

$$\Lambda^k A : \Lambda^k E \to \Lambda^k E, \ \Lambda^k(v^1 \wedge \dots \wedge v^k) := Av^1 \wedge \dots \wedge Av^k;$$

and

$$(d\Lambda^k)A : \Lambda^k E \to \Lambda^k E,$$

$$(d\Lambda^k)A(v^1 \wedge \dots \wedge v^k) := \sum_{j=1}^k \ v^1 \wedge \dots \wedge v^{j-1} \wedge Av^j \wedge v^{j+1} \wedge \dots \wedge v^k.$$

For $k = 1, (d\Lambda)A = \Lambda A$. The Ito formula for k-forms, $1 \leq k \leq n$, is due to Elworthy (Prop. 4B [27]).

Theorem 4. Let M be isometrically embedded in R^m. Let $V_0 \in \Lambda^k T_{x_0} M$. Set $V_\tau = \Lambda^k(TF_\tau)(V_0)$ Then for any differential form ϕ of degree k in $C^{1,2}(R\times M)$, $1 \leq k \leq n$,

$$\begin{aligned} \phi(V_\tau) &= \phi(V_0) + \int_0^\tau \nabla^g \phi(X(x_s)dWs)(V_s) \\ &+ \int_0^\tau \phi((d\Lambda)^k A_{x_s}(-, Y(x_s)dW_s)(V_s)) \\ &+ \int_0^\tau [\frac{\partial}{\partial s} + H_k(g, \hat{Q})]\phi(V_s)ds \end{aligned} \tag{48}$$

i.e., $\partial_\tau + H_k(g, \hat{Q})$ is the i.g. (with domain of definition the differential forms of degree k in $C^{1,2}(R \times M)$) of $\{V_\tau : \tau \geq 0\}$.

Remarks 4: Therefore, starting from the flow $\{F_\tau : \tau \geq 0\}$ of the s.d.e. (35) with i.g. given by $\partial_\tau + H_0(g, Q)$, we obtained that the derived velocity process $\{v(\tau) : \tau \geq 0\}$ given by (38) (or (35) and (39)) has $H_1(g, Q)$ as i.g.; finally, if we consider the diffusion of differential forms of degree $k \geq 1$, we get that $\partial_\tau + H_k(g, Q)$ is the i.g. of the process $\Lambda^k v(\tau)$, i.e. the exterior product of degree k ($k = 1, \ldots, n$) of the velocity process. In particular, $\partial_\tau + H_2(g, Q)$ is the i.g. of the stochastic process $\{v(\tau) \wedge v(\tau) : \tau \geq 0\}$.

Note that consistently with the notation we have that $\{\Lambda^0 v_\tau : \tau \geq 0\}$ is the position process $\{x_\tau : \tau \geq 0\}$ untop of which $\{\Lambda^k v_\tau : \tau \geq 0\}$, $(1 \leq k \leq n)$ is fibered (recall, $\Lambda^0(M) = M$). We can resume our results in the following theorem.

Theorem 5. Assume M is isometrically embedded in R^m. There is a one to one correspondance between RCW connections ∇ determined by a Riemannian metric g and trace-torsion Q with the family of gradient diffusion processes $\{\Lambda^k v_\tau : \tau \geq 0\}$ generated by $H_k(g, Q)$, $k = 0, \ldots, n$

Finally, we are now in a situation for presenting the solution of the Cauchy problem

$$\frac{\partial \phi}{\partial \tau} = H_k(g, Q_\tau)(x)\phi, \tau \in [0, T] \tag{49}$$

with the given initial condition

$$\phi(0, x) = \phi_0(x), \tag{50}$$

for ϕ and ϕ_0 k-forms on a smooth compact orientable manifold isometrically embedded in R^m. From the Ito-Elworthy formula follows that the formal solution of this problem is as follows: Consider the diffusion process on M generated by $H_0(g, Q)$: For each $\tau \in [0, T]$ consider the s.d.e. (with $s \in [0, \tau]$):

$$dx_s^{\tau,x} = X(x_s^{\tau,x})dW_s + \hat{Q}(\tau - s, x_s^{\tau,x})ds, \tag{51}$$

with initial condition

$$x_0^{\tau,x} = x, \tag{52}$$

and the derived velocity process $\{v_s^{\tau,v(x)} = x_s^{\tau,x}, \tilde{v}_s^{\tau,v(x)}, 0 \leq s \leq \tau\}$:

$$d\tilde{v}_s^{\tau,v(x)} = \nabla^g X(x_s^{\tau,x})(\tilde{v}_s^{\tau,v(x)})dW_s + \nabla^g \hat{Q}(\tau - s, x_s^{\tau,x})(\tilde{v}_s^{\tau,v(x)})ds, \tag{53}$$

with initial condition

$$\tilde{v}_0^{\tau,v(x)} = v(x). \tag{54}$$

Then, the $C^{1,2}$ (formal) solution of the Cauchy problem defined in $[0,T] \times M$ is

$$\phi(\tau, x)(\Lambda^k v(x)) = E_x[\phi_0(x_\tau^{\tau,x})(\Lambda^k \tilde{v}_\tau^{\tau,v(x)})]. \tag{55}$$

6 The Navier-Stokes Equation and Riemann-Cartan-Weyl Diffusions

In the sequel, M is a compact orientable (possible with smooth boundary ∂M) n-manifold with a Riemannian metric g. We provide M with a 1-form whose de Rham-Kodaira-Hodge (RKH for short) decomposition is

$$Q(x) = df(x) + u(x), \quad \delta u = -div(\hat{u}) = 0,$$

where f is a scalar field and u is a coclosed 1-form, weakly orthogonal to df, i.e. $\int g(df, u)vol_g = 0$. We shall assume that $u(x, 0) = u(x)$ is the initial velocity 1-form of an incompressible viscous fluid on M, and that we further have a 1-form $Q(x, \tau) = Q_\alpha(x, \tau)dx^\alpha$ whose RKH decomposition is:

$$Q(x, \tau) = df(x, \tau) + u(x, \tau), \tag{56}$$

with $\delta u_\tau(x) = \delta u(x, \tau) = 0$ (incompressibility condition), and

$$\int g(df_\tau, u_\tau)vol(g) = 0,$$

which satisfies the evolution equation on $M \times R$ (Eulerian representation of the fluid):

$$\frac{\partial Q_\alpha}{\partial \tau} + \nabla_{\hat{u}}^g Q_\alpha = -Q_\beta \nabla_\alpha^g u^\beta + \nu \triangle_1 Q_\alpha, \tag{57}$$

Here ν is the kinematical viscosity. In the above notations and in the following, all covariant operators act in the M variables only. In the formulation of Fluid Mechanics in Euclidean domains, $Q(x,\tau)$ receives the name of (Buttke) "magnetization variable" [1].

Remarks 5: We recall that to take the RKH decomposition of the velocity of a viscous fluid is a basic procedure in Fluid Mechanics [1,6,7,9]. We shall see below that Q and in particular u are related to a natural RCW geometry of the incompressible fluid. In the formulation of Quantum Mechanics and of non-linear non-equilibrium thermodynamics stemming from RCW diffusions, we have a RKH decomposition of the trace-torsion associated to a stationary state; see [10,21,22,26,49]. This decomposition allows to associate with the divergenceless term of the trace-torsion a probability current which characterizes the time-invariance symmetry breaking of the diffusion process, and is central to the construction of the ergodic theory of these flows.

Equation (57) is the gauge-invariant form of the NS for the velocity 1-form $u(x,\tau)$. Indeed, if we substitute the decomposition of $Q(x,\tau) = Q_\tau(x)$ into (57) we obtain,

$$\frac{\partial u}{\partial \tau} + \nabla^g_{\hat{u}_\tau} u_\tau = \nu \triangle_1 u_\tau - d(\frac{\partial f}{\partial \tau} + \nabla^g_{\hat{u}_\tau} f + \frac{1}{2}|u_\tau|^2 - \nu \triangle_g f). \quad (58)$$

Consider the operator P of projection of 1-forms into co-closed 1-forms: $P\omega = \alpha$ for any one-form ω whose RKH decomposition is $\omega = df + \alpha$, with $\delta\alpha = 0$. From (32) we get that

$$P\triangle_1 u_\tau = \triangle_1 u_\tau, \quad (59)$$

and further applying P to (57) we finally get the well known covariant NS (with no exterior forces; the gradient of the pressure term disappears by projecting with P [1,9])

$$\frac{\partial u}{\partial \tau} + P[\nabla^g_{\hat{u}_\tau} u_\tau] - \nu \triangle_1 u_\tau = 0. \quad (60)$$

Conversely, starting with equation (58) which is equivalent to NS we obtain (57). Note that Q_τ and u_τ differ by a differential of a function for all times. Multiplication of (58) by $I - P$ (I the identity operator) yields an equation for the evolution of f which is only arbitrary for $\tau = 0$. Now we note that the non-linearity of NS originates from applying P to the term

$$\nabla^g_{\hat{u}_\tau} u_\tau = i_{\hat{u}_\tau} du_\tau,$$

which taking in account (29) can still be written as

$$L_{\hat{u}_\tau} u_\tau - di_{\hat{u}_\tau} u_\tau = L_{\hat{u}_\tau} u_\tau - 1/2d(|u_\tau|^2). \quad (61)$$

Applying P to (61), we see that the kinetic energy term there disappears and the non-linear term in NS can be written as

$$P[\nabla^g_{\hat{u}_\tau} u_\tau] = P[L_{\hat{u}_\tau} u_\tau]. \tag{62}$$

Therefore, from (28) and (62), NS (60) takes the final concise form

$$\frac{\partial u}{\partial \tau} = PH_1(2\nu g, \frac{-1}{2\nu} u_\tau) u_\tau. \tag{63}$$

Therefore we have found that NS for the velocity of an incompressible fluid is a a non-linear diffusion equation.

Let us introduce the vorticity two-form

$$\Omega_\tau = du_\tau. \tag{64}$$

Note that also $\Omega_\tau = dQ_\tau$. Now, if we know Ω_τ for any $\tau \geq 0$, we can obtain u_τ (or still Q_τ) by inverting the definition (64). Namely, applying δ to (64) and taking in account (23) we obtain the Poisson-de Rham equation (would g be hyperbolic, it is the Maxwell-de Rham equation [10a])

$$\triangle_1 u_\tau = -\delta\Omega_\tau. \tag{65}$$

and an identical equation for Q_τ. (Note that if we know Q_τ we can reconstruct f_τ by solving $-\delta Q_\tau = div(\hat{Q}_\tau) = \triangle_g f_\tau$, for any τ.) From the Weitzenbock formula, we can write (65) showing the coupling of the Ricci metric curvature to the velocity $u = u_\alpha(x,\tau)dx^\alpha$:

$$(\nabla^g)^2 u_\tau - R_{\alpha\beta} u^\beta_\tau dx^\alpha = -\delta\Omega_\tau. \tag{66}$$

with $R^\beta_\alpha(g) = R_{\mu\alpha}{}^{\mu\beta}(g)$, the Ricci (metric) curvature tensor. Thus, the vorticity Ω_τ is a source for the velocity one-form u_τ, for all τ; in the case that M is a compact euclidean domain, equation (66) is integrated to give the Biot-Savart law of Fluid Mechanics [1,39].

Now, apply d to (63) and further RKH decompose $L_{-\hat{u}_\tau} u_\tau = \alpha_\tau + dp_\tau$ (with p_τ the pressure at time τ); in account that

$$dPL_{-\hat{u}_\tau} u_\tau = d\alpha_\tau = d(\alpha_\tau + dp_\tau) = dL_{-\hat{u}_\tau} u_\tau = L_{-\hat{u}_\tau} du_\tau = L_{-\hat{u}_\tau}\Omega_\tau,$$

and that from (31) we have that $d\triangle_1 u_\tau = \triangle_2 \Omega_\tau$, we therefore obtain the linear evolution equation

$$\frac{\partial \Omega_\tau}{\partial \tau} = H_2(2\nu g, \frac{-1}{2\nu} u_\tau)\Omega_\tau. \tag{67}$$

Thus, we have proved that the Navier-Stokes equation is a linear diffusion equation generated by a RCW connection. This connection has $2\nu g$ for the metric, and the time-dependant trace-torsion of this connection is $-u/2\nu$. Then, the drift of this process does not depend explicitly on ν, as it coincides with the vectorfield associated via g to $-u_\tau$, i.e.$-\hat{u}_\tau$. Notice that when ν tends to zero, i.e. the Euler equation, the trace-torsion becomes singular.

Theorem 6 : Given a compact orientable Riemannian manifold with metric g, the Navier-Stokes equation (63) for an incompressible fluid with velocity one-form $u = u(\tau, x)$ such that $\delta u_\tau = 0$, assuming sufficiently regular conditions, are equivalent to a linear diffusion process for the vorticity given by (67) with u_τ satisfying the Poisson-de Rham equation (65). The RCW connection on M generating this process is determined by the metric $2\nu g$ and a trace-torsion 1-form given by $-u/2\nu$.

Remarks 6: We would like to recall that in the gauge theory of gravitation [46,57] the torsion is related to the translational degrees of freedom present in the Poincare group, i.e. to the gauging of momentum. Here we find a similar, yet dynamical situation, in which the trace-torsion is related to the velocity. We would like to point out further, than on setting the skew-symmetric torsion to be zero on taking RCW connnections, due to the fact that only the trace-torsion appears in the laplacian generating generalized Brownian motions, we are setting to zero the inertial fields which can be associated with the skew-symmetric torsion. Thus, generalized Brownian motions are not generated by inertial fields [64].

7 Random Diffeomorphims and the Navier-Stokes Equations

In the following we assume additional conditions on M, namely that $f : M \rightarrow R^m$ is an isometric embedding, and that M has no boundary.

Let u denote a solution of (63) (which we assume that exists for $\tau \in [0,T]$), and consider the flow $\{F^\nu_\tau : \tau \geq 0\})$ of the s.d.e. whose i.g. is $\frac{\partial}{\partial\tau} + H_0(2\nu g, \frac{-1}{2\nu}u)$; from (35) we know that this is the flow defined by integrating the non-autonomous Ito s.d.e. with $X = \nabla f$)

$$dx^{\nu,\tau,x} = [2\nu]^{\frac{1}{2}}X(x^{\nu,\tau,x})dW(\tau) - \hat{u}(\tau, x^{\nu,\tau,x})d\tau, x^{\nu,0,x} = x, \tau \in [0,T]. \qquad (68)$$

We shall assume in the following that the diffusion tensor X and the drift $\hat{u}_\tau$ have the regularity conditions stated in Section 4, so that the randoms flows of (68) is a diffeomorphism of M of class C^m, $m \geq 1$.

Theorem 7: Equation (68) is a random Lagrangian representation for the fluid particles positions, i.e. $x(\tau)$ is the random position of the fluid particles of the incompressible fluid whose velocity obeys (63).

Remark 7: Note that the drift of the derived process $\{\tilde{v}_\tau : \tau \geq 0\}$ is minus the deformation tensor of the fluid (see equation (71) further below). This will have a crucial role in the solution for the vorticity equation as well as the kinematic dynamo problem. We further note that if in (68) we set the viscosity to zero, we get the classical flow of the Euler equation.

7.1 Cauchy Problem for the Vorticity

Let us solve the Cauchy problem for $\Omega(\tau, x)$ of class C^m in $[0,T] \times M$ satisfying (67) with initial condition $\Omega_0(x)$.

For each $\tau \in [0,T]$ consider the s.d.e. (with $s \in [0,\tau]$) (obtained by running backwards the Lagrangian representation (68) above):

$$dx_s^{\nu,\tau,x} = (2\nu)^{\frac{1}{2}} X(x_s^{\nu,\tau,x}) dW_s - \hat{u}(\tau - s, x_s^{\nu,\tau,x}) ds, \qquad (69)$$

with initial condition

$$x_0^{\nu,\tau,x} = x, \qquad (70)$$

and the derived velocity process $\{v_s^{\nu,\tau,v(x)} = (x_s^{\nu,\tau,x}, \tilde{v}_s^{\nu,\tau,v(x)}), 0 \leq s \leq \tau\}$:

$$d\tilde{v}_s^{\nu,\tau,v(x)} = (2\nu)^{\frac{1}{2}} \nabla^g X(x_s^{\nu,\tau,x})(\tilde{v}_s^{\nu,\tau,v(x)}) dW_s - \nabla^g \hat{u}(\tau - s, x_s^{\nu,\tau,x})(\tilde{v}_s^{\nu,\tau,v(x)}) ds, \qquad (71)$$

with initial condition

$$\tilde{v}_0^{\nu,\tau,v(x)} = v(x). \qquad (72)$$

Theorem 8: Let $\Omega_\tau(x)$ be a bounded $C^{1,2}$ solution of the Cauchy problem; then it follows from the Ito formula-Elworthy (48) (with $k = 2$) is

$$\tilde{\Omega}_\tau(\Lambda^2 v(x)) = E_x[\Omega_0(x_\tau^{\nu,\tau,x})(\Lambda^2 \tilde{v}_\tau^{\nu,\tau,v(x)})] \qquad (73)$$

where the expectation value at x is taken with respect to the measure on the process $\{x_\tau^{\nu,\tau,x} : \tau \in [0,T]\}$ (whenever it exists):

Proof: It follows just from applying the Ito-Elworthy formula for 2-forms.c.q.d.

Remarks 8: We would like to examine the physical interpretation of the representation (73). To determine the vorticity at time τ on a point x evaluated on a bivector $\Lambda^2 v(x)$, we run backwards the random lagrangian representations starting at time τ at x, and its Jacobian flow starting at $v(x)$ along which we transport the time-zero vorticity, and further we take the mean value along all possible random paths. Furthermore, the transport of Ω_0 along this Jacobian flow, indicates that the original vorticity is acted upon by the fluid-deformation tensor and the gradient noise term.

8 Integration of the Poisson-de Rham equation

In (68) we have that u_τ verifies (65), for every $\tau \geq 0$ which from (28) we can rewrite as

$$H_1(g,0)u_\tau = -\frac{1}{2}\delta\Omega_\tau, \text{ for any } \tau \geq 0. \quad (74)$$

This last representation together with the Ito-Elworthy formula for 1-forms, indicates automatically what the representation for the solution is: We have to construct a Jacobian process on TM originated from the derivative of the scalar diffusion with zero drift and diffusion tensor defined by g.

Thus, consider the autonomous s.d.e. generated by $H_0(g,0) = \frac{1}{2}\Delta_g$:

$$dx_s^{g,x} = X(x_s^g)dW_s, x_0^{g,x} = x. \quad (75)$$

We shall solve the Dirichlet problem in an open set U (of a partition of unity) of M given by (74) with the boundary condition $u_\tau \equiv \phi$ on ∂U, with ϕ a given 1-form. Then one can "glue" the solutions and use the strong Markov property to obtain a global solution (cf. [31]). Consider the derived velocity process $v^g(s) = (x^g(s), \tilde{v}^g(s))$ on TM, with $\tilde{v}^g(s) \in T_{x^g(s)}M$, whose i.g. is $H_1(g,0)$, i.e. from (35) we have:

$$d\tilde{v}_s^{g,v(x)}(x_s^{g,x}) = \nabla^g X(x^{g,x}(s))(\tilde{v}^{g,v(x)}(s))dW(s), \quad (76)$$

with initial velocity $\tilde{v}^{g,v(x)}(0) = v(x)$. Notice that equations (75&76) are obtained by taking $u \equiv 0$ in equations (69&71) respectively, and further rescaling by $(2\nu)^{-\frac{1}{2}}$. Then if u_τ is a solution of (74) for any fixed $\tau \in [0,T]$, applying to it the Ito-Elworthy formula and assuming further that $\delta\Omega_\tau$ is bounded, we then obtain that the formal $C^{1,2}$ solution of the Dirichlet problem is given by:

$$\tilde{u}_\tau(x)(v(x)) = E_x^B[\phi(x_{\tau_e}^{g,x})(v_{\tau_e}^{g,v(x)} + \int_0^{\tau_e} \frac{1}{2}\delta\Omega_\tau(x_s^{g,x}(v_s^{g,v(x)})ds]$$

$$= \int [\phi(x_{\tau_e}^{g,x})(v^{g,v(x)\tau_e}) + 1/2 \int_0^{\tau_e} \delta\Omega_\tau(y)(v_s^{g,v(x)}(y)ds]p^g(s,x,y)vol_g(y), \quad (77)$$

where $\tau_e = \inf\{s : x_s^{g,x} \notin U\}$, the first-exit time of U of the process $\{x_s^{g,x}\}$, and E^B denotes the expectation value with respect to $p^g(s,x,y)$, the transition density of the s.d.e. (76), i.e.the fundamental solution of the heat equation on M:

$$\partial_\tau p(y) = H_0(g,0)(y)p(y) \equiv 1/2\Delta_g p(y) \quad (78)$$

with $p(s,x,-) = \delta_x$ as $s \downarrow 0$.

Theorem 9: Assume g is uniformly elliptic, and U has a $C^{2,\epsilon}$-boundary, and furthermore $g^{\alpha\beta}$ and $\delta\Omega_\tau$ are Hoelder-continuous of order ϵ on U and u_τ is uniformly Hoelder-continuous of order ϵ, for $\tau \in [0,T]$. Then the solution of the Dirichlet problem above has a unique solution belonging to $C^{2,\epsilon}(U)$ for each $\tau \in [0,T]$ (the maximum principle) [31,47]. Assume instead that $u_\tau \in H^1(T^*U)$ for each $\tau \in [0,T]$, i.e. belongs to the Sobolev space of order 1. If $\delta\Omega_\tau \in H^{k-1}(\Lambda^1(T^*U))$, then $u_\tau \in H^{k+1}(\Lambda^1(T^*U))$, for $k \geq 1$ and $\tau \in [0,T]$ (cf. [53]).

Notice that in the representations (73&77), the local dependance on the curvature is built-in (the curvature is defined by second-order derivatives). This dependance might be exhibited through the scalar curvature term in the Onsager-Machlup lagrangian appearing in the path-integral representation of the fundamental solution of the transition densities of equations (73) and (77) [35,44]. There is further a dependance of the solution on the global geometry and topology of M appearing through the Riemannian spectral invariants of M in the short-time asymptotics of these transition densities [28,29,43].

9 Kinematic Dynamo Problem of Magnetohydrodynamics

The kinematic dynamo equation for a passive magnetic field transported by an incompressible fluid, is the system of equations [56] for the time-dependant magnetic vectorfield $B(\tau,x) = B_\tau(x)$ on M defined by $i_{B_\tau}\mu(x) = \omega_\tau(x)$ (for $\tau \geq 0$), satisfying

$$\partial_\tau \omega + (L_{\hat{u}_\tau} - \nu^m \Delta_{n-1})\omega_t = 0, \omega(0,x) = \omega(x), 0 \leq t, \tag{79}$$

where ν^m is the magnetic diffusivity, and we recall that $\mu = vol(g) = det(g)^{\frac{1}{2}} dx^1 \wedge \ldots \wedge dx^n$ is the Riemannian volume density ($(x^1,\ldots,x^n)$ a local coordinate system on M), and $\omega \in \Lambda^{n-1}(R \times T^*M)$. In (79), u is assumed given, and it may either be a solution of NS, or of the Euler equation given by setting $\nu = 0$ in (63). From the definition follows that $\text{div}B_\tau \equiv 0$, for any $\tau \geq 0$. Now we note that from (28) we can rewrite this problem as

$$\partial_\tau \omega = H_{n-1}(2\nu^m g, -\frac{1}{2\nu^m}u_\tau)\omega_\tau, \omega(0,x) = \omega(x), 0 \leq \tau, \tag{80}$$

as a linear evolution equation for a $(n-1)$-form, similar to the evolution Navier-Stokes equation for the vorticity. Now if we assume that there is an isometric embedding $f : M \to R^m$, so that the diffusion tensor $X = \nabla f$, we can take the Lagrangian representation for the scalar diffusion generated by $H_0(2\nu^m g, -\frac{1}{2\nu^m}u_\tau)$, i.e. the Ito s.d.e. given by substituting ν^m instead of ν in equation (68), and we consider the jacobian process given by $(69-72)$ and with ν^m instead of ν, then the formal $C^{1,2}$ solution of (80) defined on $[0,T]\times M$ for some $T>0$, is given by

$$\tilde{\omega}_\tau(\Lambda^{n-1}v(x)) = E_x[\omega_0(x_\tau^{\nu^m,\tau,x})(\Lambda^{n-1}\tilde{v}_\tau^{\nu^m,\tau,v(x)})] \tag{81}$$

Remarks 8: We note that similarly to the representation for the vorticity, instead of the initial vorticity, now it is the initial magnetic form which is transported backwards along the scalar diffusion, where now the parameter is the magnetic diffusivity, and along its way it is deformed by the fluid-deformation tensor and the gradient diffusion tensor noise term (this accurately represents the actual macroscopical physical phenomena yet in a microscopic approach), and finally we take the average for all those paths starting at x. For both equations as well as the Poisson-de Rham equation, we have a microspic description which clearly evokes the Feynman approach to Quantum Mechanics through a summation of the classical action of the mechanical system along non-differentiable paths. In distinction with the usual Feynman approach, these Brownian integrals are well defined and they additionally have a clear physical interpretation which coincides with actual experience.

10 Random Implicit Integration Of The Navier-Stokes Equations For Compact Manifolds

Up to this point, all our constructions have stemmed from the fact that for gradient diffusion processes, the Ito-Elworthy formula shows that the random process on Λ^2TM given by $\{\Lambda^2 v_\tau : \tau \geq 0\}$ with $\{v_\tau : \tau \geq 0\}$ the jacobian process fibered on the diffusion process $\{\Lambda^0 v_\tau \equiv x_\tau : \tau \geq 0\}$ on M given by (68) , is a random Lagrangian flow for the Navier-Stokes equation. Our previous constructions have depended on the form of the isometric embedding of M. This construction is very general, since that from a well known theorem due to J. Nash (1951), such an immersion exists of class C^1 for any smooth manifold (cf. [53]). (Furthermore, our assumption of compactness is for the obtention of a random flow which is defined for all times, and gives a global diffeomorphism of M. The removal of this condition, requires to consider the random flow up to its explosion time, so that in this case we have a local diffeomorphism of M.)

There is an alternative construction of diffusions of differential forms which does not depend on the embedding of M in Euclidean space, being thus the objective of the following section its presentation. A fortiori, we shall apply these constructions to integrate NS and the kinematic dynamo problem.

10.1 The Generalized Hessian Flow

In the following M is a complete compact orientable smooth manifold without boundary. We shall construct another flow in distinction of the derived flow of the previous sections, which depends explicitly of the curvature of the manifold, and also of the drift of the diffusion of scalars. We start by considering an autonomous drift vector field $\hat{Q}$ (further below we shall lift this condition) and we define a flow $W_\tau^{k,\hat{Q}}$ on $\Lambda^k T^*M$ $(1 \leq k \leq n)$ over the flow of (35), $\{F_\tau(x_0) : \tau \geq 0\}$, by the covariant equation

$$\frac{D^g W_\tau^{k,\hat{Q}}}{\partial \tau}(V_0) = -1/2R^k(W_\tau^{k,\hat{Q}}(V_0)) + (d\Lambda^k)(\nabla^g \hat{Q}(.))(W_\tau^{k,Q}(V_0)), \qquad (82)$$

with $V_0 \in \Lambda^k T_{x_0} M$; in this expression the operator $\frac{D^g}{\partial \tau}$ denotes parallel transport along the curves x_τ with ∇^g; R^k is the Weitzenbock term (see [14]) appearing in the Weitzenbock formula for k-forms: $\triangle_k = (\nabla^g)^2 - R^k$. Let ϕ be a k-form in $C^{1,2}(M)$, and $V_\tau = W_\tau^{k,\hat{Q}}(V_0)$ and $x_\tau = F_\tau(x_0)$; then from the Weitzenbock and general Ito formula we have the following Ito-Elworthy formula [27]:

$$\phi(V_\tau) = \phi(V_0) + \int_0^\tau \nabla^g \phi(X(x_s) dW_s) V_s + \int_0^\tau [H_k(g, Q)] \phi(V_s) ds; \qquad (83)$$

In other words, $H_k(g, Q)$ is the i.g. of $\{V_\tau\}$. In the case that Q is exact the flow V_τ is called the Hessian flow. Assume that $1/2R^k - (d\Lambda)^k(\nabla^g \hat{Q}))(.))$ is bounded below, i.e. for any $V \in \Lambda^k TM$ with $|V| = 1$ we have

$$-\infty < C^k(Q) \equiv inf \frac{1}{2} < R^k(V), V > - < (d\Lambda)^k(\nabla^g \hat{Q})(.)) V, V >, \qquad (84)$$

where we have denoted by $<,>$ the induced metric on $\Lambda^k TM$.

Proposition 2 (Elworthy [27]) Assume that $\frac{1}{2}R^k - (d\Lambda)^k(\nabla^g \hat{Q})(.))$ is bounded below. Define $P_\tau^k : L^\infty \Lambda^k T^*M \to L^\infty \Lambda^k T^*M$ by

$$P_\tau^k(\phi)(V) = E(\phi(W_\tau^{k,Q}(V))) \qquad (85)$$

for $V \in \Lambda^k T_x M, \phi \in L^\infty \Lambda^k T^*M$. Then $\{P_\tau^k : \tau \geq 0\}$ is a contraction semigroup of bounded continuous forms and is strongly continuous there with i.g. agreeing with $H_k(g, Q)$ on $C^2(M)$.

Under the above conditions we can integrate the heat equation for bounded twice differentiable k-forms of class C^2 $(0 \leq k \leq n)$ and in the general case of a non-autonomous drift vector field $\hat{Q} = \hat{Q}_\tau(x)$. Indeed, for every $\tau \geq 0$ consider the flow $V_s^\tau = W_{\tau,s}^{k,\hat{Q}}$ over the flow of $\{x_s^\tau : 0 \leq s \leq \tau\}$, given by the equation

$$dx_s^{x,\tau} = X(x_s^{x,\tau}) dW_s + \hat{Q}_{\tau-s}(x_s^{x,\tau}) ds, x_0^{x,\tau} = x,$$

obtained by integration of the equation

$$\frac{D^g V_s^\tau}{\partial s}(v_0) = -1/2 R^k(V_s^\tau(v_0)) + (d\Lambda^k)(\nabla^g \hat{Q}_{\tau - s}(.))(V_s^\tau(v_0)), \tag{86}$$

with $v_0 = V_0^\tau \in T_x M$. Then, applying the Ito-Elworthy formula we prove as before that if $\tilde{\alpha}_\tau$ is a bounded $C^{1,2}$ solution of the Cauchy problem for the heat equation for k forms:

$$\frac{\partial}{\partial \tau}\alpha_\tau = H_k(g, Q)\alpha_\tau \tag{87}$$

with initial condition $\alpha_0(x) = \alpha(x)$ a given k-form of class C^2, then the solution of the heat equation is

$$\alpha_\tau(v(x)) = E_x[\alpha_0(V_\tau^\tau(v(x))], \tag{88}$$

with $V_\tau^\tau(x)$ the generalized Hessian flow over the flow $\{F_\tau(x) : \tau \geq 0\}$ of $\{x_\tau^\tau : \tau \geq 0\}$ with initial condition $v(x)$.

To integrate the Poisson-de Rham equation we shall need to consider the so-called Ricci-flow $W_\tau^{\mathcal{R}} \equiv W_\tau^{1,0}(\omega) : TM \rightarrow TM$ over the random flow generated by $H_0(g, 0)$, obtained by integrating the covariant equation (so we fix the drift to zero and further take $k = 1$ in (82))

$$\frac{D^g W_\tau^{\mathcal{R}}}{\partial \tau}(v_0) = -\frac{1}{2}\tilde{Ric}(W_\tau^{\mathcal{R}}(v_0), -), v_0 \in T_{x_0}M \tag{89}$$

where $Ric : TM \oplus TM \rightarrow R$ is the Ricci curvature and $\tilde{Ric}(v, -) \in T_x M$ is the conjugate vector field defined by $< \tilde{Ric}(v, -), w >= Ric(v, w)$, $w \in T_x M$.

10.2 Integration of the Cauchy problem for the Vorticity on Compact Manifolds

Theorem 10: The integration of the equation (67) with initial condition $\Omega(0,) = \Omega_0$ yields

$$\Omega_\tau(v(x)) = E_x[\Omega_0(V_\tau^\tau(v(x))] \tag{90}$$

where $\{V_\tau^\tau : \tau \geq 0\}$ is the solution flow over the flow of $\{x_\tau^{\nu,\tau,x} : \tau \geq 0\}$ (see equation (69)) of the covariant equation

$$\begin{aligned} \frac{D^g W^{2,-\hat{u}_0}}{\partial \tau}(v(x)) &= -\nu R^2(W_\tau^{k,-\hat{u}_0}(v(x))) \\ &- (d\Lambda^2)(\nabla^g \hat{u}_0(.))(W_\tau^{2,-\hat{u}_0}(v(x))) \end{aligned} \tag{91}$$

with initial condition $v(x) \in T_x M$. In this expression, $\nabla^g \hat{u}_0(.)$ is a linear transformation, A, between $T_x^* M$ and $T_x M$, and $d\Lambda^2(A) : T_x M \wedge T_x M \rightarrow T_x M \wedge T_x M$ is given by $d\Lambda^2 A(v_1 \wedge v_2) = Av_1 \wedge v_2 + v_1 \wedge Av_2$, for any $v_1, v_2 \in T_x M$, $x \in M$.

10.3 Integration of the Kinematic Dynamo for Compact Manifolds

Substituting the magnetic diffusivity ν^m instead of the kinematic viscosity in (69) and we further consider $\{V_s^\tau : s \in [0,\tau]\}$ given by the solution flow over the flow of $\{x_s^{\nu^m,\tau,x} : s \in [0,\tau]\}$ (see equation (69)) of the covariant equation

$$\frac{D^g V_s^\tau}{\partial s}(v(x)) = -\nu^m R^{n-1}(V_s^\tau(v(x))) + (d\Lambda^{n-1})(\nabla^g \hat{Q}_{\tau-s}(.))(V_s^\tau(v(x))), \quad (92)$$

with $v(x) = V_0^\tau \in T_x M$. Then, the formal $C^{1,2}$ solution of (80) is

$$\omega_\tau(v(x)) = E_x[\omega_0(V_\tau^\tau(v(x))]. \quad (93)$$

with E_x denoting the expectation valued with respect to the measure on $\{x_\tau^{\nu^m,\tau,x}\}$ (whenever it exists).

10.4 Integration of the Poisson-de Rham Equation for the Velocity

With the same notations as in the case of isometrically embedded manifolds, we have a martingale problem with a bounded solution given by

$$u_\tau(v(x)) = E_x^B[\phi(W_{\tau_e}^{\mathcal{R}}(v(x))) + 1/2 \int_0^{\tau_e} \delta\Omega_\tau(W_s^{\mathcal{R}}(v(x)))ds] \quad (94)$$

11 Representations Of NS On Euclidean Space

In the case that M is euclidean space, the solution of NS is easily obtained from the solution in the general case. In this case the isometric embedding f of M is realized by the identity mapping, i.e. $f(x) = x, \forall x \in M$. Hence the diffusion tensor $X = I$, so that the metric g is also the identity. For this case we shall assume that the velocity vanishes at infinity, i.e. $u_t \to 0$ as $|x| \to \infty$. (This allows us to carry out the application of the general solution, in spite of the non-compacity of space). Furthermore, $\tau_e = \infty$. The solution for the vorticity equation results as follows. We have the s.d.e. (see (69) where we omit the kinematical viscosity, for simplicity)

$$dx_s^{\tau,x} = -u(\tau - s, x_s^{\tau,x})ds + (2\nu)^{\frac{1}{2}} dW_s, x_0^{\tau,x} = x, s \in [0,\tau]. \quad (95)$$

The derived process is given by the solution of the o.d.e. (since in (71) we have $\nabla X \equiv 0$)

$$d\tilde{v}_s^{\tau,x,v(x)} = -\nabla u(\tau - s, x_s^{\tau,x})(\tilde{v}_s^{\tau,x,v(x)})ds, v_0^{\tau,v(x)} = v(x) \in R^n, s \in [0,\tau], \quad (96)$$

Now for $n = 3$ we have that the vorticity $\Omega(\tau, x)$ is a 2-form on R^3, or still by duality has an adjoint 1-form, or still a function, which with abuse of notation we still write as $\tilde{\Omega}(\tau, .) : R^3 \to R^3$, which from (73) we can write as

$$\tilde{\Omega}(\tau, x) = E_x[\tilde{v}_\tau^{\tau,x,I}\Omega_0(x_\tau^{\tau,x})], \tag{97}$$

where E_x denotes the expectation value with respect to the measure (if it exists) on $\{x_\tau^{\tau,x} : \tau \geq 0\}$, for all $x \in R^3$, and in the r.h.s. of (97) we have matrix multiplication Thus, in this case, we have that the deformation tensor acts on the initial vorticity along the random paths. This action is the one that for $3D$ might produce the singularity of the solution. Note that in Euclidean space there is no gradient-noise contribution to the folding of the initial vorticity by the action of the fluid deformation tensor.

In the case of R^2, the vorticity can be thought as a pseudoscalar, since $\Omega_\tau(x) = \tilde{\Omega}_\tau(x)dx^1 \wedge dx^2$, with $\tilde{\Omega}_\tau : R^2 \to R$, and being the curvature identically equal to zero, the vorticity equation is (a *scalar* diffusion equation)

$$\frac{\partial \tilde{\Omega}_\tau}{\partial \tau} = H_0(2\nu I, \frac{-1}{2\nu}u_\tau)\tilde{\Omega}_\tau \tag{98}$$

so that for $\tilde{\Omega}_0 = \tilde{\Omega}$ given, the solution of the initial value problem is

$$\tilde{\Omega}(\tau, x) = E_x[\tilde{\Omega}(x_\tau^{\tau,x})] \tag{99}$$

This solution is qualitatively different from the previous case. Due to a geometrical duality argument, for $2D$ we have factored out completely the derived process in which the action of the deformation tensor on the initial vorticity is present.

Furthermore, the solution of equation (75) is (recall that $X = I$)

$$x_\tau^{g,x} = x + W_\tau, \tag{100}$$

and since $\nabla X = 0$, the derived process (see (76)) is constant

$$v_\tau^{g,x,v(x)} = v(x), \forall \tau \in [0, T]. \tag{101}$$

so that its influence on the velocity of the fluid can be factored out in the representation (77). Indeed, we have

$$\begin{aligned} \tilde{u}_\tau(x)(v(x)) &= E_x^B[\int_0^\infty \frac{1}{2}\delta\Omega_\tau(x + W_s)(\tilde{v}^{g,x,v(x)}(s))ds] \\ &= E_x^B[\int_0^\infty \frac{1}{2}\delta\Omega_\tau(x + W_s)ds(v(x))] \end{aligned}$$

for any tangent vector $v(x)$ at x, and in particular (we take $v(x) = I$) we obtain

$$\tilde{u}_\tau(x) = E_x^B[\int_0^\infty \frac{1}{2}\delta\Omega_\tau(x+W_s)ds]. \tag{102}$$

In this expression we know from (78) that the expectation value is taken with respect to the standard Gaussian function, $p^g(s,x,y) = (4\pi s)^{\frac{-n}{2}} exp(-\frac{|x-y|^2}{4s})$.

Let us describe in further detail this solution separately for each dimension. We note first that if $\Omega_\tau \in L^1 \cap C_b^1$ (where C_b^1 means continuously differentiable, bounded with bounded derivatives)

$$E[\delta\Omega_\tau(x+W_s)] = \delta E[\Omega_\tau(x+W_s)] \tag{103}$$

In case $n=2$, for a 2-form β on M we have $\delta\beta = \delta(\tilde{\beta}dx^1 \wedge dx^2) = -(\partial_2\tilde{\beta}dx^1 - \partial_1\tilde{\beta}dx^2) \equiv -\nabla^\perp\beta$. In case $n=3$, for a vorticity described by the 1-form (or a vector-valued function) $\tilde{\Omega}_\tau : R^3 \to R^3$ adjoint to the vorticity 2-form Ω_τ, we have that

$$\delta\Omega_\tau = -d\tilde{\Omega}_\tau = -\text{rot}\tilde{\Omega}_\tau. \tag{104}$$

Therefore, we have the following expressions for the velocity: When $n=2$ we have

$$u_\tau(x) = \int_0^\infty -\frac{1}{2}\nabla^\perp E_x^B[\tilde{\Omega}_\tau(x+W_s)]ds \tag{105}$$

while for $n=3$ we have

$$u_\tau(x) = \int_0^{\tau_e} \frac{-1}{2} dE_x^B[\tilde{\Omega}_\tau(x+W_s)]ds. \tag{106}$$

Now we can obtain an expression for the velocity which has no derivatives of the vorticity; this follows the basic idea in a non-geometrical construction given by Busnello who starts with the stream function (of an unbounded incompressible fluid) instead of the velocity [54]. Consider the semigroup generated by $H_0(I,0) = \frac{1}{2}\triangle$, i.e. $P_s\tilde{\Omega}_\tau(x) = E[\tilde{\Omega}_\tau(x+W_s)]$ (in the case $n=3$ this means the semigroup given on each component of $\tilde{\Omega}$). From the Elworthy-Bismut formula valid for scalar fields (see [52]) we have that (in the following $e_i, i = 1,2,3$ denotes the canonical base in R^2 or R^3)

$$\partial_i P_s\tilde{\Omega}_\tau(x) =< dP_s\tilde{\Omega}(x), e_i >= \frac{1}{s}E_x^B[\tilde{\Omega}(x+W_s)\int_0^s < e_i, dW_r >]$$

$$= \frac{1}{s}E_x^B[\tilde{\Omega}(x+W_s)\int_0^s dW_r^i] = \frac{1}{s}E[\tilde{\Omega}_\tau(x+W_s)W_s^i]. \tag{107}$$

Therefore, for $n = 2$ we have from $(105, 107)$

$$u_\tau(x) = -\int_0^{\tau_e} \frac{1}{2s} E_x^B[\tilde{\Omega}_\tau(x + W_s)W_s^\perp]ds \tag{108}$$

where $W_s^\perp = (W_s^1, W_s^2)^\perp = (W_s^2, -W_s^1)$. Instead, for $n = 3$ we have from $(106, 107)$ that

$$u_\tau(x) = -\int_0^{\tau_e} \frac{1}{2s} E_x^B[\tilde{\Omega}_\tau(x + W_s) \times W_s]ds \tag{109}$$

where $\times$ denotes the vector product and $W = (W^1, W^2, W^3) \in R^3$.

Thus, we have obtained the representations for NS in 2D and 3D:

11.1 Integration of the Kinematic Dynamo Problem in Euclidean space

With the notations in this section, the kinematic dynamo problem in $3D$ can be solved as follows. As for the vorticity, the magnetic field is for $n = 3$ is a 2-form on R^3, or still by duality has an adjoint 1-form (so the argument turns to work out as well for 2D), or still a function, which with abuse of notation we still write as $\tilde{\omega}(\tau, .) : R^3 \to R^3$, which from (81) we can write as

$$\tilde{\omega}(\tau, x) = E_x[\tilde{v}_\tau^{\tau,x,I}\omega_0(x_\tau^{\tau,x})], \tag{110}$$

where E_x denotes the expectation value with respect to the measure (if it exists) on $\{x_\tau^{\tau,x} : \tau \geq 0\}$, for all $x \in R^3$, and in the r.h.s. of (110) we have matrix multiplication Thus, in this case, we have that the deformation tensor acts on the initial vorticity along the random paths. This action is the one that for $3D$ produces the complicated topology of transported magnetic fields. This solution was obtained independently by Molchanov et al [50] and further applied in numerical simulations (see Ghill and Childress [51] and references therein).

12 The Navier-Stokes Equation is Purely Diffusive For Any Dimension Other than 1

12.1 Motivations

We have given up to now a derivation of diffusion processes starting from gauge theoretical structures, and applied this to give implicit representations for the covariant Navier-Stokes equations. These constructions were possible as they stemmed from the extremely tight relation existing between the metric-compatible Riemann-Cartan-Weyl connections, and the diffusion processes for differential forms, built untop of the diffusions for scalar fields. As we saw already this stemmed from the fact that there is a one-to-one correspondance between said RCW connections and the scalar diffusion processes $\{x_\tau : \tau \geq 0\}$ with drift given by $\hat{Q}$ and diffusion tensor X. As we can easily check from (34), this construction is valid for $n \neq 1$. This leads to conjecture that in a gauge theoretical setting and further applying stochastic analysis, one could do away with the drift, in any dimension other than 1. If this would be the case, then we could apply this construction to the Navier- Stokes equation, which thus in any dimension other than 1 would turn to be representable by random lagrangian paths which do not depend explicitly on the velocity of the fluid, since they would be purely diffusive processes.

12.2 More On Connections

Consider the map $M \times R^m \rightarrow TM \rightarrow 0$ which we assume that it has a right inverse $Y : TM \rightarrow M \times R^m$. Here, $Y = X^\dagger$ is the adjoint of X with respect to the Riemannian metric on TM induced by X, $Y = X^*$. Write $X(x) = X(x,.) : R^m \rightarrow TM$. For $u \in TM$, let $Z^u \in \Gamma(TM)$ defined by

$$Z^u(x) = X(x)Y(\pi(u))u. \tag{111}$$

Proposition 3: There is a unique linear connection $\tilde{\nabla}$ on TM such that for all $u_0 \in T_{x_0}M, x \in M$, we have that

$$\tilde{\nabla}_{v_0} Z^{u_0} = 0. \tag{112}$$

It is the pushforward connection defined as

$$\tilde{\nabla}_{v_0} Z := X(x_0)d(Y(.)Z(.))(v_0), v_0 \in T_{x_0}M, Z \in \Gamma(TM), \tag{113}$$

where d is the usual derivative of the map $Y(.)Z(.) : M \rightarrow R^m$.

Proof: The above definition defines a connection. Let $\hat{\nabla}$ be any linear connection on TM. We have

$$Z(.) = X(.)Y(.)Z(.). \tag{114}$$

Then, for $v \in T_{x_0}M$,

$$\hat{\nabla}_v Z = X(x_0)d(Y(.)Z(.))(v) + \hat{\nabla}_v[X(.)(Y(x_0)Z(x_0))] = \tilde{\nabla}_v Z + \hat{\nabla}_v Z^{Z(x_0)}. \tag{115}$$

Since $\hat{\nabla}$ is a connection by assumption, and since the map

$$TM \times TM \to TM, (v,u) \longmapsto \hat{\nabla}_v Z \tag{116}$$

gives a smooth section of the bundle $Bil(TM \times TM; TM)$, then $\hat{\nabla}$ is a smooth connection on TM. Taking $\hat{\nabla} = \tilde{\nabla}$ we obtain a connection with the desired property.

Theorem 11: Let Y be the adjoint of X with respect to the induced metric on TM by X. Then, $\tilde{\nabla}$ is metric compatible, where the metric is the one induced by X on TM, which we denote by $\tilde{g}$. Moreover, since M is finite-dimensional, any metric-compatible connection for any metric on TM can be obtained this way from such X and R^m.

Proof: We have

$$2\tilde{g}(\nabla_v U, U) = 2(X(x_0)(d(Y(.)U(.))(v), U(x_0))$$

$$= 2\tilde{g}(d(Y(.)U(.))(v), Y(x_0)U(x_0)) = d(\tilde{g}(U,U)(v), \tag{117}$$

so that $\tilde{\nabla}$ is indeed metric compatible. By the Narasimhan-Ramanan theorem on universal connections [51], any metric compatible connection arises likes this. Indeed, $\tilde{\nabla}$ is the pull-back of the universal connection over the Grassmanian $G(m,n)$ of n-planes in R^m by the map $x \mapsto [\text{Image}Y(x) : T_xM \to R^m]$. In particular, the RCW connections arise from such a construction. c.q.d.

Two connections, ∇^a and ∇^b on TM give rise to a bilinear map $D^{ab} : TM \times TM \to TM$ such that

$$\nabla^a_V U = \nabla^b_V U + D^{ab}(V,U), U, V \in \Gamma(TM). \tag{118}$$

Choose $\nabla^b = \nabla^g$, the Levi-Civita connection of a certain Riemannian metric g. Consider

$$\tilde{\nabla}_v U = \nabla^g_V U + \tilde{D}(V,U), \tag{119}$$

where we decompose $\tilde{D}$ into

$$\tilde{D}(u,v) = A(u,v) + S(u,v), \tag{120}$$

where

$$A(u,v) = -A(v,u), S(u,v) = S(v,u). \tag{121}$$

Since the torsion tensors T^a and T^b of any two connections ∇^a and ∇^b respectively, are connected through the expression

$$T^a(u,v) + T^b(u,v) = D^{ab}(u,v) - D^{ab}(v,u). \tag{122}$$

which for the case of $\nabla^b = \nabla^g$ as $T^b = 0$, we can write for $D^{ab} = \tilde{D}$, the identity

$$\tilde{T}(u,v) = \tilde{D}(u,v) - \tilde{D}(v,u) + 2A(u,v). \tag{123}$$

Thus,

$$A(u,v) = \frac{1}{2}\tilde{T}(u,v), u, v, \in \Gamma(TM), \tilde{T} \equiv T^a. \tag{124}$$

Therefore, the decomposition in (119) is written in the form

$$\tilde{\nabla}_v U = \nabla^g_v U + \frac{1}{2}\tilde{T}(u,v) + S(u,v), u, v \in \Gamma(TM). \tag{125}$$

which is nothing else than the original decomposition for a metric compatible Cartan connection given in (10&11). As we know already from the decomposition (11), we have the following lemma.

Lemma A connection $\tilde{\nabla}$ is metric-compatible if and only if the map $\tilde{D}(v,.) : TM \to TM$ is skew-symmetric for each $v \in TM$, i.e.

$$g(\tilde{D}(v,u_1),u_2) + g(\tilde{D}(v,u_2),u_1) = 0, u_1, u_2 \in \Gamma(TM). \tag{126}$$

Equivalently,

$$g(S(u_1,u_2),v) = \frac{1}{2}g(\tilde{T}(v,u_1),u_2) + \frac{1}{2}g(\tilde{T}(v,u_2),u_2). \tag{127}$$

Consequently, for U_1, U_2,V in $\Gamma(TM)$, we have

$$g(\tilde{D}(V,U_1),U_2) = \frac{1}{2}g(\tilde{T}(V,U_1),U_2) + \frac{1}{2}g(\tilde{T}(U_2,V),U_1) + \frac{1}{2}g(\tilde{T}(U_2,U_1),V), \tag{128}$$

which is decomposition (10).

Proof: Take $V, U_1, U_2 \in \Gamma(TM)$. Then,

$$d(g(U_1,U_2))(V) = g(\nabla^g_V U_1, U_2) + g(U_1, \nabla^g_V U_2)$$

$$= g(\nabla^g_V U_1, U_2) + g(U_1, \nabla^g_V U_2) - g(\tilde{D}(V,U_1),U_2) - g(U_1,\tilde{D}(V,U_2). \tag{129}$$

So, $\tilde{\nabla}$ is metric compatible if and only if

$$g(\tilde{D}(V,U_1),U_2) + g(U_1,\tilde{D}(V,U_2)) = 0. \tag{130}$$

Now, writing $\tilde{D} = A + S$ we get

$$g(A(V,U_1),U_2) + g(A(V,U_2),U_1) = -g(S(V,U_1),U_2) - g(S(V,U_2),U_1). \tag{131}$$

We now observe that for an alternating bilinear map $L : TM \times TM \to TM$,

$$Cyl[g(L(v, u_1), u_2) + g(L(v, u_2), u_1)] = 0, \tag{132}$$

where Cyl denotes cyclic sum. Taking the cyclic sum in equation (131) and apply (132) to A, we thus obtain $\text{Cyl}g(S(V, U_1), U_2) = 0$ which on further substituting in (131) we obtain

$$g(A(V, U_1), U_2) + g(A(V, U_2), U_1) = g(S(U_1, U_2), V). \tag{133}$$

12.3 The Trace-Torsion Is Dynamically Redundant in Any Dimension Other Than 1

Let us return to our original setting of Section 1. We assume a metric-compatible Cartan connection, which we now write as $\tilde{\nabla}$ with torsion tensor $\tilde{T}$. The following result is a reduction of a more general result due to Elworthy, Le Jan and Li 61].

Theorem 12: Assume M has dimension bigger than 1. Consider the laplacian on 0-forms $H_0(g, Q)$ where Q is the trace-torsion 1-form of $\tilde{\nabla}$,

$$Q(u) = \text{trace } g(\tilde{T}(-, u), -). \tag{134}$$

Assume further that we can write the laplacian $H_p(g, Q)$ on p-forms $(0 \leq p \leq n)$ in the Hormander form:

$$\frac{1}{2}\sum_{i=1}^{m} L_{V_i} L_{V_i} + L_Z \tag{135}$$

where Z is a vectorfield on M, $V : M \times R^m \to TM$ is a smooth surjection, linear in the second variable, and V_i is defined by

$$V(x, e) = V(x)e = \sum_{i=1}^{m} V^i(x) < e, e_i >, \tag{136}$$

whith $e_1, \ldots, e_m$ the standard orthonormal basis for R^m. (Since ∇^g is metric compatible, from Theorem 11 and the transformation rules between Stratonovich and Ito calculi, we can always introduce a defining map V for ∇^g that gives such decomposition (c.f. [27])). Then, there exists a map $K : M \times R^m \to TM$ linear on the second variable, such that the solution to the Stratonovich equation

$$dx_\tau = K(x_\tau) \circ dW_\tau, \tag{137}$$

has $H_0(g, Q)$ for infinitesimal generator, i.e.

$$H_0(g, Q) = \frac{1}{2}\sum_{i=1}^{m} L_{K_i} L_{K_i}. \tag{138}$$

In other words, the Ito s.d.e. given by (35) admits a driftless representation given by (137).

Proof: Set for the original drift vectorfield $\hat{Q}$ (the g-conjugate of Q), the decomposition

$$\hat{Q} = \frac{1}{2}\sum_{i=1}^{m} \nabla^g_{V^i} V^i - Z, \tag{139}$$

A connection $\tilde{\nabla}$ suitable for this is such that

$$2A(u,v) = \tilde{T}(v,u) = \frac{2}{n-1}(u \wedge v)Q(x). \tag{140}$$

Consider a bundle map $K : M \times R^m \rightarrow TM$ which gives rise to the metric compatible connection $\tilde{\nabla}$ (theorem 11). Consider the s.d.e.

$$dx_\tau = K(x_\tau) \circ dW_\tau. \tag{141}$$

Its generator is (c.f. [27])

$$\frac{1}{2}\text{trace}(\nabla^g)^2 + \frac{1}{2}\sum_{i=1}^{m} \nabla^g K^i(K^i) \tag{142}$$

while by assumption we have

$$H_0(g,Q) = \frac{1}{2}\text{trace}(\nabla^g)^2 + (\frac{1}{2}\sum_{i=1}^{m} \nabla^g_{V^i} V^i - Z) = \frac{1}{2}\text{trace}(\nabla^g)^2 + \hat{Q} \tag{143}$$

The required result follows after we show

$$\sum_{i=1}^{m} \nabla^g K^i(K^i) = -\sum_{i=1}^{m} \tilde{D}(K^i, K^i) = \text{trace}\tilde{D}(-,-) \tag{144}$$

equals $2\hat{Q}$. For this we note that for all $v \in TM$,

$$g(\sum_{i=1}^{m} \tilde{D}(K^i,K^i),v)) = g(\sum_{i=1}^{m} S(K^i,K^i),v)) = -\sum_{i=1}^{m} g(\tilde{T}(v,K^i),K^i) = 2g(\hat{Q},v) \tag{145}$$

Consequently

$$\text{trace}(\nabla^g)^2 + \frac{1}{2}\sum_{i=1}^{m} \nabla^g(K^i)(K^i) = \text{trace}(\nabla^g)^2 + \hat{Q} = H_0(g,Q) \tag{146}$$

and the K so constructed is the required map.c.q.d.

Remarks 8: Of course in the above construction, it is innecessary to start with an arbitrary metric-compatible Cartan connection, only the trace-torsion as proved already matters.

12.4 Navier-Stokes Equations Is Purely Diffusive In Any Dimension Other Than 1

We recall that for any dimension other than 1, the Navier-Stokes equation is a diffusion process which arises from a RCW connection of the form (34) with metric given by $2\nu g$, where g is the original metric defined on TM, and torsion restricted to the trace-torsion given by $\frac{-1}{2\nu}u_\tau$, where $\tau \geq 0$: we shall call this connection the Navier-Stokes connection with parameter ν, which we shall denote as $\nabla^{NS;\mu}$. Let us then consider a bundle map $K_\tau : M \times R^m \rightarrow TM$, for $\tau \geq 0$ for such a connection; from theorem 11 we know it exists. Therefore, from Theorem 12 we conclude that:

Theorem 13: For any dimension other than 1, the random lagrangian representations given by (68) admit representations as a Stratonovich s.d.e. without drift $-\hat{u}_\tau$ term

$$dx_\tau = K_\tau(x(\tau)) \circ dW_\tau. \tag{147}$$

Remarks 9: Thus, we have gauged out the velocity in the dynamical representation for the fluid particles. Of course, the new diffusion tensor K_τ ($\tau \geq 0$) depends implicitly in the velocity of the fluid as well as in the kinematical viscosity. Indeed, K can be in principle computed from the knowledge of the Navier-Stokes connection with parameter ν, by solving the equation

$$\nabla_v^{NS;\mu} Z = K(x)d(K^\dagger(.)Z(.)(v), v \in T_xM, Z \in \Gamma(TM). \tag{148}$$

Remarks 10: This last theorem deserves further examination. As well known, the Navier-Stokes equations is a mechanism of "competition" between the linear diffusion term $\nu\Delta_1 u_\tau$ which at the level of the fluid flow described by the diffusion tensor term $(2\nu)^{\frac{1}{2}}X$ and the drift non-linear term $PL_{u_\tau}u_\tau$, described at the level of the fluid flow by the drift vectorfield $-\hat{u}_\tau$. The long-time existance of the representations of Navier-Stokes depends on the prevailance of the diffusion term, so that the non-linearity which feeds the fluid with energy would fade out. The representation given by Theorem 13 assures that this is the case for any physically interesting dimension, without distinction between 2-dimensional and 3-dimensional manifolds, since we can always find a non-linear representation for the random fluid flow which incorporates both the diffusion tensor and the velocity into a new diffusion tensor! (We can think here in the dispute on pre-Socratic Greek philosophy between the followers of Parmenides and Heraclitus, and thus name (147) as the Parmenidean representation: in this representation the fluid flow does not depend explicitly on the fluid velocity.)

Remarks 11: Just like in theorem 13, we can represent the lagrangian random paths for the kinematic dynamo problem as a purely diffusive process, by putting ν^m instead of ν.

13 Final Observations:

The method of integration applied in the previous section is the extension to differential forms of the method of integration (the so-called martingale problems) of elliptic and parabolic partial differential equations for scalar fields [24,31]. In distinction with the Reynolds approach in Fluid Mechanics, which has the feature of being non-covariant, in the present approach, the invariance by the group of space-diffeomorphims has been the key to integrate the equations, in separating covariantly the fluctuations and drift terms and thus setting the integration in terms of covariant martingale problems. The role of the RCW connection is precisely to yield this separation for the diffusion of scalars and differential forms, and thus the role of the differential geometrical structure is essential.

The solution scheme we have presented gives rise to infinite particle random trajectories due to the arbitrariness of the initial point of the Lagrangian paths. This continuous infinite particle solution is *exact* and we have actually computed its expression. Actually, to integrate NS we choose a finite set of initial points and we take for Ω_0 a linear combination of 2-forms (or area elements in the 2-dimensional case) supported in balls centered in these points, the so-called many vortices solutions; one can choose the original $f_0(x)$ so that Ω_0 is supported in these balls and these localizations persists in time. Thus the role of the potential term in the Buttke magnetization 1-form in the expression (56) is to push the vortices to be confined on predetermined finite radii balls; see Chorin [1]. This requires that convergence to a solution of NS be proved in addition.

A new approach to NS as a (*random*) dynamical system appears. Given a stationary measure for the *random* diffeomorphic flow of NS given by the stationary flow of equation (67), one can construct the state space of this flow and further, its *random* Lyapunov spectra. Consequently, assuming ergodicity of this measure, one can conclude that the moment instability of the flow is related to a cohomological property of M, namely the existance of non-trivial harmonic one-forms ϕ, which are preserved by the vectorfield $\hat{u}$ of class C^2, i.e. $L_{\hat{u}}\phi = di_{\hat{u}}\phi = 0$; see page 61 in [27]. We also have the random flows $\{v_\tau \wedge v_\tau : \tau \geq 0\}$ and $W_\tau^{2,-\hat{u}_0}$ on $TM \wedge TM$ of Theorem 7 and Theorem 10 respectively, which integrate the linear equation for the vorticity. Concerning these flows, the stability theory of NS (67) requires an invariant measure on a suitable subspace of $TM \wedge TM$ and further, the knowledge of the spectrum of the one-parameter family of *linear* operators depending on ν, $H_2(2\nu g, -\frac{1}{2\nu}u_\tau)$. The latter may play the role of the Schroedinger operators in Ruelle's theory of turbulence, which were introduced by linearising NS for the velocity as the starting point for the discussion of the instability theory; see article in pages $295 - 310$ in Ruelle [48].

Finally, it has been established numerically that turbulent fluids resemble the random motion of dislocations [4]. In the differential geometric gauge theory of crystal dislocations, the torsion tensor is the dislocation tensor [12], and our presentation suggests that this analogy might be established rigorously from the perspective presented here. We would like to remark that the results presented in this article, are part of a more general program of formulation of gravitation, quantum mechanics and irreversible thermodynamics, in terms of stochastic differential geometry, developed by the author.

14 Appendix

We shall review some basic concepts of the probabilistic, analytical and geometrical realms.

Let $\{\mathcal{F}_\tau : \tau \geq 0\}$ be a family of sub σ-fields of a σ-field $\mathcal{F}$ in a probability space $(\Omega, \mathcal{F}, P)$. It is called a **filtration** of sub σ-fields if it satisfies the following three properties: i)$\mathcal{F}_s \subset \mathcal{F}_\tau$ if $s < \tau$; ii)$\cap_{\epsilon>0}\mathcal{F}_{\tau+\epsilon} = \mathcal{F}_\tau$, and iii) each $\mathcal{F}_\tau$ contains all null sets of $\mathcal{F}$

A stochastic process $x_\tau, \tau \in T$, with T a time-set, say $[0, \infty)$, the interval $[0, T]$ or the real line, is called $(\mathcal{F}_\tau)-$**adapted** if for each τ, x_τ is $\mathcal{F}_\tau$-measurable. **Predictable sets** are subsets of $[0, \infty) \times R$, which are elements of the smallest σ-algebra relative to which all real $\mathcal{F}_\tau$-adapted, right-continuous processes with left-hand limits are measurable in (τ, ω). A stochastic process $x : [0, \infty) \mapsto S$, where S is a measurable space with σ-algebra $\mathcal{B}$ is called **predictable** if, for any Borel subset $\mathcal{B} \in S$, $\{(\tau, \omega), x(\tau, \omega) \in B\}$ is predictable.

A positive random variable t is called a **stopping time** (with respect to the filtration $\{\mathcal{F}_\tau : \tau \geq 0\}$ if for all $0 \leq \tau, \{t \leq \tau\} \in \mathcal{F}_\tau$. This concept is used to indicate the ocurrence of some random event.

The **conditional expectation** of a real-valued random variable X with respecto to a sub-σ algebra $\mathcal{G}$ of $\mathcal{F}$ is denoted by $E(X|\mathcal{G})$.

Let x_τ be a $(\mathcal{F}_\tau)$-adapted real-valued process such that for each τ, x_τ is integrable. It is called a **martingale** if it satisfies: $E[x_\tau|\mathcal{F}_s] = x_s$ a.s. for any $\tau > s$. Furthermore, x_τ is called a **local martingale** if there exists an increasing sequence of stopping times $\{\tau_n\}$ such that $P(\tau_n < T) \to 0$ as $n \to \infty$, and each stopped time $x_\tau^{\tau_n} \equiv x_{\tau\wedge\tau_n}$ is a martingale, where $\tau \wedge \tau_n = min\{\tau, \tau_n\}$. A martingale is obviously a local martingale (set $\tau_n \equiv T$, for all n. Finally, x_τ is called a **semimartingale** if it can be decomposed as the sum of a localmartingale and a process of bounded variation.

This decomposition, which in the course of the presentation of the rules of stochastic analysis on manifolds for differential forms appears explicitly in the Ito-Elworthy Formula, sets the integration of Navier-Stokes equations and the kinematic dynamo problem as the solutions of the so-called **martingale problems** (after Stroock and Varadhan [30]).

Let D be a domain in Euclidean space R^d and let R^l another Euclidean space (eventually $d = l$). Let $m \in N$; denote $C^m \equiv C^m(D; R^l)$ the set of all maps $f : D \to R^l$ which are m-times continuously differentiable. For the multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in N^d$, define

$$D_x^\alpha = \frac{\partial^\alpha}{(\partial x^1)^{\alpha_1} \ldots (\partial x^d)^{\alpha^d}}, \text{with} |\alpha| = \sum_{i=1}^{d} \alpha_i. \tag{149}$$

Let K be a subset of D. Set

$$||f||_{m,K} = sup_{x \in K} \frac{f(x)}{(1 + |x|)} + \sum_{1 \leq |\alpha| \leq m} sup_{x \in K} |D^\alpha f(x)|. \tag{150}$$

Then $C^m(D; R^l)$ is a Frechet space by seminorms $\{||.||_{m;K}$: K are compacts in $D\}$. When $K = D$ we write $||.||_{m;K}$ as $||.||_K$. Now let δ such that $0 < \delta < 1$. Denote by $C^{m;\delta} \equiv C^{m;\delta}(D; R^l)$ the set of all $f \in C^m$ such that $D^\alpha f, |\alpha| = m$ are δ-Holder continuous. By the seminorms

$$||f||_{m+\delta;K} = ||f||_{m;K} + \sum_{\alpha = m} sup_{x,y \in K, x \neq y} \frac{D^\alpha f(x) - D^\alpha f(y)}{|x - y|^\delta}, \tag{151}$$

is a Frechet space, the so-called **space of δ − Holder continuous $\mathbf{C^m}$ mappings**. When $D = K$ we write $||.||_{m+\delta;K}$ as $||.||_{m+\delta}$. Denote further as $C_b^{m;\delta}$ the set $\{f \in C^{m;\delta} : ||f||_{m+\delta} < \infty\}$. A continuous mapping $f(\tau, x)$, $x \in D, \tau \in T\}$ is said to belong to the class $\mathbf{C^{m;\delta}}$ if for every $\tau, f(\tau) \equiv f(\tau, .)$ belongs to $C^{m;\delta}$ and $||f(\tau)||_{m+\delta;K}$ is integrable on T with respect to τ in any compact subset K. If the set K is replaced by D, f is said to belong to the class $\mathbf{C_b^{m;\delta}}$. Furthermore, if $||f(\tau)||_{m+\delta}$ is bounded in τ, it is said to belong to the class $\mathbf{C_{ub}^{m;\delta}}$.

Consider the **canonical Wiener space** Ω of continuous maps $\omega : R \to R^d, \omega(0) = 0$, with the canonical realization of the Wiener process $W(\tau)(\omega) = \omega(\tau)$. Let $\phi_{s,\tau}(x, \omega), s, \tau \in T, x \in R^d$ be a continous R^d-valued random field defined on $(\Omega, \mathcal{F}, P)$. Then, for almost all $\omega \in \Omega$, $\phi_{s,\tau}(\omega) \equiv \phi_{s,\tau}(., \omega)$ defines a continuous map from R^d into itself, for any s, τ. Then, let us assume that $\phi_{s,\tau}$ satisfies the conditions: (i) $\phi_{s,u}(\omega) = \phi_{\tau,u}(\omega)\phi_{s,\tau}(\omega)$, holds for all s, τ, u, where in the r.h.s. of (i) we have the composition of maps, (ii) $\phi_{s.s}(\omega) = id$, for all s, where id denotes the identity map, (iii) $\phi_{s,\tau}(\omega) : R^d \to R^d$, is an onto homeomorphism for all s, τ, and (iv) $\phi_{s,\tau}(x, \omega)$, is an onto homeomorphism with respect to x for all s, τ, and the derivatives are continuous in (s, τ, x).

Let $\phi_{s,\tau}(\omega)^{-1}$ be the inverse map of $\phi_{s,\tau}(\omega)$. Then i) and ii) imply that $\phi_{\tau,s}(\omega) = \phi_{s,\tau}(\omega)^{-1}$. This fact and condition iii) show that $\phi_{s,\tau}(\omega)^{-1}$ is also continuous in (s,τ,x) and further condition iv) implies that $\phi_{s,\tau}(\omega)^{-1}(x)$ is k-times differentiable with respect to x. Hence $\phi_{s,\tau}(\omega) : R^d \to R^d$ is actually a C^k-diffeomorphism of M, for all s,τ, the so-called random diffeomorphic flow. We can regard $\phi_{s,\tau}(\omega)^{-1}(x)$ as a random field with parameter (s,τ,x) which is often denoted as $\phi_{s,\tau}^{-1}(x,\omega)$. Therefore,

$$\phi_{s,\tau}^{-1}(x) = \phi_{\tau,s}(x) \tag{152}$$

holds for all s,τ,x a.s.When we choose the initial time s of $\phi_{s,\tau}(x,\omega)$ to be 0, we shall write $\phi_\tau(x,\omega)$.

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References

[1] A. Chorin, "Turbulence and Vorticity", Springer,New York, (1994);

[2] K. Gustafson & J. Sethian (edts.), "Vortex Methods and Vortex Motions", SIAM, Philadelphia, (1991).

[3] M. Lesieur, "Turbulence in Fluids", 3rd.ed., Kluwer, Dordrecht, (1997)

[4] M.Lesieur, "La Turbulence", Presses Univs. de Grenoble, (1994).

[5] A. Chorin & J. Marsden, "A Mathematical Introduction to Fluid Mechanics", Springer, New York/Berlin, (1993).

[6] C. Marchioro & M. Pulvirenti, "Mathematical Theory of Incompressible Nonviscous Fluids", Springer, New York/Berlin, (1994).

[7] R. Temam, "Navier-Stokes Equations", North-Holland, Amsterdam, (1977).

[8] U.Frisch, "Turbulence. The legacy of A.N.Kolmogorov", Cambridge Univ. Press, (1996).

[9] D. Ebin & J. Marsden, Ann. Math. **92**, 102-163 (1971)

[10] a. D. Rapoport, *Int. J.Theor. Physics* **35** No.10 (1987), 2127-2152; b.**30**, (1)1, 1497 (1991); c. **35**,(2),287 (1996)

[11] P. Malliavin, "Géométrie Differentielle Stochastique", Les Presses Univ. Montreal (1978).

[12] K.D. Elworthy, "Stochastic Differential Equations on Manifolds", Cambridge Univ. Press, Cambridge, (1982).

[13] J. Eells & K.D.Elworthy, Stochastic dynamical systems, in "Control Theory and topics in Functional Analysis, v.III", ICTP- Trieste, International Atomic Energy Agency, Vienna, 1976.

[14] N. Ikeda & S. Watanabe, "Stochastic Differential Equations on Manifolds", North-Holland/Kodansha, Amsterdam/Tokyo, (1981).

[15] K. Ito, The Brownian motion and tensor fields on Riemannian manifolds, in "Proc. the Intern. Congress of Mathematics", Stockholm, 536-539, 1963.

[16] O.Reynolds, On the dynamical theory of turbulent incompressible fluids and the determination of the criterion, Philosophical Transactions of the Royal Society of London A, **186**, 123-161, (1894)

[17] A. S. Monin & A. M. Yaglom, "Statistical Fluid Mechanics, vol. II", J. Lumley (ed.), M.I.T. Press, Cambridge (MA) (1975).

[18] J. Lumley, "Stochastic Tools in Turbulence", Academic Press, New York, (1970).

[19] S.A. Orszag, Statistical Theory of Turbulence, in "Fluid Dynamics, Les Houches 1973", 237-374, Eds. R. Balian & J.Peube, Gordon and Breach, New York, (1977).

[20] L.Onsager, Statistical Hydrodynamics, Nuovo Cimento **6**(2), 279-287, (Suppl.Serie IX), 1949.

[21] D. Rapoport, The Geometry of Quantum Fluctuations, the Quantum Lyapounov Exponents and the Perron-Frobenius Stochastic Semigroups, in "Dynamical Systems and Chaos", Proceedings (Tokyo, 1994), Y.Aizawa (ed.), World Sc. Publs., Singapore, 73-77,(1995).

[22] D. Rapoport, Covariant Non-linear Non-equilibrium Thermodynamics and the Ergodic theory of stochastic and quantum flows, in "Instabilities and Non-Equilibrium Structures, vol. VI", Proceedings, E. Tirapegui and W. Zeller (eds.), Kluwer, (1998).

[23] H. Kleinert, "The Gauge Theory of Defects, vols. I and II", World Scientific Publs., Singapore,(1989).

[24] H. Kunita, "Stochastic Flows and Stochastic Differential Equations", Cambridge Univ. Press, (1994).

[25] S. Kobayashi & K.Nomizu, "Foundations of Differentiable Geometry I", Interscience, New York, (1963).

[26] D. Rapoport, Torsion and non-linear quantum mechanics, in " Group XXI, Physical Applications and Mathematical Aspects of Algebras, Groups and Geometries, vol. I", Proceedings (Clausthal, 1996), H.D. Doebner et al (edts.), World Scientific, Singapore, (1997). ibidem, Riemann-Cartan-Weyl Geometries, Quantum Diffusions and the Equivalence of the free Maxwell and Dirac-Hestenes Equations, Advances in Applied Clifford Algebras, vol. **8**, No.1, p. 129-146, (1998).

[27] K.D. Elworthy, Stochastic Flows on Riemannian Manifolds, in "Diffussion Processes and Related Problems in Analysis", M.A. Pinsky et al (edts.), vol.II, Birkhauser,(1992).

[28] Fulling, S.A., "Aspects of Quantum Field Theory in Curved Space-Time", Cambridge U.P., (1989)

[29] M. Berger, P. Gauduchon & E. Mazet, "Le spectre d'une variété riemanniene", Springer LNM **170**, (1971).

[30] R. Durrett, "Brownian Motion and Martingales in Analysis", Wadsworth, Belmont,(1984). D. Stroock and S.R.S. Varadhan, Multidimensional Diffusion Processes, Springer-Verlag, 1984.

[31] R. Pinsky, "Positive Harmonic Functions and Diffusions", Cambridge University Press, (1993).

[32] B. Simon, Schroedinger Semigroups, Bull. AMS (new series) **7**, 47-526, 1982.

[33] M. Nagasawa, " Schroedinger Equations and Diffusion Theory, Birkhauser, Basel, (1994).

[34] E. Nelson, "Quantum Fluctuations", Princeton Univ. Press, Princeton, New Jersey, (1985).

[35] Y. Takahashi & S. Watanabe, The probability functionals (Onsager-Machlup functions) of diffusion processes, in "Durham Symposium on Stochastic Integrals", Springer LNM No. **851**, D. Williams (ed.), (1981).

[36] M.I.Vishik & A. Fursikov, "Mathematical Problems of Statistical Hydrodynamics", Kluwer Academic Press,

[37] L.C. Rogers & D.Williams, "Diffusions, Markov Processes and Martingales, vol. II", John Wiley, New York, 1989.

[38] M.T. Landhal & E. Mollo-Christensen, Turbulence and Random Processes in Fluid Mechanics, Cambridge Univ. Press, 1994.

[39] A. Majda, Incompressible Fluid Flow, Communications in Pure and Applied Mathematics Vol. XXXIX, S-187-220, 1986.

[40] P. Meyer, Géometrie stochastique san larmes, in "Séminaire des Probabilites XVI, Supplement",Lecture Notes in Mathematics 921, Springer-Verlag, Berlin, 165-207, 1982.

[41] T. Bohr, M. Jensen, G.Paladini & A.Vulpiani, "Dynamical Systems Approach to Turbulence", Cambridge Non-linear Series No.7, Cambridge Univ. Press, Cambridge,1998.

[42] A.B. Cruzeiro & S. Albeverio, Global flows with invariant (Gibbs) measures for Euler and Navier-Stokes two dimensional fluids, Communications in Mathematical Physics **129** (1990), 431-444; ibidem Solutions et mesures invariantes pour des equations d evolution du type de Navier-Stokes, Expo. Math. **7** (1989), p.73-82.

[43] S.A. Molchanov, Diffusion Processes and Riemannian Geometry, Russian Mathematical Surveys **30** (1975), 1-63.

[44] F. Langouche, D. Roenkarts and E. Tirapegui, Functional Integration and Semiclassical Expansions, Reidel Publs. Co., Dordrecht (1981).

[45] J. Leray, Selected Works, vol. II, Societe Mathematique de France and Springer-Verlag, Berlin, 1998.

[46] F. Hehl, J. Dermott McCrea, E. Mielke & Y. Ne'eman, Physics Reports vol. **258**, 1-157, 1995.

[47] A. Friedman, "Stochastic Differential Equations and Applications, vol. I", Academic Press, New York, (1975).

[48] D. Ruelle (editor), Turbulence, Strange Attractors and Chaos, Series A on Nonlinear Science vol. 16, World Scientific (1995).

[49] D. Rapoport, On the Geometry of Fluctuations, I,& II, in "New Frontiers of Algebras, Groups and Geometries", Proceedings (Monteroduni, 1995), G. Tsagas (ed), Hadronic Press, 1996.

[50] S.A. Molchanov, A.A. Ruzmaikin and D.D. Sokoloff, A Dynamo Theorem, Geophs. Astrophys. Fluid Dynamics 30 (1984), 242.

[51] M. Ghill and S.Childress, "Stretch, Twist and Fold ", Springer-Verlag, New York, 1995.

[52] D. Elworthy and X.M. Li, Formulae for the Derivatives of Heat Semigroups, J. of Functional Analysis 125 (1994), 252-286.

[53] M. Taylor, "Partial Differential Equations vol.I ", Springer Verlag, 1995.

[54] B.Busnello, Ph.D. Thesis, Dipartimento di Matematica, Univ.di Pisa, February 2000.

[55] P.Baxendale and K.D.Elworthy,Flows of Stochastic Dynamical Systems, Z.Wahrschein.verw.Gebiete 65, 245-267 (1983).

[56] V.I. Arnold and B. Khesin, "Topological Methods in Hydrodynamics", Springer Verlag, 1999.

[57] D. Rapoport and S. Sternberg, On the interactions of spin with torsion, Annals of Physics, vol. 154, no. 11, (1984), p.447-475

[58] D. Rapoport, On the Random Geometry of Quantum Mechanics, Gravitation and Fluid Dynamics, in "Fundamental Open Problems of Science at the End of the Millenium, vol. I ",p.248, Proceedings, Academia Sinica, Beijing, September 1998, T. Gill et al (edts.), Hadronic Press, Palm Harbor (USA), 1999.

[59] D. Rapoport, Torsion and Quantum, thermodynamical and hydrodynamical fluctuations, in "The Eighth Marcel Grossmann Meeting in General Relativity, Gravitation and Field Theory, Proceedings ", Jerusalem, June 1997,p. 73-76, vol. A, T.Piran and R.Ruffini

[60] M.S. Narasimhan and S.Ramanan, Existance of Universal Connections, American Journal of Mathematics, vol. 83, 1961. (edts.), World Scientific, Singapore, 1999.

[61] K.D. Elworthy, Y. Le Jan and X.M. Li, On the Geometry of Diffusion Operators And Stochastic Flows, Lecture Notes in Mathematics No. 1704, Springer Verlag, Berlin, 2000.

[62] D. Rapoport, Stochastic Differential Geometry and the Random Integration of the Naviera-Stokes Equations on Smooth Manifolds and the Kinematic Dynamo Problem on Smooth Compact Manifolds and Euclidean Space, to appear, Algebras, Groups and Geometries.

[63] D. Rapoport, Random diffeomorphisms and the integration of the classical Navier-Stokes equations, Trabajos de Matematicas, Instituto Argentino de Matematicas-Conicet, preprint 291, Buenos Aires, Argentina, submitted to Communications in Mathematical Physics.

[64] D.Rapoport, Torsion, Brownian Motion, Quantum Mechanics and Fluid-dynamics, submitted to Proceedings, Ninth Marcel Grossman Meeting in Relativity, Gravitation and Field Theory, Univ. of Rome, June 2000, World Scientific, Singapore, and http://icra.it:8000/