VASSILIEV INVARIANTS CLASSIFY PLANE CURVES AND DOODLES

ALEXANDER B. MERKOV *
Institute for System Analysis
of Russian Academy of Sciences,
prosp. 60-letiya Oktiabria 9,
Moscow, 117312 Russia
Email: merx@ium.ips.ras.ru

Abstract

An *ornament* is a collection of oriented closed curves in a plane or another 2-surface, none three of which intersect at the same point. Similarly a *doodle* is a collection of oriented closed curves with no triple points at all. Homotopy invariants of ornaments and doodles are natural analogues of homotopy and isotopy invariants of links respectively. The Vassiliev theory of ***finite-order invariants*** of ornaments and the series of such invariants are applied to doodles. It appears that these finite-order invariants classify doodles. Similar finite-order invariants of connected oriented closed plane curves classify the latter up to isotopy of the ambient plane.
Keywords: Vassiliev invariant, Gauss diagram, ornament, doodle, plane curve.

Contents

*Supported by INTAS grant, project #4373, Netherland Organization for Scientific Research (NWO), project 47.03.005 and by RFBR grant, project 95-01-00846

1 Introduction

The *ornaments* were introduced by V.Vassiliev in [V93]. An ornament is a finite collection of closed curves in a plane or another surface, none three of which meet at the same point. The homotopy classification of ornaments is similar to the homotopy classification of links. A general way of calculation of cohomology of the space of ornaments was described in [V93]. The calculations are very complicated even for 0-cohomology, i.e. the group of invariants. The set of explicitly shown invariants is even smaller, than the set of calculated generators of the cohomology group.

R.Fenn and P.Taylor ([FT77]) introduced *doodles* as finite collections of closed curves in the 2-sphere or plane (or in another two-dimensional manifold) with no selfintersections and no triple intersections. Later M.Khovanov ([Kh94]) redefined *doodles* as finite collections of closed curves with neither triple intersections nor triple selfintersections (generic (self-)intersections are allowed). The second definition will be used below.

V.Arnold in [A93] started a big series of works on isotopy classification of generic smooth closed plane curves, which is an analogue of the isotopy classification of knots. A generic plane curve can have a finite number of transversal selfintersections, but other singularities are forbidden. This theory is developing in another way than the classification of ornaments: no general homological calculations have been performed, but a big number of particular invariants are studied in details (and generalized to the case of Legendrian fronts).

All ornaments, doodles and curves below are oriented.

This paper is a by-product of attempts to check whether Vassiliev invariants classify ornaments. The question, whether Vassiliev invariants classify objects of some type, is interesting for different geometric objects. It is answered positively for braids and string links by D.Bar-Natan in [B-N95] and for their plane analogues by A.Merkov in [M96]; it remains open for other cases (e.g. knots).

In this paper a series of Vassiliev invariants of ornaments, constructed in [M96] and generalizing Polyak-Viro invariants via Gauss diagrams (see [PV94, P94]), is modified for doodles. The original series is not large enough to classify ornaments, but the modified one classifies doodles and, after minor changes, plane curves.

In section 2 the general theory of ornaments and doodles is reminded briefly. In section 3 the series of Vassiliev invariants of doodles is described. In section 4 it is proved, that they distinguish all non-equivalent doodles. In section 5 the same is done for classification of plane curves.

2 Vassiliev Invariants of Ornaments and Doodles

The notions 'finite-order invariant', 'finite type invariant' and 'Vassiliev invariant' coincide identically. The first one was introduced by V.Vassiliev himself, the other two are used by the majority of the other authors.

2.1 Homological Theory

What follows is an informal sketch of the theory of Vassiliev invariants of ornaments and doodles. See [V93] and [M95] for details for ornaments and [V94] for more general theory. The doodle-specific details are outlined when they differ from such of ornaments.

All invariants of ornaments, doodles, etc. can be expressed via Alexander duality in terms of closed homology of the *discriminant*, i.e. of the set of collections of curves with forbidden triple points. The *finite-order invariants* are exactly those which can be expressed in terms of a finite number of strata of the natural stratification of the discriminant induced by the classification of singular points. Because the discriminant is a very singular space, it is more convenient to study its *resolution*, which is some topological space, homotopy (and hence, homology) equivalent to the discriminant. Though these spaces are infinite-dimensional, the approximation technique developed in [V93] allows one to consider them as finite-dimensional spaces of some big dimension ∞. There exists a natural filtration of the resolution space, and the corresponding homological spectral sequence and the dual cohomological spectral sequence $E_r^{p,q}, r \geq 1, p \leq 0, p+q \geq 0$ may be written. There exists an algorithm calculating this spectral sequence. For each n the group $\oplus_{p=0}^{n} E_{\infty}^{-p,p}$ is the graded group adjoined to the group of *invariants of order n*. There is an interesting question, whether this spectral sequence converges to cohomology of the space of ornaments, or not. The question is still open for ornaments. The same question is also open for the parallel theory of invariants of knots. It was answered positively for invariants of string links by D.Bar-Natan in [B-N95].

Another problem is to describe all finite-order invariants explicitly. By now even the groups $E_1^{-p,p}$ are not calculated in general. Nor the conjecture that $E_1^{-p,p} = E_{\infty}^{-p,p}$ is proved (the similar fact for knots was proved by M.Kontsevich in [K93]).

2.2 Elementary theory

The homological definition of the order of invariants is not convenient for calculation of the order of an explicitly constructed invariant. Fortunately, it can be reformulated in elementary terms, see [V93]. To do this we need to give several definitions and introduce some notation.

2.2.1 Ornaments and quasiornaments

Denote by C_k the disjoint union $c_1 \sqcup \ldots \sqcup c_k$ of k circles.

Definition 1 (See [V93])

O. A *k-ornament* (or simply *ornament*) is a C^∞-smooth map $C_k \to \mathbf{R}^2$ such that no point in $\mathbf{R}^2$ has three different points *of different circles* in its pre-image. Two ornaments are *equivalent*, if the corresponding maps $C_k \to \mathbf{R}^2$ can be connected by a homotopy $C_k \times [0,1] \to \mathbf{R}^2$ such that for any $t \in [0,1]$ the corresponding map of $C_k \times t$ is an ornament. The ornament is *trivial*, if it is equivalent to an embedding of C_k.

D. The definition of *k-doodle* (or simply *doodle*) is exactly the same except the assertion *'of different circles'* omitted.

Definition 2 (See [V93])

O. A *k-quasiornament* (or simply *quasiornament*) is any C^∞-smooth map $C_k \to \mathbf{R}^2$.

D. So a *quasidoodle* is.

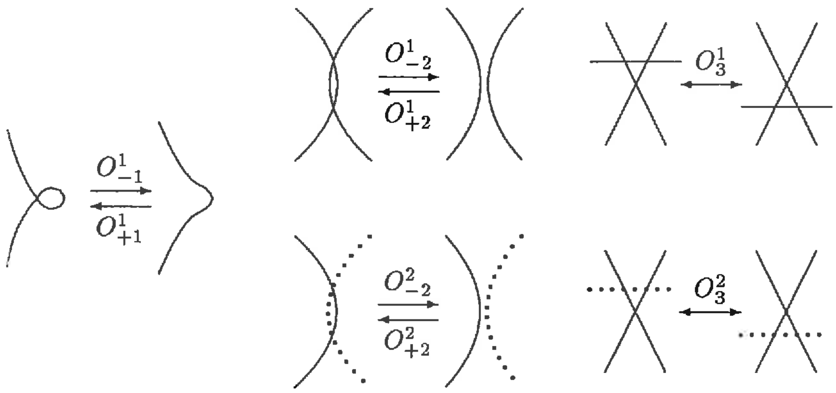

Figure 1: Allowed local moves for ornaments. The upper index indicates the number of the participating components, the lower index indicates the number of participating intersection points or its jump. Moves $O^*_{\pm *}$ are allowed for doodles, but moves O^*_3 are forbidden.

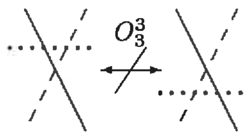

Figure 2: Forbidden local moves for ornaments and doodles

Definition 3 (See [V93]) An ornament or doodle is *regular* if it is an immersion of C_k, and all the multiple points of the image of C_k in $\mathbf{R}^2$ are simple transversal intersection points only. A quasidoodle is *regular* if it is an immersion of C_k, at any multiple point of the image all local components meet pairwise transversally. A regular quasiornament is such a regular quasidoodle that the pre-image of each multiple point either consists of two points, or contains points of at least three circles c_l [1].

Regular ornaments form a dense open set in the space of quasiornaments. Given a generic path in the space of ornaments, almost all ornaments along this path remain regular. At a finite number of singular instants one of the local moves shown in fig. 1 occurs. Below we consider regular (quasi-)ornaments (doodles) only.

Remark. Ornaments, doodles, etc. can be defined similarly in any 2-dimensional manifold, e.g. in the sphere $\mathbf{S}^2$, instead of the plane $\mathbf{R}^2$. Though the homological theory is the simplest for plane ornaments, some elementary theorems are more natural for ornaments in $\mathbf{S}^2$.

2.2.2 Examples of invariants of ornaments

Let us fix an orientation of $\mathbf{R}^2$. Let x be a point of $\mathbf{R}^2$. Recall that the index of a point $x \in \mathbf{R}^2$ with respect to a closed oriented curve $c : \mathbf{S}^1 \to \mathbf{R}^2$ not containing x is the rotation number of the vector $c(t) - x$ when t runs along $\mathbf{S}^1$. Denote by $\mathrm{ind}_l(x) = \mathrm{ind}_{\phi(c_l)}(x)$ the index of x with respect to the l-th component c_l of a regular k-quasiornament ϕ if $x \notin \phi(c_l)$, and the arithmetical mean of the values of the index in the components of the intersection of $\mathbf{R}^2 \setminus \phi(c_l)$ and a small neighborhood of x, if $x \in \phi(c_l)$[2]. Obviously, $\mathrm{ind}_l(x)$ is half-integer if x is a regular point of $\phi(c_l)$, and

[1] The last condition was not stated explicitly in [V93].

[2] In [V93] V.Vassiliev defined the index differently to make all the constructed invariants integer-valued functions of integer variables.

integer if x is a selfintersection of $\phi(c_l)$. Denote by $\mathrm{Ind}(x,\phi)$ the k-dimensional vector $(\mathrm{ind}_1(x),\dots,\mathrm{ind}_k(x))$.

If x is a simple intersection of the i-th component c_i and j-th component c_j, let $\sigma_{ij}(x)$ be equal to 1 if the orientation given by the tangent frame $(t_x(c_i), t_x(c_j))$ coincides with the orientation of $\mathbf{R}^2$, and to -1 otherwise.

To any regular k-ornament ϕ and numbers i and j, such that $1 \le i < j \le k$, there corresponds the following integer-valued function $I_{ij}(b_1,\dots,b_k)$ of half-integer arguments:

$$I_{ij}(b_1,\dots,b_k)(\phi) = \sum_{\substack{x\in\phi(c_i)\cap\phi(c_j)\\ \mathrm{ind}_l(x)=b_l, 1\le l\le k}} \sigma_{ij}(x).$$

Theorem 1 (See [V93, Theorem 3]) *For any $1 \le i < j \le k$ and $b_1,\dots,b_k \in \mathbf{Z}/2$ the function $I_{ij}(b_1,\dots,b_k)$ is an invariant of ornaments.*

In fact, $I_{ij}(b_1,\dots,b_k)$ is non-trivial only when b_i and b_j are not integer, and all the other b_l are integer.

For any k integer nonnegative exponents $\beta = (\beta_1,\dots,\beta_k)$ we can define the functions $M_{ij}(\beta)$ of k-ornaments as follows:

$$\begin{aligned} M_{ij}(\beta) &= \sum_{b_1,\dots,b_k\in\mathbf{Z}/2} {b_1}^{\beta_1}\cdot\ldots\cdot{b_k}^{\beta_k}\cdot I_{ij}(b_1,\dots,b_k) \\ &= \sum_{x\in\phi(c_i)\cap\phi(c_j)} \sigma_{ij}(x)\mathrm{ind}_1(x)^{\beta_1}\cdot\ldots\cdot\mathrm{ind}_k(x)^{\beta_k} \\ &= \sum_{x\in\phi(c_i)\cap\phi(c_j)} \sigma_{ij}(x)\mathrm{Ind}(x)^{\beta}, \end{aligned}$$

where $1 \le i < j \le k$. Obviously, $M_{ij}(\beta)$ are invariants. These *index momenta invariants* are the simplest finite-order invariants (see theorem 2 below).

Remark. Each invariant of ornaments is obviously an invariant of doodles.

2.2.3 Singular points and degeneration modes

Definition 4 (See [V93])

O. A point $x \in \mathbf{R}^2$ is a *singular point of complexity j* of a regular quasiornament $\phi : C_k \to \mathbf{R}^2$, if $\phi^{-1}(x)$ consists of exactly $j+1$ points, at least three of which belong to *different components of* C_k. The *complexity* $\mathrm{compl}(\phi)$ of a regular quasiornament ϕ is the sum of the complexities of all its singular points.

D. The definition of complexity of a regular quasidoodle is the same, except the assertion *'different components of'* omitted.

Definition 5 (See [V93])

O. Suppose that a regular quasiornament ϕ has m singular points $x^1,\dots,x^m$. A *degeneration mode* of ϕ is an order of marking of all the points of their pre-images, satisfying the following conditions. On any step we mark either some three points of $\phi^{-1}(x^l)$, belonging to *three different components of* C_k (if none of the points of $\phi^{-1}(x^l)$ is already marked), or one point (if three or more points of $\phi^{-1}(x^l)$ are already marked).

D. The definition of a degeneration mode of a regular quasidoodle is the same, except the assertion *'three different components of'* omitted.

2.2.4 Characteristic numbers

The *characteristic numbers*, assigned to regular quasiornaments and their degeneration modes by invariants, are numbers, defined inductively by the complexity of quasiornaments.

The characteristic number, assigned to an ornament (whose only degeneration mode is empty), is the value of the invariant of the ornament. The last step of the degeneration mode of the k-quasiornament is either marking of a triple of points on different components c_i, c_j and c_l, $i < j < l$, or marking a single point on c_l. Consider a local transversal perturbation of the l-th component of the k-quasiornament ϕ in a small neighborhood of one of the marked points, i.e. a shift of a small piece of the image $\phi(c_l)$ through the intersection point x, transversally to the tangent to $\phi(c_l)$ in x. The perturbed quasiornaments are of smaller complexity than the initial one, and belong to two different equivalence classes. One of them can be called 'positive' and the other 'negative'. Namely, if x is a triple intersection of $\phi(c_i)$, $\phi(c_j)$, and $\phi(c_l)$, then the 'positive' k-quasiornaments are those with greater value of $\sigma_{ij}(x)\,\mathrm{ind}_l(x)$, and if x is a more complicated intersection, then the 'positive' k-quasiornaments are those with greater value of $\mathrm{ind}_l(x)$. The characteristic number of the given k-quasiornament and the degeneration mode is defined as the difference of the characteristic numbers of the 'positive' and 'negative' k-quasiornaments for the degeneration modes, obtained from the given one by removing the last step.

The same definition remains for quasidoodles, except two additions: it is necessary to define, which of the pictures, participating in the moves O_3^1 and O_3^2 (fig. 1), are positive, and which are negative. If x is a selfintersection of the j-th component and the image of the l-th component passes through x (the move O_3^2), then the positive quasidoodle is the one with greater value of $\mathrm{ind}_l(x)$. For moves O_3^1 the routing along the component in its positive direction induces cyclic order of the edges of the little 'vanishing' triangle. Routing the triangle in this order induces the direction of each edge, which may coincide or not with its native direction (as a part of the oriented curve). Positive quasidoodles are those with even number of edges, for which the two directions coincide (see [A93]).

2.2.5 Finite-order invariants

Definition 6 (See [V93]) An invariant of ornaments (doodles) is an *invariant of order i* if all characteristic numbers, assigned by it to any regular quasiornament (quasidoodle) of complexity $> i$ vanish.

Theorem 2 (See [V93, Theorem 4]) *$M_{ij}(\beta)$ is an invariant of order $|\beta| + 1$, where $|\beta| = \beta_1 + \ldots + \beta_k$.*

3 New Vassiliev invariants of doodles

The invariants of doodles, constructed below, are just modified invariants of ornaments from [M98]. They 'generalize' both invariants of ornaments given by theorem 2 and invariants of knot and curves constructed in [PV94, P94].

3.1 Arrow diagrams

Definition 7 An *arrow diagram* or *oriented Gauss diagram* of *length n* is C_k supplied with a collection of n arrows, connecting n pairs of different points of C_k. The diagram is *non-degenerated* if all $2n$ points are different. Two diagrams are *isomorphic* if one can be mapped onto the other by a diffeomorphism $C_k \to C_k$ preserving each circle of C_k, its orientation and directions of the arrows.

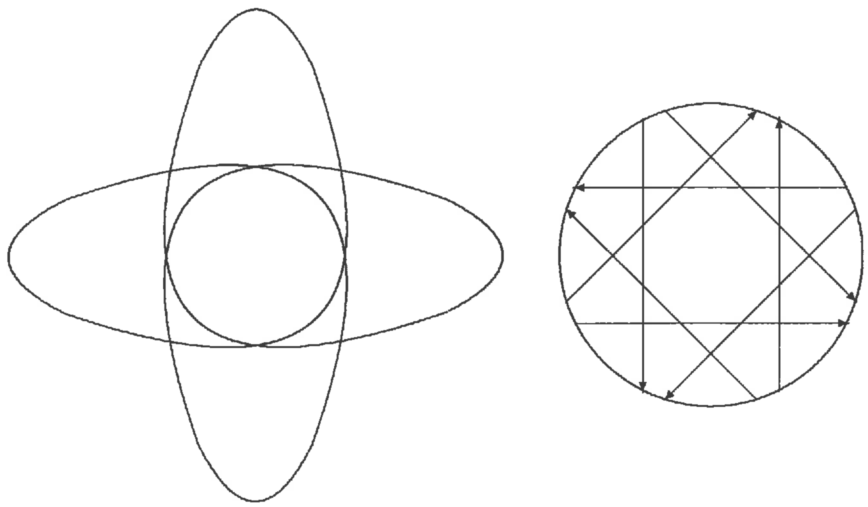

Figure 3: A non-trivial 1-doodle and its arrow diagram

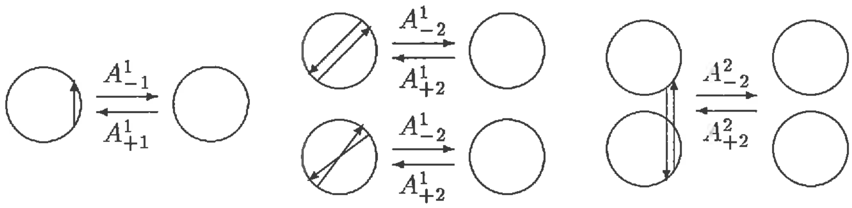

Figure 4: Allowed moves of arrow diagrams. Unchanged arrows and circles are not shown. The orientations of the circles are arbitrary.

Definition 8 The arrow diagram ADiag(ϕ) of a regular (quasi-)doodle $\phi : C_k \to \mathbf{R}^2$ is a diagram, the arrows of which goes from $x_i \in C_k$ to $x_j \in C_k$ if and only if $\phi(x_i) = \phi(x_j)$ and the tangent map ϕ_* maps the pair of positive tangent vectors $(t(x_i), t(x_j))$ to a positive tangent frame.

Definition 8 can be applied to (quasi-)doodles on any oriented surface. The arrow diagram of a regular doodle is obviously non-degenerated. The diagram of a regular quasidoodle is degenerated, but any two arrows can have at most one common endpoint. From now on by 'arrow diagram' we usually mean the class of isomorphic diagrams and omit 'non-degenerated' considering the diagrams of doodles. It will not cause any confusion. The diagrams may be also thought as 1-dimensional cell complexes with each 1-cell oriented and k oriented circles marked and numbered.

Example. A 1-doodle and its arrow diagram are shown in fig. 3.

Let us define equivalence of arrow diagrams in a way, guaranteeing that the diagrams of equivalent doodles are equivalent. The equivalence is defined by the following *allowed moves*, corresponding to allowed local moves of doodles, shown in fig. 4 and 1 respectively.

A^1_{-1} Deletion of an arrow of some of the circles, whose ends are neighboring (i.e. are not separated by the ends of other arrows).

A^1_{+1} The move opposite to A^1_{-1}.

A^1_{-2} Annihilation of two arrows of some of the circles, if the beginning of each arrow is a neighbor of the end of the other one.

A^1_{+2} The move opposite to A^1_{-2}.

A^2_{-2} Annihilation of two arrows connecting two circles, if the beginning of each arrow is a neighbor of the end of the other one.

A^2_{+2} The move opposite to A^2_{-2}.

Like in fig. 1, the upper index of a move denotes the number of participating circles, and the lower one denotes the jump of the number of arrows. The equivalence class of its arrow diagram is obviously an invariant of the doodle.

Proposition 1 (Maximum principle) *For each sequence of allowed moves of arrow diagrams there exists a sequence of moves with the same ends, such that the length of each intermediate diagram does not exceed the maximum of the lengths of the end diagrams.*

Proof. The unwanted local maximum of the length can occur between two moves M and N of the types A^*_{+*} and A^*_{-*} respectively. It is easy to see, that if none of the arrows born at M dies at N, then M and N can be permuted; otherwise M and N can be either removed from the sequence or changed to a single move. In any case the number of pairs 'birth before death' decreases. If the given sequence consists of m moves, in $\leq \left(\frac{m}{2}\right)^2$ such steps it will be transformed to a sequence of $\leq m$ moves with no inner local maximums of the length. □

Corollary 1 *Among arrow diagrams equivalent to the diagram of a doodle there is a unique diagram of a minimal length. The sequence of moves turning the diagram to such a* minimal diagram *does not contain moves of the type A^*_{+*}. There exists an algorithm producing the minimal diagram equivalent to a given one. In particular, there exists an algorithm checking the equivalence of two given diagrams.*

□

So, it is easy to check equivalence of the diagrams of doodles; but not of doodles themselves.

Example. In fig. 3 the arrow diagram is minimal, hence the doodle is non-trivial.

Remark. A doodle in $\mathbf{R}^2$ cannot be reconstructed from its full arrow diagram unless its exterior contour is marked on the diagram. A disconnected doodle cannot be reconstructed from its arrow diagram even in $\mathbf{S}^2$, see fig. 5.

3.2 Finite-order invariants, generated by arrow diagrams

The following notation is used below. An arrow diagram X is denoted by $(x^1, \ldots, x^n)$, where $x^l = [x^l_{i^l}, x^l_{j^l}]$, $l = 1, \ldots, n$ are all its arrows, $x^l_{i^l} \in c_{i^l}$ and $x^l_{j^l} \in c_{j^l}$ are the beginning and the end of the l-th arrow, $1 \leq i^l, j^l \leq k$. The length of the diagram X is denoted by $L(X) = n$.

Arrow diagrams are 1-dimensional cell complexes. Let us consider continuous maps of a non-degenerated diagram X into an arrow diagram Y, which restriction to each circle c_l is a map onto itself, homotopic to the identical map, and monotonic (maybe not strictly monotonic, if the diagram Y is degenerated), and restriction to each arrow is a homeomorphism onto another arrow. Let us call a class of homotopy equivalence of such maps a *representation* of an arrow diagram X in an arrow diagram Y.

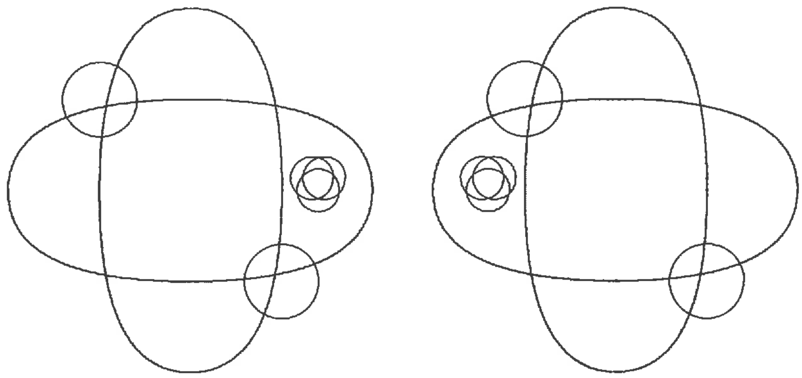

Figure 5: Nonequivalent 7-doodles with the same arrow diagram. The three-component subdoodle is moved as a whole within the four-component one.

Given a representation $\mu : X \to Y$, let $\sigma_\mu(x^l)$ be equal to 1 if the direction of the arrow $\mu(x^l)$, induced from the direction of x^l, coincides with its native direction in Y, and to -1 otherwise. Denote by $\sigma(\mu)$ the product

$$\sigma(\mu) = \prod_{l=1}^{L(X)} \sigma_\mu(x^l), \tag{1}$$

and by $\langle X, Y \rangle$ the sum

$$\langle X, Y \rangle = \sum_{\mu : X \to Y} \sigma(\mu) \tag{2}$$

over all representations $\mu : X \to Y$.

Given a regular doodle ϕ, let us define $\langle X, \phi \rangle$ by

$$\langle X, \phi \rangle = \langle X, \mathrm{ADiag}\, \phi \rangle = \sum_{\mu : X \to \mathrm{ADiag}\, \phi} \prod_{l=1}^{L(X)} \sigma_\mu(x^l) \tag{3}$$

Remark. Practically the same construction is used in [P94] for invariants of plane curves. The only difference is that instead of the arrow diagram X a chord diagram with an additional regular point (*base point*) marked on each circle is considered. In such an approach for some, but not for all, diagrams X the expression given by formula (3) is independent of the choice of the base points.

Proposition 2 *If X is a minimal arrow diagram, then $\langle X, \cdot \rangle$ is an invariant of doodles.*

Proof. We need to check, that the allowed local moves of the doodles do not change $\langle X, \cdot \rangle$. To do this, it is enough to verify, that moves of the types $A^*_{\pm *}$ (three pairs of types) of the arrow diagrams of doodles do not change $\langle X, \cdot \rangle$. This is elementary. □

Remark. Since $\langle X, \phi \rangle$ depends on $\mathrm{ADiag}(\phi)$ only, it cannot distinguish doodles with the same arrow diagram, for instance, doodles in $\mathbf{R}^2$, which are equivalent in $\mathbf{S}^2$.

This class of invariants can be easily generalized. Let X be an arrow diagram of length n and F be a function of $k \times n$ half-integer variables. Given a doodle ϕ, let us define $\langle (X, F), \phi \rangle$ as the sum

$$\langle (X, F), \phi \rangle = \sum_{\mu : X \to \mathrm{ADiag}(\phi)} \sigma(\mu) F(\mathrm{Ind}(\phi(\mu(x^1)), \phi), \ldots, \mathrm{Ind}(\phi(\mu(x^n)), \phi)) \tag{4}$$

over all representations $\mu : X \to \mathrm{ADiag}(\phi)$.

Theorem 3 *If X is a minimal arrow diagram of n arrows and F is a function of $k \times n$ half-integer variables, then $\langle (X, F), \cdot \rangle$ is an invariant of doodles.*

Proof. This theorem is a special case of theorem 4 below. □

Example. All invariants $M_{ij}(\beta)$ are defined by formula (4) for arrow diagrams of length 1.

The function $\mathrm{ind}_*(\cdot)$ of a closed curve and a point in $\mathbf{R}^2$ (see paragraph 2.2.2) can be extended by linearity to a pairing defined on pairs of 1-cycles and 0-chains (similarly, to a pairing of 1-cycles and 0-cycles in $\mathbf{S}^2$). Hence for each arrow diagram X, considered as a 1-dimensional cell complex, regular doodle ϕ and representation $\mu : X \to \mathrm{ADiag}(\phi)$ the pairing $\mathrm{ind} \circ (\phi_* \circ \mu_* \otimes \phi_* \circ \mu_*) : Z_1(X) \times C_0(X) \to \mathbf{Z}/2$ is defined. We have already used this pairing in formula (4), where the 1-cycles are circles of the arrow diagram. To extend it to arbitrary cycles, we need to generalize the definition of index (page 4) to all cycles in the image of a regular doodle.

Definition 9 Let ϕ be a regular doodle, ξ be a closed piecewise smooth curve, $\xi(\mathbf{S}^1) \subset \phi(C_k)$. Then the index $\mathrm{ind}_\xi(x)$ of a point $x \in \mathbf{R}^2$ is the same as defined on page 4 if $x \notin \xi(\mathbf{S}^1)$ or x is a regular point of ξ, and is the arithmetical mean of the indices of the points of four adjacent areas of $\mathbf{R}^2 \setminus \phi(\mathbf{S}^1)$ if x is a vertex of ξ.

This definition agrees obviously with the old definition when ξ is a component of ϕ. The value of $\mathrm{ind}_\xi(x)$ is quarter-integer in general.

Theorem 4 *Let X be a minimal arrow diagram with arrows $\{x^1, \dots, x^n\}$, and $\Xi = \{\xi_1, \dots, \xi_m\}$ a set of 1-cycles in X. If F is a function of $m \times n$ variables v_l^i, $i = 1, \dots, n$, $l = 1, \dots, m$, actually depending only on those v_l^i, for which the cycle ξ_l does not pass along the arrow x^i, then the sum*

$$\langle (X, \Xi, F), \phi \rangle = \sum_{\mu : X \to \mathrm{ADiag}(\phi)} \sigma(\mu) F(\mathrm{ind}_{\phi(\mu(\xi_1))}(\phi(\mu(x^1))), \dots, \mathrm{ind}_{\phi(\mu(\xi_m))}(\phi(\mu(x^n)))) \tag{5}$$

over all representations $X \to \mathrm{ADiag}(\phi)$ is an invariant of doodles in $\mathbf{R}^2$.

Proof. We need to check, that the allowed local moves of the doodles do not change $\langle (X, \Xi, F), \cdot \rangle$. Let us consider a move of the type O_{+2}^*. All the summands of formula (5) remain unchanged, and several new summands may appear, where μ maps at least one arrow of X to the two newborn arrows y_1 and y_2 in $\mathrm{ADiag}(\phi)$, corresponding to the newborn intersection points z_1 and z_2. In fact, μ cannot map two arrows to both y_1 and y_2, because the diagram X is minimal. On the other hand, if a representation μ_1 maps an arrow x to y_1, there exists a representation μ_2 mapping x to y_2, and coinciding with μ_1 on the other arrows. The directions of y_1 and y_2 are opposite, and for any cycle ξ not passing along x $\mathrm{ind}_{\phi(\mu_1(\xi))}(z_1) = \mathrm{ind}_{\phi(\mu_2(\xi))}(z_2)$, so the two new summands of (5) annihilate.

Similarly, a move of the type O_{+1}^1 cannot add new summands because of the minimalness of X. □

The proof of theorem 4 can be applied as well to the following theorem, where the assertion of minimalness of the diagram X is relaxed.

Theorem 5 *Let X be an arrow diagram with arrows $\{x^1, \ldots, x^n\}$, $\Xi = \{\xi_1, \ldots, \xi_m\}$ a set of 1-cycles in X, and F a function of $m \times n$ variables v_l^i, $i = 1, \ldots, n$, $l = 1, \ldots, m$, such that*

- *F depends only on those v_l^i, for which the cycle ξ_l does not pass along the arrow x^i, and*
- *for any doodle ϕ and representation $\mu : X \to \mathrm{ADiag}(\phi)$ if $\phi \circ \mu$ maps some arrow(s) to the vertex (both vertices) of a loop or segment, vanishing at moves O^*_{-*}, then $F(\mathrm{ind}_{\phi(\mu(\xi_1))}(\phi(\mu(x^1))), \ldots, \mathrm{ind}_{\phi(\mu(\xi_m))}(\phi(\mu(x^n)))) = 0$.*

Then the sum (5) is an invariant of doodles.

□

3.3 Order of invariants

Theorem 6 *If X is an arrow diagram, Ξ is a set of 1-cycles in X, and F is a polynomial, then $\langle (X, \Xi, F), \cdot \rangle$ in theorems 4 and 5 is an invariant of order* $\deg(F) + L(X)$.

Corollary 2 *If X is a minimal arrow diagram and F is a polynomial, then $\langle (X, F), \cdot \rangle$ is an invariant of order* $\deg(F) + L(X)$.

□

Corollary 3 *If X is a minimal arrow diagram, then $\langle X, \cdot \rangle$ is an invariant of order* $L(X)$.

□

Corollary 3 allows one to go from the classification of doodles by their arrow diagrams to classification by finite-order invariants.

Example. If ϕ is the 1-doodle shown in fig. 3, then the diagram $\mathrm{ADiag}(\phi)$ is minimal, and $\langle \mathrm{ADiag}(\phi), \cdot \rangle$ is an invariant of order 8, distinguishing ϕ from the trivial 1-doodle ($\langle \mathrm{ADiag}(\phi), \phi \rangle = 8$).

Proof of theorem 6. The proof essentially repeats the proof of lemma 31 and theorem 7 in [M9?], so the technical details, such as which perturbations of the quasidoodles are good enough, are omitted. Generic perturbations are good.

Each summand of the sum (5) can be considered separately. It is not necessarily an invariant of doodles, but it is a local invariant in a small neighborhood of any quasidoodle, and the characteristic numbers are defined like for global invariants.

Let ϕ' be a regular quasidoodle and ϕ a perturbation of ϕ' (see paragraph 2.2.4). Then there exists a representation $\mu_{\phi\phi'} : \mathrm{ADiag}(\phi) \to \mathrm{ADiag}(\phi')$, close to identity on C_k and preserving the directions of all arrows.

The summand

$$F(\mathrm{ind}_{\phi(\mu(\xi_1))}(\phi(\mu(x^1))), \ldots, \mathrm{ind}_{\phi(\mu(\xi_m))}(\phi(\mu(x^n)))) \qquad (6)$$

with μ close to a representation $\mu' : X \to \mathrm{ADiag}(\phi')$ defines a local invariant of regular doodles in a neighborhood of ϕ'. It vanishes on the doodles ϕ, for which there is no such representation $\mu : X \to \mathrm{ADiag}(\phi)$ ('perturbation of μ''), that $\mu_{\phi\phi'} \circ \mu = \mu'$. The sign $\sigma(\mu)$ is omitted since it is the same for all μ close to μ'.

The proof is done inductively in the complexity of quasidoodles. The inductive assumption is as follows.

For any representation $\mu : X \to \mathrm{ADiag}(\phi)$ denote by $n(\mu)$ the number of arrows of X going to simple intersection points of ϕ.

Then the characteristic number assigned to a degeneration mode of ϕ by a local invariant (6) is equal to a sum

$$\sum_j p_j(\mathrm{ind}_{\nu_j(\xi_1)}(\nu_j(x^1)),\dots,\mathrm{ind}_{\nu_j(\xi_m)}(\nu_j(x^n))), \tag{7}$$

where ν_j stands for $\phi_j \circ \mu_j$, the doodles ϕ_j are perturbations of ϕ, representations $\mu_j : X \to \mathrm{ADiag}(\phi_j)$ are perturbations of μ, and p_j are polynomials, such that for each summand in (7)

$$\deg(p_j) + n(\nu_j) \le \deg(F) + n - \mathrm{compl}(\phi), \tag{8}$$

where $n(\nu_j)$ is the number of arrows of X going to simple intersection points of ϕ_j.

Now the induction base ($\mathrm{compl}(\phi) = 0$, i.e. ϕ is an doodle) is trivial: $j = 1$, $p_1 = F$, $\phi_1 = \phi$, and $\mu_1 = \mu$. The induction step is elementary and described below. Then if $\mathrm{compl}(\phi) > \deg(F)+n$, then the left hand side of (8) is negative, and hence the sum (7) vanishes.

Each summand

$$p(\mathrm{ind}_{\nu(\xi_1)}(\nu(x^1)),\dots,\mathrm{ind}_{\nu(\xi_m)}(\nu(x^n)))$$

of (7) may be treated separately at the induction step. The step splits into several cases. First it depends on the type of the last step of degeneration.

Marking a triple of points i.e. the perturbation is a passing through a triple point of ϕ. The following subcases are possible, depending on the number of arrows, going into a small neighborhood of the triple point.

0 The characteristic number is zero.

1 Each of the $m \times n$ arguments $\mathrm{ind}_{\nu(\xi_l)}(\nu(x^i))$ of polynomial p either remains unchanged when passing from negative to positive perturbations, or changes by 1 (if x^i goes to a vertex of the vanishing triangle, and $\nu(\xi_l)$ passes it opposite side). Hence the characteristic number is equal to the value of a 'discrete partial derivative' of p, which is a polynomial of degree $\deg(p) - 1$, taken on a perturbation of the quasidoodle.

2 or 3 The representation μ and the map ν are defined only for positive or only for negative perturbations. Then the characteristic number is equal to the value of p with shifted arguments on the unperturbed quasidoodle, but $n(\nu)$ decreases at least by 2.

Marking a single point i.e. the perturbation is a passing of a local branch of a doodle through a multiple point. The following objects need to be watched:

- The multiple point q.
- The running local branch b.
- The local branches $b_1,\dots,b_s$, passing through q.
- The intersections $q_1,\dots,q_s$ of $b_1,\dots,b_s$ and b.

If ν maps no arrow to $q, q_1,\dots,q_s$, then the characteristic number is zero. If ν maps exactly one arrow to $q, q_1,\dots,q_s$, then the characteristic number is the difference of the values of the polynomial p in two points, some of whose $m \times n$ coordinates coincide, and some of the coordinates differ by 1. Such a sum of 'discrete partial derivatives' is a polynomial of degree $\deg(p) - 1$, taken on a perturbation of the quasidoodle. If ν maps more than one arrow to $q, q_1,\dots,q_s$, then it is defined only for positive or only for negative perturbations. Then the characteristic number is equal to the value of p with shifted arguments on the unperturbed quasidoodle, but $n(\nu)$ decreases at least by 1.

In any case the condition (8) remains true. □

4 Finite-order invariants classify doodles

The arrow diagram, being considered as a cell complex, can be *connected* or not. Obviously, for any doodle ϕ the diagram ADiag(ϕ) is connected if and only if the image $\phi(C_k)$ is connected.

The case of connected doodles in $\mathbf{S}^2$ is studied first. Then the case of connected doodles in $\mathbf{R}^2$ and the general case are considered.

4.1 Strongly connected doodles in $\mathbf{S}^2$

The following lemma and theorem are almost obvious (see [M98] for more details). They reduce classification of connected doodles in $\mathbf{S}^2$ to classification of their diagrams.

Lemma 1 *Let ϕ and ψ be regular doodles in the sphere $\mathbf{S}^2$ with connected images $\phi(C_k)$ and $\psi(C_k)$ and the same arrow diagram. Then the map $\psi \circ \phi^{-1} : \phi(C_k) \to \psi(C_k)$ can be extended to an orientation-preserving diffeomorphism $\mathbf{S}^2 \to \mathbf{S}^2$.*

□

Theorem 7 *A connected doodle in the sphere $\mathbf{S}^2$ is determined up to equivalence by its arrow diagram.*

□

Now we see, to what extent the equivalence of diagrams can be lifted up to equivalence of doodles.

Lemma 2 *If ϕ is a connected doodle in $\mathbf{S}^2$ and an arrow diagram X is obtained from* ADiag(ϕ) *by a move of type A^*_{-*}, then there exists an equivalent to ϕ doodle ψ with* ADiag(ψ) $= X$.

Proof. Each move of type A^*_{-*} of ADiag(ϕ) can be lifted to a homotopy of the doodle ϕ, because the two vanishing arcs (or the vanishing loop) split $\mathbf{S}^2$ into two areas, the smaller one of which is contractible and contains no points of $\phi(C_k)$. This contraction leads to a doodle with a singular point (tangency or cusp), which can be eliminated easily, and the desired doodle ψ will be obtained. □

Definition 10 The doodle is *strongly connected* if the image of each doodle equivalent to it is connected.

Lemma 3 *A doodle ϕ in $\mathbf{S}^2$ is strongly connected if and only if its minimal arrow diagram X is connected. If ϕ is strongly connected, then there exists an equivalent to ϕ doodle ψ with* ADiag(ψ) $= X$.

Proof. The arrow diagram of the doodle or of any equivalent doodle can be transformed to the minimal diagram by finite number of moves of type A^*_{-*}. Such moves transform disconnected diagrams to disconnected, so if the minimal diagram of a doodle is connected, all equivalent diagrams and doodles are connected too. On the other hand, if ϕ is strongly connected, lemma 2 is applicable to each such move, and hence the minimal diagram is a diagram of some doodle ψ equivalent to ϕ, and is connected. □

Lemma 3 follows also immediately from results of [Kh94, Section 2]. Lemma 3 gives us an efficient way to check, if the doodle is strongly connected.

For doodles and curves in $\mathbf{S}^2$ the index $\mathrm{ind}_\xi(x)$ cannot be defined, but the difference $\mathrm{ind}_\xi(x-y)= \mathrm{ind}_\xi(x)-\mathrm{ind}_\xi(y)$ is defined correctly. The following analogue of theorem 4 and corollary 2 holds.

Theorem 8 *Let X be a minimal arrow diagram with arrows $\{x^1,\dots,x^n\}$, and $\Xi=\{\xi_1,\dots,\xi_m\}$ a set of 1-cycles in X. If F is a function of $m\times\binom{n}{2}$ variables v_l^{ij}, $1\le i<j\le n$, $l=1,\dots,m$, actually depending only on those v_l^{ij}, for which the cycle ξ_l does not pass along the arrows x^i and x^j, then the sum*

$$\langle(X,\Xi,F),\phi\rangle = \sum_{\mu:X\to\mathrm{ADiag}(\phi)} \sigma(\mu)F(\mathrm{ind}_{\phi(\mu(\xi_1))}(\phi(\mu(x^2))-\phi(\mu(x^1))),\dots,$$
$$\mathrm{ind}_{\phi(\mu(\xi_m))}(\phi(\mu(x^n))-\phi(\mu(x^{n-1})))) \qquad (9)$$

over all representations $X\to\mathrm{ADiag}(\phi)$ is an invariant of doodles in $\mathbf{S}^2$. If F is a polynomial, then $\langle(X,\Xi,F),\cdot\rangle$ is an invariant of order $\deg(F)+L(X)$.

Proof. All the calculations in the proofs of theorems 4 and 6 are local and do not depend on the topology of the ambient oriented 2-manifold. □

The boundary of each component of the complement of the image of a doodle consists of several canonically oriented non-selfintersecting piecewise-smooth closed curves. The pre-image of such a curve in the arrow diagram of the doodle is a 1-cycle. The set of all such cycles is determined by the arrow diagram unambiguously. Indeed, the positive orientation of each such cycle is defined by its regular point and the direction in it in the following way. The running point moves along the components of the arrow diagram; every time it meets an endpoint of an arrow, it jumps to its other endpoint and goes on along the other (or, maybe, the same) component in the same (positive or negative) direction as before if it has jumped along the arrow, and in the opposite direction otherwise. And conversely, the directions of the arrows can be reconstructed by the set of pre-images in the arrow diagram of the canonically oriented boundary components.

The following lemma expresses these reasoning in terms of indices.

Lemma 4 *Let ξ be a connected 1-cycle in the arrow diagram $\mathrm{ADiag}(\phi)$ of a connected doodle ϕ in $\mathbf{S}^2$, passing none of the arcs the arrow diagram twice, and x^1 be an arrow out of ξ. The image of ξ is the canonically oriented boundary of a component of the complement to the image of ϕ if and only if*

$$\begin{array}{ll} \mathrm{ind}_{\phi(\xi)}(\phi(x)-\phi(x^1))=0 & \textit{for any arrow } x\notin\xi, \textit{ and} \\ \mathrm{ind}_{\phi(\xi)}(\phi(x)-\phi(x^1))=\frac{n(\xi,x)}{4} & \textit{for any arrow } x\in\xi, \end{array}$$

where $n(\xi,x)$ is how many times ξ meets x, i.e. either 1 or 2 (see fig. 6).

Proof. In other words, the image of ξ turns left at each intersection point met, and all the other intersection points remain to its right. □

The goal of this section is to express the equations of lemma 4 via polynomial function of theorem 8 and to construct a finite-order invariant, which checks whether the diagrams of two given strongly connected doodles, and hence the doodles themselves, are equivalent. Unfortunately, only half-integer values of indices $\mathrm{ind}_*(\cdot-\cdot)$ are allowed in theorem 8. A technical trick allows one to formulate a substitution of lemma 4 in terms of half-integer indices.

To keep track of the elementary technical details below it is helpful to have in mind the following well-known facts.

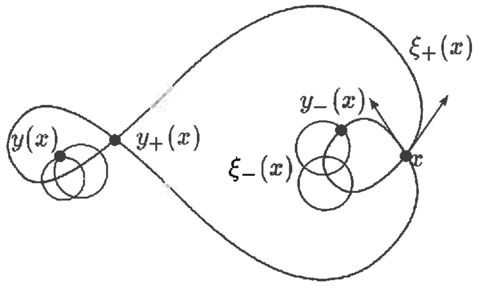

Figure 6: The boundary of an area can pass an intersection point either once or twice. The intersection point belongs to the closure of either three or four areas.

Proposition 3

The components of the complement to the image of a regular doodle ϕ can be painted in two colors in such a way, that any two components of the same color have no common arc.

If a running point goes along such a doodle and turns right or left at each intersection point it passes, then the color of the area to its left (or to its right) remains the same.

Proof. The color of a point x depends on $\sum_{l=1}^{k} \mathrm{ind}_l(x) (\mathrm{mod}\, 2)$ only. □

The intersection point of a doodle in $\mathbf{S}^2$ belongs to the closure of either 3 or 4 areas of $\mathbf{S}^2 \setminus \phi(C_k)$ the complement (see fig. 6). We call such points *3-points* and *4-points*, respectively. A 3-point is necessarily a selfintersection point, and the 'big' area approaches it in two opposite quadrants.

Given an arrow x of the diagram of a regular doodle ϕ, denote by $\xi_1(x), \xi_2(x), \dots$ the pre-images of the canonically ('counterclockwise') oriented boundaries of the neighboring to $\phi(x)$ areas. Denote by $\xi(x)$ the pre-image of the boundary of the union of these areas . If $\phi(x)$ is a 4-point, we suppose $\xi_i(x)$ numbered in a counterclockwise cyclical order, and denote by $\xi_{ij}(x)$ the boundary of the union of the i-th and the j-th one.

Proposition 4 *If the diagram* ADiag(ϕ) *is connected and minimal, then for each arrow x there exists such an arrow $y(x)$, that*

$$\mathrm{ind}_{\phi(\xi(x))}(\phi(x) - \phi(y(x))) = 1. \tag{10}$$

If $\phi(x)$ is a 4-point, then also

$$\begin{aligned} \mathrm{ind}_{\phi_{12}(\xi(x))}(\phi(x) - \phi(y(x))) = \mathrm{ind}_{\phi_{23}(\xi(x))}(\phi(x) - \phi(y(x))) &= \\ \mathrm{ind}_{\phi_{34}(\xi(x))}(\phi(x) - \phi(y(x))) = \mathrm{ind}_{\phi_{41}(\xi(x))}(\phi(x) - \phi(y(x))) &= \frac{1}{2} \end{aligned} \tag{11}$$

Proof. Since the Euler characteristics of $\mathbf{S}^2$ is equal to 2 and a connected regular doodle with n intersections has $2n$ arcs, its complement consists of $n+2$ areas. First, we eliminate the degenerated case, when $\phi(\xi(x))$ is not a curve. Indeed, since ADiag(ϕ) is minimal, there are exactly 4 different arcs incident to $\phi(x)$. All the 4 second vertices of the arcs may coincide, but then the image of ϕ is diffeomorphic to the union of two big circles on the sphere, and ADiag(ϕ) is not minimal. Hence $\phi(\xi(x))$ is a connected closed piecewise-smooth curve on $\mathbf{S}^2$, which possibly passes some arcs and vertices more than once.

Now we suppose that the cycle $\phi(\xi(x))$ is incident to all $n-1$ intersections except $\phi(x)$, calculate the total number of arcs and will come to a contradiction. $\phi(\xi(x))$ has at least $n-1$ arcs. There are at least $(n+2)-4=n-2$ areas, not incident to $\phi(x)$, and the boundary of each area consists of at least 3 arcs, because $\mathrm{ADiag}(\phi)$ is minimal. So the total number of 'left sides' of the arcs, is at least

$$\begin{array}{rcl} & 2\times 4 & \text{(the four arcs incident to } x\text{)} \\ + & n-1 & \text{(the arcs of } \phi(\xi(x))\text{)} \\ + & 3(n-2) & \text{(the other arcs)} \\ = & 4n+1 & > 2\times 2n. \text{ The contradiction.} \end{array}$$

So there exists an arrow $y(x)$, different from x and all arrows of $\xi(x)$. Then $\mathrm{ind}_{\phi(\xi(x))}(\phi(x)-\phi(y(x)))=1$ by definition of $\xi(x)$. The equalities (11) are also obvious. □

Let now $\phi(x)$ be a 3-point of a connected minimal doodle ϕ. The boundary of the neighboring to $\phi(x)$ area Δ, touching $\phi(x)$ in two quadrants, is the union of two cycles $\phi(\xi_+(x))$ and $\phi(\xi_-(x))$: 'positive' and 'negative'. The orientation of each of them is determined by the orientation of the doodle component near $\phi(x)$, and coincides or is opposite to the canonical boundary orientation for $\phi(\xi_+(x))$ or $\phi(\xi_-(x))$, respectively. Since ϕ is minimal, there exist arrows $y_+(x)$ in $\xi_+(x)$ and $y_-(x)$ in $\xi_-(x)$, different from x. The closure of the area Δ splits the rest of the sphere to 'positive' and 'negative' sets, depending on the sign of $\mathrm{ind}_{\phi(\xi_+(x)+\xi_-(x))}(\phi(x)-\cdot)$. $\phi(\xi_+(x))$ forms the clockwise oriented boundary of the positive set, and $\phi(\xi_-(x))$ forms the counterclockwise oriented boundary of the negative set. For instance, $\phi(y(x))$ belongs either to the positive or to the negative set; then either

$$\mathrm{ind}_{\phi(\xi_+(x))}(\phi(y_-(x))-\phi(y(x)))=1 \text{ and } \mathrm{ind}_{\phi(\xi_-(x))}(\phi(y_+(x))-\phi(y(x)))=0, \quad (12)$$

or

$$\mathrm{ind}_{\phi(\xi_+(x))}(\phi(y_-(x))-\phi(y(x)))=0 \text{ and } \mathrm{ind}_{\phi(\xi_-(x))}(\phi(y_+(x))-\phi(y(x)))=-1, \quad (13)$$

respectively.

Proposition 5 *Let μ be a representation of a connected minimal diagram* $\mathrm{ADiag}(\phi)$ *in a diagram* $\mathrm{ADiag}(\psi)$ *with the same number of arrows. Then μ is an isomorphism if and only if for each arrow x in* $\mathrm{ADiag}(\phi)$

$$\mathrm{ind}_{\psi(\mu(\xi_{ij}(x)))}(\psi(\mu(x))-\psi(\mu(y(x))))=\mathrm{ind}_{\phi(\xi_{ij}(x))}(\phi(x)-\phi(y(x)))$$

if $\phi(x)$ is a 4-point, and

$$\mathrm{ind}_{\psi(\mu(\xi_\pm(x)))}(\psi(\mu(y_\mp(x)))-\psi(\mu(y(x))))=\mathrm{ind}_{\phi(\xi_\pm(x))}(\phi(y_\mp(x))-\phi(y(x)))$$

if $\phi(x)$ is a 3-point.

Proof. The 'only if' part follows immediately from lemma 1. To prove the 'if' part let us consider 3-points and 4-points separately.

4-point. Let us take an arrow x in $\mathrm{ADiag}(\phi)$, such that $\phi(x)$ is a 4-point, and consider the boundaries $\psi(\mu(\xi_1(x))),\dots,\psi(\mu(\xi_4(x)))$. Being the boundaries, $\phi(\xi_i(x))$ are almost non-selfintersecting (they may pass an intersection point twice, but since both the local branches change their direction at this point, they can be smoothed to avoid the intersection), do not pass through $\phi(y(x))$, and

$$\mathrm{ind}_{\phi(\xi_i(x))}(\phi(x)-\phi(y(x)))=\frac{1}{4}.$$

Since μ induces a bijection of the set of intersections of ϕ to the set of intersections of ψ, all the same properties hold for $\psi(\mu(\xi_i(x)))$, except that

$$\mathrm{ind}_{\psi(\mu(\xi_i(x)))}(\psi(\mu(x)) - \psi(\mu(y(x)))) = \sigma_\mu(x)(\frac{1}{4} - \delta_i),$$

where $\delta_i = 1$ if $\psi(\mu(y(x)))$ belongs to the component $\Delta_i(x)$ of $\mathbf{S}^2 \setminus \psi(\mu(\xi_i(x)))$, touching $\psi(\mu(x))$ from the inner angle, and $\delta_i = 0$ otherwise.

But

$$\begin{aligned}
1 &= \mathrm{ind}_{\psi(\mu(\xi(x)))}(\psi(\mu(x)) - \psi(\mu(y(x)))) \\
&= \sum_{i=1}^{4} \mathrm{ind}_{\psi(\mu(\xi_i(x)))}(\psi(\mu(x)) - \psi(\mu(y(x)))) \\
&= \sum_{i=1}^{4} \sigma_\mu(x)(\frac{1}{4} - \delta_i) \\
&= \sigma_\mu(x)(1 - \sum_{i=1}^{4} \delta_i).
\end{aligned}$$

This means either $\sigma_\mu(x) = 1$ and $\sum_{i=1}^4 \delta_i = 0$, or $\sigma_\mu(x) = -1$ and $\sum_{i=1}^4 \delta_i = 2$.

In the latter case $\psi(\mu(y(x)))$ belongs to the intersection $\Delta_i(x) \cap \Delta_j(x)$ of two components, for which $\delta_i = \delta_j = 1$, and $\psi(\mu(x))$ does not belong. Two subcases should be considered.

- If i and j are not adjacent, then $\xi_i(x)$ and $\xi_i(x)$ cannot have common arc because of proposition 3. This means that $\psi(\mu(\xi_i(x)))$ and $\psi(\mu(\xi_i(x)))$ have no common arcs and cannot cross each other, because each of them turns right or left at each intersection point. Hence the interior parts of $\Delta_i(x)$ and $\Delta_j(x)$ have no common points, and cannot contain $\psi(\mu(y(x)))$ both.
- If i and j are adjacent, then

$$\begin{aligned}
\frac{1}{2} &= \mathrm{ind}_{\phi(\xi_{ij}(x))}(\phi(x) - \phi(y(x))) \\
&= \mathrm{ind}_{\psi(\mu(\xi_{ij}(x)))}(\psi(\mu(x)) - \psi(\mu(y(x)))) \\
&= \mathrm{ind}_{\psi(\mu(\xi_i(x)))}(\psi(\mu(x)) - \psi(\mu(y(x)))) \\
&\qquad + \mathrm{ind}_{\psi(\mu(\xi_j(x)))}(\psi(\mu(x)) - \psi(\mu(y(x)))) \\
&= \sigma_\mu(x)((\frac{1}{4} - \delta_i) + \sigma_\mu(x)((\frac{1}{4} - \delta_j) = \frac{3}{2}
\end{aligned}$$

The contradiction is obtained in the both subcases, so $\sigma_\mu(x) = 1$.

3-point. Like $\phi(\xi_i(x))$ in the previous case, $\phi(\xi_+(x))$ and $\phi(\xi_-(x))$ are almost non-selfintersecting and almost non-mutual-intersecting, so $\psi(\mu(\xi_+(x)))$ and $\psi(\mu(\xi_-(x)))$ are. The latter two cycles split the rest of sphere into three sets, depending on the sign of $\mathrm{ind}_{\psi(mu(\xi_+(x)+\xi_-(x)))}(\psi(\mu(x)) - \cdot)$. The cycles $\psi(\mu(\xi_+(x)))$ and $\psi(\mu(\xi_-(x)))$ form the boundaries of the 'positive' and 'negative' sets. The assertion of the proposition together with formula (12) or (13) guarantees, that $\psi(\mu(\xi_+(x)))$ is the boundary of the positive area. Considering a small neighborhood of the point $\psi(\mu(x))$ one can easily see, that $\psi(\mu(\xi_+(x)))$ is the boundary of the positive area if and only if $\sigma_\mu(x) = 1$.

It is shown $\sigma_\mu(x) = 1$. The arrow x is arbitrary, so μ preserves the directions of all arrows of ADiag(ϕ) and is an isomorphism. □

Remark. More careful analysis allows to relax the assertion of proposition 5: $\psi \circ \mu \circ \phi^{-1}$ should preserve only the one non-zero $\mathrm{ind}_{\phi(\xi_\pm)}(\ldots)$ for a 3-point x and three $\mathrm{ind}_{\phi(\xi_{12})}(\ldots)$, $\mathrm{ind}_{\phi(\xi_{23})}(\ldots)$ and $\mathrm{ind}_{\phi(\xi_{34})}(\ldots)$ for a 4-point.

For an arbitrary representation μ of a connected minimal diagram ADiag(ϕ) in a diagram ADiag(ψ) with the same number n of arrows, and arbitrary cycle ξ and arrows x and y in ADiag(ϕ)

$$|\,\mathrm{ind}_{\psi(\mu(\xi))}(\psi(\mu(x)) - \psi(\mu(y)))| \le n+1,$$

because the complement to the image of the doodle ψ consists of $n+2$ areas. Denote by p_n the polynomial of degree $2(n+1)$, such that $p_n(1) = 1$ and $p_n(i) = 0$ for all other integer i, $|i| \le n+1$. Similarly, denote by q_n the polynomial of degree $2(n+1)-1$, such that $q_n(\frac{1}{2}) = 1$ and $q_n(i) = 0$ for all other half-integer i, $|i| < n+1$.

Denote by $L(\phi)= L(\mathrm{ADiag}(\phi))$ the number of intersection points of ϕ. Given a representation $\mu : \mathrm{ADiag}(\phi) \to \mathrm{ADiag}(\psi)$ where ϕ is a minimal doodle $L(\phi) = n$, we define the integer number $P(\mu, \phi, \psi)$ as the product

$$\begin{aligned} P(\mu,\phi,\psi) &= \prod_{\text{4-points } x} \prod_{i=1}^{3} q_n(\mathrm{ind}_{\psi(\mu(\xi_{i(i+1)}(x)))}(\psi(\mu(x)) - \psi(\mu(y(x))))) \\ &\times \prod_{\text{3-points } x} p_n(\pm\,\mathrm{ind}_{\psi(\mu(\xi_\pm(x)))}(\psi(\mu(y_\mp(x))) - \psi(\mu(y(x))))), \quad (14) \end{aligned}$$

where the signs in the factor corresponding to the pre-image x of a 3-point are chosen depending on whether (12) or (13) holds for it. The degree of the product of all polynomials in (14) does not exceed $6n(n+1)$. $P(\mu, \phi, \psi) = 1$ if μ is an isomorphism, and $P(\mu, \phi, \psi) = 0$ if $L(\psi) = L(\phi)$ and μ is not an isomorphism.

The invariant $\langle \phi, \cdot \rangle$ is defined by a special case of formula (9):

$$\langle \phi, \psi \rangle = \sum_{\mu:\mathrm{ADiag}(\phi)\to\mathrm{ADiag}(\psi)} \sigma(\mu) P(\mu, \phi, \psi) \quad (15)$$

Proposition 6 *Let ϕ be a minimal connected doodle in $\mathbf{S}^2$, d be the order of the group of automorphisms of* ADiag(ϕ). *Then $\langle \phi, \phi \rangle = d$, and $\langle \phi, \psi \rangle = 0$ for any other (i.e. not equivalent to ϕ) doodle ψ with $L(\psi) \le L(\phi)$.*

Proof. It follows immediately from formula (15), and proposition 5. The μ-th summand of (15) is equal to 1 if μ is an isomorphism, and is equal to 0 otherwise. For instance, if $L(\psi) < L(\phi)$, there are no summands at all. □

Corollary 4 *If strongly connected doodles ϕ and ψ in $\mathbf{S}^2$ with $\le n$ intersection points are not equivalent in $\mathbf{S}^2$, they can be distinguished by invariants of order $6n(n+1)+n$.*

Proof. Let ϕ' and ψ' be the minimal doodles, equivalent to the doodles ϕ and ψ respectively. Suppose $L(\psi') \le L(\phi')$. Then $\langle \phi', \phi \rangle = \langle \phi', \phi' \rangle > 0$, $\langle \phi', \psi \rangle = \langle \phi', \psi' \rangle = 0$, $L(\phi') \le n$, and the order of $\langle \phi', \cdot \rangle$ is estimated by theorem 8. □

4.2 Doodles in $\mathbf{R}^2$

Now we refine theorem 7 and lemmas 1 and 2, to show finite-order invariants classifying doodles in $\mathbf{R}^2$.

Remind that the arrow diagram of a doodle determines the components of the boundaries of the components of the complement to the image of this doodle (see section 3.1). Let us supply this diagram with additional information about mutual position of these boundary components. This *boundary information* must contain the following data about each boundary component:

- whether the component is a piece of the inner or the outer boundary of some area (i.e. whether it is oriented clockwise or counterclockwise respectively);
- if it is a piece of the inner boundary, which component is the outer boundary of the same area (or there is no outer boundary, if the area is infinite).

Lemma 5 *Let ϕ and ψ be regular doodles in the plane $\mathbf{R}^2$ with the same arrow diagram and boundary information. Then the map $\psi \circ \phi^{-1} : \phi(C_k) \to \psi(C_k)$ can be continued to an orientation-preserving diffeomorphism $\mathbf{R}^2 \to \mathbf{R}^2$.*

Theorem 9 *A doodle in the plane $\mathbf{R}^2$ is determined by its arrow diagram and boundary information up to equivalence.*

The proofs of lemma 5 and theorem 9 repeat the proofs of lemma 1 and theorem 7 with obvious additional technics.

4.2.1 Strongly connected doodles in $\mathbf{R}^2$

Let us study the obstacle to prove lemmas 2 and 3 for strongly connected doodles in $\mathbf{R}^2$. Lemma 2 fails exactly when the vanishing in the move of the type A^*_{-*} arc(s) form(s) the boundary of the infinite area. The sequence of moves from the given strongly connected doodle to an equivalent doodle with minimal diagram in the proof of lemma 3 either ends successfully, or breaks when only such a move of type A^*_{-*} 'through infinity' is possible. The doodle obtained is called *quasiminimal*; so its arrow diagram and boundary information are. The boundary information is simple: all the boundary components are outer boundaries and consist of more than two arcs each, except one one- or two-arc component, which is the inner boundary of the infinite area.

The further reasoning repeats the one for connected doodles in $\mathbf{S}^2$ with slight changes. Here is the analogue of lemma 4

Lemma 6 *Let ξ be a connected 1-cycle in the arrow diagram $\mathrm{ADiag}(\phi)$ of a connected plane doodle ϕ, passing none of the arcs of the arrow diagram twice. The image of ξ is the canonically oriented boundary of a finite component of the complement to the image of ϕ if and only if*

$\mathrm{ind}_{\phi(\xi)}(\phi(x)) = \frac{n(\xi,x)}{4}$ *for any arrow $x \in \xi$, and*
$\mathrm{ind}_{\phi(\xi)}(\phi(x)) = 0$ *for any arrow $x \notin \xi$.*

It is the canonically oriented boundary of the infinite component of the complement to the image of ϕ if and only if

$\mathrm{ind}_{\phi(\xi)}(\phi(x)) = \frac{n(\xi,x)}{4} - 1$ *for any arrow $x \in \xi$, and*
$\mathrm{ind}_{\phi(\xi)}(\phi(x)) = -1$ *for any arrow $x \notin \xi$.*

□

Again, a trick is required to reconstruct the boundary information from the half-integer indices. Let $X_e(\phi)$ and $X_i(\phi)$ be the sets of 'external' and 'internal' arrows of the diagram $\mathrm{ADiag}(\phi)$, i.e. the pre-images of the intersections, incident and not incident, respectively, to the infinite area. Then proposition 4 and its proof remain unchanged for a quasiminimal doodle ϕ and $x \in X_e(\phi)$. For $x \in X_i(\phi)$ the 'infinity' point substitutes $\phi(y(x))$, i.e. the corresponding summand, such as $\mathrm{ind}_{\phi(\xi(x))}(\phi(y(x)))$, is omitted. Similarly, the infinity substitutes the absent $\phi(y_+(x))$ or $\phi(y_-(x))$, if any. After this substitution the analogue of proposition 5 looks exactly like proposition 5 itself.

Proposition 7 *Let μ be a representation of a connected quasiminimal diagram* $\mathrm{ADiag}(\phi)$ *in a diagram* $\mathrm{ADiag}(\psi)$ *with the same number of arrows. Then μ is an isomorphism if and only if for each arrow x in* $\mathrm{ADiag}(\phi)$

$$\mathrm{ind}_{\psi(\mu(\xi_{ij}(x)))}(\psi(\mu(x)) - \psi(\mu(y(x)))) = \mathrm{ind}_{\phi(\xi_{ij}(x))}(\phi(x) - \phi(y(x)))$$

if $\phi(x)$ is a 4-point,

$$\mathrm{ind}_{\psi(\mu(\xi_\pm(x)))}(\psi(\mu(y_\mp(x))) - \psi(\mu(y(x)))) = \mathrm{ind}_{\phi(\xi_\pm(x))}(\phi(y_\mp(x)) - \phi(y(x)))$$

if $\phi(x)$ is a 3-point.

Proof. The proof coincides with the proof of proposition 5. □

One more index is preserved, and the representation μ preserves not only the directions of the arrows, but also the boundary information.

Proposition 8 *Let μ be an isomorphism of the diagram* $\mathrm{ADiag}(\phi)$ *of a connected quasiminimal doodle to a diagram* $\mathrm{ADiag}(\psi)$ *with the same number of arrows. Let $\xi(\infty)$ be the pre-image of the canonically (clockwise!) oriented boundary of the infinite component of $\mathbf{R}^2 \setminus \phi(C_k)$, and $y(\infty)$ an arrow of* $\mathrm{ADiag}(\phi)$ *out of $\xi(\infty)$. Then $\psi \circ \mu \circ \phi^{-1}$ can be continued to the equivalence of the doodles if and only if*

$$\mathrm{ind}_{\psi(\mu(\xi(\infty)))}(\psi(\mu(y(\infty)))) = \mathrm{ind}_{\phi(\xi(\infty))}(\phi(y(\infty))) = -1$$

Proof. It follows from lemma 5, because this equation is nothing but the boundary information, absent in proposition 7: the boundary of the infinite area goes to the boundary of the infinite area. □

$P(\mu, \phi, \psi)$ can be defined for a quasiminimal doodle ϕ by the same formula (14) and has the same properties. For a quasiminimal connected doodle ϕ in $\mathbf{R}^2$ we define the invariant $\langle \phi, \cdot \rangle$ similarly to formula (15):

$$\langle \phi, \psi \rangle \quad = \sum_{\mu:\mathrm{ADiag}(\phi)\to\mathrm{ADiag}(\psi)} \sigma(\mu) P(\mu, \phi, \psi) p_n(-\,\mathrm{ind}_{\psi(\mu(\xi(\infty)))}(\psi(\mu(y(\infty))))). \quad (16)$$

If $\mathrm{ADiag}(\phi)$ is not minimal, the last factor guarantees that the product of the polynomials satisfies the assertion of theorem 5, and the sum is an invariant. The degree of the product of all polynomials in (15) does not exceed $(6n + 2)(n + 1)$.

Proposition 9 *Let ϕ be a quasiminimal (or minimal) connected doodle in $\mathbf{R}^2$, d be the order of the group of automorphisms of* $\mathrm{ADiag}(\phi)$*, which can be continued to automorphisms of ϕ (i.e. preserving the pre-image of the boundary of the infinite area). Then $\langle \phi, \phi \rangle = d$, and $\langle \phi, \psi \rangle = 0$ for any other (i.e. not equivalent to ϕ) doodle ψ with $L(\psi) \leq L(\phi)$.*

Proof. The properties of $P(\mu, \phi, \psi)$ and proposition 8 guarantee that a summand of (16) is equal 1 if $\psi \circ \mu \circ \phi^{-1}$ is an equivalence of the doodles, and equal to 0 otherwise. □

Corollary 5 *If strongly connected doodles ϕ and ψ with $\leq n$ intersection points in $\mathbf{R}^2$ are not equivalent, they can be distinguished by an invariant of order $(6n + 2)(n + 1) + n$.*

Proof. The proof coincides with the proof of corollary 4. □

4.2.2 Generic doodles in $\mathbf{R}^2$

The difference between the generic case and the case of strongly connected doodles is that the (quasi-)minimal doodles are more sophisticated. The skeleton of the proof is exactly the same.

Again, given a doodle ϕ with non-minimal arrow diagram $\mathrm{ADiag}\,\phi$, let us keep trying to apply lemma 2 to the allowed moves of type A^*_{-*} while possible. If the doodle is not connected, its connected components (and 1-cycles) are partially ordered by the following relation: $\alpha \prec \beta$ if all intersection points of α belong to a finite area of $\mathbf{R}^2 \setminus \beta$. We will process a minimal possible component. After successful lifting of a move A^*_{-*} to the corresponding move O^*_{-*}, one of the following cases occurs:

- the component processed has decayed into two non-intersecting curves;
- the component processed has decayed into two components, only one of which is a curve, not intersecting any other curve;
- the component processed has decayed into two components, none of which is a curve, not intersecting any other curve;
- the component processed has not decayed.

The connected component of a doodle, formed by one curve, bounding an empty disk, is called *trivial*. We can move each trivial component, which might appear in the first two cases, through the other components far away from all of them.

This minimization process stops when the connected components of the doodle satisfies the following conditions:

- each pair of arcs, which can vanish in some move of type A^*_{-*}, either bounds the infinite area of the complement to this component, or bounds the finite area, containing another non-trivial component of the doodle;
- each non-trivial one-curve component bounds the finite area, containing a multicurve connected component.

Such a doodle is called *quasiminimal*.

Two non-trivial one-curve connected components are called *equivalent*, if they bounds the same set of other connected components. There is insignificant, which of them bounds another one, since they can be moved one through another leaving the doodle equivalent.

The following obvious analogue of lemma 6 describes the mutual disposition of the intersection points and 1-cycles via indices.

Lemma 7 *Let ξ be a connected 1-cycle in the arrow diagram* $\mathrm{ADiag}(\phi)$ *of a doodle ϕ, passing none of the arcs the arrow diagram twice. The image of ξ is the canonically oriented outer boundary of a finite component β of the complement to the image of ϕ if and only if*

$\mathrm{ind}_{\phi(\xi)}(\phi(x)) = \frac{n(\xi,x)}{4}$ *for any arrow $x \in \xi$,*
$\mathrm{ind}_{\phi(\xi)}(\phi(x)) = 1$ *for any arrow $x \notin \xi$ in the boundary of a component $\alpha \prec \beta$,*
$\mathrm{ind}_{\phi(\xi)}(\phi(x)) = 0$ *for any other arrow x.*

It is a canonically oriented component of the inner boundary of a component (finite or infinite) of the complement to the image of ϕ if and only if

$\mathrm{ind}_{\phi(\xi)}(\phi(x)) = \frac{n(\xi,x)}{4} - 1$ *for any arrow $x \in \xi$,*
$\mathrm{ind}_{\phi(\xi)}(\phi(x)) = -1$ *for any arrow $x \notin \xi$ in the boundary of a component $\alpha \prec \beta$,*
$\mathrm{ind}_{\phi(\xi)}(\phi(x)) = 0$ *for any other arrow x.*

□

Only integer-valued conditions are added in lemma 7 to the conditions of lemma 6. This means that no new tricks are needed to encode these conditions into finite-order invariants. We need only check, that despite the quasiminimal doodle ϕ is neither connected nor minimal, for any arrow x in ADiag(ϕ) an arrow $y(x)$ (or arrows $y_+(x)$ and $y_-(x)$, see proposition 4 and below) can be either found, or substituted by infinity. It is easy. If proposition 4, cannot be applied to an arrow x of the diagram of a connected subdoodle ϕ', then either one of the areas of the complement to ϕ', not adjacent to $\phi(x)$, is infinite and $y(x)$ can be substituted by infinity, or one of them encloses another non-trivial component of ϕ, and any arrow of the latter can be taken. $y_+(x)$ and $y_-(x)$ are treated in the same way.

Let a quasiminimal doodle $\phi : C_k \to \mathbf{R}^2$ consist of m ($m \le k$) connected components $\phi^1, \ldots, \phi^m$ with $n^1, \ldots, n^m$ intersections, and 1-cycles $\xi^1(\infty), \ldots, \xi^m(\infty)$ are the pre-images of their canonically oriented outer boundaries. For each pair $\phi^i \prec \phi^j$ denote by y^{ij} and ξ^{ij} the pre-images of an intersection point of ϕ^i and of the outer boundary of the area of $\mathbf{R}^2 \setminus \phi^j$, containing ϕ^i.

We are going to construct an analogue of formula (16). For each multicurve component ϕ^i let

$$P^i(\mu, \phi, \psi) = P(\mu|_{\mathrm{ADiag}(\phi^i)}, \phi^i, \psi),$$

and

$$Q^i(\mu, \phi, \psi) = p_{n^i}(-\operatorname{ind}_{\psi(\mu(\xi^i(\infty)))}(\psi(\mu(y^i(\infty)))))$$

($P^i(\ldots)$ and $Q^i(\ldots)$ are the factors of formula 16 for ϕ^i). If $\phi^i \prec \phi^j$, let

$$R^{ij}(\mu, \phi, \psi) = p_{n^j}(\operatorname{ind}_{\psi(\mu(\xi^{ij}))}(\psi(\mu(y^{ij})))).$$

Denote by S the set of such pairs (i, j), $1 \le i, j \le m$, that $\phi^i \prec \phi^j$ and there exists no multicurve component ϕ^k, such that $\phi^i \prec \phi^k \prec \phi^j$. S consists of no more than $m-1$ pairs. Indeed, each one-curve component may participate only as the second member of a pair and only once, and in the remaining pairs each multicurve component can participate as the first member at most once, but the 'largest' component(s) do(es) not.

Now let

$$F(\mu, \phi, \psi) = \prod_{i=1}^{m} (P^i(\mu, \phi, \psi) Q^i(\mu, \phi, \psi)) \times \prod_{(i,j)\in S} R^{ij}(\mu, \phi, \psi).$$

$F(\mu, \phi, \psi)$ satisfies the requirement of theorem 5. It is a polynomial of different indices of shape $(\operatorname{ind}_{\psi(\mu(\xi))}(\psi(\mu(x)))$ of degree no more than

$$\begin{aligned}\sum_{i=1}^{m}(6n^i+2)(n^i+1) + 2(m-1)(n+1) &\le (6n+8)n + 2m + 2(m-1)(n+1)\\ &= 6n(n+1) + 2m(n+2) - 2\\ &\le 6n(n+1) + 2k(n+2) - 2.\end{aligned}$$

For a quasiminimal doodle ϕ in $\mathbf{R}^2$ we define the invariant $\langle \phi, \cdot \rangle$ as follows:

$$\langle \phi, \psi \rangle = \sum_{\mu:\mathrm{ADiag}(\phi)\to\mathrm{ADiag}(\psi)} \sigma(\mu) F(\mu, \phi, \psi) \qquad (17)$$

If the doodle ϕ is connected, this formula coincides with formula (16).

Proposition 10 *Let ϕ be a quasiminimal doodle in $\mathbf{R}^2$, d be the order of the group of automorphisms of* ADiag(ϕ)*, which can be continued to automorphisms of ϕ. Then $\langle \phi, \phi \rangle = d$, and $\langle \phi, \psi \rangle = 0$ for any other (i.e. not equivalent to ϕ) doodle ψ with $L(\psi) \le L(\phi)$.*

Corollary 6 *If k-doodles ϕ and ψ with $\leq n$ intersection points in $\mathbf{R}^2$ are not equivalent, they can be distinguished by a finite-order invariant of order $6n(n+1)+2k(n+2)-2+n$.*

The proofs of proposition 10 and corollary 6 repeat the proofs of proposition 9 and corollary 5, respectively.

5 Finite-order invariants of plane curves

By a *plane curve* we mean a smooth map $\phi : \mathbf{S}^1 \to \mathbf{R}^2$. The image of a generic plane curve can have a finite number of double points, where two local branches of the curve meet transversally; all the other points are regular. The generic homotopy of a curve is an isotopy of its image everywhere except a finite number of instants, when the curve has a cusp, a selftangency point or a triple point. Near these instants a local move of the type $O^1_{\pm 1}$, $O^1_{\pm 2}$, or O^1_3 respectively occurs (see fig. 1).

The classification of immersions of a circle to a plane was described by H.Whitney in [W37]. The result is simple: the only invariant of a curve ϕ is its Whitney index (= rotation number) $\mathrm{ind}(\phi)$. The study of homotopy classification of immersions of a circle to a plane with no selftangencies and triple points was started by V.Arnold (see [A93]) and is not finished yet. Numerous invariants, most of which are or can be expressed via finite-order invariants, were constructed. The most famous of them, namely Arnold invariants J^+, J^- and St, are invariants of order 1. [3]

On the other hand, the direct application of Vassiliev theory and calculation of the homology groups of the discriminant has not been undertaken. The difference between this situation and the situation of ornaments can be explained easily. There is only one type of forbidden singularity and five types of allowed moves for ornaments, and three (in fact, four, since there are two kinds of selftangencies: parallel and antiparallel) types of forbidden singularities and no allowed moves for curves. So it is easier to construct an invariant of curves, but more difficult to study the discriminant.

Here we are going to show that Vassiliev invariants classify plane curves. Like in the case of doodles, a modification of the construction of invariants, defined by M.Polyak in [P94], is used. Two things should be done: the classification of singularities of curves and the definition of the order of invariants be given (at least in the amount of section 2), and the desired invariants constructed. The second thing is a simple job, and the first one is long and hard.

To keep the tradition of [A93] and to save ourselves extra troubles, we consider immersed curves only. The order of singularities, and hence the order of invariants, defined below, is rather arbitrary, but it allows to apply the theory of ornaments and doodles with minimal changes.

A singular immersed curve in a finite stratum of the discriminant has finitely many singular points, each of which has finitely many pre-images: 2 for a generic selftangency point, and 3 or more otherwise. Let $s(x)$ be the number of pre-images of such a singular point x, and $t(x)$ be the number of tangent lines to the curve at the point x. Then the complexity $\mathrm{compl}(x)$ of the point x is equal to $2s(x)-t(x)-1$ (i.e. the complexity of a selftangency and of a triple point is 2, the complexity of a triple point with two tangent local branches and of a quadruple point is 3, etc.),

[3] There is no canonical definition of the order of invariants, or equally, the complexity of singular points. Some authors define the complexity of any singular point as 1 like in the theory of Vassiliev knot invariants, where only simple transversal intersections are to be considered. Such a definition seems to be too rough. More naturally, in the spirit of general Vassiliev theory, is to define the complexity of a singularity as (a multiple of) the codimension of the subspace of objects (knots, doodles, curves, ...) with this singularity. Then the complexities of cusps, selftangencies and triple points are 2, 3 and 4 (or 1, $\frac{3}{2}$ and 2), respectively.

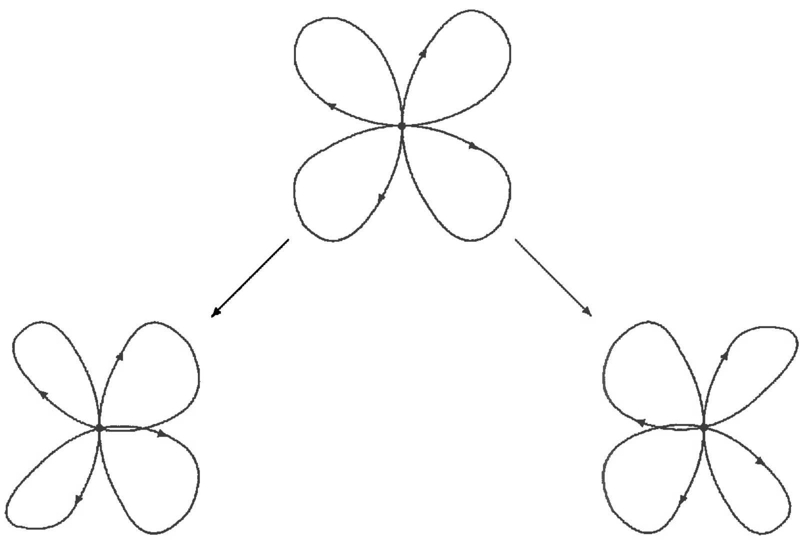

Figure 7: Both the perturbations of the singular immersion with a quadruple point with two selftangencies are equivalent.

and the complexity of the curve is equal to the sum of complexities of all its singular points.

Now we are going to define the degeneration modes and characteristic numbers. Denote by $Pj^1\phi : \mathbf{S}^1 \to \mathbf{R}^2 \times \mathbf{R}P^1$ the projectivization of the 1-jet extension of the immersion ϕ. Two kinds of marks, namely *s-marks* and *t-marks*, are used below. There are 5 types of degeneration steps:

1. marking with s-marks a triple of points on the circle $\mathbf{S}^1$, which ϕ maps to the same singular point, none of whose pre-images is already marked;

2. marking with both s-marks and t-marks a couple of points on the circle $\mathbf{S}^1$, which $Pj^1\phi$ maps to the same point, none of whose pre-images is already marked;

3. marking with s-mark a point on the circle $\mathbf{S}^1$, which the curve ϕ maps to a point, some of whose pre-images are already marked;

4. marking with t-marks a couple of points on the circle $\mathbf{S}^1$, already marked with s-marks, which $Pj^1\phi$ maps to the same point, none of whose pre-images is already marked with t-marks;

5. marking with t-marks a point on the circle $\mathbf{S}^1$, already marked with s-marks, which $Pj^1\phi$ maps to a point, some of whose pre-images are already marked with t-marks.

To define the characteristic numbers as it is done in section 2.2.4 for doodles, one need to consider the corresponding 5 types of perturbations of singular immersions, and decide, which perturbations are positive, and which are negative. For degeneration steps of the types 1 and 3 this is defined in section 2.2.4; for steps of the type 2 it is defined in [A93]. For steps of the type 5 it can be done easily, but for steps of the type 4 it is impossible in general: see fig. 7. This means that the corresponding stratum of the discriminant is not coorientable in the previous stratum. To bypass this obstacle it is enough to choose any local coorientation and add this choice to the definition of the degeneration step. The details are omitted, because the definition of the order of invariants coincides with definition 6 and does not depend on the particular choice of the coorientations.

A *strict representation* μ of an arrow diagram X in an arrow diagram Y is a diffeomorphism of their base circles, mapping arrows of X to arrows of Y, preserving the orientation of the base circle and the directions of arrows, defined up to homotopy equivalence (cf. the definition of representation in section 3.2, where the directions of the arrows can be changed). Given an arrow diagram X with $n = L(X)$ arrows $\{x^1, \ldots, x^n\}$, a set $\Xi = \{\xi_1, \ldots, \xi_m\}$ of 1-cycles in it, and function F of $m \times n$ variables, for any regular curve ϕ we define $\langle (X, \Xi, F), \phi \rangle$ as the sum

$$\langle (X, \Xi, F), \phi \rangle = \sum_{\mu : X \to \mathrm{ADiag}(\phi)} F(\mathrm{ind}_{\phi(\mu(\xi_1))}(\phi(\mu(x^1))), \ldots, \mathrm{ind}_{\phi(\mu(\xi_m))}(\phi(\mu(x^n)))) \tag{18}$$

over all strict representations $X \to \mathrm{ADiag}(\phi)$ (cf. formula (5): the signs $\sigma(\mu)$ are omitted). Note, that there are absolutely no restrictions, such as minimalness of the diagram X, since no topological moves are allowed for the curves.

Theorem 10 *$\langle (X, \Xi, F), \cdot \rangle$ is an invariant of plane curve. If F is a polynomial, $\langle (X, \Xi, F), \cdot \rangle$ is an invariant of order $\leq \deg(F) + L(X)$.*

The proof repeats the proof of theorems 6. The rather strange definition of complexity of the singular immersion was chosen especially to preserve the estimate of the order in this theorem.

A curve can be considered as a strongly connected 1-doodle. Like in section 4, the pre-images of the boundaries of the areas of the complement to the curve in its arrow diagram are determined uniquely. All these areas has connected boundaries. For a plane curve the only boundary information (cf. subsection 4.2.1) missed in its arrow diagram is which of the boundaries is the boundary of the infinite area. We are going to reconstruct this boundary information from the values of finite-order invariants of the curve. The boundary information is expressed in via indices in lemma 6. In fact only one of the equations of this lemma is enough to distinguish the curves.

Let ϕ be a generic curve with $n = L(\psi)$ selfintersections, ξ_∞ be the pre-image in $\mathrm{ADiag}(\phi)$ of the canonically oriented boundary of the infinite area of $\mathbf{R}^2 \setminus \phi(\mathbf{S}^1)$, x_∞ be an arrow in ξ_∞. For any cycle ξ in $\mathrm{ADiag}(\phi)$, passing none of the arcs twice, and any arrow x, $\mathrm{ind}_{\phi(\xi)}(\phi(x)) \in \mathbf{Z}/4$ and $|\mathrm{ind}_{\phi(\xi)}(\phi(x))| < n+1$. Denote by p_ϕ the polynomial of degree $2n+1$, such that $p_\phi(\mathrm{ind}_{\phi(\xi_\infty)}(\phi(x_\infty))) = 1$ and $p_\phi(t) = 0$ for any other t of shape $\mathrm{ind}_{\phi(\xi_\infty)}(\phi(x_\infty)) + i$, where i integer and $|t| < n+1$. We define the invariant of curves $\langle \phi, \cdot \rangle$ by the following special case of formula (18):

$$\langle \phi, \psi \rangle = \sum_{\mu : \mathrm{ADiag}(\phi) \to \mathrm{ADiag}(\psi)} p_\phi(\mathrm{ind}_{\psi(\mu(\xi_\infty))}(\psi(\mu(x_\infty)))).$$

By theorem 10 $\langle \phi, \cdot \rangle$ is an invariant of order $3n+1$.

Theorem 11 *Finite-order invariants $\langle \phi, \cdot \rangle$ and Whitney index classify plane curves. Two non-equivalent curves with $\leq n$ selfintersections in each can be distinguished by an invariant of order $3n+1$.*

Proof. The proof coincides with the proofs of proposition 6 and corollary 4. Whitney index is required only to distinguish two curves with no selfintersections (clockwise and counterclockwise oriented). □

References

[A93] V.I.Arnold, *Plane curves, their invariants, perestroikas and classifications.* Preprint ETH, Zürich, May 1993; *Singularities and Bifurcations*, AMS, Advances in Soviet Math., vol. 21 (ed. V.I.Arnold), Providence, R.I., 1994.

[B-N95] D.Bar-Natan, *Vassiliev homotopy string link invariants*, Jour. of Knot Theory and its Ramifications, v. 4, No. 1, 1995, pp. 13–32.

[FT77] R.Fenn, P.Taylor, Introducing doodles. In: Topology of low-dimensional manifolds, R.Fenn (ed.), Lect. Notes Math., v.722, 1977, p. 37–43.

[Kh94] M.Khovanov, *Doodle groups*, preprint Yale Univ., 1994; Trans. AMS, v.349, no.6, 1997, p.2297–2315.

[K93] M.Kontsevich, *Vassiliev's knot invariants*, Adv. in Sov. Math., vol.16, part 2, Amer. Math. Soc., Providence, RI, 1993, p.137–150.

[M94] A.B.Merkov, *On classification of ornaments*, *Singularities and Bifurcations*, AMS, Advances in Soviet Math., vol. 21 (ed. V.I.Arnold), Providence, R.I., 1994, p.199–211.

[M95] A.B.Merkov, *Finite-order invariants of ornaments*, preprint 1995; J. of Math. Sciences, vol.90, no.4 (1998), pp.2215–2273.

[M96] A.B.Merkov, *Vassiliev invariants classify flat braids*, preprint 1996; AMS Transl, (2) Vol.190, AMS, Providence RI, 1999, p.83–102.

[P94] M.Polyak, *Invariants of plane curves via Gauss diagrams*, preprint Max-Plank-Institut MPI/116-94, Bonn, 1994.

[PV94] M.Polyak, O.Viro, *Gauss diagram formulas for Vassiliev invariants*, Int. Math. Res. Notices 11 (1994) 445–453.

[V93] V.A.Vassiliev, *Invariants of ornaments*, Preprint Maryland Univ., March 1993; *Singularities and Bifurcations*, AMS, Advances in Soviet Math., vol. 21 (ed. V.I.Arnold), Providence, R.I., 1994, p. 225–261.

[V94] V.A.Vassiliev, *Complements of discriminants of smooth maps: topology and applications* (revised edition), Transl. of Math. Monographs, v.98, AMS, Providence, Rhode Island, 1994.

[W37] H.Whitney, *On regular closed curves in the plane*, Composition Math. 4 (1937), 276–284.

Index