ON THE STABILITY OF A RIGID BODY IN A MAGNETOSTATIC EQUILIBRIUM

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ABSTRACT

In this paper we study the stability of the equilibria of the dynamical system consisting of a rigid body and a conducting, inviscid (or viscous), incompressible fluid. We assume that the body is perfectly conducting. First we show that the total energy of the system has a critical point in a static equilibrium, then we calculate the second variation of the energy at the critical point and prove that the positive definiteness of the second variation gives both necessary and sufficient condition for linear stability. The general theory is applied to two particular problems, namely: the stability of a circular cylinder in a magnetostatic equilibrium with circular lines of the magnetic field and the stability of an arbitrary cylinder in an irrotational magnetic field (which is homogeneous at infinity). Finally, we extend the theory to the case of a viscous fluid with finite conductivity and show that the results obtained for an ideal fluid remain valid in this case.

1. INTRODUCTION

In this paper we study the stability of equilibria of the dynamical system that consists of a rigid body and a conducting incompressible fluid in magnetic field. For simplicity, we consider the two-dimensional problem. We assume that the body is perfectly conducting. The fluid may be inviscid or viscous. We consider both the perfectly conducting fluid and the fluid with finite conductivity. External forces (such, for example, as gravity force) may be applied to the system as a whole.

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We study the stability of static equilibria of this mechanical system. For the stability analysis we use the energy method of Arnold (Arnold 1965): first we show that the total energy of the system has a critical point in a static equilibrium, then the second variation of the energy at the critical point is calculated. It is well-known that this equilibrium is stable to small perturbations if the second variation of the energy evaluated in this equilibrium is positive definite. This assertion is an analogue of the Lagrange theorem of classical mechanics.

We also prove the converse Lagrange theorem stating that an equilibrium of the system is unstable provided that the second variation of the energy (evaluated in this equilibrium) can take negative values. The proof is constructive rather than abstract: we explicitly construct a functional that grows exponentially with time by virtue of linearized equations of motion. We obtain an explicit formula that gives the dependence of the perturbation growth rate upon the equilibrium considered and the initial data for the perturbation. We obtain also the upper and lower bounds for growing solutions of the linearized problem and identify the initial data corresponding to the perturbation with maximum growth rate.

Thus, the stability analysis leads to the conclusion that the positive definiteness of the second variation is both necessary and sufficient condition for (linear) stability.

The general theory developed in the paper is applied to two simple particular problems. We consider (i) a circular cylinder in a magnetostatic equilibrium with circular lines of magnetic field and (ii) the equilibrium of an arbitrary (two-dimensional) body in an irrotational magnetic field (which is homogeneous at infinity). In the former case, the system is always stable. In the latter case, the stability of the system depends on the orientation of the body relative to the magnetic field at infinity: if in the equilibrium the principal axis of the virtual-mass tensor which corresponds to a minimum virtual mass is parallel to the magnetic field at infinity, the equilibrium is stable and if this axis is perpendicular to the magnetic field at infinity, the equilibrium is unstable.

The effects of viscosity and finite conductivity on the stability of the system is also discussed. We show that the stability criteria obtained for an inviscid fluid remain valid for a viscous (but still perfectly conducting) fluid and for a viscous fluid with finite conductivity. In particular, we prove the converse Lagrange theorem for a viscous fluid and give *a priori* estimates of (exponentially) growing solutions of the linearized problem.

The results of the present paper may be viewed as development and generalization of the ideas and results contained in the papers by Arnold (1965, 1966), Moffatt (1986), Vladimirov & Rumyantsev (1989, 1990), Vladimirov & Ilin (1994, 1998, 1999), Davidson (1998) and Gubarev (1995, 1999).

We conclude this introduction with a simple example that inspired us to develop this theory.

Consider an elliptic cylinder in a magnetostatic equilibrium of an inviscid, incompressible, and perfectly conducting fluid. We assume that in this equilibrium, the magnetic field is irrotational and homogeneous at infinity. The lines of the magnetic field coincide with the streamlines of the analogous irrotational flow of an inviscid fluid past the elliptic cylinder (with zero circulation of the velocity around it). We assume that the centre of the ellipse is fixed, but the ellipse is free to rotate. It is evident from the symmetry of the problem that there are two equilibrium positions of the cylinder: (i) the longer axis is parallel to the magnetic field at infinity (see Fig. 1a) and (ii) the longer axis is perpendicular to the magnetic field at infinity (see Fig. 1b). These equilibria correspond to zero torque exerted by the fluid on the cylinder.

The natural question is whether these equilibria are stable or unstable. The answer to this question is *a priori* unclear. However, as we show below, it can be obtained using simple physical arguments.

First, we consider the case (i) (Fig. 1a). If we slightly perturb the equilibrium by turning the ellipse about its centre, the magnetic field lines around the ellipse stretch, so that the magnetic field in magnetic tubes near the body increases due to magnetic flux conservation. Therefore, this perturbation results in an increase in the magnetic energy. This means that the energy has a local minimum in the basic equilibrium and, as a consequence, this equilibrium is stable.

In the case (ii) (Fig. 1b), a similar perturbation leads to contraction of the magnetic field lines, thus reducing the magnetic field. This corresponds to a decrease in the magnetic energy and, therefore, to a local maximum of the magnetic energy in the basic equilibrium. The latter, in turn, may result in instability.

Thus, the above arguments provide certain evidence that the equilibrium of an elliptic cylinder with the longer axis parallel to the magnetic field at infinity is stable to small perturbations and the equilibrium of the cylinder with the longer axis perpendicular to the magnetic field at infinity is unstable. This conclusion is based on purely physical arguments and, certainly, needs mathematical justification. First, our arguments deal with the static situation. To prove the stability, we need to consider dynamic perturbations with nonzero velocity. Second, the absence of the energy minimum in an equilibrium does not automatically imply instability and requires more careful treatment (a classical example from finite-dimensional mechanics when the system has no minimum in an equilibrium but this equilibrium is stable may be found in Arnold, 1978). The general theory that we develop in this paper is applied, in particular, to the problem of stability of an elliptic cylinder and gives a rigorous proof of the above-formulated stability criterion.

The outline of the paper is as follows. First, we consider the case of a perfectly conducting inviscid fluid. In §2 we discuss the equations of motion for the dynamical system considered. In §3 we show that the energy of the system has a stationary value on the set of all 'isomagnetic' states of the system. In §4 we calculate the corresponding second variation of the energy, discuss the relation between this second variation and the linearized equations, and obtain general sufficient conditions for linear stability. In §5 we show that if the condition obtained in §4 is not satisfied, the equilibrium considered is unstable. Two particular examples of stable and unstable equilibria of the system are presented in §6. The effect of viscosity is discussed in §7. Here we show that all the results obtained in §§4–6 are also valid for viscous fluid. Finally, in §8 we consider the case of a fluid with finite conductivity and prove that, for the basic equilibria that are compatible with the equations of resistive magnetohydrodynamics, the results of §§4–7 remain valid for fluids with finite conductivity.

2. BASIC EQUATIONS

Consider a dynamical system consisting of an incompressible, homogeneous, inviscid and perfectly conducting fluid and a perfectly conducting rigid body. For simplicity, we consider the two-dimensional problem. Let $\mathcal{D}$ be a domain on (x, y)-plane that contains both the fluid and the rigid body, and let $\mathcal{D}_b(t)$ be a domain occupied by the body ($\mathcal{D}_b(t) \subset \mathcal{D}$). The domain $\mathcal{D}_f(t) = \mathcal{D} - \mathcal{D}_b(t)$ is completely filled with a fluid, its boundary $\partial\mathcal{D}_f(t)$ consists of two parts: the inner boundary $\partial\mathcal{D}_b(t)$ representing the surface of the rigid body and the outer boundary $\partial\mathcal{D}$ which is fixed in space.

Motion of the body is described by the radius-vector $\mathbf{R}(t)$ of its centre of mass and by the angle $\phi(t)$ which determines orientation of the body on the plane.

The equations of motion of the fluid are the standard equations of the ideal magnetohydrodynamics which, with help of the flux function $a(x, y, t)$ ($\mathbf{h} = \nabla a \times \mathbf{k}$), may be written as

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p - (\nabla^2 a)\nabla a\,, \tag{2.1}$$

$$a_t + (\mathbf{u} \cdot \nabla)a = 0\,, \quad \nabla \cdot \mathbf{u} = 0\,. \tag{2.2}$$

Motion of the rigid body obeys Newton's equations of classical mechanics:

$$M\dot{\mathbf{w}} = M\ddot{\mathbf{R}} = \int_{\partial\mathcal{D}_b} \Big(p + (\nabla a)^2/2\Big)\mathbf{n}\,\mathrm{d}l - \partial\Pi/\partial\mathbf{R}\,, \tag{2.3}$$

$$I\dot{\Omega} = I\ddot{\phi} = \int_{\partial\mathcal{D}_b} \mathbf{k} \cdot \{(\mathbf{x} - \mathbf{R}) \times \mathbf{n}\}\Big(p + (\nabla a)^2/2\Big)\mathrm{d}l - \partial\Pi/\partial\phi\,. \tag{2.4}$$

Here M is the mass of the body, I is the moment of inertia of the body, $\mathbf{n}$ is the unit normal vector which is always directed outward with respect to domain $\mathcal{D}_f$ occupied by the fluid, $\Pi(\mathbf{R}, \phi)$ is the potential of an external force applied to the body.

Boundary conditions are

$$\begin{aligned} &\mathbf{u} \cdot \mathbf{n} = 0\,, \quad \boldsymbol{\sigma} \cdot \nabla a = 0 \quad \text{on } \partial\mathcal{D}\,, \\ &\mathbf{u} \cdot \mathbf{n} = \{\mathbf{w} + \Omega[\mathbf{k} \times (\mathbf{x} - \mathbf{R})]\} \cdot \mathbf{n}\,, \quad \boldsymbol{\sigma} \cdot \nabla a = 0 \quad \text{on } \partial\mathcal{D}_b\,. \end{aligned} \tag{2.5}$$

Here $\boldsymbol{\sigma}$ is the unit vector tangent to the corresponding boundary.

The conserved total energy of the system is given by

$$E = E_f + E_b = const\,, \tag{2.6a}$$

$$E_f \equiv \frac{1}{2}\int_{\mathcal{D}_f} \left(\mathbf{u}^2 + (\nabla a)^2\right) \mathrm{d}x\mathrm{d}y\,, \tag{2.6b}$$

$$E_b \equiv \frac{1}{2}M\mathbf{w}^2 + \frac{1}{2}I\Omega^2 + \Pi(\mathbf{R}, \phi)\,. \tag{2.6c}$$

BASIC STATE. An exact steady solution of (2.1)–(2.5) whose stability will be investigated represents a static equilibrium of the system is given by

$$\mathbf{R} = \mathbf{w} = 0\,, \quad \phi = \Omega = 0\,; \tag{2.7a}$$

$$\mathbf{u} = 0\,, \quad a = A(\mathbf{x}) \quad \text{in} \quad \mathcal{D}_{f0}\,. \tag{2.7b}$$

The flux function $A(\mathbf{x})$ in (2.7b) is a solution of the problem:

$$\nabla P + (\nabla^2 A)\nabla A = 0 \ \text{ in } \ \mathcal{D}_{f0}\,, \quad \boldsymbol{\sigma}\cdot\nabla A = 0 \ \text{ on } \ \partial\mathcal{D}_{b0} \ \text{ and } \ \partial\mathcal{D}\,. \tag{2.8}$$

Taking curl of (2.8), we obtain

$$\mathbf{k}\cdot\Big(\nabla(\nabla^2 A)\times\nabla A\Big) = 0\,. \tag{2.9}$$

This implies that the functions $\nabla^2 A(\mathbf{x})$ and $A(\mathbf{x})$ are (at least) locally dependent in $\mathcal{D}_{f0}$.

Note that, in view of (2.8),

$$P = \text{const} \quad \text{on } \partial\mathcal{D}_{b0} \ \text{ and } \ \partial\mathcal{D}\,. \tag{2.10}$$

In the equilibrium (2.7), the total force and torque exerted on the body by the fluid are balanced by the external force and torque applied to the body:

$$\partial\Pi/\partial\mathbf{R}\,\Big|_{\mathbf{R},\phi=0} = \int_{\partial\mathcal{D}_{b0}} \frac{1}{2}(\nabla A)^2\mathbf{n}\,\mathrm{d}l\,, \tag{2.11a}$$

$$\partial\Pi/\partial\phi\,\Big|_{\mathbf{R},\phi=0} = \int_{\partial\mathcal{D}_{b0}} \frac{1}{2}(\nabla A)^2\mathbf{k}\cdot(\mathbf{x}\times\mathbf{n})\mathrm{d}l\,. \tag{2.11b}$$

3. VARIATIONAL PRINCIPLE

In this section, following the procedure of Vladimirov & Ilin (1999) we shall show that the equilibrium state (2.7) has an extremal value of the energy in comparison with all *isomagnetic* states of the system. Isomagnetic states of the system are, by definition, the states that can be obtained by displacement of fluid particles from their position in the basic state (2.7), the value of the flux function $a(\mathbf{x})$ in every fluid particle being unchanged. This variational principle may be viewed as the generalization of Moffatt's variational principle (Moffatt 1986) to the case of a variable flow domain.

For convenience, we introduce the notation $\mathbf{q} = (q_1, q_2, q_3) = (R_1, R_2, \phi)$ and $\dot{\mathbf{q}} = (\dot{q}_1, \dot{q}_2, \dot{q}_3) = (w_1, w_2, \Omega)$ which will be used along with $\mathbf{R}$, $\mathbf{w}$, ϕ, and Ω. Greek indices α and β take values 1, 2, and 3. In what follows, summation over repeated Greek indices is implied.

To formulate the variational principle, we introduce the family of transformations

$$\tilde{\mathbf{x}} = \tilde{\mathbf{x}}(\mathbf{x}, \epsilon)\,, \quad \tilde{q}_\alpha = \tilde{q}_\alpha(\epsilon)\,, \tag{3.1}$$

depending on a parameter $\epsilon \geq 0$. Functions $\tilde{\mathbf{x}}(\mathbf{x},\epsilon)$ and $\tilde{q}_\alpha(\epsilon)$ are twice differentiable with respect to ϵ and the value $\epsilon = 0$ corresponds to the equilibrium (2.7):

$$\tilde{\mathbf{x}}(\mathbf{x},0) = \mathbf{x}, \qquad \tilde{q}_\alpha(0) = 0. \tag{3.2}$$

The transformations defined by equations (3.1)–(3.2) may be interpreted as a 'virtual motion' of the system 'body + fluid' where ϵ plays the role of a 'virtual time', $\tilde{\mathbf{x}}(\mathbf{x},\epsilon)$ is the position vector at the moment of 'time' ϵ of a fluid particle whose position at the initial instant $\epsilon = 0$ was $\mathbf{x}$ (in other words, $\mathbf{x}$ ($\mathbf{x} \in \mathcal{D}_{f0}$) serves as a label to identify the fluid particle, while $\tilde{\mathbf{x}}(\mathbf{x},\epsilon)$ represents its trajectory) and where the functions $\tilde{q}_\alpha(\epsilon)$ determine the position and orientation of the rigid body at the moment of 'time' ϵ. In such a motion, the domain $\mathcal{D}_{f0} = \tilde{\mathcal{D}}_f(0)$ evolves to a new one $\tilde{\mathcal{D}}_f(\epsilon)$ which is completely determined by the position and orientation of the rigid body, i.e. by $\tilde{q}_\alpha(\epsilon)$.

Functions $\tilde{q}_\alpha(\epsilon)$ are arbitrary, function $\tilde{\mathbf{x}}(\mathbf{x},\epsilon)$ represents integral curves of the ordinary differential equation

$$\frac{d\tilde{\mathbf{x}}}{d\epsilon} = \mathbf{f}(\tilde{\mathbf{x}},\epsilon), \tag{3.3}$$

i.e. $\tilde{\mathbf{x}}(\mathbf{x},\epsilon)$ denotes the solution $\tilde{\mathbf{x}}(\epsilon)$ of (3.3) with initial value $\tilde{\mathbf{x}}(0) = \mathbf{x}$. In (3.3), $\mathbf{f}(\tilde{\mathbf{x}},\epsilon)$ is a vector field that satisfies the conditions

$$\tilde{\nabla}\cdot\mathbf{f} = 0 \ \text{ in } \tilde{\mathcal{D}}_f, \quad \mathbf{f}\cdot\mathbf{n} = 0 \ \text{ on } \partial\tilde{\mathcal{D}}, \quad \mathbf{f}\cdot\mathbf{n} = \left[\tilde{\mathbf{R}}_\epsilon + \tilde{\phi}_\epsilon\mathbf{k}\times(\tilde{\mathbf{x}} - \tilde{\mathbf{R}})\right]\cdot\mathbf{n} \ \text{ on } \partial\tilde{\mathcal{D}}_b \tag{3.4}$$

and is otherwise arbitrary. From here on, subscript ϵ denotes ordinary (or partial, if appropriate) derivative with respect to ϵ. In terms of 'virtual motion', functions $\mathbf{f}(\tilde{\mathbf{x}},\epsilon)$, $\tilde{\mathbf{R}}_\epsilon$ and $\tilde{\phi}_\epsilon$ have a natural interpretation as the 'virtual velocities' of the fluid and the rigid body. The conditions (3.4) mean that in the 'virtual motion' the fluid remains incompressible and that there is no fluid 'flow' through the rigid boundaries.

Also, we introduce functions $\tilde{\mathbf{u}}(\tilde{\mathbf{x}},\epsilon)$, $\tilde{a}(\tilde{\mathbf{x}},\epsilon)$, and $\dot{\tilde{q}}_\alpha(\epsilon)$ such that the value $\epsilon = 0$ corresponds to the equilibrium (2.7), i.e.

$$\tilde{\mathbf{u}}(\tilde{\mathbf{x}},\epsilon)\Big|_{\epsilon=0} = 0, \quad \tilde{a}(\tilde{\mathbf{x}},\epsilon)\Big|_{\epsilon=0} = A(\mathbf{x}), \quad \dot{\tilde{q}}_\alpha(\epsilon)\Big|_{\epsilon=0} = 0. \tag{3.5}$$

Functions $\dot{\tilde{q}}_\alpha(\epsilon)$ are arbitrary, and $\tilde{\mathbf{u}}(\tilde{\mathbf{x}},\epsilon)$ is an arbitrary vector field satisfying the conditions

$$\tilde{\nabla}\cdot\tilde{\mathbf{u}} = 0 \ \text{ in } \tilde{\mathcal{D}}_f, \quad \tilde{\mathbf{u}}\cdot\mathbf{n} = 0 \ \text{ on } \partial\tilde{\mathcal{D}}, \quad \tilde{\mathbf{u}}\cdot\mathbf{n} = \left(\tilde{\mathbf{w}} + \tilde{\Omega}\mathbf{k}\times(\tilde{\mathbf{x}} - \tilde{\mathbf{R}})\right)\cdot\mathbf{n} \ \text{ on } \partial\tilde{\mathcal{D}}_b. \tag{3.6}$$

Function $\tilde{a}(\tilde{\mathbf{x}},\epsilon)$ is defined as a solution of the equation

$$\tilde{a}_\epsilon + \mathbf{f}\cdot\tilde{\nabla}\tilde{a} = 0, \tag{3.7}$$

such that $\tilde{a}(\tilde{\mathbf{x}},\epsilon)|_{\epsilon=0} = A(\mathbf{x})$. Equation (3.7) means that the flux function $\tilde{a}$ is considered as a passive scalar advected by the 'virtual flow', so that the 'virtual evolution' of the magnetic field is the same as its evolution in a real flow. Another meaning of the equation

(3.7) is that the value of the flux function in every fluid particle is conserved in the 'virtual motion'. Solutions of Eq. (3.7) define *isomagnetic* states of the system.

Assuming that ϵ is small, we define the first and the second variations of the fluid velocity $\mathbf{u}$, the flux function a and the coordinates and velocities of the rigid body $\mathbf{R}$, ϕ, $\mathbf{w}$, Ω as follows

$$\delta\mathbf{x} \equiv \mathbf{f}\Big|_{\epsilon=0}\epsilon\,, \quad \delta\mathbf{u} \equiv \tilde{\mathbf{u}}_\epsilon\Big|_{\epsilon=0}\epsilon\,, \quad \delta^2\mathbf{u} \equiv \frac{1}{2}\tilde{\mathbf{u}}_{\epsilon\epsilon}\Big|_{\epsilon=0}\epsilon^2\,,$$
$$\delta a \equiv a_\epsilon\Big|_{\epsilon=0}\epsilon\,, \quad \delta^2 a \equiv \frac{1}{2}\tilde{a}_{\epsilon\epsilon}\Big|_{\epsilon=0}\epsilon^2\,, \qquad \text{etc.} \tag{3.8}$$

Note that $\delta\mathbf{x}$ has a clear physical meaning: it is the infinitesimal Lagrangian displacement of the fluid element whose position in the unperturbed state was $\mathbf{x}$. The first and the second variations of the energy (2.6) considered as a functional of $\tilde{\mathbf{u}}(\tilde{\mathbf{x}},\epsilon)$, $a(\tilde{\mathbf{x}},\epsilon)$, $\tilde{\mathbf{R}}(\epsilon)$, $\tilde{\mathbf{w}}(\epsilon)$, $\tilde{\phi}(\epsilon)$, and $\tilde{\Omega}(\epsilon)$ are, by definition,

$$\delta E \equiv dE/d\epsilon\Big|_{\epsilon=0}\epsilon\,, \qquad \delta^2 E \equiv \frac{1}{2}d^2E/d\epsilon^2\Big|_{\epsilon=0}\epsilon^2\,. \tag{3.9}$$

The first variation of E is

$$\delta E = \delta E_f + \delta E_b\,.$$

¿From (2.6c) it follows that

$$\delta E_b = M\mathbf{w}\cdot\delta\mathbf{w} + I\Omega\,\delta\Omega + \frac{\partial\Pi}{\partial\mathbf{R}}\cdot\delta\mathbf{R} + \frac{\partial\Pi}{\partial\phi}\,\delta\phi\,. \tag{3.10}$$

Since $\mathbf{w}=0$ and $\Omega=0$ in the equilibrium (2.7), we obtain

$$\delta E_b = \frac{\partial\Pi}{\partial\mathbf{R}}\cdot\delta\mathbf{R} + \frac{\partial\Pi}{\partial\phi}\,\delta\phi. \tag{3.11}$$

To calculate δE_f, we first note that

$$\frac{d}{d\epsilon}\int_{\tilde{\mathcal{D}}_f(\epsilon)} F(\tilde{\mathbf{x}},\epsilon)\mathrm{d}x\mathrm{d}y = \int_{\tilde{\mathcal{D}}_f(\epsilon)} F_\epsilon \mathrm{d}x\mathrm{d}y + \int_{\partial\tilde{\mathcal{D}}_b(\epsilon)} F(\mathbf{f}\cdot\mathbf{n})\mathrm{d}l$$

for any function $F(\tilde{\mathbf{x}},\epsilon)$ (this follows from the formula for the rate of change of material volume integral; see e.g. Batchelor, 1967). Using this formula, we find

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} E_f = \int_{\mathcal{D}_{f0}} \nabla A\cdot\nabla\tilde{a}_\epsilon \mathrm{d}x\mathrm{d}y + \int_{\partial\mathcal{D}_{b0}} \frac{1}{2}(\nabla A)^2(\mathbf{f}\cdot\mathbf{n})\mathrm{d}l\,.$$

Substituting (3.7) into this equation, we obtain

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} E_f = \int_{\mathcal{D}_{f0}} \Big(-\nabla A\cdot\nabla(\mathbf{f}\cdot\nabla A)\Big)\mathrm{d}x\mathrm{d}y + \int_{\partial\mathcal{D}_{b0}} \frac{1}{2}(\nabla A)^2(\mathbf{f}\cdot\mathbf{n})\mathrm{d}l\,.$$

By using Eqs. (2.8), Green's theorem and the boundary condition (3.4), this can be transformed to

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} E_f = -\int_{\partial\mathcal{D}_{b0}} \frac{1}{2}(\nabla A)^2 \left[\tilde{\mathbf{R}}_\epsilon + \tilde{\phi}_\epsilon(\mathbf{k}\times\mathbf{x})\right]\cdot\mathbf{n}\,dl\,. \tag{3.12}$$

Finally, from (3.8), (3.11), and (3.12), it follows that

$$\delta E = \frac{\partial\Pi}{\partial\mathbf{R}}\cdot\delta\mathbf{R} + \frac{\partial\Pi}{\partial\phi}\delta\phi - \int_{\partial\mathcal{D}_{b0}} \frac{1}{2}(\nabla A)^2\left[\delta\mathbf{R} + \delta\phi\,(\mathbf{k}\times\mathbf{x})\right]\cdot\mathbf{n}\,dl. \tag{3.13}$$

The comparison of (3.13) with (2.11) shows that $\delta E = 0$. Thus, we have proved that *the equilibrium state (2.7) has an stationary value of the energy in comparison with all isomagnetic states of the system.* This result is a natural generalization of the variational principle of Moffatt (1986) who considered magnetostatic equilibria of a perfectly conducting fluid in a fixed domain.

4. THE SECOND VARIATION

Let us now calculate the second variation of the energy at the critical point. We have

$$\delta^2 E = \delta^2 E_f + \delta^2 E_b\,. \tag{4.1}$$

For $\delta^2 E_b$, we obtain

$$\delta^2 E_b = \frac{1}{2}M(\delta\mathbf{w})^2 + \frac{1}{2}I(\delta\Omega)^2 + \delta^2\Pi\,. \tag{4.2}$$

where

$$\delta^2\Pi = \frac{1}{2}\frac{\partial^2\Pi}{\partial q_\alpha\partial q_\beta}\delta q_\alpha\delta q_\beta. \tag{4.3}$$

Here the second order partial derivatives of $\Pi(\mathbf{q})$ are evaluated at $\mathbf{q}=0$.

It is shown in Appendix that

$$\begin{aligned}\delta^2 E_f =& \frac{1}{2}\int_{\mathcal{D}_{f0}}\left\{(\delta\mathbf{u})^2 + (\nabla\delta a)^2 + (\delta\mathbf{x}\cdot\nabla A)(\delta\mathbf{x}\cdot\nabla)\nabla^2 A\right\}\mathrm{d}x\mathrm{d}y\\ &-\frac{1}{2}\int_{\partial\mathcal{D}_{b0}}\left\{(\delta\mathbf{r}\cdot\mathbf{n})(\delta\mathbf{r}\cdot\nabla)\frac{\mathbf{H}^2}{2} - \frac{\mathbf{H}^2}{2}\delta\phi(\delta\mathbf{R}\cdot\boldsymbol{\sigma})\right\}\mathrm{d}l,\end{aligned} \tag{4.4}$$

where $\delta\mathbf{r} = \delta\mathbf{R} + \delta\phi\,(\mathbf{k}\times\mathbf{x})$ and $\delta a = -\delta\mathbf{x}\cdot\nabla A$. Combining Eqs. (4.2) and (4.4), we obtain

$$\begin{aligned}\delta^2 E =& \frac{1}{2}\int_{\mathcal{D}_{f0}}\left\{(\delta\mathbf{u})^2 + (\nabla\delta a)^2 + (\delta\mathbf{x}\cdot\nabla A)(\delta\mathbf{x}\cdot\nabla)\nabla^2 A\right\}\mathrm{d}x\mathrm{d}y\\ &-\frac{1}{2}\int_{\partial\mathcal{D}_{b0}}\left\{(\delta\mathbf{r}\cdot\mathbf{n})(\delta\mathbf{r}\cdot\nabla)\frac{\mathbf{H}^2}{2} - \frac{\mathbf{H}^2}{2}\delta\phi(\delta\mathbf{R}\cdot\boldsymbol{\sigma})\right\}\mathrm{d}l\\ &+\frac{1}{2}M(\delta\mathbf{w})^2 + \frac{1}{2}I(\delta\Omega)^2 + \frac{1}{2}\frac{\partial^2\Pi}{\partial q_\alpha\partial q_\beta}\delta q_\alpha\delta q_\beta.\end{aligned} \tag{4.5}$$

One can see that the second variation (4.5) is split into parts: the quadratic form of the variations of the coordinates and velocities of the rigid body and the volume integral containing the variations of the velocity and the magnetic field. Unlike to the case of a rigid body in a steady flow of an ideal fluid, the second variation does not involve the cross terms in which the variations of both the body and the fluid are present.

Circular cylinder. If the body is a circular cylinder, then no torque is exerted on the body by the fluid, so that angular coordinate ϕ can be ignored. In this case, the second variation (4.5) simplifies to

$$\delta^2 E = \frac{1}{2}\int_{\mathcal{D}_{f0}} \Big\{(\delta\mathbf{u})^2 + (\nabla\delta a)^2 + (\delta\mathbf{x}\cdot\nabla A)(\delta\mathbf{x}\cdot\nabla)\nabla^2 A\Big\}\mathrm{d}x\mathrm{d}y$$
$$-\frac{1}{2}\int_{\partial\mathcal{D}_{b0}} (\delta\mathbf{R}\cdot\mathbf{n})(\delta\mathbf{R}\cdot\nabla)\frac{\mathbf{H}^2}{2}\,\mathrm{d}l + \frac{1}{2}M(\delta\mathbf{w})^2 + \frac{1}{2}\frac{\partial^2\Pi}{\partial R_i\partial R_j}\delta R_i\delta R_j\,. \qquad (4.6)$$

Linearized equations. The remarkable fact is that if variations $\delta\mathbf{x}$, $\delta\mathbf{u}$, $\delta\mathbf{R}$ and $\delta\phi$ are considered as the infinitesimal perturbations of the equilibrium (2.7), whose evolution is governed by appropriate linearized equations, then $\delta^2 E$ is an invariant of these equations (Arnold 1965a,b, 1966; see also Holm et al. 1985, Vladimirov 1987). To make this statement more precise, we shall formulate the linearized problem.

Let $\boldsymbol{\xi}(\mathbf{x},t)$, $\mathbf{R}(t)$, and $\phi(t)$ be an infinitesimal perturbation of the equilibrium (2.7). The perturbation velocity of the fluid $\mathbf{u}(\mathbf{x},t)$, the perturbation flux function $a(\mathbf{x},t)$, and the perturbation velocities of the body $\mathbf{w}(t)$ and $\Omega(t)$ are given by

$$\mathbf{u} = \boldsymbol{\xi}_t, \quad a = -\boldsymbol{\xi}\cdot\nabla A, \quad \mathbf{w} = \dot{\mathbf{R}}, \quad \Omega = \dot{\phi}.$$

The evolution of the perturbation is governed by the equations of motion linearized in a neighbourhood of the equilibrium (2.7):

$$\mathbf{u}_t = -\nabla p - (\nabla^2 A)\nabla a - (\nabla^2 a)\nabla A\,, \quad a = -\boldsymbol{\xi}\cdot\nabla A\,, \quad \nabla\cdot\mathbf{u} = \nabla\cdot\boldsymbol{\xi} = 0\,, \qquad (4.7)$$

$$M\ddot{R}_i = \int_{\partial\mathcal{D}_{b0}} \Big\{p + \nabla A\cdot\nabla a + a\nabla^2 A + (\mathbf{r}\cdot\nabla)\frac{\mathbf{H}^2}{2}\Big\} n_i \mathrm{d}l$$
$$-\int_{\partial\mathcal{D}_{b0}} \frac{\mathbf{H}^2}{2}\phi\,\sigma_i \mathrm{d}l - \frac{\partial^2\Pi}{\partial R_i\partial R_k}R_k - \frac{\partial^2\Pi}{\partial R_i\partial\phi}\phi, \qquad (4.8)$$

$$I\ddot{\phi} = \int_{\partial\mathcal{D}_{b0}} \Big\{p + \nabla A\cdot\nabla a + a\nabla^2 A + (\mathbf{r}\cdot\nabla)\frac{\mathbf{H}^2}{2}\Big\}[\mathbf{n}\cdot(\mathbf{k}\times\mathbf{x})]\mathrm{d}l$$
$$-\frac{\partial^2\Pi}{\partial\phi\partial R_k}R_k - \frac{\partial^2\Pi}{\partial\phi^2}\phi, \qquad (4.9)$$

where $\mathbf{r} = \mathbf{R} + \phi(\mathbf{k}\times\mathbf{x})$. Boundary conditions for $\boldsymbol{\xi}$ are

$$\boldsymbol{\xi}\cdot\mathbf{n} = \mathbf{r}\cdot\mathbf{n} \text{ on } \partial\mathcal{D}_{b0}\,, \quad \boldsymbol{\xi}\cdot\mathbf{n} = 0 \text{ on } \partial\mathcal{D}\,. \qquad (4.10)$$

As was mentioned above, the second variation (4.6) is conserved by the linearized equations. This means that we have the following invariant of (4.7)–(4.10) (this fact may be verified by direct calculation of the time-derivative of $\tilde{E}$):

$$\tilde{E} = \tilde{T} + \tilde{\Pi},$$
$$\tilde{T} = \frac{1}{2} M \dot{\mathbf{R}}^2 + \frac{1}{2} I \dot{\phi}^2 + \frac{1}{2} \int_{\mathcal{D}_{fo}} \boldsymbol{\xi}_t^2 \, \mathrm{d}x\mathrm{d}y$$
$$\tilde{\Pi} = \frac{1}{2} \int_{\mathcal{D}_{fo}} \Big((\nabla a)^2 + (\boldsymbol{\xi} \cdot \nabla A)(\boldsymbol{\xi} \cdot \nabla) \nabla^2 A \Big) \mathrm{d}x\mathrm{d}y$$
$$- \frac{1}{2} \int_{\partial \mathcal{D}_{bo}} \left\{ (\mathbf{r} \cdot \mathbf{n})(\mathbf{r} \cdot \nabla) \frac{\mathbf{H}^2}{2} - \frac{\mathbf{H}^2}{2} \phi (\mathbf{R} \cdot \boldsymbol{\sigma}) \right\} \mathrm{d}l + \frac{1}{2} \frac{\partial^2 \Pi}{\partial q_\alpha \partial q_\beta} \delta q_\alpha \delta q_\beta. \qquad (4.11)$$

Here $\tilde{T}$ and $\tilde{\Pi}$ correspond to the second variations of the kinetic energy and the potential energy of the system (the magnetic energy being considered as a part of the potential energy). For brevity, we shall call them the kinetic energy and the potential energy of the linearized equations.

If $\tilde{E}(t)$ is positive definite, then $\sqrt{\tilde{E}(t)}$ can be taken as the norm of the perturbation. In this case, the conservation of $\tilde{E}(t)$ implies the stability of the equilibrium (2.7): $\tilde{E}(t) = \tilde{E}(0)$.

Note that the kinetic energy $\tilde{T}$ is always positive definite, so that the positive definiteness of $\tilde{E}$ is completely determined by the potential energy $\tilde{\Pi}$. The linear stability problem thus reduces to the analysis of the potential energy of the linearized equations.

Remark. The theory developed above deals with isomagnetic perturbations for which the perturbation flux function $a(\mathbf{x}, t)$ is given in terms of $\boldsymbol{\xi}(\mathbf{x}, t)$ by the equation $a = -\boldsymbol{\xi} \cdot \nabla A$. This restriction may, however, be discarded. It can be shown that the integral similar to (4.11) and given by

$$\tilde{I} = \tilde{T} + \tilde{\Pi},$$
$$\tilde{T} = \frac{1}{2} M \dot{\mathbf{R}}^2 + \frac{1}{2} I \dot{\phi}^2 + \frac{1}{2} \int_{\mathcal{D}_{fo}} \mathbf{u}^2 \, \mathrm{d}x\mathrm{d}y$$
$$\tilde{\Pi} = \frac{1}{2} \int_{\mathcal{D}_{fo}} \Big((\nabla a)^2 + \frac{\nabla(\nabla^2 A)}{\nabla A} a^2 \Big) \mathrm{d}x\mathrm{d}y$$
$$- \frac{1}{2} \int_{\partial \mathcal{D}_{bo}} \left\{ (\mathbf{r} \cdot \mathbf{n})(\mathbf{r} \cdot \nabla) \frac{\mathbf{H}^2}{2} - \frac{\mathbf{H}^2}{2} \phi (\mathbf{R} \cdot \boldsymbol{\sigma}) \right\} \mathrm{d}l + \frac{1}{2} \frac{\partial^2 \Pi}{\partial q_\alpha \partial q_\beta} \delta q_\alpha \delta q_\beta \qquad (4.11a)$$

is conserved by the linearized equation that are the same as Eqs. (4.7)–(4.9) except that the equation $a = -\boldsymbol{\xi} \cdot \nabla A$ is replaced by

$$a_t + \mathbf{u} \cdot \nabla A = 0.$$

In fact, up to the change of notation, this expression coincides with the second variation of the conserved (Energy–Casimir) functional

$$I = \frac{1}{2} \int_{\mathcal{D}_f} \{ \mathbf{u}^2 + (\nabla a)^2 + F(a) \} \, \mathrm{d}x\mathrm{d}y + \frac{1}{2} M \mathbf{w}^2 + \frac{1}{2} I \Omega^2 + \Pi(\mathbf{R}, \phi)$$

where $F(a)$ is a certain, specially chosen function.

Therefore, most of the results obtained below are valid not only for isomagnetic perturbation but also for nonisomagnetic perturbations.

Potential energy. Consider now the functional $\tilde{\Pi}(t)$. It may be written in the form

$$\tilde{\Pi} = I + Q, \quad Q = \frac{1}{2} Q_{\alpha\beta} q_\alpha q_\beta$$
$$I = \frac{1}{2} \int_{\mathcal{D}_{f0}} \left\{ (\nabla a)^2 + \frac{\nabla(\nabla^2 A)}{\nabla A} a^2 \right\} \mathrm{d}x\mathrm{d}y, \tag{4.12a}$$

where the coefficients $Q_{\alpha\beta}$ $(\alpha, \beta = 1, 2, 3)$ are

$$Q_{ij} = -\int_{\partial\mathcal{D}_{b0}} n_i \frac{\partial}{\partial x_j} \frac{\mathbf{H}^2}{2} \mathrm{d}l + \frac{\partial^2 \Pi}{\partial R_i \partial R_j} \quad \text{for} \quad i, j = 1, 2;$$
$$Q_{i3} = Q_{3i} = \int_{\partial\mathcal{D}_{b0}} \left\{ \sigma_i \frac{\mathbf{H}^2}{2} - n_i \{(\mathbf{k} \times \mathbf{x}) \cdot \nabla\} \frac{\mathbf{H}^2}{2} - (\boldsymbol{\sigma} \cdot \mathbf{x}) \frac{\partial}{\partial x_i} \frac{\mathbf{H}^2}{2} \right\} \mathrm{d}l$$
$$+ 2 \frac{\partial^2 \Pi}{\partial R_i \partial \phi} \quad \text{for} \quad i = 1, 2;$$
$$Q_{33} = -\int_{\partial\mathcal{D}_{b0}} (\boldsymbol{\sigma} \cdot \mathbf{x}) \{(\mathbf{k} \times \mathbf{x}) \cdot \nabla\} \frac{\mathbf{H}^2}{2} \mathrm{d}l + \frac{\partial^2 \Pi}{\partial \phi^2}. \tag{4.12b}$$

Now we decompose the perturbation flux function a into two parts:

$$a = \psi + \chi. \tag{4.13}$$

Here ψ corresponds to the acyclic irrotational part of the magnetic field and is the unique solution of the following boundary-value problem:

$$\nabla^2 \psi = 0 \quad \text{in} \quad \mathcal{D}_{f0}, \quad \boldsymbol{\sigma} \cdot \nabla \psi = 0 \quad \text{on} \quad \partial\mathcal{D},$$
$$\psi = -\mathbf{r} \cdot \nabla A \quad \text{on} \quad \partial\mathcal{D}_{b0}, \quad \oint_{\partial\mathcal{D}_{b0}} \mathbf{n} \cdot \nabla \psi \, \mathrm{d}l = 0, \tag{4.14}$$

and χ is an arbitrary function satisfying the boundary conditions: $\boldsymbol{\sigma} \cdot \nabla \chi = 0$ on $\partial\mathcal{D}$ and $\chi = 0$ on $\partial\mathcal{D}_{b0}$. This decomposition is the same as that which is usually used to prove Kelvin's minimum energy theorem in fluid mechanics (see e.g. Batchelor 1967).

Consider now the integral

$$I_1 = \frac{1}{2} \int_{\mathcal{D}_{f0}} (\nabla a)^2 \mathrm{d}x\mathrm{d}y.$$

Substituting (4.13) into this integral, integrating by parts, and using (4.14) and the boundary conditions for χ, we obtain

$$I_1 = \frac{1}{2} \int_{\mathcal{D}_{f0}} \left\{ (\nabla \psi)^2 + (\nabla \chi)^2 \right\} \mathrm{d}x\mathrm{d}y. \tag{4.15}$$

¿From (4.14), we have

$$\int_{\mathcal{D}_{f0}} (\nabla\psi)^2 \mathrm{d}x\mathrm{d}y = \int_{\partial\mathcal{D}_{b0}} \psi(\mathbf{n}\cdot\nabla\psi). \tag{4.16}$$

It follows from (4.14) that function ψ can be presented in the form

$$\psi = \psi_\alpha q_\alpha. \tag{4.17}$$

with ψ_α $(\alpha = 1, 2, 3)$ being the solutions of the problems

$$\begin{aligned} &\nabla^2\psi_\alpha = 0 \quad \text{in} \quad \mathcal{D}_{f0}, \quad \boldsymbol{\sigma}\cdot\nabla\psi_\alpha = 0 \quad \text{on} \quad \partial\mathcal{D}, \\ &\psi_\alpha = g_\alpha \quad \text{on} \quad \partial\mathcal{D}_{b0}, \quad \oint_{\partial\mathcal{D}_{b0}} \mathbf{n}\cdot\nabla\psi_\alpha \, \mathrm{d}l = 0. \end{aligned} \tag{4.18}$$

Here

$$g_i = -n_i(\mathbf{n}\cdot\nabla A) \quad \text{for} \quad i = 1, 2; \quad g_3 = -(\boldsymbol{\sigma}\cdot\mathbf{x})(\mathbf{n}\cdot\nabla A). \tag{4.19}$$

Substitution of (4.17) into (4.16) yields

$$\int_{\mathcal{D}_{f0}} (\nabla\psi)^2 \mathrm{d}x\mathrm{d}y = C_{\alpha\beta} q_\alpha q_\beta, \quad C_{\alpha\beta} = \int_{\partial\mathcal{D}_{b0}} \psi_\alpha(\mathbf{n}\cdot\nabla\psi_\beta). \tag{4.20}$$

Note that the 3×3 matrix of the coefficients $C_{\alpha\beta}$ is symmetric and depends only on the form of the body and the magnetic field in the basic equilibrium (2.7).

It follows from Eqs. (4.12), (4.15), and (4.20) that the potential energy $\tilde{\Pi}(t)$ can be written as

$$\begin{aligned} &\tilde{\Pi} = \tilde{I} + \tilde{Q}, \\ &\tilde{I} = \frac{1}{2}\int_{\mathcal{D}_{f0}} \Big((\nabla\chi)^2 + \frac{\nabla(\nabla^2 A)}{\nabla A} a^2\Big) \mathrm{d}x\mathrm{d}y, \\ &\tilde{Q} = \frac{1}{2}\tilde{Q}_{\alpha\beta} q_\alpha q_\beta = \frac{1}{2}(Q_{\alpha\beta} + C_{\alpha\beta}) q_\alpha q_\beta. \end{aligned} \tag{4.21}$$

Evidently, the sufficient condition for $\tilde{\Pi}$ to be positive definite is

$$\frac{\nabla(\nabla^2 A)}{\nabla A} \geq 0 \quad \text{in} \quad \mathcal{D}_{f0}, \tag{4.22a}$$

$$D_1 \geq 0, \quad D_2 \geq 0, \quad D_3 \geq 0, \tag{4.22b}$$

where D_1, D_2, and D_3 are the principal minors of the matrix of the coefficients $\tilde{Q}_{\alpha\beta}$, i.e.

$$D_1 = \tilde{Q}_{11}, \quad D_2 = \det\begin{pmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} \\ \tilde{Q}_{21} & \tilde{Q}_{22} \end{pmatrix}, \quad D_3 = \det\begin{pmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} & \tilde{Q}_{13} \\ \tilde{Q}_{21} & \tilde{Q}_{22} & \tilde{Q}_{23} \\ \tilde{Q}_{31} & \tilde{Q}_{32} & \tilde{Q}_{33} \end{pmatrix}.$$

Thus, we have obtained the following general stability criterion: *the equilibrium state (2.7) is linearly stable if the inequalities (4.22) are satisfied.*

5. INSTABILITY OF THE SYSTEM IF $\delta^2 E$ HAS NO SIGN

The result of the preceding section says that the sufficient condition for linear stability of the equilibrium state (2.7) is that the second variation of the energy in this state or, equivalently, the potential energy of the linearized equations is positive definite. An important question which arises in this context is whether this condition is necessary for stability. In other words, the question is whether the equilibrium is unstable under the condition that the second variation can take negative values. In this section, we shall answer this question.

Consider the functional

$$W = W_f + W_b\,, \quad W_f = \int_{\mathcal{D}_{f0}} \boldsymbol{\xi}\cdot\boldsymbol{\xi}_t\,\mathrm{d}x\mathrm{d}y\,, \quad W_b = M\mathbf{R}\cdot\dot{\mathbf{R}} + I\phi\dot{\phi}, \tag{5.1}$$

which is an analogue of the virial of classical mechanics.

Let us differentiate W with respect to t assuming that $\boldsymbol{\xi}(\mathbf{x},t)$, $\mathbf{R}(t)$, and $\phi(t)$ represent a solution of Eqs. (4.7)–(4.10). We have

$$\begin{aligned}
\dot{W}_f &= \int_{\mathcal{D}_{f0}} \Big\{\boldsymbol{\xi}_t^2 + \boldsymbol{\xi}\cdot\big(-\nabla p - (\nabla^2 A)\nabla a - (\nabla^2 a)\nabla A\big)\Big\}\mathrm{d}x\mathrm{d}y \\
&= \int_{\mathcal{D}_{f0}} \Big(\boldsymbol{\xi}_t^2 - \nabla\cdot(p\boldsymbol{\xi}) - \nabla\cdot(a\nabla^2 a\boldsymbol{\xi}) + a(\boldsymbol{\xi}\cdot\nabla)\nabla^2 A + \nabla a\cdot\nabla(\boldsymbol{\xi}\cdot\nabla A)\Big)\mathrm{d}x\mathrm{d}y \\
&= \int_{\mathcal{D}_{f0}} \Big(\boldsymbol{\xi}_t^2 - (\nabla a)^2 - (\boldsymbol{\xi}\cdot\nabla A)(\boldsymbol{\xi}\cdot\nabla)\nabla^2 A\Big)\mathrm{d}x\mathrm{d}y \\
&\qquad + \int_{\partial\mathcal{D}_{b0}} (\mathbf{r}\cdot\mathbf{n})\Big(-p - \nabla A\cdot\nabla a - a\nabla^2 a\Big)\mathrm{d}l\,.
\end{aligned} \tag{5.2}$$

Also, we have

$$\begin{aligned}
\dot{W}_b &= \int_{\partial\mathcal{D}_{b0}} \Big(p + \nabla A\cdot\nabla a - (\mathbf{R}\cdot\nabla A)\nabla^2 A + (\mathbf{R}\cdot\nabla)\frac{\mathbf{H}^2}{2}\Big)(\mathbf{r}\cdot\mathbf{n})\mathrm{d}l \\
&\quad - \int_{\partial\mathcal{D}_{b0}} \frac{\mathbf{H}^2}{2}\phi(\mathbf{R}\cdot\boldsymbol{\sigma})\mathrm{d}l + M\dot{\mathbf{R}}^2 + I\dot{\phi}^2 - \frac{\partial^2\Pi}{\partial R_i\partial R_k}R_iR_k - 2\frac{\partial^2\Pi}{\partial R_i\partial\phi}R_i\phi - \frac{\partial^2\Pi}{\partial\phi^2}\phi^2.
\end{aligned} \tag{5.3}$$

It follows from (5.2) and (5.3) that

$$\begin{aligned}
\dot{W} &= \int_{\mathcal{D}_{f0}} \boldsymbol{\xi}_t^2\,\mathrm{d}x\mathrm{d}y - \int_{\mathcal{D}_{f0}} \Big((\nabla a)^2 + (\boldsymbol{\xi}\cdot\nabla A)(\boldsymbol{\xi}\cdot\nabla)\nabla^2 A\Big)\mathrm{d}x\mathrm{d}y \\
&\quad + \int_{\partial\mathcal{D}_{b0}} \Big\{(\mathbf{R}\cdot\mathbf{n})(\mathbf{R}\cdot\nabla)\frac{\mathbf{H}^2}{2} - \frac{\mathbf{H}^2}{2}\phi\,(\mathbf{R}\cdot\boldsymbol{\sigma})\Big\}\,\mathrm{d}l + M\dot{\mathbf{R}}^2 \\
&\quad + I\dot{\phi}^2 - \frac{\partial^2\Pi}{\partial R_i\partial R_k}R_iR_k - 2\frac{\partial^2\Pi}{\partial R_i\partial\phi}R_i\phi - \frac{\partial^2\Pi}{\partial\phi^2}\phi^2.
\end{aligned} \tag{5.4}$$

Comparing (4.11) and (5.4), we find that

$$\dot{W} = 2(\tilde{T} - \tilde{\Pi}) = 4\tilde{T} - 2\tilde{E}. \tag{5.5}$$

¿From (5.1), it is evident that W can be presented in the form

$$W = \dot{L}/2$$

where

$$L \equiv M\mathbf{R}^2 + I\phi^2 + \int_{\mathcal{D}_{f0}} \boldsymbol{\xi}^2 \, dxdy, \tag{5.6}$$

so that Eq. (5.5) can be written as

$$\ddot{L} = 8\tilde{T} - 4\tilde{E}. \tag{5.7}$$

Now we assume that $\tilde{E}$ is not positive definite, i.e. there exist a perturbation $\boldsymbol{\xi}$, $\mathbf{R}$, $\dot{\mathbf{R}}$, ϕ, $\dot{\phi}$ such that $\tilde{E} < 0$. It follows from (5.7) that for this perturbation,

$$\ddot{L} > 8\tilde{T}. \tag{5.8}$$

Using the Schwartz inequality, one can obtain

$$\dot{L}^2 \geq 8\,\tilde{T}\,L. \tag{5.9}$$

Combining inequalities (5.8) and (5.9), we find that

$$\ddot{L} > \frac{\dot{L}^2}{L} \quad \text{or, equivalently,} \quad \frac{d}{dt}\left(\frac{\dot{L}}{L}\right) > 0\,. \tag{5.10}$$

Finally, integrating (5.9) over time twice, we obtain

$$L(t) \geq L(0)e^{\lambda t} \tag{5.11}$$

with

$$\lambda \equiv \dot{L}(0)/L(0)\,. \tag{5.12}$$

Further, we note that initial values for $\boldsymbol{\xi}_t$, $\dot{\mathbf{R}}$ and $\dot{\phi}$ can be chosen independent of initial values for $\boldsymbol{\xi}$, $\mathbf{R}$ and ϕ. Now we choose them such that λ, given by (5.12), is positive: $\lambda > 0$. Then, inequality (5.11) shows that the quantity L, which is quadratic in the perturbation, grows at least exponentially with time, and this proves instability of the equilibrium (2.7), provided that the second variation of the energy (4.11) can take negative values. This assertion is nothing but the converse Lagrange theorem for this dynamical system.

6. EXAMPLES

6.1 The stability of a circular cylinder in a magnetostatic equilibrium with circular lines of magnetic field

Let $\mathcal{D}_f$ be an annulus which in polar coordinates is defined by the inequality $r_1 < r < r_2$. In the basic state (2.7), the magnetic field has only azimuthal component

$$\mathbf{H} = (0, H_0(r)) \quad \text{or, equivalently,} \quad a = A(r) \quad \text{with} \quad A'(r) = H_0(r). \tag{6.1}$$

And let

$$\boldsymbol{\xi} = (\xi, \eta). \tag{6.2}$$

We also assume that $\Pi(\mathbf{R}) = 0$. Then, the energy of the linearized equations (4.12) reduces to

$$\tilde{E} = \frac{1}{2}\int_{\mathcal{D}_{f0}} \{\xi_t^2 + \eta_t^2 + g(r)(\xi_\theta^2 + \eta_\theta^2) + rg'(r)\xi^2\}\, dxdy + \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\pi H_0^2(a)\mathbf{R}^2\,, \tag{6.3}$$

where $g(r) = H_0^2/r^2$.

We introduce a 'stream function' $\chi(r,\theta)$ for the field $\boldsymbol{\xi}(r,\theta)$:

$$\xi = \frac{1}{r}\chi_\theta\,, \quad \eta = -\chi_r\,, \tag{6.4}$$

and present it in the form of Fourier series in θ

$$\chi(r,\theta) = \sum_{m=0}^{\infty} \left(\chi_m(r)e^{im\theta} + \chi_m^*(r)e^{-im\theta}\right)\,. \tag{6.5}$$

Substitution of this into (6.3) and standard manipulations yield the expression

$$\tilde{E} = \int_{\mathcal{D}_{f0}} \sum_{m=0}^{\infty} \Big(|\dot{\chi}_m'|^2 + \frac{m^2}{r^2}|\dot{\chi}_m|^2 + \frac{m^2}{r^2}H_0^2|\chi_m' - \chi_m/r|^2 + (m^2-1)\frac{m^2}{r^2}H_0^2|\chi_m|^2\Big)\, dxdy + \frac{1}{2}M\dot{\mathbf{R}}^2\,. \tag{6.6}$$

Here primes denote derivatives with respect to r, while dots denote derivatives with respect to time.

Evidently, $\tilde{E}$ given by (6.6) is always positive. We, therefore, may conclude that the circular cylinder in the equilibrium with circular $\mathbf{H}$-lines is always stable.

6.2 The stability of a body in an irrotational magnetic field

Consider a perfectly conducting cylinder of arbitrary cross-section placed in a magnetostatic equilibrium with homogeneous magnetic field $\mathbf{H}_0 = -H_0\mathbf{i}$ ($H_0 = \text{const}$, $H_0 > 0$).

Since the cylinder is perfectly conducting, the magnetic field cannot penetrate into it, so that the initial (homogeneous) magnetic field is distorted by the field produced by surface currents in the cylinder to satisfy the boundary condition of no normal magnetic field at the body surface. The resulting magnetic field is irrotational, i.e. there are no currents in the fluid. We assume that the total electric current through the cylinder (i.e. in the direction perpendicular to the (x, y) plane) is zero and, therefore, the circulation of the magnetic field along any closed curve in the (x, y) plane that encircles the cylinder is also zero. In addition, we assume that the centre of the ellipse is fixed, but the ellipse is free to rotate about the centre.

The equilibrium magnetic field $\mathbf{H}(\mathbf{x})$ can be presented as

$$\mathbf{H} = \mathbf{H}_0 + \nabla\varphi_0$$

where the potential $\varphi(\mathbf{x})$ is the (unique) solution of the following boundary-value problem:

$$\begin{aligned}
&\nabla^2\varphi_0 = 0 \quad \text{in } \mathcal{D}_{f0},\\
&\mathbf{n}\cdot\nabla\varphi_0 = -\mathbf{n}\cdot\mathbf{H}_0 \quad \text{on } \partial\mathcal{D}_{f0},\\
&|\nabla\varphi_0| \to 0, \quad \text{as } |\mathbf{x}| \to \infty,\\
&\int_{\partial\mathcal{D}_{b0}} \nabla\varphi_0\cdot \mathrm{d}l = 0.
\end{aligned} \tag{6.7}$$

The boundary value problem (6.7) is exactly the same as the hydrodynamic problem of an irrotational flow past a cylinder moving with the constant velocity $-\mathbf{H}_0$ provided that φ_0 is treated as the potential for the velocity field. It is well-known (see e.g. Batchelor 1967) that no force is exerted on the body by the fluid if the circulation of velocity is zero. It can be shown that this is also true for the analogous magnetostatic problem. Using this analogy, it can be shown that the total torque exerted on the body by the magnetic pressure is zero provided that one of the principal axes of body's virtual-mass tensor is parallel to the magnetic field at infinity $\mathbf{H}_0$.

We study the stability of the equilibrium without external forces (i.e. $\Pi = 0$). And, therefore, we assume that one of the principal axes of body's virtual-mass tensor is parallel to $\mathbf{H}_0$. After standard but lengthy manipulations using the irrotational character of the basic state and the boundary condition for the perturbation magnetic field

$$\mathbf{h}\cdot\mathbf{n} = \boldsymbol{\sigma}\cdot\nabla\left[(\mathbf{r}\cdot\mathbf{n})(\mathbf{H}\cdot\boldsymbol{\sigma})\right] \quad \text{on } \partial\mathcal{D}_{b0}$$

(which is equivalent to the condition $a = -\mathbf{r}\cdot\nabla A$), formula (4.21) for the 'potential energy' $\tilde{\Phi}$ can be simplified to

$$\tilde{\Pi} = \frac{1}{2}\int_{\mathcal{D}_{f0}} (\nabla\chi)^2 \mathrm{d}x\mathrm{d}y + \frac{1}{2}H_0^2(\mu_{22} - \mu_{11})\phi^2, \tag{6.8}$$

where μ_{11} and μ_{22} are the virtual-mass coefficients corresponding to a motion of the body along the x- and y-axes respectively.

It follows from (6.8) that *if in the equilibrium the principal axis of the virtual-mass tensor which corresponds to a minimum virtual mass is parallel to the magnetic field at infinity, then this equilibrium is stable; if this axis is perpendicular to the magnetic field at infinity, the equilibrium is unstable.* In particular, this result confirms the heuristic conclusion on the stability of an elliptic cylinder formulated in Introduction.

7. EFFECT OF VISCOSITY

In this section we shall show that the results obtained in previous sections are also valid for viscous (but still perfectly conducting) fluids. In this case, the governing equation are

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p - (\nabla^2 a)\nabla a + \nu\nabla^2\mathbf{u}\,, \tag{7.1}$$

$$a_t + (\mathbf{u}\cdot\nabla)a = 0, \quad \nabla\cdot\mathbf{u} = 0, \tag{7.2}$$

$$M\dot{\mathbf{w}} = M\ddot{\mathbf{R}} = \int_{\partial\mathcal{D}_b}\Big\{\Big(p + (\nabla a)^2/2\Big)\mathbf{n} - \nu(\mathbf{n}\cdot\nabla)\mathbf{u}\Big\}\,\mathrm{d}l - \partial\Pi/\partial\mathbf{R}, \tag{7.3}$$

$$I\dot{\Omega} = I\ddot{\phi} = \int_{\partial\mathcal{D}_b}\Big\{\Big(p + (\nabla a)^2/2\Big)\mathbf{n} - \nu(\mathbf{n}\cdot\nabla)\mathbf{u}\Big\}\cdot[\mathbf{k}\times(\mathbf{x}-\mathbf{R})]\,\mathrm{d}l - \partial\Pi/\partial\phi, \tag{7.4}$$

where ν is the kinematic viscosity.

Boundary conditions (2.5) for velocity are replaced by no-slip conditions on rigid boundaries

$$\mathbf{u} = 0 \quad \text{on } \partial\mathcal{D}, \quad \mathbf{u} = \mathbf{w} + \Omega[\mathbf{k}\times(\mathbf{x}-\mathbf{R})] \quad \text{on } \partial\mathcal{D}_b, \tag{7.5}$$

while boundary conditions for magnetic field remain the same (see Eqs. (2.5)).

The corresponding linearized (in a neighbourhood of the equilibrium (2.7)) equations are given by (cf. Eqs. (5.1)-(5.3))

$$\mathbf{u}'_t = -\nabla p' - (\nabla^2 A)\nabla a' - (\nabla^2 a')\nabla A + \nu\nabla^2\mathbf{u}', \quad a'_t = -\,\mathbf{u}\cdot\nabla A, \quad \nabla\cdot\mathbf{u}' = 0, \tag{7.6}$$

$$\begin{aligned} M\ddot{R}'_i = &\int_{\partial\mathcal{D}_{b0}}\Big(p' + \nabla A\cdot\nabla a' - (\mathbf{r}'\cdot\nabla A)\nabla^2 A + (\mathbf{r}'\cdot\nabla)\frac{\mathbf{H}^2}{2}\Big)n_i\mathrm{d}l \\ &- \int_{\partial\mathcal{D}_{b0}}\frac{\mathbf{H}^2}{2}\phi'\sigma_i\mathrm{d}l - \nu\int_{\partial\mathcal{D}_{b0}}(\mathbf{n}\cdot\nabla)u'_i\,\mathrm{d}l - \frac{\partial^2\Pi}{\partial R_i\partial R_k}R'_k - \frac{\partial^2\Pi}{\partial R_i\partial\phi}\phi', \end{aligned} \tag{7.7}$$

$$\begin{aligned} I\ddot{\phi} = &\int_{\partial\mathcal{D}_{b0}}\Big(p' + \nabla A\cdot\nabla a' - (\mathbf{r}'\cdot\nabla A)\nabla^2 A + (\mathbf{r}'\cdot\nabla)\frac{\mathbf{H}^2}{2}\Big)\mathbf{n}\cdot(\mathbf{k}\times\mathbf{x})\mathrm{d}l \\ &- \nu\int_{\partial\mathcal{D}_{b0}}(\mathbf{k}\times\mathbf{x})\cdot(\mathbf{n}\cdot\nabla)\mathbf{u}'\,\mathrm{d}l - \frac{\partial^2\Pi}{\partial\phi\partial R_k}R'_k - \frac{\partial^2\Pi}{\partial\phi^2}\phi'. \end{aligned} \tag{7.8}$$

Here primes refer to infinitesimal perturbations of the corresponding quantities, $\mathbf{r}' = \mathbf{R}' + \phi'(\mathbf{k}\times\mathbf{x})$. Boundary conditions for the linearized equations (7.6)–(7.8) are

$$\begin{aligned} &\mathbf{u}' = 0\,, \quad \boldsymbol{\sigma}\cdot\nabla a' = 0 \quad \text{on } \partial\mathcal{D}\,, \\ &\mathbf{u}' = \dot{\mathbf{R}}' + \dot{\phi}'[\mathbf{k}\times\mathbf{x}]\,, \quad a' = -\mathbf{r}'\cdot\nabla A \quad \text{on } \partial\mathcal{D}_b\,. \end{aligned} \tag{7.9}$$

¿From here on, primes will be dropped for simplicity of the notation.

Quantity $\tilde{E}$ given by Eq. (4.11a) is not an invariant of the linearized equations (7.6)-(7.9). It can be shown, however, that $\tilde{E}$ satisfies the equation

$$\dot{\tilde{E}} = -D, \quad D = \nu \int_{\mathcal{D}_{f0}} \frac{\partial u_i}{\partial x_k} \frac{\partial u_i}{\partial x_k} \mathrm{d}x\mathrm{d}y. \tag{7.10}$$

Integral D represents the rate of the energy dissipation due to viscosity and, evidently, is always nonnegative. Therefore, $\tilde{E}(t) \leq \tilde{E}(0)$, so that if $\tilde{E}$ is positive definite for a given equilibrium (2.7), then this equilibrium is linearly stable. Thus, *the equilibria of the system that are stable in the framework of inviscid model are also stable in the case of a viscous fluid.*

Let us show now that the results of Section 5 also remain valid, i.e. inviscid instability implies viscous instability, which is *a priori* unclear, because, usually, sufficiently large viscosity stabilizes fluid flows.

First, we introduce the field of (infinitesimal) Lagrangian displacements of fluid particles by the formula

$$\boldsymbol{\xi}_t = \mathbf{u} \quad \text{in } \mathcal{D}_{f0}. \tag{7.11a}$$

For viscous fluid, the field $\boldsymbol{\xi}(\mathbf{x}, t)$ must satisfy the conditions (cf. Eq. (4.10))

$$\boldsymbol{\xi} = 0 \quad \text{in } \partial\mathcal{D}, \quad \boldsymbol{\xi} = \mathbf{R} + \phi[\mathbf{k} \times \mathbf{x}] \quad \text{on } \partial\mathcal{D}_b. \tag{7.11b}$$

Further, it can be shown that, for Eqs. (7.6)-(7.9), (7.11), the virial equality (5.7) takes the form

$$\ddot{L} = 8\tilde{T} - 4\tilde{E} - \dot{G}, \tag{7.12}$$

where

$$G = \nu \int_{\mathcal{D}_{f0}} \frac{\partial \xi_i}{\partial x_k} \frac{\partial \xi_i}{\partial x_k} \mathrm{d}x\mathrm{d}y. \tag{7.13}$$

Following Vladimirov and Rumyantsev (1990), we introduce the functional

$$X = \dot{L} + G. \tag{7.14}$$

Then Eq. (7.12) is rewritten as

$$\dot{X} = 8\tilde{T} - 4\tilde{E}. \tag{7.15}$$

Multiplying Eq. (7.15) by a constant factor $(-s/2)$ and adding the result to Eq. (7.10), we obtain

$$\dot{E}_s = 2sE_s - 4sT_s - D_s, \tag{7.16}$$

where

$$E_s = T_s + \Pi_s, \quad \Pi_s = \tilde{\Pi} + \frac{1}{2}sG + \frac{1}{2}s^2 L, \tag{7.17}$$

$$T_s = \tilde{T} - \frac{1}{2}s\dot{L} + \frac{1}{2}s^2 L = \frac{1}{2}M(\dot{\mathbf{R}} - s\mathbf{R})^2 + \frac{1}{2}I(\dot{\phi} - s\phi)^2 + \frac{1}{2}\int_{\mathcal{D}_{f0}} (\boldsymbol{\xi}_t - s\boldsymbol{\xi})^2 \mathrm{d}x\mathrm{d}y, \tag{7.18}$$

$$D_s = D - s\dot{G} + s^2 G = \nu \int_{\mathcal{D}_{f0}} \left(\frac{\partial u_i}{\partial x_k} - s\frac{\partial \xi_i}{\partial x_k}\right)\left(\frac{\partial u_i}{\partial x_k} - s\frac{\partial \xi_i}{\partial x_k}\right) \mathrm{d}x\mathrm{d}y. \tag{7.19}$$

Let $s > 0$. Then, since T_s and D_s are always non-negative, it follows from (7.16) that

$$E_s \leq 2sE_s.$$

Integrating this inequality over time, we obtain

$$E_s(t) \leq E_s(0)e^{2st}. \tag{7.20}$$

Note that (7.20) holds for any solution of the linearized problem (7.6)-(7.9) and for any positive s.

As in section 5, we assume that $\tilde{E}$ can take negative values, i.e. there exists a set $\mathcal{Z}$ such that

$$\begin{aligned} \tilde{\Pi} < 0 \quad &\text{for} \quad \{\boldsymbol{\xi}(\mathbf{x}), \mathbf{R}, \phi\} \in \mathcal{Z}, \\ \tilde{\Pi} \geq 0 \quad &\text{for} \quad \{\boldsymbol{\xi}(\mathbf{x}), \mathbf{R}, \phi\} \notin \mathcal{Z}. \end{aligned} \tag{7.21}$$

We shall show that under this condition there exist solutions of the linearized problem (7.6)-(7.9) that grow with time, and we shall obtain a lower bound for these solutions.

If the set $\mathcal{Z}$ is not empty, we can take the initial values for $\boldsymbol{\xi}(\mathbf{x}, t)$, $\mathbf{R}(t)$, $\phi(t)$ such that

$$\{\boldsymbol{\xi}(\mathbf{x}, 0), \mathbf{R}(0), \phi(0)\} \in \mathcal{Z},$$

and therefore,

$$\tilde{\Pi}(0) \leq 0. \tag{7.22}$$

Let us show that under condition (7.22) it is always possible to choose $E_s(0) < 0$ (if it is so then exponential growth of perturbations follows directly from inequality (7.20)). According to Eqs.(7.17)-(7.19) we have

$$E_s(0) = s^2 M(0) + sA(0) + \tilde{E}(0)\,, \quad A(0) \equiv \frac{1}{2}(G(0) - \dot{L}(0))\,.$$

We choose the initial data $\mathbf{u}(\mathbf{x}, 0)$ for the velocity field such that $\tilde{T}(0) < |\tilde{\Pi}(0)|$, and hence $\tilde{E}(0) < 0$. Then $E_s(0)$ is a quadratic polynomial of s with a positive coefficient $L(0)$ at s^2 and with a negative constant term $\tilde{E}(0)$. Therefore the conditions $s > 0$ and $E_s(0) < 0$ determine the interval of admissible values of s:

$$0 < s < S_1, \tag{7.23a}$$

where

$$S_1 \equiv -\frac{A(0)}{2L(0)} + \left[\left(\frac{A(0)}{2L(0)}\right)^2 - \frac{\tilde{E}(0)}{L(0)}\right]^{\frac{1}{2}}. \tag{7.23b}$$

Obviously, $S_1 > 0$ for any initial data which are consistent with condition $\tilde{E}(0) < 0$.

We now show that $E_s(0) < 0$ implies exponential growth with time of the solutions of the problem (7.6)-(7.9). From the fact that $T_s \geq 0$ and the definition of Π_s, it follows that

$$E_s(t) \equiv T_s(t) + \Pi_s(t) > \tilde{\Pi}(t).$$

This and the inequality (7.20) yield

$$\tilde{\Pi}(t) < E_s(0)\exp(2st). \tag{7.24}$$

For any s from the interval defined by (7.23) this inequality means that the potential energy $\tilde{\Pi}(t)$ is exponentially decreasing with time from its negative initial value $\tilde{\Pi}(0)$, so that, in absolute value, $\tilde{\Pi}$ is growing. Evidently, the condition (7.22) can be satisfied only if either I or Q (which enter the expression (4.12) for $\tilde{\Pi}$), or both of them can take negative values.

In the first case, there exist a subdomain $\mathcal{D}_1$ of $\mathcal{D}_{f0}$ ($\mathcal{D}_1 \subset \mathcal{D}_{f0}$) such that

$$\begin{aligned} \frac{\nabla(\nabla^2 A)}{\nabla A} &< 0 \quad \text{for} \quad \mathbf{x} \in \mathcal{D}_1, \\ \frac{\nabla(\nabla^2 A)}{\nabla A} &\geq 0 \quad \text{for} \quad \mathbf{x} \in \mathcal{D}_{f0} \setminus \mathcal{D}_1. \end{aligned} \tag{7.25}$$

We introduce the function $F(\mathbf{x})$ such that

$$F = \begin{cases} \nabla(\nabla^2 A)/\nabla A, & \mathbf{x} \in \mathcal{D}_1, \\ 0 & \mathbf{x} \in \mathcal{D}_{f0} \setminus \mathcal{D}_1. \end{cases} \tag{7.26}$$

(Note that $F(\mathbf{x})$ can be identically equal to zero if the set $\mathcal{D}_1$ is empty.)

Since $F(\mathbf{x}) \leq 0$, we have

$$I \geq = -\frac{1}{2}\int_{\mathcal{D}_{f0}} |F(\mathbf{x})|\, (\boldsymbol{\xi}\cdot\nabla A)^2 \mathrm{d}x\mathrm{d}y. \tag{7.27}$$

It is well-known that any real quadratic form can be transformed to its principal axes by a certain orthogonal matrix N, i.e.

$$Q = \frac{1}{2} Q_{\alpha\beta} q_\alpha q_\beta = \frac{1}{2}\tilde{Q}_{\alpha\beta}\tilde{q}_\alpha\tilde{q}_\beta, \quad \mathbf{q} = N\tilde{\mathbf{q}},$$

where

$$\tilde{Q} = N^T Q N = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

and λ_1, λ_2, and λ_3 are eigenvalues of the matrix with elements $Q_{\alpha\beta}$. If Q can take negative values, then at least one of the eigenvalues is negative. Let

$$Q_- \equiv -\frac{1}{2}\sum_{\alpha=1}^{l} |\lambda_\alpha| \tilde{q}_\alpha^2. \tag{7.28}$$

Here l is the number of negative eigenvalues, and we assumed that the eigenvalues are indexed starting from negative ones.

Evidently,

$$Q \geq Q_- . \tag{7.29}$$

It follows from (4.12a), (7.27)–(7.29) that

$$\Pi \geq -\frac{1}{2}\int_{\mathcal{D}_{f0}} |F(\mathbf{x})| \, (\boldsymbol{\xi}\cdot\nabla A)^2 \mathrm{d}x\mathrm{d}y - \frac{1}{2}\sum_{\alpha=1}^{l} |\lambda_\alpha| \tilde{q}_\alpha^2 . \tag{7.30}$$

Combining inequalities (7.24) and (7.30), we find that

$$\mathcal{J}(t) \geq |E_s(0)| \exp(2st) \tag{7.31}$$

with

$$\mathcal{J} \equiv \frac{1}{2}\int_{\mathcal{D}_{f0}} |F(\mathbf{x})| \, (\boldsymbol{\xi}\cdot\nabla A)^2 \mathrm{d}x\mathrm{d}y + \frac{1}{2}\sum_{\alpha=1}^{l} |\lambda_\alpha| \tilde{q}_\alpha^2 . \tag{7.32}$$

Inequality (7.32) holds for any s from the interval (7.23) and gives us the lower bound for the solutions of the linearized problem. Inequality (7.32) means that a positive definite quadratic (in perturbations) functional $\mathcal{J}$ grows exponentially with time, and this fact, in turn, implies linear instability of the equilibrium (2.7). This proves the converse Lagrange theorem for this dynamical system.

It also follows from Eq. (7.32) that the lower bound for growth rate of the solutions of the problem (7.6)-(7.9) is given by the constant $S_1 - \delta$ where δ is any number from the interval (7.23); in particular, δ may be an arbitrarily small number. Note that S_1 is completely determined by the initial data for perturbations. The upper limit for the growth rate corresponds to the maximum value of the parameter S_1 for all possible initial fields $\boldsymbol{\xi}(\mathbf{x},0)$, $\mathbf{R}(0)$ and $\phi(0)$. Following exactly the same procedure as that in Vladimirov & Rumyantsev (1990) and Vladimirov & Ilin (1998), it can be shown that the maximum growth rate S^* (that corresponds to the most unstable perturbation) is given by the formula

$$S^* \equiv \sup_{\{\boldsymbol{\xi},\mathbf{R},\phi\}\in\mathcal{Z}} S_2, \tag{7.33a}$$

where

$$S_2 \equiv -\frac{G(0)}{2M(0)} + \left[\left(\frac{G(0)}{2M(0)}\right)^2 - \frac{2\Pi_2(0)}{M(0)}\right]^{\frac{1}{2}} . \tag{7.33b}$$

The problem of maximizing S_2 is a quite complicated one. However, it can be solved numerically for any given particular system.

8. Effect of finite conductivity

In this section we consider a more realistic situation when the fluid has finite conductivity (but the body is still perfectly conducting). In this case, the governing equation are the same as Eqs. (7.1)–(7.4) except for the equation for the flux function a which now has the form

$$a_t + (\mathbf{u}\cdot\nabla)a = \frac{1}{\sigma}\nabla^2 a \tag{8.1}$$

where σ is the electric conductivity of the fluid.

Boundary conditions for velocity and magnetic field (7.5) are still valid. In addition to these conditions, for the fluid with finite conductivity we must prescribe two more boundary conditions on perfectly conducting rigid surfaces $\partial\mathcal{D}_b$ and $\partial\mathcal{D}_b$. These additional conditions are given by

$$\mathbf{n}\times(\mathbf{E}+\mathbf{V}\times\mathbf{h}) = 0 \quad \text{on} \quad \partial\mathcal{D}_b, \tag{8.2a}$$

$$\mathbf{n}\times\mathbf{E} = 0 \quad \text{on} \quad \partial\mathcal{D}, \tag{8.2a}$$

where $\mathbf{E}$ is the electric field and $\mathbf{V}$ is the velocity of the body surface (note that, in view of (7.5), $\mathbf{V}=\mathbf{u}$). This conditions follow from the condition that (in the reference frame relative to which the surface is fixed) the tangential component of the electric field is zero. Combining (8.2) with Ohm's law for the fluid

$$\mathbf{j} = \frac{1}{\sigma}(\mathbf{E}+\mathbf{u}\times\mathbf{h}),$$

we find that $\mathbf{n}\times\mathbf{j}=0$ on $\partial\mathcal{D}_b$ and $\partial\mathcal{D}$ or, in terms of flux function a,

$$\nabla^2 a = 0 \quad \text{on} \quad \partial\mathcal{D}_b \quad \text{and} \quad \partial\mathcal{D}_b. \tag{8.3}$$

In the case of finite conductivity, the equilibrium flux function $A(\mathbf{x})$ satisfies the equation

$$\nabla^2 A = 0 \quad \text{in} \quad \mathcal{D}_{f0} \tag{8.4}$$

rather than Eq. (2.8), so that only irrotational equilibrium magnetic fields are possible. In particular, the equilibria with irrotational magnetic field that were considered in Section 6 are possible for the fluid of finite conductivity.

The linearized equations are given by (cf. Eqs. (7.6)–(7.8))

$$\mathbf{u}'_t = -\nabla p' - (\nabla^2 a')\nabla A + \nu\nabla^2\mathbf{u}', \quad a'_t = -\mathbf{u}'\cdot\nabla A + \frac{1}{\sigma}\nabla^2 a', \quad \nabla\cdot\mathbf{u}'=0, \tag{8.5a}$$

$$M\ddot{R}'_i = \int_{\partial\mathcal{D}_{b0}}\Big(p' + \nabla A\cdot\nabla a' + (\mathbf{r}'\cdot\nabla)\frac{\mathbf{H}^2}{2}\Big)n_i\mathrm{d}l - \int_{\partial\mathcal{D}_{b0}}\frac{\mathbf{H}^2}{2}\phi'\sigma_i\mathrm{d}l - \nu\int_{\partial\mathcal{D}_{b0}}(\mathbf{n}\cdot\nabla)u'_i\,\mathrm{d}l - \frac{\partial^2\Pi}{\partial R_i\partial R_k}R'_k - \frac{\partial^2\Pi}{\partial R_i\partial\phi}\phi', \tag{8.5b}$$

$$I\ddot{\phi} = \int_{\partial\mathcal{D}_{b0}}\Big(p' + \nabla A\cdot\nabla a' + (\mathbf{r}'\cdot\nabla)\frac{\mathbf{H}^2}{2}\Big)\mathbf{n}\cdot(\mathbf{k}\times\mathbf{x})\mathrm{d}l - \nu\int_{\partial\mathcal{D}_{b0}}(\mathbf{k}\times\mathbf{x})\cdot(\mathbf{n}\cdot\nabla)\mathbf{u}'\,\mathrm{d}l - \frac{\partial^2\Pi}{\partial\phi\partial R_k}R'_k - \frac{\partial^2\Pi}{\partial\phi^2}\phi'. \tag{8.5c}$$

Here, as before, primes refer to infinitesimal perturbations of the corresponding quantities, $\mathbf{r}' = \mathbf{R}' + \phi'(\mathbf{k}\times\mathbf{x})$. Boundary conditions for $\mathbf{u}'$ and a' are the conditions (7.9) supplemented by the conditions

$$\nabla^2 a' = 0 \quad \text{on} \quad \partial\mathcal{D}_b \quad \text{and} \quad \partial\mathcal{D}_b. \tag{8.6}$$

¿From here on, primes will be omitted.

It can be shown by direct calculation that the 'energy' $\tilde{E}$ given by Eq. (4.11a) satisfies the equation

$$\dot{\tilde{E}} = -D, \quad D = \nu \int_{\mathcal{D}_{f0}} \frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_k} \mathrm{d}x\mathrm{d}y + \frac{1}{\sigma}\int_{\mathcal{D}_{f0}} \left(\nabla^2 a\right)^2 \mathrm{d}x\mathrm{d}y. \tag{8.7}$$

This equation is the same as Eq. (7.10) except that now D contains an additional term due to resistive dissipation of energy.

Now we introduce the field of Lagrangian displacements of fluid particles defined by Eqs. (7.11). In the case of finite conductivity, the relation $a = -\boldsymbol{\xi}\cdot\nabla A$ is no longer valid. Formal integration of the equation for a yields

$$a(\mathbf{x},t) = -\boldsymbol{\xi}(\mathbf{x},t)\cdot\nabla A(\mathbf{x}) + \frac{1}{\sigma} g(\mathbf{x},t) \tag{8.8}$$

where

$$g(\mathbf{x},t) = \int_0^t \nabla^2 a(\mathbf{x},t')\mathrm{d}t' + g_0(\mathbf{x}). \tag{8.9}$$

Note that, according to (8.8) and (8.9),

$$g_0(\mathbf{x}) = a(\mathbf{x},0) + \boldsymbol{\xi}(\mathbf{x},0)\cdot\nabla A(\mathbf{x}).$$

It can be shown that, for Eqs. (7.6)-(7.9), (7.11), the virial equality (7.12) has the same form

$$\ddot{L} = 8\tilde{T} - 4\tilde{E} - \dot{G}, \tag{8.10}$$

except that now G is given by

$$G = \nu \int_{\mathcal{D}_{f0}} \frac{\partial \xi_i}{\partial x_k}\frac{\partial \xi_i}{\partial x_k} \mathrm{d}x\mathrm{d}y + \frac{1}{\sigma}\int_{\mathcal{D}_{f0}} g^2 \mathrm{d}x\mathrm{d}y. \tag{8.11}$$

Using Eqs. (8.7)–(8.10), one can repeat the arguments of Section 7 to obtain the inequality (7.31), this proves that the equilibrium (2.7) is linearly unstable provided the 'potential energy' (4.12) can take negative values.

Thus, we have shown that necessary and sufficient conditions for linear stability obtained in Section 6 for a body in an irrotational magnetic field are also valid in the case of a viscous fluid with finite conductivity.

9. CONCLUSION

In this paper we have established the variational principle for equilibria of a rigid perfectly conducting body in an inviscid, perfectly conducting fluid with magnetic field. We have calculated the corresponding second variation of the energy of the system and showed that an equilibrium is stable if the second variation is positive definite and unstable if the second variation is indefinite in sign, so that the positive definiteness of the second variation gives us the necessary and sufficient condition for linear stability. As an application of the general theory, we have considered two simple particular examples. We have shown (i) that a circular cylinder in a magnetostatic equilibrium with circular lines of magnetic field is always stable and (ii) that the equilibrium of an arbitrary cylinder in an irrotational magnetic field (which is homogeneous at infinity) is stable if the principle axis of its virtual-mass tensor which corresponds to a minimum virtual mass is parallel to the magnetic field at infinity and unstable if it is perpendicular to the magnetic field at infinity.

Then, we have extended the theory to the case of viscous (but still perfectly conducting) fluid and showed that that the stability results obtained for ideal fluid remain valid for this case. Finally, we have proved that for equilibria of the system that are compatible with the governing equation for a fluid with finite conductivity, the stability criteria obtained for a perfectly conducting fluid are also valid for a fluid with finite conductivity. For example, this is true for equilibria of a cylinder in irrotational magnetic field.

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APPENDIX. DERIVATION OF FORMULA (4.4)

Here we shall calculate $\delta^2 E_f$. First, we show that for any function $F(\mathbf{x},\epsilon)$ the following equation holds

$$\frac{d^2}{d\epsilon^2}\int_{\tilde{\mathcal{D}}_f} F(\tilde{\mathbf{x}},\epsilon)\,\mathrm{d}\tilde{x}\mathrm{d}\tilde{y} = \int_{\tilde{\mathcal{D}}_f} F_{\epsilon\epsilon}\,\mathrm{d}\tilde{x}\mathrm{d}\tilde{y} + \int_{\partial\tilde{\mathcal{D}}_b}\Big(2F_\epsilon + \tilde{\mathbf{r}}_\epsilon\cdot\tilde{\nabla}F\Big)\big(\tilde{\mathbf{r}}_\epsilon\cdot\mathbf{n}\big)\mathrm{d}l+ \\ -\int_{\partial\tilde{\mathcal{D}}_b} F(\tilde{\mathbf{R}}_\epsilon\cdot\boldsymbol{\sigma})\tilde{\phi}_\epsilon \mathrm{d}l + \int_{\partial\tilde{\mathcal{D}}_b} F\Big(\tilde{\mathbf{R}}_{\epsilon\epsilon} + \tilde{\phi}_{\epsilon\epsilon}[\mathbf{k}\times(\tilde{\mathbf{x}}-\tilde{\mathbf{R}})]\Big)\cdot\mathbf{n}\,\mathrm{d}l \qquad (A1)$$

where

$$\tilde{\mathbf{r}}_\epsilon \equiv \tilde{\mathbf{R}}_\epsilon + \tilde{\phi}_\epsilon[\mathbf{k}\times(\tilde{\mathbf{x}}-\tilde{\mathbf{R}})]\,. \qquad (A2)$$

It is well known that the rate of change of a material volume integral is given by the formula (see e.g. Batchelor 1967)

$$\frac{d}{d\epsilon}\int_{\tilde{\mathcal{D}}_f(\epsilon)} F(\tilde{\mathbf{x}},\epsilon)\,\mathrm{d}\tilde{x}\mathrm{d}\tilde{y} = \int_{\tilde{\mathcal{D}}_f(\epsilon)}\Big(F_\epsilon + \tilde{\nabla}\cdot(F\mathbf{f})\Big)\mathrm{d}\tilde{x}\mathrm{d}\tilde{y} \qquad (A3)$$

where $F(\tilde{\mathbf{x}},\epsilon)$ is an arbitrary sufficiently smooth function and $F_\epsilon \equiv \partial F/\partial\epsilon$. Using this formula one more time, we obtain

$$\frac{d^2}{d\epsilon^2}\int_{\tilde{\mathcal{D}}_f(\epsilon)} F(\tilde{\mathbf{x}},\epsilon)\,\mathrm{d}\tilde{x}\mathrm{d}\tilde{y} = \int_{\tilde{\mathcal{D}}_f(\epsilon)}\Big\{F_{\epsilon\epsilon}+\tilde{\nabla}\cdot\Big(\big[2F_\epsilon+\tilde{\nabla}\cdot(F\mathbf{f})\big]\mathbf{f}\Big)+\tilde{\nabla}\cdot\big(F\mathbf{f}_\epsilon\big)\Big\}\mathrm{d}\tilde{x}\mathrm{d}\tilde{y}. \qquad (A4)$$

To proceed further, we need boundary conditions for the function $\mathbf{f}_\epsilon(\mathbf{x},\epsilon)$. Differentiation with respect to ϵ of the boundary conditions (3.4) yields

$$\mathbf{f}_\epsilon\cdot\mathbf{n} = 0 \quad \text{on } \partial\tilde{\mathcal{D}}. \qquad (A5a)$$

$$\mathbf{f}_\epsilon\cdot\mathbf{n} = \tilde{\mathbf{r}}_{\epsilon\epsilon}\cdot\mathbf{n} + \tilde{\phi}_\epsilon(\tilde{\mathbf{r}}_\epsilon-\mathbf{f})\cdot(\mathbf{k}\times\mathbf{n}) - \mathbf{n}\cdot(\tilde{\mathbf{r}}_\epsilon\cdot\tilde{\nabla})\mathbf{f} \quad \text{on } \partial\tilde{\mathcal{D}}_b(\epsilon). \qquad (A5b)$$

Here we have used the obvious formula $\mathbf{n}_\epsilon = \tilde{\phi}_\epsilon(\mathbf{k}\times\mathbf{n})$; $\tilde{\mathbf{r}}_\epsilon$ is given by Eq. (A2), and

$$\begin{aligned}\tilde{\mathbf{r}}_{\epsilon\epsilon} &= \tilde{\mathbf{R}}_{\epsilon\epsilon} + \tilde{\phi}_{\epsilon\epsilon}\,\mathbf{k}\times(\tilde{\mathbf{x}}-\tilde{\mathbf{R}}) + \tilde{\phi}_\epsilon\,\mathbf{k}\times(\tilde{\mathbf{x}}_\epsilon-\tilde{\mathbf{R}}_\epsilon)\\ &= \tilde{\mathbf{R}}_{\epsilon\epsilon} + \tilde{\phi}_{\epsilon\epsilon}\,\mathbf{k}\times(\tilde{\mathbf{x}}-\tilde{\mathbf{R}}) + \tilde{\phi}_\epsilon^2\,\mathbf{k}\times\big[\mathbf{k}\times(\tilde{\mathbf{x}}-\tilde{\mathbf{R}})\big].\end{aligned} \qquad (A6)$$

Applying now the divergence theorem to the integral on the right side of (A4) and taking account of the boundary conditions (3.4) and (A5), we obtain

$$\begin{aligned}\frac{d^2}{d\epsilon^2}\int_{\tilde{\mathcal{D}}_f(\epsilon)} F(\tilde{\mathbf{x}},\epsilon)\,\mathrm{d}\tilde{x}\mathrm{d}\tilde{y} &= \int_{\tilde{\mathcal{D}}_f(\epsilon)} F_{\epsilon\epsilon}\,\mathrm{d}\tilde{x}\mathrm{d}\tilde{y} + \int_{\partial\tilde{\mathcal{D}}_b(\epsilon)}\big(2F_\epsilon+\mathbf{f}\cdot\nabla F\big)(\tilde{\mathbf{r}}_\epsilon\cdot\mathbf{n})\mathrm{d}l\\ &+ \int_{\partial\tilde{\mathcal{D}}_b(\epsilon)} F\Big\{\tilde{\mathbf{r}}_{\epsilon\epsilon}\cdot\mathbf{n} + \tilde{\phi}_\epsilon(\tilde{\mathbf{r}}_\epsilon-\mathbf{f})\cdot(\mathbf{k}\times\mathbf{n}) - \mathbf{n}\cdot(\tilde{\mathbf{r}}_\epsilon\cdot\nabla)\mathbf{f}\Big\}\mathrm{d}l.\end{aligned} \qquad (A7)$$

Consider now the following integral (which appears in Eq. (A7))

$$\mathcal{I}_1 = \int_{\partial\tilde{\mathcal{D}}_b(\epsilon)}\Big\{(\mathbf{f}\cdot\tilde{\nabla}F)(\tilde{\mathbf{r}}_\epsilon\cdot\mathbf{n}) - F\mathbf{n}\cdot(\tilde{\mathbf{r}}_\epsilon\cdot\tilde{\nabla})\mathbf{f}\Big\}\mathrm{d}l.$$

Note first that

$$Fn_k\tilde{r}_{i\epsilon}\tilde{\partial}_i f_k = n_k\tilde{r}_{i\epsilon}\tilde{\partial}_i(Ff_k) - n_k f_k\tilde{r}_{i\epsilon}\tilde{\partial}_i F.$$

Therefore, the integrand in $\mathcal{I}_1$ can be written in the form

$$(\mathbf{f}\cdot\tilde{\nabla}F)(\tilde{\mathbf{r}}_\epsilon\cdot\mathbf{n}) - F\mathbf{n}\cdot(\tilde{\mathbf{r}}_\epsilon\cdot\tilde{\nabla})\mathbf{f} = (\tilde{\mathbf{r}}_\epsilon\cdot\tilde{\nabla}F)(\tilde{\mathbf{r}}_\epsilon\cdot\mathbf{n}) - B$$

where

$$\begin{aligned}B &\equiv \tilde{r}_{i\epsilon}n_k\tilde{\partial}_i(Ff_k) - n_k\tilde{r}_{k\epsilon}f_i\tilde{\partial}_i F\\ &= n_k\tilde{\partial}_i\big[(\tilde{r}_{i\epsilon}f_k - \tilde{r}_{k\epsilon}f_i)F\big] - Fn_kf_k\tilde{\partial}_i\tilde{r}_{i\epsilon} + Fn_kf_i\tilde{\partial}_i\tilde{r}_{k\epsilon}.\end{aligned}$$

It follows from Eq. (A2) that

$$\tilde{\partial}_i\tilde{r}_{i\epsilon} = 0\,, \qquad \tilde{\partial}_i\tilde{r}_{k\epsilon} = \tilde{\phi}_\epsilon\, e_{ikl}e_{zl}\,.$$

Therefore,

$$B = \mathbf{n}\cdot \mathrm{curl}\Big(F(\mathbf{f}\times\tilde{\mathbf{r}}_\epsilon)\Big) + \tilde{\phi}_\epsilon F\mathbf{f}\cdot(\mathbf{n}\times\mathbf{k}).$$

The first term in this expression integrated over $\int_{\partial\tilde{\mathcal{D}}_b(\epsilon)}$ yields zero, and $\mathcal{I}_1$ simplifies to

$$\mathcal{I}_1 = \int_{\partial\tilde{\mathcal{D}}_b(\epsilon)} \Big\{\big(\tilde{\mathbf{r}}_\epsilon\cdot\nabla F\big)\big(\tilde{\mathbf{r}}_\epsilon\cdot\mathbf{n}\big) - F\tilde{\phi}_\epsilon\mathbf{f}\cdot\big(\mathbf{n}\times\mathbf{k}\big)\Big\}\mathrm{d}l. \tag{A8}$$

Finally, after substitution of (A8) in (A7) and some manipulations using Eq. (A6), we arrive at formula (A1).

Now we use formula (A1) to obtain

$$\begin{aligned}\frac{d^2}{d\epsilon^2}\bigg|_{\epsilon=0} E_f = &\int_{\mathcal{D}_{f0}} \Big\{\tilde{\mathbf{u}}_\epsilon^2 + (\nabla\tilde{a}_\epsilon)^2 + \nabla A\cdot\nabla\tilde{a}_{\epsilon\epsilon}\Big\}\mathrm{d}x\mathrm{d}y+\\ &+\int_{\partial\mathcal{D}_{b0}} \Big\{2\nabla A\cdot\nabla\tilde{a}_\epsilon + (\tilde{\mathbf{r}}_\epsilon\cdot\nabla)(\nabla A)^2/2\Big\}(\tilde{\mathbf{r}}_\epsilon\cdot\mathbf{n})\mathrm{d}l+\\ &+\frac{1}{2}\int_{\partial\mathcal{D}_{b0}} (\nabla A)^2\Big\{-(\tilde{\mathbf{R}}_\epsilon\cdot\boldsymbol{\sigma})\tilde{\phi}_\epsilon + \Big(\tilde{\mathbf{r}}_{\epsilon\epsilon} + \tilde{\phi}_{\epsilon\epsilon}(\mathbf{k}\times\mathbf{x})\Big)\cdot\mathbf{n}\Big\}\mathrm{d}l.\end{aligned} \tag{A9}$$

¿From Eq. (3.7), we have

$$\tilde{a}_{\epsilon\epsilon}\bigg|_{\epsilon=0} = -\mathbf{f}_\epsilon\cdot\nabla A + \mathbf{f}\cdot\nabla(\mathbf{f}\cdot\nabla A)\,.$$

Substituting this into the volume integral in (A9) and integrating by parts, we obtain

$$\mathcal{I} \equiv \int_{\mathcal{D}_{f0}} \nabla A\cdot\nabla\tilde{a}_{\epsilon\epsilon}\,\mathrm{d}x\mathrm{d}y = \mathcal{I}_1 + \mathcal{I}_2, \tag{A10a}$$

where

$$\begin{aligned}\mathcal{I}_1 = &\int_{\mathcal{D}_{f0}} (\mathbf{f}\cdot\nabla A)(\mathbf{f}\cdot\nabla)\nabla^2 A\mathrm{d}x\mathrm{d}y\\ &+\int_{\partial\mathcal{D}_{b0}} \Big\{(\mathbf{n}\cdot\nabla A)(\mathbf{f}\cdot\nabla)(\mathbf{f}\cdot\nabla A) - (\tilde{\mathbf{r}}_\epsilon\cdot\mathbf{n})(\mathbf{f}\cdot\nabla A)\nabla^2 A\Big\}\mathrm{d}l\end{aligned} \tag{A10b}$$

and

$$\mathcal{I}_2 = -\int_{\partial\mathcal{D}_{b0}} \Big(P + (\nabla A)^2\Big)(\mathbf{f}_\epsilon\cdot\mathbf{n})\mathrm{d}l. \tag{A10c}$$

Substituting (A5b) into (A10c) and using (2.10), we find that

$$\begin{aligned}\mathcal{I}_2 = &-\int_{\partial\mathcal{D}_{b0}} (\nabla A)^2\Big\{\Big(\tilde{\mathbf{R}}_{\epsilon\epsilon} + \tilde{\phi}_{\epsilon\epsilon}\mathbf{k}\times\mathbf{x}\Big)\cdot\mathbf{n} - \tilde{\phi}_\epsilon^2\mathbf{x}\cdot\mathbf{n}\Big\}\,\mathrm{d}l\\ &-\int_{\partial\mathcal{D}_{b0}} \Big(P + (\nabla A)^2\Big)\Big(\tilde{\phi}_\epsilon(\mathbf{f}\cdot\boldsymbol{\sigma} - \tilde{\mathbf{R}}_\epsilon\cdot\boldsymbol{\sigma}) - \mathbf{n}\cdot(\tilde{\mathbf{r}}_\epsilon\cdot\nabla)\mathbf{f}\Big)\mathrm{d}l.\end{aligned} \tag{A11}$$

It follows from (A8) that the equality

$$\int_{\partial\mathcal{D}_{b0}} \Big\{ (\mathbf{f}\cdot\nabla F)(\tilde{\mathbf{r}}_\epsilon\cdot\mathbf{n}) - F\mathbf{n}\cdot(\tilde{\mathbf{r}}_\epsilon\cdot\nabla)\mathbf{f} \Big\} \mathrm{d}l = \int_{\partial\mathcal{D}_{b0}} \Big\{ (\tilde{\mathbf{r}}_\epsilon\cdot\nabla F)(\tilde{\mathbf{r}}_\epsilon\cdot\mathbf{n}) - F(\mathbf{f}\cdot\boldsymbol{\sigma})\tilde{\phi}_\epsilon \Big\} \mathrm{d}l$$

holds for any function $F(\mathbf{x})$. Using this, we find that

$$\begin{aligned} \mathcal{I}_2 = &- \int_{\partial\mathcal{D}_{b0}} (\nabla A)^2 \Big(\tilde{\mathbf{R}}_{\epsilon\epsilon} + \tilde{\phi}_{\epsilon\epsilon}\mathbf{k}\times\mathbf{x} \Big) \cdot \mathbf{n}\,\mathrm{d}l \\ &+ \int_{\partial\mathcal{D}_{b0}} \Big\{ (\nabla A)^2 \tilde{\phi}_\epsilon (\tilde{\mathbf{R}}_\epsilon\cdot\boldsymbol{\sigma}) + (\tilde{\mathbf{r}}_\epsilon\cdot\mathbf{n})\left((\mathbf{f}-\tilde{\mathbf{r}}_\epsilon)\cdot\nabla\right)(\nabla A)^2 \Big\} \mathrm{d}l. \end{aligned} \qquad (A12)$$

Here we have used (2.10). ¿From (A9), (A10a,b), and (A12), we obtain

$$\begin{aligned} \frac{d^2}{d\epsilon^2}\bigg|_{\epsilon=0} E_f = &\int_{\mathcal{D}_{f0}} \Big(\tilde{\mathbf{u}}_\epsilon^2 + (\nabla\tilde{a}_\epsilon)^2 + (\mathbf{f}\cdot\nabla A)(\mathbf{f}\cdot\nabla)\nabla^2 A \Big) \mathrm{d}x\mathrm{d}y \\ &+ \frac{1}{2}\int_{\partial\mathcal{D}_{b0}} \frac{\mathbf{H}^2}{2}\tilde{\phi}_\epsilon(\tilde{\mathbf{R}}_\epsilon\cdot\boldsymbol{\sigma})\,\mathrm{d}l + \int_{\partial\mathcal{D}_{b0}} (\tilde{\mathbf{r}}_\epsilon\cdot\mathbf{n})\ \{(\mathbf{f}-\tilde{\mathbf{r}}_\epsilon)\cdot\nabla\}\,\mathbf{H}^2\,\mathrm{d}l \\ &- \int_{\partial\mathcal{D}_{b0}} \frac{\mathbf{H}^2}{2}\Big(\tilde{\mathbf{R}}_{\epsilon\epsilon} + \tilde{\phi}_{\epsilon\epsilon}\mathbf{k}\times\mathbf{x} \Big)\cdot\mathbf{n}\,\mathrm{d}l + \mathcal{I}_3. \end{aligned} \qquad (A13)$$

where $\mathbf{H} = \nabla A\times\mathbf{k}$ is the magnetic field in the equilibrium (2.7) and

$$\begin{aligned} \mathcal{I}_3 = \int_{\partial\mathcal{D}_{b0}} \Big\{ &(\mathbf{H}\cdot\mathbf{f})\,(\boldsymbol{\sigma}\cdot\nabla)(\mathbf{f}\cdot\nabla A) + (\tilde{\mathbf{r}}_\epsilon\cdot\mathbf{n})(\nabla A\cdot\nabla\tilde{a}_\epsilon) \\ &- (\tilde{\mathbf{r}}_\epsilon\cdot\mathbf{n})(\mathbf{f}\cdot\nabla A)\nabla^2 A + (\tilde{\mathbf{r}}_\epsilon\cdot\mathbf{n})(\tilde{\mathbf{r}}_\epsilon\cdot\nabla)\frac{\mathbf{H}^2}{2} \Big\} \mathrm{d}l, \end{aligned} \qquad (A14)$$

Since on $\partial\mathcal{D}_{b0}$

$$\begin{aligned} \nabla A\cdot\nabla\tilde{a}_\epsilon &= \mathbf{H}\cdot\nabla\times(\mathbf{f}\times\mathbf{H}) = \mathbf{H}\cdot(\mathbf{H}\cdot\nabla)\mathbf{f} - \mathbf{H}\cdot(\mathbf{f}\cdot\nabla)\mathbf{H} \\ &= (\mathbf{H}\cdot\boldsymbol{\sigma})(\boldsymbol{\sigma}\cdot\nabla)(\mathbf{H}\cdot\mathbf{f}) - (\mathbf{f}\cdot\nabla)\mathbf{H}^2 + (\mathbf{f}\cdot\nabla A)\nabla^2 A, \end{aligned}$$

we obtain

$$\mathcal{I}_3 = -\int_{\partial\mathcal{D}_{b0}} (\tilde{\mathbf{r}}_\epsilon\cdot\mathbf{n})(\tilde{\mathbf{r}}_\epsilon\cdot\nabla)\frac{\mathbf{H}^2}{2}\,\mathrm{d}l + \int_{\partial\mathcal{D}_{b0}} (\tilde{\mathbf{r}}_\epsilon\cdot\mathbf{n})\,\{(\tilde{\mathbf{r}}_\epsilon-\mathbf{f})\cdot\nabla\}\mathbf{H}^2\,\mathrm{d}l.$$

Finally, substituting this into (A13), we arrive at the formula

$$\begin{aligned} \delta^2 E_f = &\frac{1}{2}\int_{\mathcal{D}_{f0}} \Big\{ (\delta\mathbf{u})^2 + (\nabla\delta a)^2 + (\delta\mathbf{x}\cdot\nabla A)(\delta\mathbf{x}\cdot\nabla)\nabla^2 A \Big\} \mathrm{d}x\mathrm{d}y \\ &- \frac{1}{2}\int_{\partial\mathcal{D}_{b0}} \Big\{ (\delta\mathbf{r}\cdot\mathbf{n})(\delta\mathbf{r}\cdot\nabla)\frac{\mathbf{H}^2}{2} - \frac{\mathbf{H}^2}{2}\delta\phi(\delta\mathbf{R}\cdot\boldsymbol{\sigma}) \Big\}\,\mathrm{d}l. \end{aligned} \qquad (A15)$$

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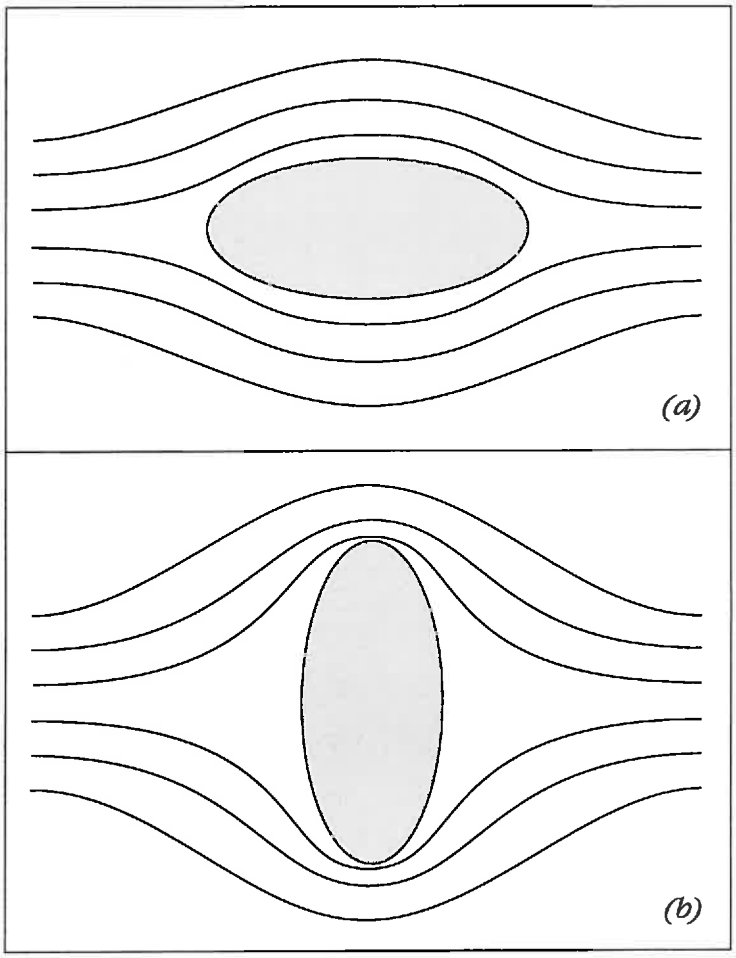

Figure 1