

A LECTURE ABOUT CLASSIFICATION OF LORENTZIAN KAC-MOODY ALGEBRAS OF THE RANK THREE

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ABSTRACT. This is the announcement and partly the review of our results about classification of Lorentzian Kac-Moody algebras of the rank three. One of our results gives the classification of Lorentzian Kac-Moody algebras with denominator identity functions which are holomorphic reflective automorphic forms with respect to paramodular groups. Almost all of them were found in our paper "Automorphic forms and Lorentzian Kac-Moody algebras. Parts I, II", Intern. J. Math. 9 (1998), 153–275. Our main result gives classification of all reflective meromorphic automorphic products with respect to paramodular groups.

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0. INTRODUCTION

In this paper we study Lorentzian Kac-Moody algebras with the root lattice S_t^* , where $S_t = U \oplus \langle 2t \rangle$, $t \in \mathbb{N}$ (see definitions and notations below), and with the symmetry group

$$\widehat{O}^+(L_t) = \{g \in O^+(L_t) \mid g \text{ is trivial on } L_t^*/L_t\}$$

where $L_t = U \oplus S_t$. This case is especially interesting because any hyperbolic lattice S of the rank three and representing zero has an equivariant embedding to S_t or to its m -duals, $m|2t$. Any lattice L of the signature $(3, 2)$ having a two dimensional isotropic sublattice has an equivariant embedding to L_t or to its m -duals, $m|2t$. Moreover, one can consider only square-free t here. Thus, studying this case, we at the same time study Lorentzian Kac-Moody algebras with root lattices S^* and symmetry groups $G \subset O^+(L)$ (of finite index) where S has an equivariant embedding to S_t (or its m -duals), and L has an equivariant embedding to L_t (or its m -duals).

1. SOME GENERAL DEFINITIONS ON LORENTZIAN KAC-MOODY ALGEBRAS

We consider the hyperbolic lattice $S_t := U \oplus \langle 2t \rangle$, $t \in \mathbb{N}$. Here and in what follows a *lattice* means an integral symmetric bilinear form which is usually denoted by (\cdot, \cdot) . The lattice $U = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, and the lattice $\langle A \rangle$ is given by the symmetric integral matrix A ; we denote by \oplus the orthogonal sum of lattices. The lattice U is

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hyperbolic, it has the signature $(2, 1)$. We also consider the lattice $L_t = U \oplus S_t = U \oplus U \oplus \langle 2t \rangle$ of the signature $(3, 2)$.

In this situation a Lorentzian Kac–Moody algebra \mathfrak{g} is given by the *holomorphic automorphic form* $\Phi(z)$ with respect to the subgroup $G \subset O^+(L_t)$ of finite index with the Fourier expansion of a very special form

$$\Phi(z) = \sum_{w \in W} \epsilon(w) \left(\exp(-2\pi i(w(\rho), z)) - \sum_{a \in S_t^* \cap \mathbb{R}_{++} \mathcal{M}} m(a) \exp(-2\pi i(w(\rho + a), z)) \right) \quad (1.1)$$

where all coefficients $m(a)$ should be integral; $W \subset O^+(S_t)$ is the Weyl group; $\epsilon : W \rightarrow \{\pm 1\}$ is its quadratic character (e. g. $\epsilon = \det$ for usual cases); $\mathcal{M} \subset \mathcal{L}(S_t) = V^+(S_t)/\mathbb{R}_{++}$ is a fundamental chamber of W ; $\rho \in S_t \otimes \mathbb{Q}$ is the Weyl vector for the set $P(\mathcal{M}) \subset (S_t)^*$ of simple real roots. Additionally, the automorphic form $\Phi(z)$ should be *reflective*, i. e. it should have zeros only in *rational quadratic divisors* which are orthogonal to the set $G(P(\mathcal{M}))$ of roots of L_t where we always assume that $W \subset G$.

The automorphic form Φ automatically has the infinite product expansion

$$\Phi(z) = \exp(-2\pi i(\rho, z)) \prod_{\alpha \in \Delta_+} \left(1 - \exp(-2\pi i(\alpha, z)) \right)^{\text{mult}(\alpha)}. \quad (1.2)$$

where $\text{mult}(\alpha) \in \mathbb{Z}$ and $\Delta_+ \subset S_t^*$ is the set of positive roots of the algebra \mathfrak{g} .

The sum part (1.1) defines the Lorentzian Kac–Moody algebra \mathfrak{g} by generators

$$e_a, f_a, h_a, \quad a \in P(\mathcal{M}) \cup 2P(\mathcal{M})_{\bar{1}} \cup \left(\bigcup_{a \in S_t^* \cap \mathbb{R}_{++} \mathcal{M}} m(a)a \right) \quad (1.3)$$

(this is the set of simple roots) and defining relations (of Chevalley and Serre) in a usual way. The algebra \mathfrak{g} is graded by the dual lattice S_t^* :

$$\mathfrak{g}(A) = \bigoplus_{\alpha \in S_t^*} \mathfrak{g}_\alpha = \mathfrak{g}_0 \bigoplus \left(\bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha \right) \bigoplus \left(\bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_{-\alpha} \right), \quad \mathfrak{g}_0 = S_t \otimes \mathbb{C}. \quad (1.4)$$

The product part (1.2) defines multiplicities

$$\text{mult}(\alpha) := \dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{\alpha, \bar{0}} - \dim \mathfrak{g}_{\alpha, \bar{1}}.$$

In the standard terminology, the Lie algebra \mathfrak{g} is a *generalised Kac–Moody (or Borcherds) superalgebra* (an infinite-dimensional Lie superalgebra). See [B1], [B2], [B3], [GN1], [GN5], [K], [N], [R] for details. Obviously, the product part (1.2) can be also used to define the algebra \mathfrak{g} since it defines the sum (1.1). The identity (1.1)=(1.2) is called the *denominator identity* for the corresponding algebra. We call the automorphic form (1.1)=(1.2) the *denominator identity function*.

Below we explain more about the terminology above.

The Weyl group $W \subset W(S_t)$ is a *reflection subgroup* of the lattice S_t . It is generated by reflections in some roots $\alpha \in S_t^* = \text{Hom}(S_t, \mathbb{Z})$. Here $\alpha \in S_t^*$ is called root if $\alpha^2 > 0$ and $\alpha^2 \mid 2(S_t^*, \alpha)$. The same definition of roots is used for

any other lattice. The reflection group W has a fundamental chamber \mathcal{M} in the hyperbolic space $\mathcal{L}(S_t) = V^+(S_t)/\mathbb{R}_{++}$ where $V^+(S_t)$ is a half of the light cone $V(S_t) = \{x \in S_t \otimes \mathbb{R} \mid x^2 < 0\}$. We denote by $P(\mathcal{M})$ the set of orthogonal to \mathcal{M} roots directed outwards of \mathcal{M} . The set $P(\mathcal{M})$ defines \mathcal{M} and the group W which is generated by reflections in roots of $P(\mathcal{M})$.

The Weyl vector $\rho \in S_t \otimes \mathbb{Q}$ is defined by the property.

$$(\rho, \alpha) = -\frac{\alpha^2}{2}. \quad (1.5)$$

Existence of the Weyl vector is a very strong restriction on W and $P(\mathcal{M})$.

The set Δ_+ in (1.2) is called *the set of positive roots*. It is union of the set of positive imaginary roots $S \cap \overline{V^+(S)}$ and the set

$$\{\alpha \in \mathbb{N} \cdot W(P(\mathcal{M})) \mid \alpha \text{ is a root and } (\alpha, \mathcal{M}) < 0\} \quad (1.6)$$

of positive real roots.

The variable z in (1.1) and (1.2) belongs to the complexified light cone

$$\Omega(V^+(S_t)) = S_t \otimes \mathbb{R} + iV^+(S_t).$$

It is canonically identified with the Hermitian symmetric domain

$$\Omega(L_t) = \{\mathbb{C}\omega \subset L_t \otimes \mathbb{C} \mid (\omega, \omega) = 0 \ \& \ (\omega, \bar{\omega}) < 0\}_0, \quad (1.7)$$

of the type IV as follows: $z \in \Omega(V^+(S_t))$ defines the element $\mathbb{C}\omega_z \in \Omega(L_t)$ where $\omega_z = ((z, z)/2)f_1 + f_{-1} \oplus z \in L_t \otimes \mathbb{C}$ and f_1, f_{-1} is the bases of the lattice U with the Gram matrix U above. The coordinate z is the affine coordinate of the connected symmetric domain $\Omega(L_t)$ in the neighbourhood of the cusp f_1 of the arithmetic group $O^+(L_t)$ which is the subgroup of $O(L_t)$ of index two acting on $\Omega(L_t)$.

A holomorphic function $\Phi(z)$ on $\Omega(V^+(S))$ is an *automorphic form of the weight k* , $k \in \mathbb{Z}/2$, with respect to a subgroup $G \subset O^+(L_t)$ of finite index if the function $\tilde{\Phi}(\lambda\omega_z) = \lambda^{-k}\Phi(z)$, $\lambda \in \mathbb{C}^*$, on the homogeneous cone $\widetilde{\Omega(L_t)}$ over $\Omega(L_t)$, satisfies the relation $\tilde{\Phi}(g\omega) = \chi(g)\tilde{\Phi}(\omega)$ for any $\omega \in \widetilde{\Omega(L_t)}_0$ and any $g \in G$. Here χ is a finite character or a multiplier system.

The rational quadratic divisor orthogonal to an element $\alpha \in L_t \otimes \mathbb{Q}$ with $\alpha^2 > 0$ is equal to

$$D_\alpha = \{\mathbb{C}\omega \in \Omega(L_t) \mid (\omega, \alpha) = 0\}. \quad (1.8)$$

Thus, we suppose that the divisor of $\Phi(z)$ is union of rational quadratic divisors orthogonal to roots from $G(P(\mathcal{M}))$.

In this definition, one can replace the lattice S_t by arbitrary hyperbolic (i.e. of signature $(m, 1)$) lattice, and the lattice L_t by a lattice of signature $(m+1, 2)$ with an isotropic primitive vector $e \in L$ such that $e^\perp = S$. See [B4], [GN5]. The lattices S_t and L_t above are especially interesting (for the the rank three case) because they are maximal and have maximal automorphism groups for square-free t .

The kernel $\widehat{O}^+(L_t)$ of the action of the group $O^+(L_t)$ on the discriminant group $A_{L_t} = (L_t^*/L_t)$ of the lattice L_t is called the *extended paramodular group*. We want to describe all automorphic forms of the form (1.1) with the properties listed

above which are automorphic with respect to the subgroup $G \subset O^+(L_t)$ such that G contains the extended paramodular group $\widehat{O}^+(L_t)$. It will give the classification of the Lorentzian Kac–Moody algebras of the rank three with the denominator identity function (1.1)=(1.2) above which is an automorphic form with respect to the extended paramodular group.

We have

Theorem 1.1. *There are exactly 29 automorphic forms with respect to the extended paramodular group $\widehat{O}^+(L_t)$ defining Lorentzian Kac–Moody algebras of the rank three. They exist only for*

$$t = 1 \text{ (three)}, 2 \text{ (seven)}, 3 \text{ (seven)}, 4 \text{ (seven)}, 8 \text{ (one)}, 9 \text{ (one)}, \\ 12 \text{ (one)}, 16 \text{ (one)}, 36 \text{ (one)}.$$

where for each t we show in brackets the number of forms. All these 29 forms are given in Sect. 6 below.

All these 29 forms were given in [GN1], [GN2], [GN4],[GN5] together with their sum and product expansions. Formally, three forms for $t = 8, 12, 16$ are new but they coincide with some forms for $t = 2, t = 3$ and $t = 4$ respectively after appropriate change of variable. Thus, in fact, they are not new. Perhaps, Theorem 1.1 is the first result where a large class of Lorentzian Kac–Moody algebras was classified. We describe all 29 forms of Theorem 1.1 in Sect. 6 below.

To construct automorphic forms of Theorem 1.1, we used the arithmetic lifting [G1] of Jacobi forms which gives the Fourier expansion (1.1). Also we used a variant of the Borcherds lifting [B4] which we applied to Jacobi forms [GN5]. It gives the infinite product expansion (1.2) of the forms. We consider this variant of the Borcherds lifting below.

2. A VARIANT OF BORCHERDS PRODUCTS FROM [GN5]

We use a general result from [GN5] which permits to construct many automorphic products of the form similar to (1.2) and which are automorphic with respect to the extended paramodular group.

We use bases f_2, f_{-2} for U and f_3 for $\langle 2t \rangle$. Together they give a bases f_2, f_3, f_{-2} for the lattice $S_t = U \oplus \langle 2t \rangle$. The dual lattice S_t^* has the bases $f_2, \widehat{f}_3 = f_3/2t, f_{-2}$. We denote

$$\alpha = (n, l, m) := nf_2 - l\widehat{f}_3 + mf_{-2} \in S_t^*,$$

where $\alpha^2 = -2nm + \frac{l^2}{2t}$ and $D(\alpha) = 2t\alpha^2 = -4tnm + l^2$ is the *discriminant or norm* of α . For the dual lattice we usually use this form. Thus, we get the lattice $S^*(2t)$. We denote

$$z = z_3f_2 + z_2f_3 + z_1f_{-2} \in \Omega(V^+(S_t)).$$

Then

$$\exp(-2\pi i(\alpha, z)) = q^n r^l s^m$$

where $q = \exp(2\pi iz_1)$, $r = \exp(2\pi iz_2)$, $s = \exp(2\pi iz_3)$.

In [GN5] a variant of the Borcherds lifting was proved. We formulate that in Theorem 2.1 below.

Let

$$\phi_{0,t}(\tau, z) = \sum_{k,l \in \mathbb{Z}} f(k, l) q^k r^l \in J_{0,t}^{nh} \quad (2.1)$$

be a nearly holomorphic Jacobi form of weight 0 and index t (i.e. n might be negative in the Fourier expansion), where $q = \exp(2\pi i\tau)$, $\text{im}\tau > 0$, $r = \exp(2\pi iz)$. It is automorphic with respect to the Jacobi group $H(\mathbb{Z}) \rtimes SL_2(\mathbb{Z})$ where $H(\mathbb{Z})$ is the integral Heisenberg group which is the central extension

$$0 \rightarrow \mathbb{Z} \rightarrow H(\mathbb{Z}) \rightarrow \mathbb{Z} \times \mathbb{Z} \rightarrow 0.$$

Let

$$\phi_{0,t}^{(0)}(z) = \sum_{l \in \mathbb{Z}} f(0, l) r^l \quad (2.2)$$

be the q^0 -part of $\phi_{0,t}(\tau, z)$. The Fourier coefficient $f(k, l)$ of $\phi_{0,t}$ depends only on the norm $4tk - l^2$ of (k, l) and $l \pmod{2t}$. Moreover, $f(k, l) = f(k, -l)$. From the definition of nearly holomorphic forms, it follows that the norm $4tk - l^2$ of indices of non-zero Fourier coefficients $f(k, l)$ are bounded from below.

Theorem 2.1 ([GN5, Part II, Theorem 2.1]). *Assume that the Fourier coefficients $f(k, l)$ of a Jacobi form $\phi_{0,t}$ from (2.1) are integral. Then the infinite product*

$$B_\phi(z) = q^A r^B s^C \prod_{\substack{n,l,m \in \mathbb{Z} \\ (n,l,m) > 0}} (1 - q^n r^l s^m)^{f(nm,l)}, \quad (2.3)$$

where

$$A = \frac{1}{24} \sum_l f(0, l), \quad B = \frac{1}{2} \sum_{l>0} l f(0, l), \quad C = \frac{1}{4t} \sum_l l^2 f(0, l),$$

and $(n, l, m) > 0$ means that either $m > 0$ or $m = 0$ and $n > 0$ or $m = n = 0$ and $l < 0$, defines a meromorphic modular form of weight $\frac{f(0,0)}{2}$ with respect to $\widehat{O}^+(L_t)$ with a character (or a multiplier system if the weight is half-integral) induced by $v_\eta^{24A} \times v_H^{4tC}$. All divisors of $B_\phi(z)$ are the rational quadratic divisors orthogonal to $\alpha = (a, b, c) > 0$ of the discriminant $D = -4tac + b^2$ with multiplicities

$$m_{ac,b} = \sum_{n>0} f(n^2 ac, nb). \quad (2.4)$$

Moreover

$$B_\phi(V_t(z)) = (-1)^D B_\phi(z) \quad \text{with} \quad D = \sum_{\substack{k<0 \\ l \in \mathbb{Z}}} \sigma_1(-k) f(k, l) \quad (2.5)$$

where $\sigma_1(k) = \sum_{d|k} d$. See details in [GN5, Part II, Theorem 2.1].

All 29 automorphic forms of Theorem 1.1 are given by some automorphic products of Theorem 2.1. Thus, to give all these 29 automorphic forms, we should give the corresponding 29 Jacobi forms. We give all these forms in Sect. 6, Table 2.

More generally, we describe all reflective automorphic forms which are given by the Theorem 2.1 where a meromorphic automorphic form on the domain $\Omega(L_t)$

is called *reflective* if its divisor is union of rational quadratic divisors orthogonal to roots of L_t . We remind that an element $\alpha \in L_t$ is called *root* if $\alpha^2 \mid 2(\alpha, L_t)$ and $\alpha^2 > 0$. One can prove that the infinite product B_ϕ given by a Jacobi form $\phi = \phi_{0,t}$ is reflective if and only if each non-zero Fourier coefficient $f(k, l)$ of $\phi_{0,t}$ with negative norm $4tk - l^2 < 0$ satisfies

$$4tk - l^2 \mid (4t, 2l).$$

Let us denote by RJ_t the space of all Jacobi forms of the index t of Theorem 2.1 which give reflective automorphic products. It is natural to call these Jacobi forms *reflective* either. The space RJ_t of all reflective Jacobi forms is a free \mathbb{Z} -module with respect to addition.

Main Theorem 2.2. *For $t \in \mathbb{N}$ the space RJ_t of reflective Jacobi forms of the index t is not trivial (i.e. is not equal to zero) if and only if t is equal to*

$$\begin{aligned} &1(2), 2(3), 3(3), 4(3), 5(3), 6(4), 7(2), 8(3), 9(3), 10(3), 11(1), 12(4), 13(2), \\ &14(3), 15(2), 16(2), 17(1), 18(3), 20(3), 21(3), 22(1), 24(2), 25(1), 26(1), \\ &28(1), 30(3), 33(1), 34(2), 36(3), 39(2), 42(1), 45(1), 48(1), 63(1), 66(1). \end{aligned}$$

where in brackets we also show the rank of the corresponding \mathbb{Z} -module RJ_t of reflective Jacobi forms.

In Sect. 5, Table 1 below we give the bases of the module RJ_t for $t = 1, 2, 3, 4, 8, 9, 12, 16, 36$ when the subspace RJ_t also contains a Jacobi form which gives the denominator identity for a Lorentzian Kac–Moody algebra (i.e. it gives the forms of Theorem 1.1). For all other t the list of all Jacobi forms of the theorem will be given in the forthcoming publication.

All automorphic forms of Theorem 1.1 are characterised by the property that they have multiplicity 1 for components of their divisors. So, it is not hard to find all these forms from the corresponding full Table of reflective Jacobi forms of Main Theorem 2.2 since the theorem 2.1 also gives the multiplicities of divisors. See Table 1 for $t = 1, 2, 3, 4, 8, 9, 12, 16, 36$.

Potentially, Main Theorem 2.2 contains information about all reflective automorphic forms with infinite product expansion of the type of Theorem 2.1 for all equivariant sublattices $L \subset L_t$ of finite index. Here *equivariant* means that $O(L) \subset O(L_t)$; in particular, every root of L is multiple to a root of L_t . If L has a reflective automorphic form Φ with respect to $O(L)$ with an infinite product expansion, then its symmetrization

$$\prod_{g \in O(L) \setminus O(L_t)} g^* \Phi$$

is a reflective automorphic form with an infinite product expansion for the lattice L_t .

Thus, potentially, Main Theorem 2.2 contains important information about reflective automorphic forms with infinite products and about automorphic forms of denominator identities of Lorentzian Kac–Moody algebras with the lattices L instead of L_t and the corresponding hyperbolic lattices $S = S_t \cap L$ instead of S_t . Moreover, one can possibly consider more general class of Lie algebras for which reflective forms of Main Theorem 2.2 may give some denominator identities. Results from [GaL] and [KW] give a hope.

3. THE PROOF OF MAIN THEOREM 2.2
AND REFLECTIVE HYPERBOLIC LATTICES.

To classify finite-dimensional semi-simple or affine Lie algebras, one needs to classify corresponding finite or affine root systems. To prove Main Theorem 2.2 one needs description of appropriate hyperbolic root systems.

Let S be a hyperbolic (i.e. of the signature $(m, 1)$) lattice, $W(S)$ its reflection group and $\mathcal{M} \subset V^+(S)/\mathbb{R}_{++}$ its fundamental chamber and $A(\mathcal{M})$ is the symmetry group of the fundamental chamber. Thus, $O^+(S) = W(S) \rtimes A(\mathcal{M})$. The lattice S is called *reflective* if \mathcal{M} has a *generalised Weyl vector* $\rho \in S \otimes \mathbb{Q}$. It means that $\rho \neq 0$ and the orbit $A(\mathcal{M})(\rho)$ is finite. A reflective lattice is called *elliptically reflective* if it has a generalised Weyl vector ρ with $\rho^2 < 0$. It is called *parabolically reflective* if it is not elliptically reflective but has a generalised Well vector ρ with $\rho^2 = 0$. It is called *hyperbolically reflective* if it is not elliptically or parabolically reflective, but it has a generalised Weyl vector ρ with $\rho^2 > 0$.

Suppose that $B_\phi(z)$ is a reflective automorphic form of Theorem 2.1. The inequality $(n, m, l) > 0$ of Theorem 2.1 is a variant of choosing a fundamental chamber \mathcal{M} of $W(S_t)$. It follows that if the vector $\rho = (A, B, C)$ is not zero, then it defines a generalised Weyl vector for $A(\mathcal{M})$. The vector ρ is invariant with respect to the group $\widehat{A}(\mathcal{M}) = A(\mathcal{M}) \cap \widehat{O}(S_t)$ which has finite index in $A(\mathcal{M})$.

If the form $B_\phi(z)$ has a zero Weyl vector $\rho = (A, B, C)$, one can change it by other form which will have a non-zero Weyl vector, considering reflections in roots. Thus, we get

Lemma 3.1. *If the space RJ_t of reflective Jacobi forms is not zero, then the lattice S_t is reflective.*

It is interesting that the space RJ_t may really have a Jacobi form with zero Weyl vector $\rho = (A, B, C)$. It happens for $t = 6$ and $t = 12$ when $\text{rk } RJ_t = 4$.

In [N5], for the rank three, all maximal reflective hyperbolic lattices were classified. More generally, all reflective hyperbolic lattices S with the square-free determinant $\det(S)$ were classified for the rank three. In the same paper we gave estimate for invariants of any reflective hyperbolic lattice of the rank three. Using this estimate, one can obtain classification of reflective hyperbolic lattices $S_t = U \oplus \langle 2t \rangle$ for all $t \in \mathbb{N}$ (see also calculations in [N4]). As a result, we have

Theorem 3.2. *The lattice $S_t = U \oplus \langle 2t \rangle$ is reflective for the following and the only following $t \in \mathbb{N}$, where in brackets we put the type of reflectivity of the lattice, (e) for elliptic, (p) for parabolic and (h) for the hyperbolic type:*

$$\begin{aligned}
 t = 1 & - 22(e), 23(h), 24 - 26(e), 28(e), 29(h), 30(e), 31(h), \\
 & 33(e), 34(e), 35(h), 36(e), 37(h), 38(h), 39(e), 40(h), 42(e), \\
 & 44(h), 45(e), 46(h), 48(h), 49(e), 50(e), 52(e), 55(e), 56(h), \\
 & 57(h), 60(h), 63(h), 66(e), 70(h), 72(h), 78(h), 84(h), 90(h), \\
 & 100(h), 105(h).
 \end{aligned}$$

To prove Main Theorem 2.2 and to find the bases of RJ_t , one needs to analyse only the t of Theorem 3.2. These can be done using the generators of the graded ring of the weak Jacobi forms with integral Fourier coefficients found in [G2] and

[G3]. We give these bases for $t = 1, 2, 3, 4, 8, 9, 12, 16, 36$ in Sect. 5, Table 1 below.

One can also use arguments which we give below for the proof of Theorem 1.1.

We mention that in [N5] the classification of all reflective hyperbolic lattices with square-free determinant (and their duals) is given for the rank three. In particular, it contains classification of all maximal reflective hyperbolic lattices for the rank three. E.g. this classification contains $122 (e) + 66 (h)$ main hyperbolic lattices with square-free determinant, only $23 (e) + 11 (h)$ of them represent 0 and are given in Theorem 3.2. Moreover, in [N5], there are estimates on invariants of all reflective hyperbolic lattices for the rank three. Thus, potentially, one can use these results for classification of all reflective automorphic forms with infinite products of the type (see below) of Main Theorem 2.2, for the rank three.

4. PROOF OF THEOREM 1.1 AND REFLECTIVE HYPERBOLIC LATTICES WITH WEYL VECTOR.

Here we sketch the proof of Theorem 1.1 to emphasise importance of reflective hyperbolic lattices with Weyl vector.

For this case, the fundamental polyhedron \mathcal{M} and the set $P(\mathcal{M})$ of roots orthogonal to \mathcal{M} have the Weyl vector ρ (satisfying (1.5)). They are all invariant with respect to the group $\widehat{A}(\mathcal{M})$ (we use notation $\widehat{G} = G \cap \widehat{O}^+(L_t)$). For all reflective lattices S_t the fundamental polyhedron \mathcal{M}_0 for the full reflection group $W(S_t)$ of the lattice S_t can be calculated and is known (for $t = 1, 2, 3, 4, 8, 9, 12, 16, 36$ these calculations are presented in Table 1). Thus, the fundamental chamber \mathcal{M} is composed from the known polyhedron \mathcal{M}_0 by some reflections. Using this information, we can find all possible \mathcal{M} , $P(\mathcal{M})$, ρ and predict the divisor of the reflective automorphic form $\Phi(z)$. Looking at the list of 29 forms of Theorem 1.1, one can see that one of the forms (for the corresponding t) of Theorem 1.1 has the same divisor. By Koecher principle, the form $\Phi(z)$ is equal to that form.

Similar arguments can be used to classify all *reflective meromorphic automorphic forms with infinite product* of the form similar to (1.2) and with a generalised Weyl vector ρ . Like the product (1.2), this product is related with a reflection subgroup $W \subset W(S_t)$, its fundamental chamber \mathcal{M} , the set $P(\mathcal{M})$ of orthogonal roots to \mathcal{M} (they define Δ_+) and a generalised Weyl vector $\rho \in S_t \otimes \mathbb{Q}$, i. e. $\rho \neq 0$, the orbit $A(\mathcal{M})(\rho)$ is finite and $W \rtimes A(\mathcal{M})$ has finite index in $O(S_t)$. The function $\text{mult}(\alpha)$, $\alpha \in \Delta_+$, should be integral and invariant with respect to $\widehat{A}(\mathcal{M})$. The product should converge in a neighbourhood of the cusp $\text{im}(z)^2 \ll 0$. All definitions are the same. We have

Theorem 4.1. *All reflective meromorphic automorphic forms with respect to the extended paramodular group and with infinite product, where ρ is a non-zero generalised Weyl vector, belong to the list of Main Theorem 2.2.*

Applying to the forms of Theorem 4.1 reflections with respect to roots in S_t , one can get some automorphic forms with zero Weyl vector and with infinite product. They appear only for $t = 6$ or $t = 12$.

5. THE LIST OF ALL REFLECTIVE JACOBI FORMS FROM RJ_t FOR $t = 1, 2, 3, 4, 8, 9, 12, 16, 36$.

For these t we give the bases $\xi_{0,t}^{(1)}, \dots, \xi_{0,t}^{(rk)}$ of the \mathbb{Z} -module RJ_t showing the

leading parts of their Fourier expansions which define the Jacobi form uniquely. We give all their Fourier coefficients with the negative norm (up to equivalence); the corresponding negative norm is shown in brackets $[\cdot]$. We also give expressions of $\xi_{0,t}^{(i)}$ using basic Jacobi forms. In these formulae $E_4 = E_4(\tau)$ and $\Delta_{12} = \Delta(\tau)$ are the Eisenstein series of weight 4 and the Ramanujan function of weight 12 for $SL_2(\mathbb{Z})$, $E_{4,m}$ ($m = 1, 2, 3$) are Eisenstein-Jacobi series of weight 4 and index m (see [EZ]), and $\phi_{0,1}, \phi_{0,2}, \phi_{0,3}, \phi_{0,4}$ are the four generators of the graded ring of the weak Jacobi forms of weight zero with integral Fourier coefficients (see [G2] and [GN5]).

We give the set \overline{R} of primitive roots in S_t^* up to equivalence (up to the action of the group $\pm\widehat{O}(S_t)$). Up to this equivalence, a root $\alpha = (n, l, m)$ is defined by its norm $-2t\alpha^2 = -4nm + l^2$ and $l \pmod{2t}$. We also give the matrix

$$Mul(\overline{R}, \xi) = mul(\gamma_i, \xi_{0,t}^{(j)}),$$

where $mul(\gamma_i, \xi_{0,t}^{(j)})$ is the multiplicity of the form $\xi_{0,t}^{(j)}$ in a rational quadratic divisor which is orthogonal to the root from the equivalence class $\gamma_i \in \overline{R}$.

We give the set $P(\mathcal{M}_0)$ of primitive roots in S_t^* which is orthogonal to the fundamental chamber \mathcal{M}_0 of the reflection group $W(S_t)$ (this is equivalent to the ordering $(n, l, m) > 0$ used in Theorem 2.1), and their Gram matrix

$$G(P(\mathcal{M}_0)) = 2t((\alpha, \beta)), \quad \alpha, \beta \in P(\mathcal{M}_0).$$

Thus, we identify the dual lattice S^* with the integral lattice $S^*(2t) = U(2t) \oplus \langle 1 \rangle$ to make it integral.

All these data are given in Table 1 below.

Table 1. The spaces RJ_t for $t = 1, 2, 3, 4, 8, 9, 12, 16, 36$.

Case $t = 1$.

The space RJ_1 has the bases

$$\begin{aligned} \xi_{0,1}^{(1)} &= \phi_{0,1} = \\ &= (r[-1] + 10 + r^{-1}[-1]) + O(q); \\ \xi_{0,1}^{(2)} &= E_4^2 E_{4,1} / \Delta_{12} - 57\phi_{0,1} = \\ &= q^{-1}[-4] + (r^2[-4] - r[-1] + 60 - r^{-1}[-1] + r^{-2}[-4]) + O(q). \end{aligned}$$

We have $\overline{R} = \overline{P(\mathcal{M}_0)}$ and

$$P(\mathcal{M}_0) = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \equiv \begin{bmatrix} 4, \overline{0} \\ 1, \overline{1} \\ 4, \overline{0} \end{bmatrix}; \quad Mul(P(\mathcal{M}_0), \xi) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix};$$

$$G(P(\mathcal{M}_0)) = \begin{pmatrix} 4 & -2 & -2 \\ -2 & 1 & 0 \\ -2 & 0 & 4 \end{pmatrix}.$$

Case $t = 2$.

The space RJ_2 has the bases

$$\begin{aligned} \xi_{0,2}^{(1)} &= \phi_{0,2} = \\ &= (r[-1] + 4 + r^{-1}[-1]) + O(q); \\ \xi_{0,2}^{(2)} &= (\phi_{0,1})^2 - 21\phi_{0,2} = \\ &= (r^2[-4] - r[-1] + 18 - r^{-1}[-1] + r^{-2}[-4]) + O(q); \\ \xi_{0,2}^{(3)} &= E_4^2 E_{4,2} / \Delta_{12} - 14(\phi_{0,1})^2 + 216\phi_{0,2} = \\ &= q^{-1}[-8] + 24 + O(q). \end{aligned}$$

We have $\bar{R} = \overline{P(\mathcal{M}_0)}$ where

$$P(\mathcal{M}_0) = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \equiv \begin{bmatrix} 4, \bar{2} \\ 1, \bar{1} \\ 8, \bar{0} \end{bmatrix}; \quad \text{Mul}(P(\mathcal{M}_0), \xi) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

$$G(P(\mathcal{M}_0)) = \begin{pmatrix} 4 & -2 & -4 \\ -2 & 1 & 0 \\ -4 & 0 & 8 \end{pmatrix}.$$

Case $t = 3$.

The space RJ_3 has the bases

$$\begin{aligned} \xi_{0,3}^{(1)} &= \phi_{0,3} = \\ &= (r[-1] + 2 + r^{-1}[-1]) + O(q); \\ \xi_{0,3}^{(2)} &= \phi_{0,1}\phi_{0,2} - 15\phi_{0,3} = \\ &= (r^2[-4] - r[-1] + 12 - r^{-1}[-1] + r^{-2}[-4]) + O(q); \\ \xi_{0,3}^{(3)} &= E_4^2 E_{4,3} / \Delta_{12} - 2(\phi_{0,1})^3 + 33\phi_{0,1}\phi_{0,2} + 90\phi_{0,3} = \\ &= q^{-1}[-12] + 24 + O(q). \end{aligned}$$

We have $\bar{R} = \overline{P(\mathcal{M}_0)}$ where

$$P(\mathcal{M}_0) = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \equiv \begin{bmatrix} 4, \bar{2} \\ 1, \bar{1} \\ 12, \bar{0} \end{bmatrix}; \quad \text{Mul}(P(\mathcal{M}_0), \xi) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

$$G(P(\mathcal{M}_0)) = \begin{pmatrix} 4 & -2 & -6 \\ -2 & 1 & 0 \\ -6 & 0 & 12 \end{pmatrix}.$$

Case $t = 4$.

The space RJ_4 has the bases

$$\begin{aligned} \xi_{0,4}^{(1)} &= \phi_{0,4} = \\ &= (r[-1] + 1 + r^{-1}[-1]) + O(q); \\ \xi_{0,4}^{(2)} &= (\phi_{0,2})^2 - 9\phi_{0,4} = \\ &= (r^2[-4] - r[-1] + 9 - r^{-1}[-1] + r^{-2}[-4]) + O(q); \\ \xi_{0,4}^{(3)} &= E_4 E_{4,1} E_{4,3} / \Delta_{12} - 2(\phi_{0,1})^2 \phi_{0,2} + 20\phi_{0,1} \phi_{0,3} + 16\phi_{0,4} = \\ &= q^{-1}[-16] + 24 + O(q). \end{aligned}$$

We have $\bar{R} = \overline{P(\mathcal{M}_0)}$ where

$$P(\mathcal{M}_0) = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \equiv \begin{bmatrix} 4, \bar{2} \\ 1, \bar{1} \\ 16, \bar{0} \end{bmatrix}; \quad \text{Mul}(P(\mathcal{M}_0), \xi) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

$$G(P(\mathcal{M}_0)) = \begin{pmatrix} 4 & -2 & -8 \\ -2 & 1 & 0 \\ -8 & 0 & 16 \end{pmatrix}.$$

Case $t = 8$.

The space RJ_8 has the bases

$$\begin{aligned} \xi_{0,8}^{(1)} &= (\phi_{0,2})^2 \phi_{0,4} - \phi_{0,2}(\phi_{0,3})^2 - (\phi_{0,4})^2 = \\ &= (2r[-1] - 1 + 2r^{-1}[-1]) + (-r^6[-4] - 2r^5 + 4r^4 - 4r^3 + r^2 + 6r - 8 + \\ &\quad 6r^{-1} + r^{-2} - 4r^{-3} + 4r^{-4} - 2r^{-5} - r^{-6}[-4])q + O(q^2); \\ \xi_{0,8}^{(2)} &= \phi_{0,2}(\tau, 2z) = \phi_{0,1} \phi_{0,3} \phi_{0,4} + \phi_{0,2}(\phi_{0,3})^2 - 2(\phi_{0,2})^2 \phi_{0,4} - 2(\phi_{0,4})^2 = \\ &= (r^2[-4] + 4 + r^{-2}[-4]) + (r^6[-4] - 8r^4 = r^2 + 16 - r^{-2} = 8r^{-4} + r^{-6}[-4])q + O(q^2); \\ \xi_{0,8}^{(3)} &= E_4 E_{4,3} (E_{4,2} \phi_{0,3} - E_{4,1} \phi_{0,4}) / \Delta_{12} - 3(\phi_{0,1})^2 (\phi_{0,3})^2 + \\ &\quad 2(\phi_{0,1})^2 \phi_{0,2} \phi_{0,4} + 8\phi_{0,1} \phi_{0,3} \phi_{0,4} - 16(\phi_{0,4})^2 = \end{aligned}$$

$$\begin{aligned}
&= q^{-1}[-32] + 24 + \\
&\quad (8r^6[-4] + 256r^5 + 2268r^4 + 9472r^3 + 23608r^2 + 39424r + 46812 + \\
&\quad 39424r^{-1} + 23608r^{-2} + 9472r^{-3} + 2268r^{-4} + 256r^{-5} + 8r^{-6}[-4])q + O(q^2).
\end{aligned}$$

We have $\overline{R} = \overline{P(\mathcal{M}_0)}$ where

$$P(\mathcal{M}_0) = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \\ 1 & 6 & 1 \end{pmatrix} \equiv \begin{bmatrix} 4, \overline{2} \\ 1, \overline{1} \\ 32, \overline{0} \\ 4, \overline{6} \end{bmatrix}; \quad \text{Mul}(P(\mathcal{M}_0), \xi) = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 8 \end{pmatrix};$$

$$G(P(\mathcal{M}_0)) = \begin{pmatrix} 4 & -2 & -16 & -4 \\ -2 & 1 & 0 & -6 \\ -16 & 0 & 32 & 0 \\ -4 & -6 & 0 & 4 \end{pmatrix}.$$

Case $t = 9$.

The space RJ_9 has the bases

$$\begin{aligned}
\xi_{0,9}^{(1)} &= -\phi_{0,1}(\phi_{0,4})^2 + 6\phi_{0,2}\phi_{0,3}\phi_{0,4} - 5(\phi_{0,3})^3 = \\
&= (3r[-1] - 2 + 3r^{-1}[-1]) + (-4r^6 + 6r^5 - 12r^4 + 22r^3 - 30r^2 + 36r - 36 + \\
&\quad 36r^{-1} - 30r^{-2} + 22r^{-3} - 12r^{-4} + 6r^{-5} - 4r^{-6})q + \\
&\quad (-r^9[-9] - 6r^8 + 15r^7 - 36r^6 + 72r^5 - 120r^4 + 171r^3 - 216r^2 + 255r - \\
&\quad 268 + 255r^{-1} - 216r^{-2} + 171r^{-3} - 120r^{-4} + 72r^{-5} - 36r^{-6} + 15r^{-7} - \\
&\quad 6r^{-8} - r^{-9}[-9])q^2 + O(q^3);
\end{aligned}$$

$$\begin{aligned}
\xi_{0,9}^{(2)} &= \phi_{0,1}(\phi_{0,4})^2 - 5\phi_{0,2}\phi_{0,3}\phi_{0,4} + 4(\phi_{0,3})^3 = \\
&= (r^2[-4] - r[-1] + 4 - r[-1] + r^{-2}[-4]) + (3r^6 - 8r^5 + 9r^4 - 24r^3 + \\
&\quad 31r^2 - 32r + 42 - 32r^{-1} + 31r^{-2} - 24r^{-3} + 9r^{-4} - 8r^{-5} + 3r^{-6})q + \\
&\quad (r^9[-9] + 7r^8 - 15r^7 + 33r^6 - 80r^5 + 110r^4 - 177r^3 + 219r^2 - 241r + 286 - \\
&\quad 241r^{-1} + 219r^{-2} - 177r^{-3} + 110r^{-4} - 80r^{-5} + 33r^{-6} - 15r^{-7} + 7r^{-8} + \\
&\quad r^{-9}[-9])q^2 + O(q^3);
\end{aligned}$$

$$\begin{aligned}
\xi_{0,9}^{(3)} &= E_{4,2}E_{4,3}(E_{4,1}\phi_{0,3} - E_4\phi_{0,4})/\Delta_{12} - 3\phi_{0,1}\phi_{0,2}(\phi_{0,3})^2 + \\
&\quad 2(\phi_{0,1})^2\phi_{0,3}\phi_{0,4} = 30\phi_{0,1}(\phi_{0,4})^2 + 27\phi_{0,2}\phi_{0,3}\phi_{0,4} + 9(\phi_{0,3})^3 =
\end{aligned}$$

$$\begin{aligned}
 &= q^{-1}[-36] + 24 + (33r^6 + 486r^5 + 3159r^4 + 10758r^3 + 24057r^2 + 37908r + \\
 &44082 + 37908r^{-1} + 24057r^{-2} + 10758r^{-3} + 3159r^{-4} + 486r^{-5} + 33r^{-6})q + \\
 &(2r^9[-9] + 243r^8 + 5346r^7 + 44055r^6 + 204120r^5 + 642978r^4 + 1483416r^3 + \\
 &2632905r^2 + 3679020r + 4109590 + 3679020r^{-1} + 2632905r^{-2} + \\
 &1483416r^{-3} + 642978r^{-4} + 204120r^{-5} + 44055r^{-6} + 5346r^{-7} + \\
 &243r^{-8} + 2r^{-9}[-9])q^2 + O(q^3).
 \end{aligned}$$

We have $\bar{R} = \overline{P(\mathcal{M}_0)}$ where

$$\begin{aligned}
 P(\mathcal{M}_0) &= \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \\ 2 & 9 & 1 \end{pmatrix} \equiv \begin{bmatrix} 4 & \bar{2} \\ 1 & \bar{1} \\ 36 & \bar{0} \\ 9 & \bar{9} \end{bmatrix}; \quad \text{Mul}(P(\mathcal{M}_0), \xi) = \begin{pmatrix} 0 & 1 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 3 \end{pmatrix}; \\
 G(P(\mathcal{M}_0)) &= \begin{pmatrix} 4 & -2 & -18 & 0 \\ -2 & 1 & 0 & -9 \\ -18 & 0 & 36 & -18 \\ 0 & -9 & -18 & 9 \end{pmatrix}.
 \end{aligned}$$

The case $t = 12$.

The space RJ_{12} has the bases

$$\begin{aligned}
 \xi_{0,12}^{(1)} &= (\vartheta(\tau, z)/\eta(\tau))^{12} = \\
 &= (r^8[-16] - 8r^7[-1] + 24r^6 - 24r^5 - 36r^4 + 120r^3 - \\
 &88r^2 - 88r + 198 - 88r^{-1} - 88r^{-2} + \\
 &120r^{-3} - 36r^{-4} - 24r^{-5} + 24r^{-6} - 8r^{-7}[-1] + r^{-8}[-16])q + \\
 &(-4r^{10}[-4] + 24r^9 - 32r^8 - 104r^7 + 396r^6 - 352r^5 - 512r^4 + 1440r^3 - \\
 &904r^2 - 1008r + 2112 - 1008r^{-1} - 904r^{-2} + 1440r^{-3} - 512r^{-4} - \\
 &352r^{-5} + 396r^{-6} - 104r^{-7} - 32r^{-8} + 24r^{-9} - 4r^{-10}[-4])q^2 + O(q^3); \\
 \xi_{0,12}^{(2)} &= 3\phi_{0,2}(\phi_{0,3})^2\phi_{0,4} - (\phi_{0,2})^2(\phi_{0,4})^2 - 2(\phi_{0,3})^4 - (\phi_{0,4})^3 = \\
 &= (r[-1] - 1 + r^{-1}[-1]) + (-r^7[-1] + r^6 - r^5 + r^4 - r^2 + 2r - 2 + \\
 &2r^{-1} - r^{-2} + r^{-4} - r^{-5} + r^{-6} - r^{-7}[-1])q + (-r^{10}[-4] + r^8 - 2r^7 + 3r^6 - \\
 &3r^5 + 2r^4 - 2r^2 + 5r - 6 + 5r^{-1} - 2r^{-2} + 2r^{-4} - 3r^{-5} + 3r^{-6} - \\
 &2r^{-7} + r^{-8} - r^{-10}[-4])q + O(q^3); \\
 \xi_{0,12}^{(3)} &= 2(\phi_{0,2})^2(\phi_{0,4})^2 - 5\phi_{0,2}(\phi_{0,3})^2\phi_{0,4} + 3(\phi_{0,3})^4 + (\phi_{0,4})^3 =
 \end{aligned}$$

$$\begin{aligned}
&= (r^2[-4] - r[-1] + 3 - r^{-1}[-1] + r^{-2}[-4]) + (r^7[-1] - 3r^6 + r^5 - 3r^4 + \\
&\quad 3r^3 - 2r + 6 - 2r^{-1} + 3r^{-2} - 3r^{-4} + r^{-5} - 3r^{-6} + r^{-7}[-1])q + \\
&\quad (2r^{10}[-4] - 3r^8 + 2r^7 - 9r^6 + 3r^5 - 6r^4 + 7r^2 - 5r + 18 - 5r^{-1} + \\
&\quad 7r^{-2} - 6r^{-4} + 3r^{-5} - 9r^{-6} + 2r^{-7} - 3r^{-8} + 2r^{-10}[-4])q^2 + O(q^3);
\end{aligned}$$

$$\begin{aligned}
\xi_{0,12}^{(4)} &= E_{4,3} \left(E_{4,1} E_{4,2} (\phi_{0,3})^2 - 2E_4 E_{4,2} \phi_{0,3} \phi_{0,4} + E_4 E_{4,1} (\phi_{0,4})^2 \right) / \Delta_{12} - \\
&\quad 2(\phi_{0,1})^2 \phi_{0,2} (\phi_{0,4})^2 + 5(\phi_{0,1})^2 (\phi_{0,3})^2 \phi_{0,4} - 3\phi_{0,1} \phi_{0,2} (\phi_{0,3})^3 - \\
&\quad 36\phi_{0,1} \phi_{0,3} (\phi_{0,4})^2 + 24\phi_{0,2} (\phi_{0,3})^2 \phi_{0,4} + 9(\phi_{0,3})^4 + 16(\phi_{0,4})^3 =
\end{aligned}$$

$$\begin{aligned}
&= q^{-1}[-48] + 24 + (24r^7[-1] + 264r^6 + 1608r^5 + 5610r^4 + 13464r^3 + \\
&\quad 24312r^2 + 34056r + 38208 + 34056r^{-1} + 24312r^{-2} + 13464r^{-3} + \\
&\quad 5610r^{-4} + 1608r^{-5} + 264r^{-6} + 24r^{-7}[-1])q + \\
&\quad (12r^{10}[-4] + 440r^9 + 5544r^8 + 34104r^7 + 135388r^6 + 395808r^5 + \\
&\quad 902352r^4 + 1667360r^3 + 2550552r^2 + 3276240r + 3558160 + 3276240r^{-1} + \\
&\quad 2550552r^{-2} + 1667360r^{-3} + 902352r^{-4} + 395808r^{-5} + 135388r^{-6} + \\
&\quad 34104r^{-7} + 5544r^{-8} + 440r^{-9} + 12r^{-10}[-4])q^2 + O(q^3).
\end{aligned}$$

We have

$$\begin{aligned}
P(\mathcal{M}_0) &= \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \\ 1 & 8 & 1 \end{pmatrix} \equiv \begin{bmatrix} 4 & \bar{2} \\ 1 & \bar{1} \\ 48 & \bar{0} \\ 16 & \bar{8} \end{bmatrix}; \quad G(P(\mathcal{M}_0)) = \begin{pmatrix} 4 & -2 & -24 & -8 \\ -2 & 1 & 0 & -8 \\ -24 & 0 & 48 & 0 \\ -8 & -8 & 0 & 16 \end{pmatrix}. \\
\bar{R} &= \begin{bmatrix} 4, \bar{2} \\ 1, \bar{1} \\ 48, \bar{0} \\ 16, \bar{8} \\ 1, \bar{7} \\ 4, \bar{10} \end{bmatrix}; \quad \text{Mul}(\bar{R}, \xi) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ -4 & -1 & 2 & 12 \\ -12 & -2 & 3 & 36 \end{pmatrix};
\end{aligned}$$

Case $t = 16$.

The space RJ_{16} has the bases

$$\begin{aligned}
\xi_{0,16}^{(1)} &= \phi_{0,4}(\tau, 2z) = \\
&= (r^2[-4] + 1 + r^{-2}[-4]) + (-r^8 - r^6 + r^2 + 2 + r^{-2} - r^{-6} - \\
&\quad r^{-8})q + (-r^{10} - 2r^8 - 2r^6 + 3r^2 + 4 + 3r^{-2} - 2r^{-6} - 2r^{-8} - r^{-10})q^2 + \\
&\quad (r^{14}[-4] - 2r^{10} - 4r^8 - 4r^6 + 5r^2 + 8 + 5r^{-2} - 4r^{-6} - 4r^{-8} - \\
&\quad 2r^{-10} + r^{-14}[-4])q^3 + O(q^4);
\end{aligned}$$

$$\begin{aligned}
 \xi_{0,16}^{(2)} &= E_{4,3} \left(E_4 E_{4,1} (\phi_{0,3})^4 - (E_4)^2 (\phi_{0,3})^3 \phi_{0,4} - \right. \\
 &\quad 2E_{4,1} E_{4,2} (\phi_{0,3})^2 \phi_{0,4} + E_4 E_{4,2} \phi_{0,3} (\phi_{0,4})^2 - E_{4,1} E_{4,2} (\phi_{0,3})^2 \phi_{0,4} + \\
 &\quad \left. 2E_4 E_{4,2} \phi_{0,3} (\phi_{0,4})^2 - E_4 E_{4,1} (\phi_{0,4})^3 \right) / \Delta_{12} + \\
 &\quad 2(\phi_{0,1})^3 (\phi_{0,3})^3 \phi_{0,4} - 3(\phi_{0,1})^2 \phi_{0,2} (\phi_{0,3})^4 - 7(\phi_{0,1})^2 (\phi_{0,3})^2 (\phi_{0,4})^2 - \\
 &\quad 31\phi_{0,1} \phi_{0,2} (\phi_{0,3})^3 \phi_{0,4} + 46\phi_{0,1} (\phi_{0,3})^5 + 72\phi_{0,1} \phi_{0,3} (\phi_{0,4})^3 + \\
 &\quad 7(\phi_{0,2})^3 (\phi_{0,3})^2 \phi_{0,4} - 72\phi_{0,2} (\phi_{0,3})^2 (\phi_{0,4})^2 - 197(\phi_{0,3})^4 \phi_{0,4} + \\
 &\quad 2(\phi_{0,1})^2 \phi_{0,2} (\phi_{0,4})^3 + 21(\phi_{0,3})^4 \phi_{0,4} - 26(\phi_{0,4})^4 + \\
 &\quad 2\phi_{0,1} (\phi_{0,2})^2 \phi_{0,3} (\phi_{0,4})^2 - (\phi_{0,2})^2 (\phi_{0,3})^4 - 4(\phi_{0,2})^2 (\phi_{0,4})^3 - 2(\phi_{0,2})^4 (\phi_{0,4})^2 = \\
 &= q^{-1}[-64] + ((8r[-1] + 14 + 8r^{-1}[-1]) + (21r^8 + 200r^7 + 1036r^6 + 3360r^5 + \\
 &\quad 8100r^4 + 15240r^3 + 23604r^2 + 30352r + 33058 + 30352r^{-1} + 23604r^{-2} + \\
 &\quad 15240r^{-3} + 8100r^{-4} + 3360r^{-5} + 1036r^{-6} + 200r^{-7} + 21r^{-8})q + \\
 &\quad (56r^{11} + 1008r^{10} + 7336r^9 + 32932r^8 + 108800r^7 + 283504r^6 + 610344r^5 + \\
 &\quad 1112832r^4 + 1750728r^3 + 2401952r^2 + 2896688r^1 + 3081400 + \dots)q^2 + \\
 &\quad (4r^{14}[-4] + 560r^{13} + 8092r^{12} + 58328r^{11} + 283784r^{10} + 1042328r^9 + \\
 &\quad 3082176r^8 + 7616904r^7 + 16136000r^6 + 29802144r^5 + 48582612r^4 + \\
 &\quad 70497736r^3 + 91619124r^2 + 107054192r + 112732002 + \dots)q^3 + O(q^4).
 \end{aligned}$$

We have $\overline{R} = \overline{P(\mathcal{M}_0)}$ where

$$\begin{aligned}
 P(\mathcal{M}_0) &= \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \\ 5 & 32 & 3 \\ 3 & 14 & 1 \end{pmatrix} \equiv \begin{bmatrix} 4 & \overline{2} \\ 1 & \overline{1} \\ 64 & \overline{0} \\ 64 & \overline{0} \\ 4 & \overline{14} \end{bmatrix}; \quad \text{Mul}(P(\mathcal{M}_0), \xi) = \begin{pmatrix} 1 & 0 \\ 1 & 8 \\ 0 & 1 \\ 0 & 1 \\ 1 & 4 \end{pmatrix}; \\
 G(P(\mathcal{M}_0)) &= \begin{pmatrix} 4 & -2 & -32 & -32 & -4 \\ -2 & 1 & 0 & -32 & -14 \\ -32 & 0 & 64 & -64 & -64 \\ -32 & -32 & -64 & 64 & 0 \\ -4 & -14 & -64 & 0 & 4 \end{pmatrix}.
 \end{aligned}$$

Case $t = 36$.

The space RJ_{36} has the bases

$$\begin{aligned}
 \xi_{0,36}^{(1)} &= (-3r[-1] + 5 - 3r^{-1}[-1]) + (r^{12} + 3r^{11} + \dots)q + \\
 &\quad (r^{18}[-36] - 3r^{17}[-1] + 9r^{16} + \dots)q^2 + (6r^{20} - 3r^{19} + \dots)q^3 + \\
 &\quad (4r^{24} - 15r^{22} + \dots)q^4 + \\
 &\quad ((3r^{27}[-9] - 9r^{26} + 3r^{25} + \dots)q^5 + (3r^{29} + 6r^{28} + \dots)q^6 + \\
 &\quad (3r^{32}[-16] - 25r^{30} + 9r^{29} + \dots)q^7 + (-3r^{33} + 33r^{32} + \dots)q^8 + O(q^9);
 \end{aligned}$$

$$\begin{aligned}
\xi_{0,36}^{(2)} = & ((r^2[-4] - r[-1] + 1 - r[-1] + r^2[-4]) + (-r^{12} + r^{11} - r^{10} + \dots)q + \\
& (-r^{17}[-1] + r^{16} - r^{15} + \dots)q^2 + (-r^{19} + 2r^{18} - 3r^{17} + \dots)q^3 + \\
& (-r^{21} + 2r^{20} - 4r^{19} + \dots)q^4 + \\
& ((r^{27}[-9] - r^{26} + r^{25} + \dots)q^5 + (r^{29} - 2r^{28} + 3r^{27} + \dots)q^6 + \\
& (r^{32}[-16] - r^{30} + 3r^{29} + \dots)q^7 + \\
& (r^{34}[-4] - r^{33} + r^{32} - 3r^{30} + \dots)q^8 + O(q^9);
\end{aligned}$$

$$\begin{aligned}
\xi_{0,36}^{(3)} = & q^{-1}[-144] + 24 + (24r^{12} + 72r^{11} + \dots)q + \\
& 4r^{18}[-36] + 144r^{16} + 672r^{15} + \dots)q^2 + (144r^{20} + 1008r^{19} + \dots)q^3 + \\
& (24r^{24} + 288r^{23} + \dots)q^4 + \\
& (8r^{27}[-9] + 216r^{26} + 3096r^{25} + \dots)q^5 + \\
& (72r^{29} + 1584r^{28} + 15720r^{27} + \dots)q^6 + \\
& (9r^{32}[-16] + 288r^{31} + 5304r^{30} + \dots)q^7 + \\
& (672r^{33} + 12096r^{32} + \dots)q^8 + O(q^9).
\end{aligned}$$

We have

$$P(\mathcal{M}_0) = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \\ 2 & 18 & 1 \\ 5 & 27 & 1 \\ 7 & 32 & 1 \end{pmatrix}; \quad G(P(\mathcal{M}_0)) = \begin{pmatrix} 4 & -2 & -72 & -36 & -18 & -8 \\ -2 & 1 & 0 & -18 & -27 & -32 \\ -72 & 0 & 144 & -72 & -288 & -432 \\ -36 & -18 & -72 & 36 & -18 & -72 \\ -18 & -27 & -288 & -18 & 9 & 0 \\ -8 & -32 & -432 & -72 & 0 & 16 \end{pmatrix};$$

$$\bar{R} = \begin{bmatrix} 1, \bar{1} \\ 1, \bar{17} \\ 4, \bar{2} \\ 4, \bar{34} \\ 9, \bar{27} \\ 16, \bar{32} \\ 36, \bar{18} \\ 144, \bar{0} \end{bmatrix}; \quad \text{Mul}(\bar{R}, \xi) = \begin{pmatrix} -3 & 0 & 0 \\ -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 4 & 1 & 12 \\ 3 & 1 & 9 \\ 1 & 0 & 4 \\ 0 & 0 & 1 \end{pmatrix}.$$

6. THE LIST OF ALL LORENTZIAN KAC-MOODY ALGEBRAS WITH THE DENOMINATOR IDENTITY FUNCTION WHICH IS AUTOMORPHIC WITH RESPECT TO THE EXTENDED PARAMODULAR GROUP $\widehat{O}(L_t)$.

In Table 2 below we give the list of Lorentzian Kac-Moody algebras from Theorem 1.1. The product part of their denominator identities is defined by the infinite products of Theorem 2.1 for some Jacobi forms ξ from the spaces RJ_t which were described in Table 1 by their basis. These products are characterised by the property that multiplicities of divisors of the infinite product B_ξ are equal to 0 or 1 for any rational quadratic divisor which is orthogonal to a root of L_t . Since B_ξ is reflective, it is the whole divisor of B_ξ . We denote the corresponding Lorentzian Kac-Moody algebra as $\mathfrak{g}(\xi)$ since it is defined by the Jacobi form ξ .

We describe the fundamental polyhedron \mathcal{M} and the set $P(\mathcal{M})$ of orthogonal roots to \mathcal{M} defining the Weyl group and the set of simple real roots of the Algebra

$\mathfrak{g}(\xi)$. We also give the subset $P(\mathcal{M})_{\bar{1}} \subset P(\mathcal{M})$ of super roots. It means that the corresponding generators $e_\alpha, f_\alpha, \alpha \in P(\mathcal{M})_{\bar{1}}$, should be super. If we don't mention the set $P(\mathcal{M})_{\bar{1}}$, it is empty. We also give the generalised Cartan matrix

$$A = \left(\frac{2(\alpha_i, \alpha_j)}{\alpha_i^2} \right), \quad \alpha_i, \alpha_j \in P(\mathcal{M}),$$

which is the main invariant of the algebra. We also give the Weyl vector ρ .

All these polyhedra \mathcal{M} are composed from the fundamental polyhedron \mathcal{M}_0 for $W(S_t)$ using some group of symmetries of the polyhedron \mathcal{M} . We use these symmetries to describe the sets $P(\mathcal{M})$ and $P(\mathcal{M})_{\bar{1}}$ using the set $P(\mathcal{M}_0)$. We numerate as $\alpha_1, \dots, \alpha_k$ the elements of $P(\mathcal{M})$ as they are given in Table 1. We denote by s_α the reflection in the root α . It is given by the formula

$$s_\alpha : x \rightarrow x - \frac{2(x, \alpha)}{\alpha^2} \alpha, \quad x \in S_t^*.$$

We denote by $[g_1, \dots, g_k]$ the group generated by g_1, \dots, g_k

Table 2. The list of all 29 Lorentzian Kac–Moody algebras of the rank three with the root lattice S_t^* and the symmetry group $\widehat{O}^+(L_t)$ (from Theorem 1.1).

Case $t = 1$

The Algebra $\mathfrak{g}(\xi_{0,1}^{(1)})$. The fundamental chamber $\mathcal{M} = [s_{\alpha_1}, s_{\alpha_3}](\mathcal{M}_0)$ is the right triangle with zero angles. We have

$$P(\mathcal{M}) = [s_{\alpha_1}, s_{\alpha_3}](\alpha_2)$$

with the group of symmetries $[s_{\alpha_1}, s_{\alpha_3}]$ which is D_3 . The generalised Cartan matrix is

$$A_{1,II} = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}.$$

The Weyl vector $\rho = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. The corresponding automorphic form $B_{\xi_{0,1}^{(1)}}$ is Δ_5 of the weight 5 which is product of ten even theta-constants of genus 2. See [GN1], [GN2], [GN5].

The Algebra $\mathfrak{g}(\xi_{0,1}^{(2)})$. The chamber $\mathcal{M} = \mathcal{M}_0$ is a triangle with angles $\pi/3, 0, \pi/2$. The set $P(\mathcal{M}) = P(\mathcal{M}_0), P(\mathcal{M})_{\bar{1}} = \{\alpha_2\}$. The generalised Cartan matrix is

$$A_{1,I,\bar{0}} = \begin{pmatrix} 2 & -1 & -1 \\ -4 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

The Weyl vector $\rho = (\frac{5}{2}, \frac{1}{2}, \frac{3}{2})$. The corresponding automorphic form is Igusa's [Ig] modular form $\Delta_{30} = \Delta_{35}/\Delta_5$ of the weight 30. See [GN4], [GN5].

The Algebra $\mathfrak{g}(\xi_{0,1}^{(1)} + \xi_{0,1}^{(2)})$. The chamber $\mathcal{M} = \mathcal{M}_0$ and

$$P(\mathcal{M}) = \{\alpha_1, 2\alpha_2, \alpha_3\}.$$

The generalised Cartan matrix is

$$A_{1,0} = \begin{pmatrix} 2 & -2 & -1 \\ -2 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

The Weyl vector $\rho = (3, 1, 2)$. The automorphic form is Igusa's [Ig] modular form Δ_{35} of the weight 35. See [GN4] (for a new simple construction of this form) and [GN5].

Case $t = 2$.

The Algebra $\mathfrak{g}(\xi_{0,2}^{(1)})$. The chamber $\mathcal{M} = [s_{\alpha_1}, s_{\alpha_3}](\mathcal{M}_0)$ is the right quadrangle with zero angles;

$$P(\mathcal{M}) = [s_{\alpha_1}, s_{\alpha_3}](\alpha_2)$$

with the group of symmetries $[s_{\alpha_1}, s_{\alpha_3}]$ which is D_4 . The generalised Cartan matrix is

$$A_{2,II} = \begin{pmatrix} 2 & -2 & -6 & -2 \\ -2 & 2 & -2 & -6 \\ -6 & -2 & 2 & -2 \\ -2 & -6 & -2 & 2 \end{pmatrix}.$$

The Weyl vector $\rho = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$. The corresponding automorphic form is Δ_2 of the weight 2. See [GN1] and [GN5].

The Algebra $\mathfrak{g}(\xi_{0,2}^{(2)})$. The chamber $\mathcal{M} = [s_{\alpha_3}](\mathcal{M}_0)$ is a triangle with angles $0, 0, \pi/2$. The sets

$$P(\mathcal{M}) = \{\alpha_1, \alpha_2, s_{\alpha_3}(\alpha_1)\}, \quad P(\mathcal{M})_{\bar{1}} = \{\alpha_2\}$$

with the group of symmetries $[s_{\alpha_3}]$ which is D_1 . The generalised Cartan matrix is

$$A_{2,I,\bar{0}} = \begin{pmatrix} 2 & -1 & 0 \\ -4 & 2 & -4 \\ 0 & -1 & 2 \end{pmatrix}.$$

The Weyl vector $\rho = (\frac{3}{4}, \frac{1}{2}, \frac{3}{4})$. The corresponding automorphic form is $\Delta_9 = \Delta_{11}/\Delta_2$ of the weight 9. See [GN4], [GN5].

The Algebra $\mathfrak{g}(\xi_{0,2}^{(1)} + \xi_{0,2}^{(2)})$. The polygon $\mathcal{M} = [s_{\alpha_3}](\mathcal{M}_0)$ is a triangle with angles $0, 0, \pi/2$ (the same as for $\xi_{0,2}^{(2)}$); the set

$$P(\mathcal{M}) = \{\alpha_1, 2\alpha_2, s_{\alpha_3}(\alpha_1)\}$$

with the group of symmetries $[s_{\alpha_3}]$ which is D_1 . The generalised Cartan matrix is

$$A_{2,0} = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}.$$

The Weyl vector $\rho = (1, 1, 1)$. The corresponding automorphic form is Δ_{11} of the weight 11. See [GN4], [GN5].

The Algebra $\mathfrak{g}(\xi_{0,2}^{(3)})$. The chamber $\mathcal{M} = [s_{\alpha_1}, s_{\alpha_2}](\mathcal{M}_0)$ is an infinite polygon with angles $\pi/2$ and which is touching a horosphere with the centre at the Weyl vector $\rho = (1, 0, 0)$. The set

$$P(\mathcal{M}) = [s_{\alpha_1}, s_{\alpha_2}](\alpha_3)$$

with the group of symmetries $[s_{\alpha_1}, s_{\alpha_2}]$ which is D_∞ . The generalised Cartan matrix is a symmetric matrix

$$A_{2,\bar{1}} = \left(\frac{(\alpha, \alpha')}{4} \right), \quad \alpha, \alpha' \in P(\mathcal{M}).$$

The corresponding automorphic form is $\Psi_{12}^{(2)}$ of the weight 12. See [GN5].

The Algebra $\mathfrak{g}(\xi_{0,2}^{(1)} + \xi_{0,2}^{(3)})$. The polygon $\mathcal{M} = [s_{\alpha_1}](\mathcal{M}_0)$ is a quadrangle with angles $0, \pi/2, \pi/2, \pi/2$; the set

$$P(\mathcal{M}) = [s_{\alpha_1}]\{\alpha_2, \alpha_3\} = \{\alpha_2, \alpha_3, (1, 4, 1), (1, 1, 0)\}$$

with the group of symmetries $[s_{\alpha_1}]$ which is D_1 . The generalised Cartan matrix is

$$A_{2,II,\bar{1}} = \begin{pmatrix} 2 & 0 & -8 & -2 \\ 0 & 2 & 0 & -1 \\ -1 & 0 & 2 & 0 \\ -2 & -8 & 0 & 2 \end{pmatrix}.$$

The Weyl vector $\rho = (\frac{5}{4}, \frac{1}{2}, \frac{1}{4})$. The automorphic form is $\Delta_{14} = \Delta_2 \cdot \Psi_{12}^{(2)}$ of the weight 14. (We must correct the case $(2, II, \bar{1})$ in [GN5, page 264] in this way.)

The Algebra $\mathfrak{g}(\xi_{0,2}^{(2)} + \xi_{0,2}^{(3)})$. The polygon $\mathcal{M} = \mathcal{M}_0$ is the triangle with angles $0, \pi/2, \pi/4$; the set

$$P(\mathcal{M}) = P(\mathcal{M}_0), \quad P(\mathcal{M})_{\bar{1}} = \{\alpha_2\}$$

with the trivial group of symmetries and with the generalised Cartan matrix

$$A_{2,I,\bar{1}} = \begin{pmatrix} 2 & -1 & -2 \\ -4 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

The Weyl vector $\rho = (\frac{7}{4}, \frac{1}{2}, \frac{3}{4})$. The automorphic form is $\Delta_9 \cdot \Psi_{12}^{(2)}$ of the weight 21. See [GN5].

The Algebra $\mathfrak{g}(\xi_{0,2}^{(1)} + \xi_{0,2}^{(2)} + \xi_{0,2}^{(3)})$. The polygon $\mathcal{M} = \mathcal{M}_0$ is the triangle with angles $0, \pi/2, \pi/4$ (the same as for the $\mathfrak{g}(\xi_{0,2}^{(2)} + \xi_{0,2}^{(3)})$); the set

$$P(\mathcal{M}) = \{\alpha_1, 2\alpha_2, \alpha_3\}.$$

with the trivial group of symmetries and with the generalised Cartan matrix

$$A_{2,0,\bar{1}} = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

The Weyl vector $\rho = (2, 1, 1)$. The automorphic form is $\Delta_2 \cdot \Delta_9 \cdot \Psi_{12}^{(2)}$ of the weight 23. See [GN5].

Case $t = 3$.

The Algebra $\mathfrak{g}(\xi_{0,3}^{(1)})$. The chamber $\mathcal{M} = [s_{\alpha_1}, s_{\alpha_3}](\mathcal{M}_0)$ is the right hexagon with zero angles,

$$P(\mathcal{M}) = [s_{\alpha_1}, s_{\alpha_3}](\alpha_2)$$

with the group of symmetries $[s_{\alpha_1}, s_{\alpha_3}]$ which is D_6 . The generalised Cartan matrix is

$$A_{3,II} = \begin{pmatrix} 2 & -2 & -10 & -14 & -10 & -2 \\ -2 & 2 & -2 & -10 & -14 & -10 \\ -10 & -2 & 2 & -2 & -10 & -14 \\ -14 & -10 & -2 & 2 & -2 & -10 \\ -10 & -14 & -10 & -2 & 2 & -2 \\ -2 & -10 & -14 & -10 & -2 & 2 \end{pmatrix}.$$

The Weyl vector $\rho = (\frac{1}{6}, \frac{1}{2}, \frac{1}{6})$. The corresponding automorphic form is Δ_1 of the weight 1. See [GN4], [GN5].

The Algebra $\mathfrak{g}(\xi_{0,3}^{(2)})$. The chamber $\mathcal{M} = [s_{\alpha_3}](\mathcal{M}_0)$ is a triangle with angles $0, 0, \pi/3$. The sets

$$P(\mathcal{M}) = \{\alpha_1, \alpha_2, s_{\alpha_3}(\alpha_1)\}, \quad P(\mathcal{M})_{\bar{1}} = \{\alpha_2\}$$

with the group of symmetries $[s_{\alpha_3}]$ which is D_3 . The generalised Cartan matrix is

$$A_{3,I,\bar{0}} = \begin{pmatrix} 2 & -1 & -1 \\ -4 & 2 & -4 \\ -1 & -1 & 2 \end{pmatrix}.$$

The Weyl vector $\rho = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. The corresponding automorphic form is D_6 of the weight 6. See [GN4], [GN5].

The Algebra $\mathfrak{g}(\xi_{0,3}^{(1)} + \xi_{0,3}^{(2)})$. The polygon $\mathcal{M} = [s_{\alpha_3}](\mathcal{M}_0)$ is a triangle with angles $0, 0, \pi/3$ (the same as for $\xi_{0,3}^{(2)}$); the set

$$P(\mathcal{M}) = \{\alpha_1, 2\alpha_2, s_{\alpha_3}(\alpha_1)\}$$

with the group of symmetries $[s_{\alpha_3}]$ which is D_2 . The generalised Cartan matrix is

$$A_{3,0} = \begin{pmatrix} 2 & -2 & -1 \\ -2 & 2 & -2 \\ -1 & -2 & 2 \end{pmatrix}.$$

The Weyl vector $\rho = (\frac{2}{3}, 1, \frac{2}{3})$. The corresponding automorphic form is $\Delta_1 \cdot D_6$ of the weight 7. See [GN4], [GN5].

The Algebra $\mathfrak{g}(\xi_{0,3}^{(3)})$. The chamber $\mathcal{M} = [s_{\alpha_1}, s_{\alpha_2}](\mathcal{M}_0)$ is an infinite polygon with angles $\pi/3$ and which is touching a horosphere with the centre at the Weyl vector $\rho = (1, 0, 0)$. The set

$$P(\mathcal{M}) = [s_{\alpha_1}, s_{\alpha_2}](\alpha_3)$$

with the group of symmetries $[s_{\alpha_1}, s_{\alpha_2}]$ which is D_∞ . The generalised Cartan matrix is a symmetric matrix

$$A_{3, \bar{1}} = \left(\frac{(\alpha, \alpha')}{6} \right), \quad \alpha, \alpha' \in P(\mathcal{M}).$$

The corresponding automorphic form is $\Psi_{12}^{(3)}$ of the weight 12. See [GN5].

The Algebra $\mathfrak{g}(\xi_{0,3}^{(1)} + \xi_{0,3}^{(3)})$. The polygon $\mathcal{M} = [s_{\alpha_1}](\mathcal{M}_0)$ is a quadrangle with angles $0, \pi/2, \pi/3, \pi/2$; the set

$$P(\mathcal{M}) = [s_{\alpha_1}]\{\alpha_2, \alpha_3\} = \{\alpha_2, \alpha_3, (2, 6, 1), (1, 1, 0)\}$$

with the group of symmetries $[s_{\alpha_1}]$ which is D_1 . The generalised Cartan matrix is

$$A_{3, II, \bar{1}} = \begin{pmatrix} 2 & 0 & -12 & -2 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -2 & -12 & 0 & 2 \end{pmatrix}.$$

The Weyl vector $\rho = (\frac{7}{6}, \frac{1}{2}, \frac{1}{6})$. The automorphic form is $\Delta_1 \cdot \Psi_{12}^{(3)}$ of the weight 13. (We must correct the case $(3, II, \bar{1})$ in [GN5, page 264] in this way.)

The Algebra $\mathfrak{g}(\xi_{0,3}^{(2)} + \xi_{0,3}^{(3)})$. The polygon $\mathcal{M} = \mathcal{M}_0$ is the triangle with angles $0, \pi/2, \pi/6$; the set

$$P(\mathcal{M}) = P(\mathcal{M}_0), \quad P(\mathcal{M})_{\bar{1}} = \{\alpha_2\}$$

with the trivial group of symmetries and with the generalised Cartan matrix

$$A_{3, I, \bar{1}} = \begin{pmatrix} 2 & -1 & -3 \\ -4 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

The Weyl vector $\rho = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$. The automorphic form is $D_6 \cdot \Psi_{12}^{(3)}$ of the weight 18. See [GN5].

The Algebra $\mathfrak{g}(\xi_{0,3}^{(1)} + \xi_{0,3}^{(2)} + \xi_{0,3}^{(3)})$. The polygon $\mathcal{M} = \mathcal{M}_0$ is the triangle with angles $0, \pi/2, \pi/6$ (the same as for the $\mathfrak{g}(\xi_{0,3}^{(2)} + \xi_{0,3}^{(3)})$); the set

$$P(\mathcal{M}) = \{\alpha_1, 2\alpha_2, \alpha_3\}.$$

with the trivial group of symmetries and with the generalised Cartan matrix

$$A_{3, 0, \bar{1}} = \begin{pmatrix} 2 & -2 & -3 \\ -2 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

The Weyl vector $\rho = (\frac{5}{3}, 1, \frac{2}{3})$. The automorphic form is $\Delta_1 \cdot D_6 \cdot \Psi_{12}^{(3)}$ of the weight 19. See [GN5].

Case $t = 4$.

The Algebra $\mathfrak{g}(\xi_{0,4}^{(1)})$. The chamber $\mathcal{M} = [s_{\alpha_1}, s_{\alpha_3}](\mathcal{M}_0)$ is the infinite polygon with zero angles touching a horosphere with the centre at the Weyl vector $\rho = (\frac{1}{8}, \frac{1}{2}, \frac{1}{8})$;

$$P(\mathcal{M}) = [s_{\alpha_1}, s_{\alpha_3}](\alpha_2)$$

with the group of symmetries $[s_{\alpha_1}, s_{\alpha_3}]$ which is D_∞ . The generalised Cartan matrix is

$$A_{4,II,\bar{0}} = (2(\alpha, \alpha')), \quad \alpha, \alpha' \in P(\mathcal{M}).$$

The corresponding automorphic form is $\Delta_{1/2}$ of the weight $1/2$ which is the theta-constant of the genus 2. See [GN5].

The Algebra $\mathfrak{g}(\xi_{0,4}^{(2)})$. The chamber $\mathcal{M} = [s_{\alpha_3}](\mathcal{M}_0)$ is a triangle with angles $0, 0, 0$. The sets

$$P(\mathcal{M}) = \{\alpha_1, \alpha_2, s_{\alpha_3}(\alpha_1)\}; \quad P(\mathcal{M})_{\bar{1}} = \{\alpha_2\}$$

with the group of symmetries $[s_{\alpha_3}]$ which is D_1 . The generalised Cartan matrix is

$$A_{4,I,\bar{0}} = \begin{pmatrix} 2 & -1 & -2 \\ -4 & 2 & -4 \\ -2 & -1 & 2 \end{pmatrix}$$

The Weyl vector $\rho = (\frac{3}{8}, \frac{1}{2}, \frac{3}{8})$. The corresponding automorphic form is $\Delta_5^{(4)}/\Delta_{1/2}$ of the weight $\frac{9}{2}$. See [GN5].

The Algebra $\mathfrak{g}(\xi_{0,4}^{(1)} + \xi_{0,4}^{(2)})$. The polygon $\mathcal{M} = [s_{\alpha_3}](\mathcal{M}_0)$ is a triangle with angles $0, 0, 0$ (the same as for $\xi_{0,1}^{(1)}$ and $\xi_{0,4}^{(2)}$); the set

$$P(\mathcal{M}) = \{\alpha_1, 2\alpha_2, s_{\alpha_3}(\alpha_1)\}$$

with the group of symmetries $[s_{\alpha_3}]$ which is D_1 . The generalised Cartan matrix is

$$A_{4,0,\bar{0}} = A_{1,II} = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}$$

(it is the same as for $\xi_{0,1}^{(1)}$). The Weyl vector $\rho = (\frac{1}{2}, 1, \frac{1}{2})$. The corresponding automorphic form is $\Delta_5^{(4)}(z_1, z_2, z_3) = \Delta_5(z_1, 2z_2, z_3)$ of the weight 5. This case is equivalent to the case $\mathfrak{g}(\xi_{0,1}^{(1)})$ above. See [GN5].

The Algebra $\mathfrak{g}(\xi_{0,4}^{(3)})$. The chamber $\mathcal{M} = [s_{\alpha_1}, s_{\alpha_2}](\mathcal{M}_0)$ is an infinite polygon with zero angles which is touching a horosphere with the centre at the Weyl vector $\rho = (1, 0, 0)$. The set

$$P(\mathcal{M}) = [s_{\alpha_1}, s_{\alpha_2}](\alpha_3)$$

with the group of symmetries $[s_{\alpha_1}, s_{\alpha_2}]$ which is D_∞ . The generalised Cartan matrix is a symmetric matrix

$$A_{4, \bar{1}} = \left(\frac{(\alpha, \alpha')}{8} \right), \quad \alpha, \alpha' \in P(\mathcal{M}).$$

The corresponding automorphic form is $\Psi_{12}^{(4)}$ of the weight 12. See [GN5].

The Algebra $\mathfrak{g}(\xi_{0,4}^{(1)} + \xi_{0,4}^{(3)})$. The polygon $\mathcal{M} = [s_{\alpha_1}](\mathcal{M}_0)$ is a quadrangle with angles $0, \pi/2, 0, \pi/2$; the set

$$P(\mathcal{M}) = [s_{\alpha_1}]\{\alpha_2, \alpha_3\} = \{\alpha_2, \alpha_3, (3, 8, 1), (1, 1, 0)\}$$

with the group of symmetries $[s_{\alpha_1}]$ which is D_1 . The generalised Cartan matrix is

$$A_{4, II, \bar{1}} = \begin{pmatrix} 2 & 0 & -16 & -2 \\ 0 & 2 & -2 & -1 \\ -1 & -2 & 2 & 0 \\ -2 & -16 & 0 & 2 \end{pmatrix}.$$

The Weyl vector is $(\frac{9}{8}, \frac{1}{2}, \frac{1}{8})$. The automorphic form is $\Delta_{1/2} \cdot \Psi_{12}^{(4)}$ of the weight $\frac{25}{2}$. (We must correct the case $(2, II, \bar{1})$ in [GN5, page 264] in this way.)

The Algebra $\mathfrak{g}(\xi_{0,4}^{(2)} + \xi_{0,4}^{(3)})$. The polygon $\mathcal{M} = \mathcal{M}_0$ is the triangle with angles $0, \pi/2, 0$; the set

$$P(\mathcal{M}) = P(\mathcal{M}_0), \quad P(\mathcal{M})_{\bar{1}} = \{\alpha_2\}$$

with the trivial group of symmetries and with the generalised Cartan matrix

$$A_{4, I, \bar{1}} = \begin{pmatrix} 2 & -1 & -4 \\ -4 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

The Weyl vector $\rho = (\frac{11}{8}, \frac{1}{2}, \frac{3}{8})$. The automorphic form is $\Psi_{12}^{(4)} \cdot \Delta_5^{(4)} / \Delta_{1/2}$ of the weight $\frac{33}{2}$. See [GN5].

The Algebra $\mathfrak{g}(\xi_{0,4}^{(1)} + \xi_{0,4}^{(2)} + \xi_{0,4}^{(3)})$. The polygon $\mathcal{M} = \mathcal{M}_0$ is the triangle with angles $0, \pi/2, \pi/4$ (the same as for the $\mathfrak{g}(\xi_{0,4}^{(2)} + \xi_{0,4}^{(3)})$); the set

$$P(\mathcal{M}) = \{\alpha_1, 2\alpha_2, \alpha_3\}.$$

with the trivial group of symmetries and with the generalised Cartan matrix

$$A_{4, 0, \bar{1}} = \begin{pmatrix} 2 & -2 & -4 \\ -2 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

The Weyl vector $\rho = (\frac{3}{2}, 1, \frac{1}{2})$. The automorphic form is $\Delta_5^{(4)} \cdot \Psi_{12}^{(4)}$ of the weight 17. See [GN5].

The Algebra $\mathfrak{g}(\xi_{0,8}^{(2)})$. The chamber $\mathcal{M} = [s_{\alpha_3}](\mathcal{M}_0)$ is the right quadrangle with zero angles; the set

$$P(\mathcal{M}) = [s_{\alpha_3}](\alpha_1, 2\alpha_2, \alpha_4) = \{\alpha_1, 2\alpha_2, s_{\alpha_3}(\alpha_1), \alpha_4\}$$

with the group of symmetries $[s_{\alpha_3}]$ which is D_1 . The generalised Cartan matrix is $A_{2,II}$ (the same as for $\mathfrak{g}(\xi_{0,2}^{(1)})$ for $t = 2$). The Weyl vector $\rho = (\frac{1}{4}, 1, \frac{1}{4})$. The automorphic form is $\Delta_2^{(8)}(z_1, z_2, z_3) = \Delta_2(z_1, 2z_2, z_3)$ of the weight 2 where Δ_2 corresponds to $\mathfrak{g}(\xi_{0,2}^{(1)})$. This case is equivalent to $\mathfrak{g}(\xi_{0,2}^{(1)})$.

Case $t = 9$.

The algebra $\mathfrak{g}(\xi_{0,9}^{(2)})$. The chamber $\mathcal{M} = [s_{\alpha_3}](\mathcal{M}_0)$ is the pentagon with angles $0, 0, \pi/2, 0, \pi/2$; the set

$$P(\mathcal{M}) = [s_{\alpha_3}](\alpha_1, \alpha_2, \alpha_4) = \{\alpha_1, \alpha_2, s_{\alpha_3}(\alpha_1), s_{\alpha_3}(\alpha_4), \alpha_4\}, \quad P(\mathcal{M})_{\bar{1}} = \{\alpha_2\},$$

with the group of symmetries $[s_{\alpha_2}]$ which is D_1 . The generalised Cartan matrix is

$$\begin{pmatrix} 2 & -1 & -7 & -9 & 0 \\ -4 & 2 & -4 & -18 & -18 \\ -7 & -1 & 2 & 0 & -9 \\ -4 & -2 & 0 & 2 & -2 \\ 0 & -2 & -4 & -2 & 2 \end{pmatrix}.$$

The Weyl vector is $(\frac{1}{6}, \frac{1}{2}, \frac{1}{6})$. The automorphic form is D_2 of the weight 2. See [GN5].

Case $t = 12$.

The Algebra $\mathfrak{g}(\xi_{0,12}^{(2)} + \xi_{0,12}^{(3)})$. The chamber $\mathcal{M} = [s_{\alpha_3}, s_{\alpha_4}](\mathcal{M}_0)$ is the right hexagon with zero angles; the set

$$P(\mathcal{M}) = [s_{\alpha_3}, s_{\alpha_4}](\alpha_1, 2\alpha_2) = \{\alpha_1, 2\alpha_2, s_{\alpha_3}(\alpha_1), s_{\alpha_4}s_{\alpha_3}(\alpha_1), s_{\alpha_4}(2\alpha_2), s_{\alpha_4}(\alpha_1)\}$$

with the group of symmetries $[s_{\alpha_3}, s_{\alpha_4}]$ which is D_4 . The generalised Cartan matrix is $A_{3,II}$ (the same as for $\mathfrak{g}(\xi_{0,3}^{(1)})$). The Weyl vector is $(\frac{1}{6}, 1, \frac{1}{6})$. The automorphic form is $\Delta_1^{(12)}(z_1, z_2, z_3) = \Delta_1(z_1, 2z_2, z_3)$ of the weight 1 where Δ_1 corresponds to $\mathfrak{g}(\xi_{0,3}^{(1)})$. This case is equivalent to $\mathfrak{g}(\xi_{0,3}^{(1)})$ above.

Case $t = 16$.

The Algebra $\mathfrak{g}(\xi_{0,16}^{(1)})$. The chamber is the infinite polygon $\mathcal{M} = [s_{\alpha_3}, s_{\alpha_4}](\mathcal{M}_0)$ with zero angles touching a horosphere with the centre at the Weyl vector $\rho = (\frac{1}{8}, 1, \frac{1}{8})$. The set

$$P(\mathcal{M}) = [s_{\alpha_3}, s_{\alpha_4}](\alpha_1, 2\alpha_2, \alpha_5)$$

with the group of symmetry $[s_{\alpha_3}, s_{\alpha_4}]$ which is D_∞ . The generalised Cartan matrix is

$$\left(\frac{(\alpha, \alpha')}{2} \right), \quad \alpha, \alpha' \in P(\mathcal{M}),$$

which is the same as for $\mathfrak{g}(\xi_{0,4}^{(1)})$. The automorphic form is $\Delta_{1/2}^{(16)}(z_1, z_2, z_3) = \Delta_{1/2}(z_1, 2z_2, z_3)$ of the weight $1/2$ where $\Delta_{1/2}$ corresponds to $\mathfrak{g}(\xi_{0,4}^{(1)})$. This case is equivalent to $\mathfrak{g}(\xi_{0,4}^{(1)})$.

Case $t = 36$.

The Algebra $\mathfrak{g}(\xi_{0,36}^{(2)})$. The chamber $\mathcal{M} = [s_{\alpha_3}, s_{\alpha_4}](\mathcal{M}_0)$ is the infinite periodic polygon with angles $\dots, 0, \pi/2, 0, 0, 0, 0, \pi/2, 0, \dots$, with the centre at the Weyl vector $\rho = (\frac{1}{24}, \frac{1}{2}, \frac{1}{24})$ at infinity. The set

$$P(\mathcal{M}) = [s_{\alpha_3}, s_{\alpha_4}](\alpha_1, \alpha_2, \alpha_5, \alpha_6), \quad P(\mathcal{M})_{\bar{1}} = [s_{\alpha_3}, s_{\alpha_4}](\alpha_2)$$

with the group of symmetry $[s_{\alpha_3}, s_{\alpha_4}]$ which is D_∞ . The generalised Cartan matrix is

$$\left(\frac{2(\alpha, \alpha')}{(\alpha, \alpha)} \right), \quad \alpha, \alpha' \in P(\mathcal{M}).$$

The automorphic form is $D_{1/2}$ of the weight $1/2$. See [GN5].

7. POSSIBLE PHYSICAL APPLICATIONS

It would be interesting to find some Quantum Systems with symmetries corresponding to the Lorentzian Kac-Moody algebras of Theorem 1.1 and to possible algebras with denominator identities which are reflective automorphic forms of Main Theorem 2.2. See [B3], [DMVV], [DVV], [G2], [G3], [HM], [GN3], [GN6], [Ka], [KaY], [M] about some attempts in this direction.

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