UPPER BOUNDS FOR COMPLEXITY OF SOME 3-DIMENSIONAL MANIFOLDS

S. Anisov

Abstract. We construct pseudominimal spines of T^2-fibered over S^1 spaces $M(\mathcal{A})$ with monodromy $\mathcal{A} \in \mathrm{SL}(2, \mathbb{Z})$ that have $c(\mathcal{A}) + 5$ vertices, which seems to be the smallest possible number; this is equivalent to triangulating $M(\mathcal{A})$ into $c(\mathcal{A}) + 5$ tetrahedra. The function $c(\mathcal{A})$ is an integer-valued lift to $\mathrm{SL}(2, \mathbb{Z})$ of the (unique) epimorphism of the modular group to $\mathbb{Z}_2$. Algebraic properties of the complexity $c(\mathcal{A})$ are discussed and an algorithm for its calculating is presented. As a byproduct, we construct pseudominimal special spines of lens spaces, which have small number of vertices.

§1. Introduction

The notion of *complexity* of three-dimensional manifolds was introduced by S. Matveev in 1990, see [7]. This complexity is a natural "filtration" on the set of compact 3-manifolds. It is additive with respect to taking the connected sum of manifolds, and for any $k \in \mathbb{Z}$ there are only finitely many compact prime 3-manifolds of complexity at most k; they can be enumerated by a simple algorithm (however, most of them appear many times in the list obtained, for example, in the list in [9, §5.2]). Note that for any compact prime 3-manifold M different from S^3, $\mathbb{R}P^3$, $L_{3,1}$, and $S^2 \times S^1$ (the last one contains a nontrivial but non-splitting sphere), the complexity $c(M)$ is nothing but the minimal possible number of tetrahedra in a singular triangulation of M. On the other hand, these four manifolds are the only closed prime manifolds of complexity 0.

The problem of evaluating the complexity of 3-manifolds is unresolved and appears to be very difficult. The only manifolds of known complexity are those with complexity less than or equal to 7. Their lists in [9] and [11] are obtained by enumeration of all special spines of closed orientable 3-manifolds of small complexity (by the algorithm mentioned above), followed by determining which of the spines obtained are equivalent (that is, are spines of the same manifold). This "equivalence problem" is difficult.

Obviously, any almost simple spine (or singular triangulation) of a manifold M provides an upper bound for $c(M)$. There is an algorithm for simplification of a given spine, see [8]; for all manifolds from [9] and [11], this algorithm is efficient, that is, stops at a minimal spine of a manifold. There is no proof of efficiency of this algorithm in the general case, although one can, of course, use it to find quite reasonable upper bounds for $c(M)$.

This research was supported by FCT-Portugal through the Research Units Pluriannual Funding Program, and by EPSRC, Grant GR K99015, via Isaac Newton Institute, Cambridge.

Much less is known about lower bounds. Clearly, $c(M) > 7$ whenever M is not homeomorphic to any manifold from tables in [9, 11]. Also, one can easily show that $c(M) \geq b_1(M, G) - 1$ for any commutative group G; here b_1 is the first Betti number. However, in most cases these estimates are very inadequate. Up to now, the only known way to prove that $c(M) = k$ for some $k > 0$, where M is a closed prime three-manifold, is to construct a special spine of M with k vertices (or a singular triangulation of M with k tetrahedra) and verify that M is not homeomorphic to any manifold of lower complexity.

Here we study 3-manifolds that can be fibered over the circle with torus fiber, and lens spaces. We present "reasonable" special spines of these manifolds, thence giving upper bounds for their complexity. The conjecture that arised about the complexity of the total spaces of torus fiberings is very similar to S. Matveev's conjecture about the complexity of the lens spaces. Further discussion of these conjectures will appear soon.

The paper is organized as follows: in §2, we give necessary definitions, following mainly [9]. In §3 we deal with a purely algebraic statement arised from a conjecture about complexity of the lens spaces. To construct "good" spines of the T^2-fibered spaces in §6, we need some preliminary results on θ-curves in the torus discussed in §4 and some auxiliary algebraic results presented in §5. As a byproduct, the techniques developed in §§4–6 allows us to construct in §7 pseudominimal spines of lens spaces in a clearer way than it was done in [8, 9].

Acknowledgements. The idea to relate θ-curves and their flips (studied in [2]) to spines of fibered spaces and their complexity is due to V. Turaev. R. Fernandes suggested a simplification of the original proof of Theorem 3. The author would also like to thank Yu. Baryshnikov, Yu. Burman, V. Nikulin, M. Polyak, V. Vassiliev, and especially S. Matveev for many useful discussions. This paper was started at Instituto Superior Técnico (Lisbon, Portugal) and finished at Isaac Newton Institute (Cambridge, U.K.). The author thanks both Institutes for their kind hospitality, and the former Institute for the support of S. Matveev's visit, which was very important for this work.

§2. Definitions

In this section we follow the papers [7, 9]. By K denote the 1-dimensional skeleton of the tetrahedron, which is nothing but the clique (that is, the total graph) with 4 vertices. Note that K is homeomorphic to a circle with three radii.

Definition 1. A compact 2-dimensional polyhedron is called *almost simple* if the link of each of its points can be embedded in K. An almost simple polyhedron P is said to be *simple* if the link of each point of P is homeomorphic to either a circle or a circle with a diameter or the whole graph K. A point of an almost simple polyhedron is *non-singular* if its link is homeomorphic to a circle, it is said to be a *triple point* if its link is homeomorphic to a circle with a diameter, and it is called a *vertex* if its link is homeomorphic to K. The set of singular points of a simple polyhedron P (i.e., the union of the vertices and the triple lines) is called its *singular graph* and is denoted by SP.

It is easy to see that any compact subpolyhedron of an almost simple polyhedron is almost simple as well. Neighborhoods of non-singular and triple points of a simple polyhedron are shown in Fig. 1 a, b; Fig. 1 c–f represent four equivalent ways of looking at vertices.

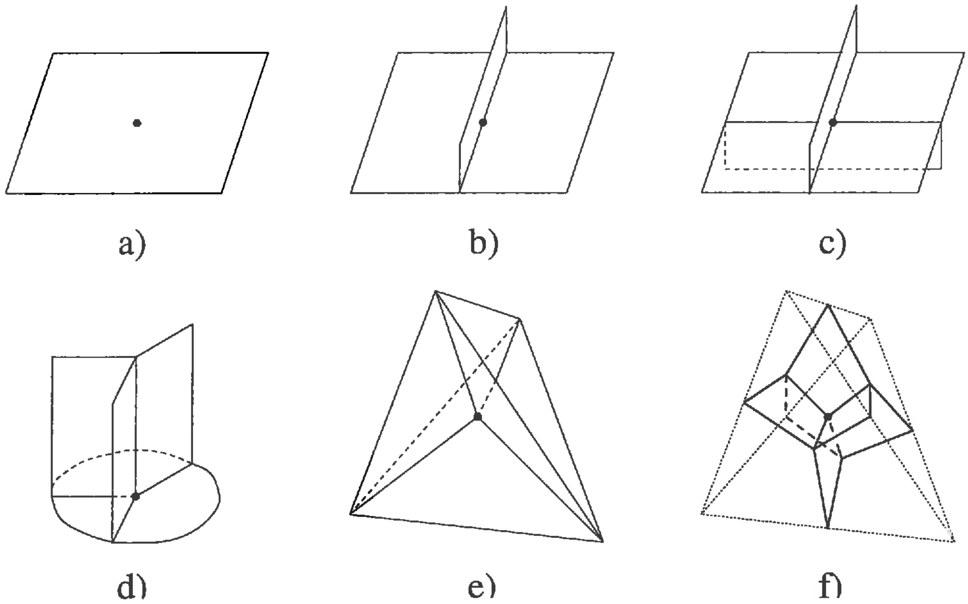

FIGURE 1. Nonsingular (a) and triple (b) points; ways of looking at vertices (c–f)

Definition 2. A simple polyhedron P with at least one vertex is said to be *special* if it contains no closed triple lines (wihtout vertices) and every connected component of $P \setminus SP$ is a 2-dimensional cell.

Definition 3. A polyhedron $P \subset \operatorname{Int} M$ is called a *spine* of a compact 3-dimensional manifold M if $M \setminus P$ is homeomorphic to $\partial M \times (0, 1]$ if $\partial M \neq 0$ or to an open 3-cell if $\partial M = 0$. In the other words, P is a spine of M if a manifold M with boundary (or punctured at one point closed manifold M) can be collapsed onto P. A spine P of a 3-manifold M is said to be *almost simple*, *simple*, or *special* if it is an almost simple, simple, or special polyhedron, respectively.

Definition 4. The *complexity* $c(M)$ of a complact 3-manifold M is the minimal possible number of vertices of an almost simple spine of M. An almost simple spine with the minimal possible number of vertices is said to be a *minimal* spine.

Theorem 1 [3]. *Any compact 3-manifold has a special spine.*

Theorem 2 [7]. *Let M be a compact orientable prime 3-manifold with incompressible (or empty) boundary and without essential annuli. If $c(M) > 0$ (that is, if M is different from (possibly punctured) S^3, $\mathbb{R}P^3$, $L_{3,1}$, and $S^2 \times S^1$), then any minimal almost simple spine of M is special.*

Recall that a 3-manifold M is called prime if it cannot be represented as a connected sum $M = M_1 \# M_2$ with M_1, M_2 both different from S^3.

Remark 1. In this paper, we consider lens spaces $L_{p,q}$, $q > 3$, and the total spaces of torus bundles over the circle. All these manifolds satisfy the assumptions of Theorem 2.

Remark 2. Starting from a special spine P of a manifold M, one can triangulate M into n tetrahedra, where n is the number of vertices of P. This singular triangulation has the only vertex somewhere inside $M \setminus P$, its edges are dual to the 2-cells of P, and triangles are dual to the edges of P. On the other hand, given a singular triangulation of M containing n tetrahedra, one can easily obtain a special spine of the manifold M punctured at all vertices of the triangulation. It was shown in [7] that puncturing does not affect the complexity. Thus for a manifold M satisfying assumptions of Theorem 2 (in particular, for any prime manifold without boundary), its complexity $c(M)$ is equal to the minimal possible number of tetrahedra in a singular triangulation of M, provided that $c(M) > 0$.

Remark 3. Let a special spine P of a manifold M without boundary have n vertices. Since each vertex of the graph SP has degree 4, P contains $2n$ edges. Since the Euler characteristic of any 3-manifold equals zero and $M \setminus P$ is a 3-cell, we have the equality $n - 2n + f - 1 = 0$, which implies $f = n + 1$, where f stands for the number of 2-dimensional "faces" of P. It follows from the construction of Remark 2 that the groups $\pi_1(M)$ and $H_1(M)$ have at most f generators. Therefore, $f - 1 = c(M) \geq b_1(M) - 1$.

§3. Example: lens spaces

Definition 5. Let p, q be coprime positive integers. The *Euclid complexity* $E(p,q)$ is the number of subtractions (not divisions!) that the Euclid algorithm takes to convert the pair (p,q) into the pair $(0,1)$. It is easy to see that $E(p,q)$ equals the sum of the denominators of the continued fraction representing any of the rational numbers p/q and q/p.

A good exposition of the Euclid algorithm and continued fraction theory can be found in [5, 17].

Conjecture 1 [7, 9]. *The complexity of the lens space $L_{p,q}$ is equal to $c(L_{p,q}) = E(p,q) - 3$.*

Pseudominimal special spines of the spaces $L_{p,q}$ with $E(p,q) - 3$ vertices were constructed in [8, 9]. Pseudominimality of a spine means that no simplification move can be applied to it; for exact definitions, see [9]. In §7 we present another construction of these spines. Note that the manifolds $L_{p,q}$ and $L_{p,p-q}$ are homeomorphic, and so are the manifolds $L_{p,q}$ and $L_{p,r}$, where $0 < q, r < p$ and $qr \equiv 1 \mod p$. So Conjecture 1 implies that $E(p,q) = E(p,p-q)$ and $E(p,q) = E(p,r)$ for p, q, r as above; if these corollaries did not hold, Conjecture 1 would fail automatically. However, they are true. Indeed, $E(p,q) = E(q,p-q)+1$ and $E(p,p-q) = E(p-q,q)+1$, which implies $E(p,q) = E(p,p-q)$. The second corollary is a true statement, too, but this is far less obvious.

Theorem 3. *Let $0 < q, r < p$ and $qr \equiv \pm 1 \mod p$. Then $E(p,q) = E(p,r)$.*

Proof. We can suppose that $p \geq 3$. Let us introduce two *row transformation* matrices

$$R_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Obviously, we have

$$R_1^{\pm 1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \pm c & b \pm d \\ c & d \end{pmatrix} \quad \text{and} \quad R_2^{\pm 1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ a \pm c & b \pm d \end{pmatrix}.$$

Consider the expansion of p/q in a continued fraction

$$\frac{p}{q} = n_1 + \cfrac{1}{n_2 + \cfrac{1}{\ddots + \cfrac{1}{n_k}}}.$$

Set $U = R_2^{-1} R_\varepsilon^{-n_k+1} \dots R_1^{-n_3} R_2^{-n_2} R_1^{-n_1}$, where $\varepsilon = 1$ for k odd and $\varepsilon = 2$ for k even; note that $n_1 \geq 1$ and $n_k \geq 2$. It is easy to see that U takes vector $(p,q)^{\mathrm{T}}$ to $(1,0)^{\mathrm{T}}$ (where T stands for transposing): $R_1^{-n_1}$ takes $(p,q)^{\mathrm{T}}$ to $(p - n_1 q, q)^{\mathrm{T}}$ and so forth, according to the Euclid algorithm, the only exception being that at the last step we apply R_2^{-1} to $(1,1)^{\mathrm{T}}$, not R_1^{-1}, in order to convert $(1,1)^{\mathrm{T}}$ to $(1,0)^{\mathrm{T}}$, not to $(0,1)^{\mathrm{T}}$.

First suppose that $qr \equiv -1 \mod p$, thus $qr = sp - 1$ for some positive integer s. Since $1 \leq q < p$ and $1 \leq r < p$, we have $s \leq q$ and $s \leq r$. Let us consider the inverse matrix

$$U^{-1} = R_1^{n_1} R_2^{n_2} R_1^{n_3} \dots R_\varepsilon^{n_k - 1} R_2. \tag{1}$$

We proclaim that

$$U^{-1} = \begin{pmatrix} p & r \\ q & s \end{pmatrix}.$$

Indeed, equation (1) implies that U^{-1} has the following properties:

1) the first column of U^{-1} is $(p,q)^{\mathrm{T}}$;
2) the determinant of U^{-1} equals 1;
3) the second column entries of U^{-1} are positive, and
4) they do not exceed the corresponding first column entries.

Property 1 follows from the construction of the sequence involved in (1), and property 2 is obvious. It is clear that both second column entries of U^{-1} are nonnegative; they are both positive, because the right hand side of (1) contains both R_1 and R_2. The last property holds for R_2 and survives under left multiplications by R_1 and R_2. The first two properties imply that the second column of U^{-1} is $(r + mp, s + mq)^{\mathrm{T}}$ for some $s \in \mathbb{Z}$, and the last two properties show that in fact $m = 0$.

Note that $E(p,q) = n_1 + n_2 + \ldots + n_k$ equals the sum of the exponents in (1). Since $R_1^{\mathrm{T}} = R_2$ and $R_2^{\mathrm{T}} = R_1$, for the transposed inverse matrix $(U^{-1})^{\mathrm{T}}$ we have

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} = (U^{-1})^{\mathrm{T}} = R_1 R_{3-\varepsilon}^{n_k - 1} \dots R_2^{n_3} R_1^{n_2} R_2^{n_1}, \tag{2}$$

where $n_k \geq 2$ and $n_1 \geq 1$. This yields the sequence of $n_1 + n_2 + \ldots + n_k = E(p,q)$ subtractions that converts the pair (p,r) into the pair $(1,0)$. Such a sequence is unique up to a possible application of several subtractions R_1 to the column $(1,0)^{\mathrm{T}}$ (clearly, R_1 does not change it); it is nothing but the Euclid algorithm provided that there are no those "fake" subtractions. This condition is satisfied, because the last subtraction (the inversed rightmost one in (2), or the leftmost one in a similar expression for U^{T}) is R_2, not R_1. Therefore $E(p,q) = E(p,r)$ for $qr \equiv -1 \mod p$. In the case $qr \equiv 1 \mod p$, note that $q(p-r) \equiv -1 \mod p$ and $0 < p - r < p$. Consequently, $E(p,q) = E(p, p-r) = E(p,r)$. $\square$

Theorem 3 means that Conjecture 1 passes a nontrivial "sanity test".

§4. θ-CURVES

Let us recall some definitions and results from [2][1].

Definition 6. A *θ-curve* $L \subset T^2$ is a graph with two vertices and three edges (not loops) connecting these vertices, embedded in T^2 in such a way that the edges are pairwise non-homotopic; this is equivalent to the condition that the complement $T^2 \setminus L$ is a 2-dimensional cell.

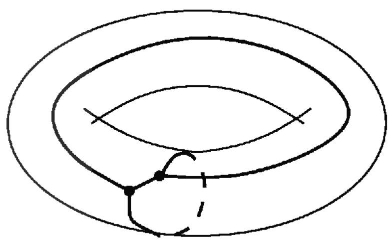

FIGURE 2. A θ-curve

Up to an isotopy, any two θ-curves can be taken to one another by a linear automorphism of the torus, see [2]. Another way to change the isotopy class of a θ-curve is to apply a sequence of flips.

Definition 7. A *flip* along an edge of a trivalent graph (in particular, of a θ-curve) is an invertible restructuring of the graph that acts on a neighborhood of this edge as shown on Fig. 3. A flip does not change the graph outside of this neighborhood.

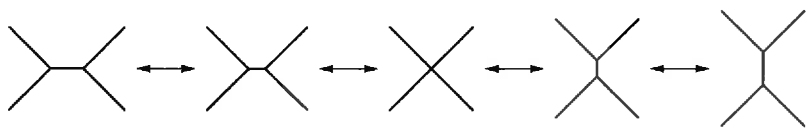

FIGURE 3. A flip

For any two θ-curves L_1, L_2, there exists a sequence of flips (and isotopies) that takes L_1 to L_2, see [1, 2]. Now let us recall how one can find the minimal number of flips required for such a sequence.

For a θ-curve L, there are three unoriented (or six oriented) cycles in $\pi_1(T^2)$ formed by pairs of edges of L. These cycles also can be represented by the three 1-cells of the singular triangulation of T^2 dual to the cell decomposition defined by L. The six points in the lattice $\mathbb{Z}^2 = \pi_1(T^2) = H_1(T^2, \mathbb{Z})$ corresponding to these cycles are the vertices of some convex centrally symmetric hexagon $W(L)$.

Definition 8. The hexagon $W(L)$ is said to be *associated* to a θ-curve L. A hexagon W with the vertices in $\mathbb{Z}^2$ is called *admissible* if it is associated to some θ-curve. The *standard hexagon* is the hexagon W_0 with the vertices $\pm(1,0)$, $\pm(0,1)$, and $\pm(1,-1)$, see Fig. 4. It is associated to the θ-curve shown on Fig. 2 (under a natural choice of a parallel and a meridian of T^2 as a basis of $H_1(T^2) = Z^2$).

[1] We do not follow the notation of [2] here.

Theorem 4 [2].

1) *A hexagon W (with vertices at lattice points), centrally symmetric with respect to the origin O, is admissible if and only if it has the following properties:*

a) *if X and Y are nonopposite vertices of W, then the area of the triangle OXY equals $1/2$;*

b) *if X, Y, and Z are three consecutive vertices of W, then $\overrightarrow{OY} = \overrightarrow{OX} + \overrightarrow{OZ}$.*

Properties a) and b) are equivalent. The origin is the only interior lattice point of an admissible hexagon W. The vertices of W are the only lattice points on its boundary.

2) *Two θ-curves L and L' are isotopic if and only if $W(L) = W(L')$. For any two θ-curves L and L' there exists an operator $A \in \mathrm{SL}(2,\mathbb{Z})$ such that AL is isotopic to L' and $A(W(L)) = W(L')$.*

3) *Let θ-curves L and L' differ by the flip along an edge e. Then associated hexagons W and W' have in common two pairs of opposite vertices that correspond to cycles σ and μ dual to two other edges. The remaining pair of the vertices is $\pm(\sigma + \mu)$ for one of the hexagons W, W' and $\pm(\sigma - \mu)$ for the other one.*

Since a θ-curve has three edges, three different flips can be applied. Fig. 4 shows how they change the standard hexagon W_0. The result of a flip transformation of an arbitrary admissible hexagon can be represented by the same picture with another coordinate system, because any admissible hexagon is $\mathrm{SL}(2,\mathbb{Z})$-equivalent to W_0.

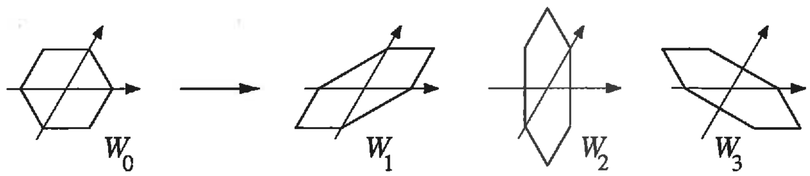

FIGURE 4. Action of three flips on the standard hexagon

According to the second part of Theorem 4, we can study sequences of flips that convert an admissible hexagon W_1 into another admissible hexagon W_2 rather than sequences of flips that take a θ-curve L_1 into another θ-curve L_2. So it is natural to introduce a graph Γ that has the admissible hexagons as its vertices and flips as its edges; the number of flips required to convert W_1 into W_2 equals the distance between the corresponding vertices of Γ. Clearly, Γ is a trivalent graph. It turns out that Γ is a tree, see [2]. More information about Γ can be found at [15, Ch. II, §1].

Definition 9. A *leading vertex* of an admissible hexagon W is its vertex that is the most distant from the origin with respect to the quadratic form $Q(x,y) = x^2 + xy + y^2$.

The standard hexagon W_0 is a unit regular hexagon with respect to $Q(x,y)$. Any other admissible hexagon has only one pair of opposite leading vertices.

Theorem 5 [2]. *Let (p,q) be a leading vertex of an admissible hexagon $W \neq W_0$. Suppose that $p > 0$ and $q > 0$. Then $d(W,W_0) = E(p,q)$, where $d(W,W_0)$ stands for the distance between W and W_0 in Γ and $E(p,q)$ is the Euclid complexity, see Definition 5 above.*

In fact, the steps (flips) of the only way from (the vertex of Γ corresponding to) W_0 to (the vertex corresponding to) W in Γ are in a natural one-to-one correspondence with the steps (subtractions) of the Euclid algorithm applied to the pair (p, q). There is an algorithm that constructs the path from W to W_0: start at W and apply the flip that decreases the length of a hexagon; such a flip is unique unless the hexagon is W_0. The numbers p, q are coprime by virtue of the first part of Theorem 4, and for any pair of coprime numbers (p, q), there exists exactly one admissible hexagon with a leading vertex at (p, q), cf. [2]. For the detailed proof of Theorem 5, see [2].

If the coordinates (p, q) of a leading vertex of W are both negative, one should consider the opposite vertex, or rotate the coordinate system by π. If $pq < 0$, one should rotate the coordinate system by $\pm\pi/3$ (we consider "triangular" coordinates shown on Fig. 4 rather than rectangular coordinates), that is, to apply the coordinate change $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ or its inverse, to make both coordinates of one of the leading vertices positive. Then Theorem 5 may be applied. This gives the following simple answer: $d(W, W_0) = E(|p|, |q|) - 1$ whenever $pq < 0$, where (p, q) is a leading vertex of W. The -1 summand can be explained as follows: if, for example, $p < 0$ and $q > 0$, the process of converting W into W_0 by flips corresponds to the converting of the unordered pair $(-p, q)$ to the pair $(1, 1)$, not to $(0, 1)$, by subtractions according to the Euclid algorithm (since we can consider $(-1, 1)$ as a leading vertex of W_0); of course, this takes one subtraction less.

Definition 10. For a matrix $A \in \mathrm{SL}(2, \mathbb{Z})$ we define its *complexity* $c(A)$ by putting $c(A) = d(W_0, AW_0)$.

To calculate the number $c(A)$, find a leading vertex (p, q) of the hexagon AW_0. Then we get $c(A) = E(|p|, |q|)$ if $pq > 0$ or $c(A) = E(|p|, |q|) - 1$ if $pq < 0$. In particular, if $AW_0 = W_0$ (there are six matrices A with this property), we get $c(A) = 0$.

§5. On $\mathrm{SL}(2, \mathbb{Z})$, $c(A)$, and $c(\mathcal{A})$

Since all admissible hexagons are $\mathrm{SL}(2, \mathbb{Z})$-equivalent and the action of this group preserves the existence of a flip connecting two given hexagons (thus, there is an action of $\mathrm{SL}(2, \mathbb{Z})$ on the graph Γ), we have $d(W_1, W_2) = d(BW_1, BW_2)$ for $B \in \mathrm{SL}(2, \mathbb{Z})$. In particular,

$$c(A^{-1}) = d(W_0, A^{-1}W_0) = d(AW_0, AA^{-1}W_0) = d(W_0, AW_0) = c(A) \qquad (3)$$

and $c(A) = d(W_0, AW_0) = d(BW_0, BAW_0)$, which may be different from $d(BW_0, ABW_0) = d(W, AW)$. Thus we have $c(A) = d(W, BAB^{-1}W)$ for any admissible hexagon $W = BW_0$.

It is not true that $c(A) = d(W, AW)$ for any admissible hexagon W. This means that the number $c(A)$ is not a conjugacy class invariant, contrary to a statement contained implicitly in [2].

Example. Let $A = \begin{pmatrix} 171 & 100 \\ -289 & -169 \end{pmatrix}$ and $B = \begin{pmatrix} 10 & -17 \\ -17 & 29 \end{pmatrix}$. Note that $A, B \in \mathrm{SL}(2, \mathbb{Z})$. By a straightforward calculation, we obtain $c(A) = 13$, while for a conjugate matrix $A' = B^{-1}AB = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ we have $c(A') = 1$.

The following problem arises: find the minimal value of $c(A)$ over the whole conjugacy class of A in $\mathrm{SL}(2,\mathbb{Z})$.

Definition 11. The *complexity* of an operator $\mathcal{A} \in \mathrm{SL}(2,\mathbb{Z})$ is the minimal possible complexity of matrices that represent $\mathcal{A}$ in all possible bases of the lattice $\mathbb{Z}^2$:

$$c(\mathcal{A}) = \min_{A \sim A_0} c(A) = \min_{B \in \mathrm{SL}(2,\mathbb{Z})} c(B^{-1} A_0 B),$$

where A_0 is any matrix representing $\mathcal{A}$ and $\sim$ denotes conjugacy in $\mathrm{SL}(2,\mathbb{Z})$. In other words, $c(\mathcal{A}) = \min d(W, \mathcal{A}W)$ over all admissible hexagons W. A matrix A of an operator $\mathcal{A}$ in some basis is said to be a *minimal* matrix of $\mathcal{A}$ if $c(A) = c(\mathcal{A})$, that is, if $c(A) \le c(A')$ for any matrix A' conjugated to A. An admissible hexagon W is called *minimal* for $\mathcal{A}$ if $d(W, \mathcal{A}W) = c(\mathcal{A})$.

In this section, we study the sequence $\{c(\mathcal{A}^k)\}$. Properties of this sequence depend on the trace of $\mathcal{A}$. Recall (see [4, §0]) that the operator $\mathcal{A}$ is called *elliptic* if $|\operatorname{tr}\mathcal{A}| < 2$. In this case $\operatorname{tr}\mathcal{A} = 0$ or $\operatorname{tr}\mathcal{A} = \pm 1$, and the equation $\mathcal{A}^2 - (\operatorname{tr}\mathcal{A})\mathcal{A} + (\det\mathcal{A})I = 0$ implies either $\mathcal{A}^2 = -I$ or $\mathcal{A}^3 \pm I = 0$, because $\det\mathcal{A} = 1$. Thus elliptic operators are periodic of period 3, 4 or 6, and so are the sequences $\{c(\mathcal{A}^k)\}$; both eigenvalues of an elliptic operator are roots of unity. If $\operatorname{tr}\mathcal{A} = \pm 2$, we say that $\mathcal{A}$ is a *parabolic* operator. In this case $(\mathcal{A} \pm I)^2 = 0$ and $\mathcal{A}$ is $\mathrm{SL}(2,\mathbb{Z})$ conjugated to either $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} -1 & n \\ 0 & -1 \end{pmatrix}$, where $n \in \mathbb{Z}$. So $\mathcal{A}$ is either a periodic operator (if $n = 0$) or, up to a sign, a power of the Jordan block $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$; both eigenvalues of a parabolic operator equal ± 1. Finally, $\mathcal{A}$ is *hyperbolic* if $|\operatorname{tr}\mathcal{A}| > 2$. In this case the eigenvalues of $\mathcal{A}$ are different real numbers, and $\mathcal{A}$ is a hyperbolic rotation. Also see [13, §5].

Lemma 1. *$c(AB) \le c(A) + c(B)$. Moreover, $c(AB) \equiv c(A) + c(B) \mod 2$.*

Proof. By definition, $c(A) = d(W_0, AW_0)$, $c(AB) = d(W_0, ABW_0)$, and $c(B) = d(W_0, BW_0) = d(AW_0, ABW_0)$. Since d is a metric on Γ, the triangle inequality $d(W_0, ABW_0) \le d(W_0, AW_0) + d(AW_0, ABW_0)$ holds. Since the graph Γ is a tree, we have $d(W_0, ABW_0) = d(W_0, AW_0) + d(AW_0, ABW_0) - 2k$, where $k \in \mathbb{Z}_{\ge 0}$, see Fig. 5. □

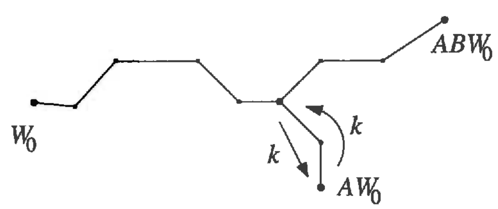

FIGURE 5. Additivity of parity for the distance on a tree

Theorem 6. *Reduction of $c(A)$ modulo 2 coincides with the unique epimorphism of the modular group $\mathrm{SL}(2,\mathbb{Z})/\{\pm I\}$ to $\mathbb{Z}_2$.*

Proof. Lemma 1 implies that this reduction is a homorphism of $\mathrm{SL}(2,\mathbb{Z})$ to $\mathbb{Z}_2$. It is an epimorphism since it takes $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ to 1. It is an epimorphism of the modular group $G = \mathrm{SL}(2,\mathbb{Z})/\{\pm I\}$, because $c(A) = c(-A)$: the hexagon AW_0 is centrally symmetric, whence $AW_0 = -AW_0$. Such an epimorphism φ is unique. Indeed, it is defined by its values $\varphi(S)$ and $\varphi(T)$ on the elements $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, which generate the modular group, see [14, Ch. VII, §1]. The relation $(ST)^3 = 1$ in G implies that $\varphi(S) = \varphi(T)$ in Z_2. Since φ is not identically zero, we have $\varphi(S) = \varphi(T) = 1$. □

Let us explain how to find a minimal matrix of an operator $\mathcal{A}$. Let A be its matrix in some basis, W_0 be the standard hexagon in this basis, $W = AW_0$, and $W_0, W_1, \ldots, W_{c(A)} = W$ be the shortest path from W_0 to W in Γ. Then $W = AW_0, AW_1, \ldots, AW_{c(A)} = AW = A^2W_0$ is the shortest path from W to AW in Γ. Both $W_{c(A)-1}$ and AW_1 are neighbors of a trivalent vertex W of Γ. Compare them.

Theorem 7. *Any matrix A with $c(A) \leq 1$ is minimal. A matrix A with $c(A) > 1$ is minimal if and only if $W_{c(A)-1} \neq AW_1$.*

Proof. If $c(A) = 0$, the matrix A is minimal. Let $c(A) = 1$, whence $c(\mathcal{A}) \leq 1$. It follows from eq. (3) and Lemma 1 that $c(B^{-1}AB) \equiv c(A) + 2c(B) \equiv c(A) = 1 \pmod 2$. So $c(\mathcal{A})$ is odd, thus $c(\mathcal{A}) \neq 0$ and A is minimal.

Suppose that $W_{c(A)-1} = AW_1$. The operator $\mathcal{A}$ takes W_1 to $AW_1 = W_{c(A)-1}$. We have $c(\mathcal{A}) \leq d(W_1, AW_1) = d(W_1, W_{c(A)-1}) = c(A) - 2$ (unless $c(A) \leq 1$), i. e., the matrix A is not minimal. This proves the "only if" part of the Theorem.

Suppose that the matrix A is not minimal. This implies that the standard hexagon W_0 is not minimal. Let V be a minimal hexagon for the operator $\mathcal{A}$. By γ_0 denote the path from V to AV in Γ of length $c(\mathcal{A})$. For any $k \in \mathbb{Z}$ let $\gamma_k = A^k\gamma$ be the path from A^kV to $A^{k+1}V$ in Γ (recall that Γ carries an action of $\mathrm{SL}(2,\mathbb{Z})$). Put $\gamma = \bigcup_{k\in\mathbb{Z}} \gamma_k$. Note that any vertex of Γ on γ represents a minimal hexagon, so $W_0 \in \Gamma \setminus \gamma$. We have to consider three cases.

Case 1: $c(\mathcal{A}) > 1$. Then two vertices of γ neighboring with A^kV, $k \in \mathbb{Z}$, are different because of the "only if" statement proven above. So γ is homeomorphic to a line, because two neighbors of any interior vertex of any γ_k are different, too (since γ_k is the shortest path from A^kV to $A^{k+1}V$), and the graph Γ is a tree.

By W_0U_0 denote the shortest path from W_0 to γ (that is, U_0 is the first point of γ belonging to any path from W_0 to a point of γ). Then $W\,U$ and $AW\,AU$ are the shortest paths from W and AW to γ, where $W = AW_0$ and $U = AU_0$. Obviously, these paths end at different points of γ: if $U_0 \in \gamma_k$, then $U \in \gamma_{k+1}$ and $AU \in \gamma_{k+2}$. Since Γ is a tree, the paths $W_0\,U_0$, $W\,U$, and $AW\,AU$ are mutually disjoint. This means that $W_0\,U_0\,U\,W$ is the shortest path from W_0 to W and $W\,U\,AU\,AW$ is the shortest path from W to AW, see Fig. 6. The leg $WU = A(W_0U_0)$ of this path is not empty. Consequently, the penultimate vertex of the path W_0W coincides with the first (different from W) vertex of the path $W\,AW$, which proves the statement of the "if" part of the Theorem.

Case 2: $c(\mathcal{A}) = 1$. Let V be a minimal hexagon for $\mathcal{A}$. If $A^2V \neq V$, then $A^{k+2}V \neq A^kV$ for any $k \in \mathbb{Z}$, all the γ_k are different, γ is homeomorphic to a line, and we can repeat the argument of Case 1. If $A^2V = V$, we have $A^kV = V$ for k even and $A^kV = AV$ for k odd. Without loss of generality, it can be assumed that the path from W_0 to V does not pass through AV. The transformation $\mathcal{A}$ takes this

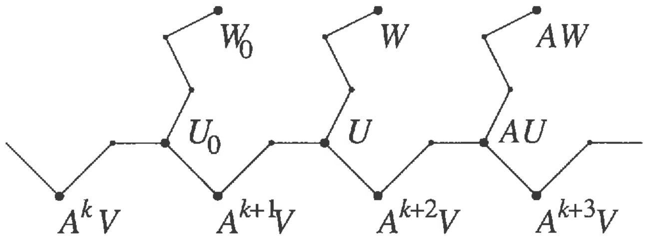

FIGURE 6. Paths W_0W and $W\,AW$ overlap

path to the path from $W = AW_0$ to AV, which does not pass through V. So the shortest path from W_0 to W is $W_0\,V\,AV\,W$, and thus it contains the edge $V\,AV$. Similarly, the path from W to AW also contains that edge. Therefore, these two paths overlap, which proves the Theorem in the case $c(\mathcal{A}) = 1$.

Case 3: $c(\mathcal{A}) = 0$. There exists an admissible hexagon V such that $AV = V$, but $AW_0 \neq W_0$ because A is not a minimal matrix. Consider paths W_0V and WV in Γ, where $W = AW_0$. Let U be the first (most distant from V) common point of this paths. Recall that A acts on Γ and takes the path W_0V to WV. Thus $AU = U$, $W_0 \neq U$, and A takes the path W_0U to WU and the path WU to $AW\,U$. So the paths $W_0W = W_0UW$ and $W\,AW = WU\,AW$ overlap over the leg WU. □

Now we can present an algorithm that finds the number $c(\mathcal{A})$ and a minimal matrix A of an operator $\mathcal{A}$. Start with any matrix A representing this operator. Apply the criterion of Theorem 7. If either $c(A) \leq 1$ or $W_{c(A)-1} \neq AW_1$, the matrix A is minimal. Otherwise, let V be the last common vertex of the paths $W\,W_0$ and $W\,AW$. Then V lies on γ and is a minimal hexagon for $\mathcal{A}$. Choose a basis so that V is the standard hexagon. The matrix of $\mathcal{A}$ in this basis is minimal, and $c(\mathcal{A})$ is equal to complexity of this matrix.

Corollary. *The subgraph $\gamma \subset \Gamma$ constructed in the proof of Theorem 7 is a line if and only if the operator $\mathcal{A}$ is not periodic.* □

This condition holds if and only if either $c(\mathcal{A}) \geq 2$ or $c(\mathcal{A}) = 1$ and $\mathcal{A}^2 \neq -I$. The line γ is the "mainstream" of the action of $\mathcal{A}$ on Γ. Minimal hexagons for $\mathcal{A}$ are exactly those corresponding to the vertices of the subgraph $\gamma \subset \Gamma$. If γ is a line, there are at most $3\,c(\mathcal{A})$ different minimal matrices for $\mathcal{A}$, because any minimal hexagon yields 6 different bases ($OX_1X_2, OX_2X_3, \ldots, OX_6X_1$, where $X_1, \ldots, X_6$ are the vertices of a hexagon and O is the origin), bases that differ by a central symmetry give the same matrix expression of $\mathcal{A}$, and hexagons V and AV lead to the same set of matrices of $\mathcal{A}$.

If $c(\mathcal{A}) = 1$, then the minimal matrix for $\mathcal{A}$ is either one of Jordan blocks $\begin{pmatrix} \pm1 & 1 \\ 0 & \pm1 \end{pmatrix}$, $\begin{pmatrix} \pm1 & -1 \\ 0 & \pm1 \end{pmatrix}$ or the $\pm\pi/2$ rotation matrix $\begin{pmatrix} 0 & \mp1 \\ \pm1 & 0 \end{pmatrix}$. These six matrices belong to six different conjugacy classes in $\mathrm{SL}(2,\mathbb{Z})$. In the case of a Jordan block, there are three minimal matrices for $\mathcal{A}$ and an infinite number of minimal hexagons (which lie on the line γ). For a rotation, there are three different minimal matrices (namely, $\begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}$, and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in the case of counterclockwise rotation) and only two minimal hexagons, which have in common the four lattice points where a positively definite integer quadratic form $Q_{\mathcal{A}}(\overrightarrow{v}) = \det(\overrightarrow{v}, \mathcal{A}\overrightarrow{v})$ (for the clockwise rotation, we set $Q_{\mathcal{A}}(\overrightarrow{v}) = -\det(\overrightarrow{v}, \mathcal{A}\overrightarrow{v})$) attains its minimal positive value 1.

If $c(\mathcal{A}) = 0$ and $\mathcal{A} \neq \pm I$, the minimal hexagon is unique; its six vertices are the six lattice points where a positively definite integer quadratic form $Q_{\mathcal{A}}(\overrightarrow{v}) = \pm\det(\overrightarrow{v}, \mathcal{A}\overrightarrow{v})$ attains value 1. The minimal matrix for $\mathcal{A}$ is also unique. Finally, for $\mathcal{A} = \pm I$, any admissible hexagon is minimal, while the only minimal matrix is, of course, $\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We omit the proofs of the statements of this paragraph and two preceding ones; most of them are straightforward.

Theorem 8. *Let $\mathcal{A}$ be a non-periodic operator. Then:*

1) *for any integer $k \neq 0$, A^k is a minimal matrix for $\mathcal{A}^k$ if and only if A is a minimal matrix for $\mathcal{A}$;*
2) *$c(\mathcal{A}^k) = |k| c(\mathcal{A})$ for any $k \in \mathbb{Z}$;*
3) *for any integer $k \neq 0$, we have $c(A^k) = |k| c(A) + b$, where $b = c(A) - c(\mathcal{A})$ is a nonnegative even number.*

Proof. It follows from the proof of Theorem 7 and Corollary that the path from W_0 to $A^k W_0$ consists of three legs $W_0 U_0$, $U_0\, A^k U_0$, and $A^k U_0\, A^k W_0 = A^k(U_0 W_0)$. If the first leg (and, simultaneously, the last one) is empty (contains no edges), matrices A and A^k are both minimal. Otherwise, neither A nor A^k is minimal. This proves the first statement and shows that the mainstreams of $\mathcal{A}$ and $\mathcal{A}^k$ coincide: $\gamma(\mathcal{A}^k) = \gamma(\mathcal{A})$. The second statement of the Theorem follows from the first one whenever $k \neq 0$; the case $k = 0$ is trivial. The third statement follows from the description of the path from W_0 to $A^k W_0$ in Γ, see above. $\square$

We conclude the section on $\mathrm{SL}(2, \mathbb{Z})$, $c(A)$, and $c(\mathcal{A})$ with one more way of looking at the mainstream $\gamma(\mathcal{A})$. Let $\mathcal{A} \in \mathrm{SL}(2, \mathbb{Z})$ be a hyperbolic rotation, so $|\operatorname{tr} \mathcal{A}| > 2$ and eigenvalues of $\mathcal{A}$ are different real numbers. Since admissible hexagons are centrally symmetric, we may assume that the eigenvalues of $\mathcal{A}$ are positive. The eigenvalues and the slopes of eigenvectors are quadratic irrationals, because the discriminant of the characteristic equation $\lambda^2 - (\operatorname{tr}\mathcal{A})\lambda + \det\mathcal{A} = 0$ equals $(\operatorname{tr}\mathcal{A})^2 - 4$ and is not equal to a square of an integer whenever $|\operatorname{tr}\mathcal{A}| > 2$. Draw through the origin two lines l_1, l_2 parallel to eigenvectors. They divide the plane into four parts. For each of these parts, consider the convex hull of the lattice points inside it. Since $\mathcal{A}$ preserves l_i, it preserves the convex hulls $h_1, \ldots, h_4$, as well as their boundaries, which are infinite sequences of segments. The group of the integers acts on ∂h_i by taking $x \in \partial h_i$ to $\mathcal{A}^k x \in \partial h_i$ for $k \in \mathbb{Z}$. An admissible hexagon is minimal for $\mathcal{A}$ (i.e., belongs to $\gamma(\mathcal{A})$) if and only if its leading vertex (and hence all its other vertices) lies on some ∂h_i. The path from a minimal hexagon $W \in \gamma$ to its image $\mathcal{A}W$ corresponds to the period of the continued fraction expansion of the slope α of l_i, which is a quadratic irrational number. An $\mathrm{SL}(2, \mathbb{Z})$ coordinate change does not affect this period: it takes α to $\dfrac{a\alpha + b}{c\alpha + d}$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$, changing the beginning of the continued fraction expansion of a quadratic irrational number without affecting its periodic part. We leave the proofs of these statements to the reader.

Example. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. This is a minimal matrix of a hyperbolic operator of complexity 2. The boundaries of the convex hulls h_i are represented on Fig. 7 by dotted lines. The coordinates of their corners in this case are, up to signs, the

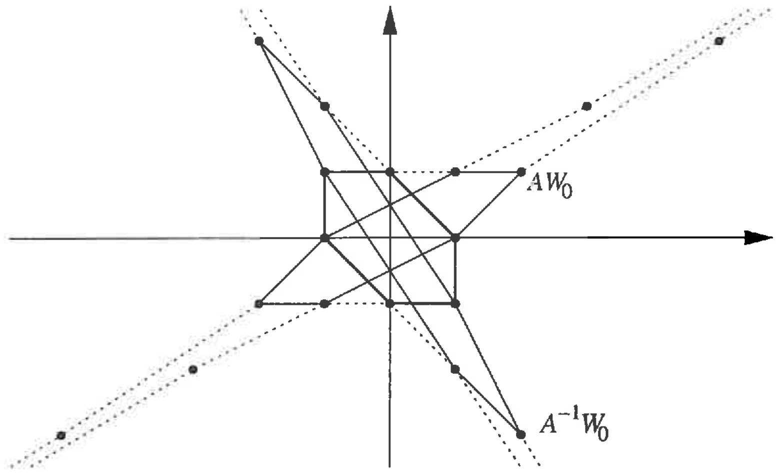

FIGURE 7. The mainstream for a hyperbolic rotation

pairs of consecutive Fibonacci numbers. The standard hexagon W_0, drawn in a bold line, is a minimal hexagon for $\mathcal{A}$. The hexagons $A^{-1}W_0$ and AW_0 also belong to the mainstream $\gamma(\mathcal{A})$. Since $c(\mathcal{A}) = 2$, there are two orbits of the action of the group $\mathbb{Z}$ on $\gamma(\mathcal{A})$ defined by the rule $k(W) = A^kW$ for $k \in \mathbb{Z}$ and $W \in \gamma(\mathcal{A})$. All three hexagons on Fig. 7 belong to one of the orbits. Hexagons of the other orbit can be obtained from them by the $\pi/2$ rotation around the origin (the eigenvectors of A, which direct l_1 and l_2, are orthogonal, because $A^{\mathrm{T}} = A$). As $k \to -\infty$, the hexagon A^kW looks more and more like the line l_1; as $k \to \infty$, it looks more and more like l_2. Directions of these lines are points of the projective line $\mathbb{R}P^1 = S^1$, which is the absolute in the Poincaré circle model of the Lobachevskii plane. We encourage the reader to find the relation between $\mathcal{A}$, $\gamma(\mathcal{A})$, and the geodesic line that connects these two absolute points. Hint: see [10, §17].

§6. SPINES OF TORUS BUNDLES

From now on, M denotes the total space of an orientable T^2-bundle over the circle and $\mathcal{A} \in \mathrm{SL}(2, Z)$ is the monodromy operator (acting on the one-dimensional homology group of the fiber containing the base point of M) of the bundle. By $M(\mathcal{A})$ denote the manifold M corresponding to the monodromy operator $\mathcal{A}$.

In this section, we construct a pseudominimal special spine of $M(\mathcal{A})$ with $c(\mathcal{A})+5$ vertices if $c > 0$ or with 6 vertices if $c = 0$.

Definition 12 [9]. A 2-dimensional component α of a special polyhedron has a *counterpass* if its boundary $\partial\alpha$ passes along some edge of SP in both directions; it is called a *component with short boundary* if $\partial\alpha$ passes through at most 3 vertices and visits any of them only once. A special spine of a 3-dimensional manifold is said to be *pseudominimal* if it contains neither components with counterpasses nor components with short boundaries.

If a special spine P is not pseudominimal, it is not minimal, because one can apply simplification moves (see [9]) to P and get an almost simple spine with a smaller number of vertices. For example, Figure 8 shows the effect of a simplification move applied to a special spine with a triangular component (the middle horizontal triangle in the left part of Fig. 8); it is easy to see that the neighborhood of a 2-cell with short boundary of length 3 in a special polyhedron P looks like the left hand

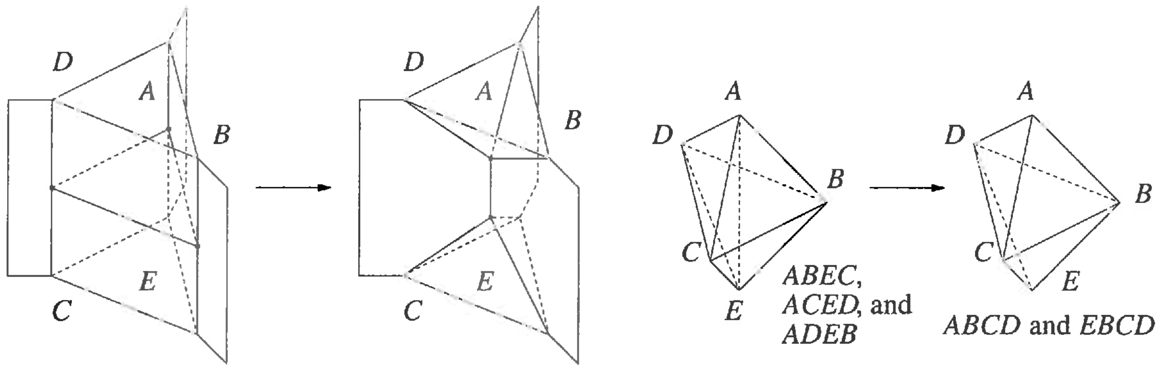

FIGURE 8. A simplification move (left) and the corresponding Pachner move (right)

side of Fig. 8. This move does not change the spine outside of the fragment shown on Fig. 8. Note that the spine obtained is special again: the move produces neither closed triple lines nor non-cellular 2-dimensional components.

Remark. Consider the singular triangulation dual to a special spine with a triangular component. Then the simplification move shown on Fig. 8 corresponds to the three-dimensional $(3,2)$ Pachner move [12], which replaces three tetrahedra by two tetrahedra. In the two-dimensional case, a flip (see Fig. 3) corresponds to the $(2,2)$ Pachner move, which switches the diagonal in a quadrilateral formed by two neighboring triangles. Recall that the move shown on Fig. 8 and its inverse are sufficient to convert any two special spines of the same compact three-dimensional manifold to one another, see [6]; this fact is crucial for the construction of the Turaev–Viro invariants [16].

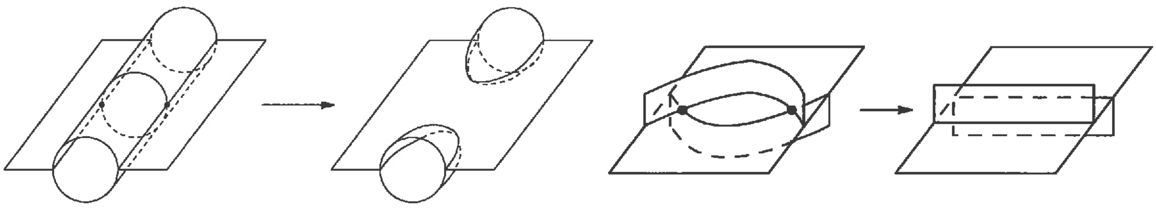

FIGURE 9. Two presentations of another simplification move

Figure 9 represents another simplification move, which is applicable to spines containing a component with short boundary of length 2; clearly, the neighborhood of this component looks like the left hand side of Fig. 9. This simplification move yields a simple, but not necessarily special, spine of the same manifold (provided that the move had been applied to a simple spine).

To construct a spine of $M(\mathcal{A})$, consider a fiber $T^2 \times \{0\}$ and choose a θ-curve L_0 in it; by doing so, we also fix a θ-curve L_1 in $T^2 \times \{1\}$; note that $W(L_1) = \mathcal{A}W(L_0)$. This choice is equivalent to the choice of some basis in the lattice $H_1(T^2, \mathbb{Z})$; by A denote the matrix of $\mathcal{A}$ in this basis. Construct a continuous family L_t transforming L_0 into L_1 by isotopy and $c(A)$ flips. Set $P_0 = \bigcup_{t\in[0,1]} L_t$; we assume that each L_t is embedded in $T^2 \times \{t\}$. Note that P_0 is a simple polyhedron, which is a spine of some punctured torus bundle. Two-dimensional components of P_0 come from edges

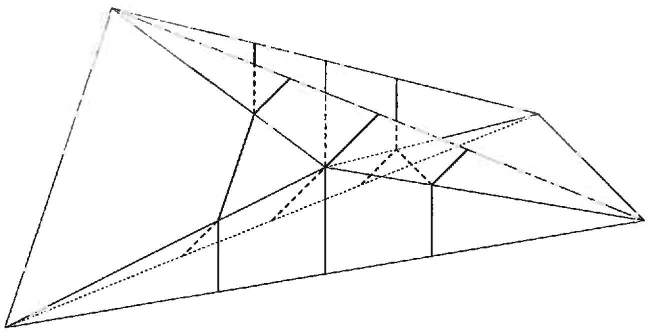

FIGURE 10. Vertices correspond to flips

of L_t as t varies; similarly, one-dimensional components of P_0 come from vertices of L_t. The $c(A)$ flips correspond to the vertices of P_0, see Fig. 10.

To minimize the number of vertices, it is natural to choose a basis in $H_1(T^2, \mathbb{Z})$ so that the operator A is minimal for $\mathcal{A}$. If the operator A is not minimal, Theorem 7 guarantees that the last flip of the first round along the base circle of the fibering and the first flip of the second round are mutually inverse. This means that the second simplification move (see Fig. 9) is applicable. Apply it until it is no longer possible. This process is nothing but the construction of a minimal hexagon for $\mathcal{A}$ by the algorithm described below the proof of Theorem 7. In the following, we suppose that A is a minimal matrix.

Examples. 1. The only periodic operators $\mathcal{A}$ with $c(\mathcal{A}) > 0$ have the minimal matrices $\begin{pmatrix} 0 & \mp 1 \\ \pm 1 & 0 \end{pmatrix}$. This is a very interesting case. The polyhedron P_0 constructed above has one vertex. Consider the two-sheeted covering of the base S^1 of the fibering. It induces the two-sheeted covering of the total space by the manifold $M(\mathcal{A}^2) = M(-I)$. The preimage of P_0 under the covering is a polyhedron in $M(-I)$ with two vertices, which can be cancelled by the second simplification move in two different ways. This is the only (up to a sign and conjugacy) operator such that $c(\mathcal{A}^k) < |k|c(\mathcal{A})$ for $k \in \mathbb{Z}$, $|k| \geq 2$, see Theorem 8; the other periodic operators are of complexity 0.

2. If $c(A) = 0$, there are no flips at all. In this case P_0 contains no vertices and consists of three orientable annuli and three edges if $A = I$, of three nonorientable annuli and one edge if $A = -I$, of one nonorientable annulus and one edge if A is equal to the standard hexagon rotation matrix $R_{\pi/3} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ or its inverse (note that $R^6_{\pi/3} = I$), and of one orientable annulus and two edges if A equals $R^2_{\pi/3}$ or $R^4_{\pi/3}$. This exhausts the case $c(A) = 0$.

The polyhedron P_0 is not a spine of $M(\mathcal{A})$, because the fibered space $M(\mathcal{A})$ admits a section that does not intersect P_0. This section represents a nontrivial element of the group $\pi_1(M(\mathcal{A}))$, while the complement to a spine of a closed manifold is a cell and hence cannot contain nontrivial loops. Let us put $P_1 = P_0 \cup (T^2 \times \{0\})$.

Lemma 2. *P_1 is a spine of $M(\mathcal{A})$.*

Proof. It is sufficient to show that $M(\mathcal{A}) \setminus P_1$ is a 3-dimensional cell. We have $M(\mathcal{A}) \setminus P_1 = T^2 \times (0,1) \setminus P_0 = T^2 \times (0,1) \setminus \bigcup_{t\in(0,1)} L_t = \bigcup_{t\in(0,1)} (T^2 \times \{t\} \setminus L_t)$, and

Lemma follows. □

Note that P_1 is not a simple polyhedron. Indeed, its part $T^2 \times \{0\}$ contains a singular subset L_0, which is more complicated than a triple line: three edges of L_0 yield three lines of transversal intersection of two surfaces, and any of two vertices of L_0 gives rise to a transversal intersection of a triple line with one extra surface.

Let us modify the previous construction by gluing $T^2 \times \{0\}$ with $T^2 \times \{1\}$ along a homeomorphism $\mathcal{A}+\delta$, where δ is a small shift of the torus in a direction transversal to the edges of L_0, see Fig. 11. Put $P_2 = \bigcup_{t\in[0,1]} L_t \cup (T^2 \times \{0\})$. Again, P_2 is a spine of $M(\mathcal{A})$.

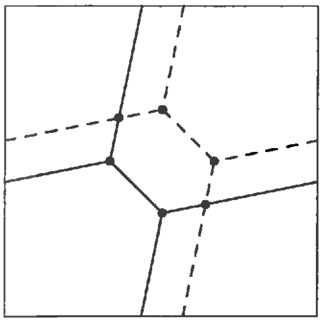

FIGURE 11. θ-curve L_0 and its δ-shift L_1 (dashed line) in the fiber $T^2 \times \{0\}$ of $M(\mathcal{A})$

Lemma 3. *P_2 is a special spine of $M(\mathcal{A})$ with $c(\mathcal{A}) + 6$ vertices.*

Proof. It is clear from the construction that P_2 is a simple polyhedron. Its triple lines are the "trajectories" (as t varies) of the vertices of L_t and the ten segments of L_0 and L_1 shown on Fig. 11, where the torus is represented by a square with the opposite sides to be identified. There are $c(\mathcal{A})$ vertices of P_2 that correspond to $c(\mathcal{A})$ flips between L_0 at $t = 0$ and L_1 at $t = 1$, and six other vertices thar are drawn on Fig. 11. Two of them arise from $T^2 \times \{0\}$ and L_t, $0 \le t < \varepsilon$, and their neighborhoods in P_2 look like Fig. 1 d. Two other vertices on Fig. 11 arise from $T^2 \times \{0\} = T^2 \times \{1\}$ and L_t, $1 - \varepsilon < t \le 1$; their neighborhoods look like the horizontal mirror reflection of Fig. 1 d. The last two vertices on Fig. 11 correspond to two intersection points of L_0 and L_1, and their neighborhoods look like Fig. 1 c. Thus P_2 is a simple spine of $M(\mathcal{A})$ with $c(\mathcal{A}) + 6$ vertices.

It remains to prove that SP_2 contains no closed triple lines and all connected components of $P_2 \setminus SP_2$ are 2-dimensional cells, cf. Definition 2. First group of triple lines of SP_2 is formed by ten arcs in $T^2 \times \{0\}$ shown on Fig. 11. Obviously, they are not closed. The rest $2c(A) + 2$ triple lines are swept by the vertices of the θ-curves $L_t \subset T^2 \times \{t\}$, $0 < t < 1$. They end at vertices of P_2, too, and thus are not closed.

Connected components of $P_2 \setminus SP_2$ also belong to two groups. Four of them, two hexagonal and two quadrilateral, lie in the fiber $T^2 \times \{0\}$, see Fig. 11. They are cells. Any other connected component of $P_2 \setminus SP_2$ intersects any fiber $T^2 \times \{t\}$, $a < t < b$ (where a is equal to either 0 or one of the flip moments, and b is either one of the flip moments or is equal to 1), along one edge of L_t, and does not intersect other fibers; this implies that this component is a cell. We have proved that the polyhedron P_2 is special. □

Corollary. $c(M(\mathcal{A})) \leq c(\mathcal{A}) + 6$. $\square$

Example. Three-dimensional torus can be represented as $M(I)$, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Since $c(I) = 0$, the construction above gives a special spine of T^3 with six vertices. The manifold $T^3 = M(I)$ is contained in Table 7 of the preprint [9] under the name 6_{71}. It is shown in [9] that all manifolds of complexity at most 5 are different from T^3. So we have $c(T^3) = 6$. The spine that we constructed here does not differ from the spine 6_{71} from [9, §5.2], while our way of presenting spines differs significantly from the one used in [9].

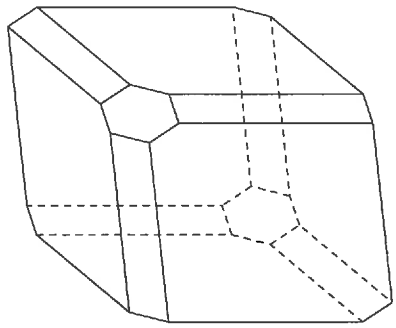

FIGURE 12. The complement $T^3 \setminus P_2$ of the minimal spine P_1 of T^3

The torus T^3 can be obtained from the cube by gluing its opposite faces. This yields a natural cell decomposition of T^3 with one vertex, three edges, three "square" 2-dimensional cells and one 3-dimensional cell. The 2-dimensional skeleton $\mathrm{sk}_2(T^3)$ has singular points more complicated than triple lines and vertices of simple polyhedra. However, the minimal spine of T^3 can be obtained as a small perturbation of $\mathrm{sk}_2(T^3)$.

Let the θ-curves L_t, $t \in [0,1]$, be very close to the bouquet of a parallel and a meridian of $T^2 \times \{t\}$, and let the shift δ involved in the construction of P_2 be very small. Then the 3-dimensional cell $T^3 \setminus P_2$ is very close to the 3-dimensional cube. Figure 12 represents this cell. If we identify opposite faces of this polyhedron by parallel transports (or, equivalently, tessellate $\mathbb{R}^3$ into parallel copies of this polyhedron and consider a quotient over the appropriate lattice $\mathbb{Z}^3$), we get the torus T^3; the image of the boundary of the polyhedron under this gluing is the minimal spine of T^3 close to $\mathrm{sk}_2(T^3)$.

The same construction gives special spines with six vertices for the manifolds $M(-I) = 6_{70}$,

$$M\left(\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}\right) = M\left(\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\right) = 6_{67},$$

and

$$M\left(\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\right) = M\left(\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}\right) = 6_{65}.$$

The spines constructed in this way are minimal spines of these manifolds, because all of them are of complexity 6; in fact, all manifolds of complexity up to 5 are quotient spaces of the sphere S^3, see [9].

However, in all other cases (that is, if $c(\mathcal{A}) > 0$) the spines with $c(\mathcal{A})+6$ vertices are not minimal spines of the manifolds $M(\mathcal{A})$. For example, the spaces

$$6_{66} = M\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right), \quad 6_{68} = M\left(\begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}\right), \quad \text{and} \quad 6_{69} = M\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\right)$$

are manifolds of complexity 6, while

$$c\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = c\left(\begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}\right) = c\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\right) = 1,$$

and our construction gives their spines with 7 vertices.

This happens because some of the spines with $c(\mathcal{A}) + 6$ vertices constructed above are not pseudominimal whenever $c(\mathcal{A}) > 0$. Namely, they have a triangular component, and the first simplification move (see Fig. 8) can be applied.

Let us return to Fig. 11. Assume that the first flip in the sequence taking L_0 to L_1 involves the short edge of L_0, that is, the edge that does not intersect dashed lines on Fig. 11. This condition can be satisfied by an appropriate choice of the shift δ involved in the construction of P_2. Then the 2-dimensional cell of P_2 adjacent to this edge and not contained in $T^2 \times \{0\}$ is a triangle, and we can apply the first simplification move, which gives a spine of $M(\mathcal{A})$ with a smaller number of vertices. This spine can be described in other words as follows. Let L' be the θ-curve obtained after the first of $c(\mathcal{A})$ flips converting L_0 into L_1. Glue the square from Fig. 13 into the torus $T^2 \times \{0\} \in M(\mathcal{A})$ and embed L_t in $T^2 \times \{t\}$ for all $t \in (0,1)$, where the family L_t contains $c(\mathcal{A})-1$ flips and connects L' with L_1. Note that the first of $c(\mathcal{A}) - 1$ flips converting L' into L_1 is performed along a long edge of L', because a flip along the short edge would annihilate with the flip converting L_0 to L'. The new spine P_3 has $6 + c(\mathcal{A}) - 1 = c(\mathcal{A}) + 5$ vertices. So we have $c(M(\mathcal{A})) \leq c(\mathcal{A}) + 5$ whenever $c(\mathcal{A}) > 0$.

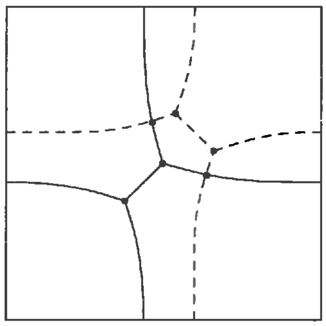

FIGURE 13. θ-curves L' and L_1 (dashed) in $T^2 \times \{0\}$

The proof that the spine P_3 is special repeats the proof of Lemma 3.

Theorem 9.
1) $c(M(\mathcal{A})) \leq \max(6, c(\mathcal{A}) + 5)$.
2) *The spine P_3 constructed above is pseudominimal.*

Compare the first statement with Corollary of Lemma 3.

Proof. If $c(\mathcal{A}) = 0$, then $c(M(\mathcal{A})) = 6$. So we may suppose that $c(\mathcal{A}) > 0$. Since P_3 is an almost simple (and even special) spine of $M(\mathcal{A})$ with $c(\mathcal{A})+5$ vertices, the first statement is obvious. The argument similar to the proof of Lemma 3 shows that two-dimensional cells of P_3 have no counterpasses. So we only have to show that P_3 has no components with short boundaries. The four cells contained in $T^2 \times \{0\}$ are pentagons, see Fig. 13. The cells that have no boundary edges in $T^2 \times \{0\}$ have even numbers of edges, namely, $2k-2$, where k is the number of flips from the vertex where the cell appears to the vertex where the cell disappears (including both the first flip and the last one). The matrix A involved in the construction of P_3 is a minimal matrix of $\mathcal{A}$. This implies that any two consecutive flips in the sequence involved in the construction are not inverse to one another, that is, $k > 2$ for any cell considered above.

It remains to consider at most 6 two-dimensional cells that have an edge in $T^2 \times \{0\} = T^2 \times \{1\}$ (if A is, up to a sign, a power of a Jordan block, there are only 5 cells of this type; otherwise, no 2-cell touches both $T^2 \times \{0\}$ along an edge of L' and $T^2 \times \{1\}$ along an edge of L_1, so three edges of L' and three edges of L_1 belong to six different cells of $P_3 \setminus T^2 \times \{0\}$). Two cells of $P_3 \setminus T^2 \times \{0\}$ are adjacent to the long edges of L' (the edges that intersect dashed lines on Fig. 13). Each of these cells has at least 4 boundary edges: two segments of a long edge of L' and two edges (transversal to fibers) that arise from the vertices of L'. The same argument works for two cells of $P_3 \setminus T^2 \times \{0\}$ adjacent to the long edges of L_1. Consider the cell of $P_3 \setminus T^2 \times \{0\}$ adjacent to the short edge of L'. Of course, it has at least 3 edges: the short edge of L' and two edges that are trajectories of the vertices of L' as t varies. It has at least one more edge: otherwise, the first flip in the sequence of flips converting L' to L_1 is performed along the short edge of L', which is impossible by the construction of P_3, see above. For the same reason, the last flip (which results in L_1) cannot be performed along the short edge of L_1: by the minimality of the matrix A, it cannot be cancelled with the flip connecting L_0 with L', see Theorem 7. This means that the 2-dimensional cell of $P_3 \setminus T^2 \times \{1\}$ adjacent to the short edge of L_1 also has more than 3 edges, and P_3 contains no components with short boundaries. The Theorem is proved. □

Conjecture 2. *The pseudominimal spines of the manifolds $M(\mathcal{A})$ constructed above are in fact their minimal spines, and the upper bound for complexity given in Theorem 9 is in fact its exact value:* $c(M(\mathcal{A})) = \max(6, c(\mathcal{A}) + 5)$ *for any monodromy operator* $\mathcal{A} \in \mathrm{SL}(2, \mathbb{Z})$. *In other words, any singular triangulation of $M(\mathcal{A})$ involves at least $c(\mathcal{A})+5$ tetrahedra if $c(\mathcal{A}) > 0$ and 6 tetrahedra if $c(\mathcal{A}) = 0$.*

§7. Spines of lens spaces

Pseudominimal special spines of the lens spaces $L_{p,q}$, $p > 3$, with exactly $E(p,q) - 3$ vertices were constructed in [9]. In that paper, spines are presented by drawing the neighborhood of the singular graph of a spine. This allows to draw spines on the plane; however, it remains unclear how the spines are embedded into corresponding manifolds.

In this section, we construct pseudominimal special spines of $L_{p,q}$, $p > 3$, with $E(p,q) - 3$ vertices, making use of the techniques developed in §§4–6. We omit some details and proofs.

Consider two solid tori. The meridians of their boundary tori are well defined, while the parallels are defined modulo meridians only. Let μ_0, μ_1 be the meridians of the tori and σ_0, σ_1 be their parallels such that the pair of the oriented cycles (σ_0, μ_0) defines the positive orientation of the boundary of the first torus and the pair (σ_1, μ_1) defines the negative orientation of the boundary of the second torus. There is a unique pair of positive integer numbers (r, s) such that $r < p$, $s < p$, and $qs - pr = 1$. Put $A = \begin{pmatrix} s & p \\ r & q \end{pmatrix}$ and attach the solid tori to one another so that the induced homomorphism of the one-dimensional homology groups of their boundary tori has the matrix A (in the bases (σ_0, μ_0) and (σ_1, μ_1)). We get a closed orientable 3-manifold that is nothing but $L_{p,q}$.

Note that $A \in \mathrm{SL}(2, \mathbb{Z})$, $c(A) = E(p, q)$, and the parallels σ_0 and σ_1 represent nontrivial elements of $\pi_1(L_{p,q}) = \mathbb{Z}_p$. This implies that any spine of $L_{p,q}$ intersects these loops. Let us shift σ_0 in the interior of the first solid torus and consider the tubular neighborhood U_0 of the shifted curve. Obviously, U_0 is a solid torus. Similarly, construct U_1 as a tubular neighborhood of σ_1 shifted inside of the interior of the second torus. We may assume U_0 and U_1 to be disjoint. Then $L_{p,q} = U_0 \cup (T^2 \times [0, 1]) \cup U_1$. Let L_i, $i = 0, 1$, be standard (with respect to the bases (σ_i, μ_i)) θ-curves in the tori $T_i^2 = \partial U_i$; they are defined up to isotopy. Following the construction of §6, consider a continuous family $L_t \subset T^2 \times \{t\}$ connecting L_0 to L_1 with $c(A)$ flips. Put $P_0 = \bigcup_{t\in[0,1]} L_t$. Let D_i, $i = 0, 1$, be meridional disks of the U_i intersecting L_i transversally at one point. Put $P_1 = D_0 \cup T_0^2 \cup P_0 \cup T_1^2 \cup D_1$.

Lemma 4. *The polyhedron P_1 is a special spine of $L_{p,q}$ with three punctures. It has $E(p, q) + 6$ vertices.*

Proof. The complement $L_{p,q} \setminus P_1$ consists of three cells $U_0 \setminus D_0$, $(T^2 \times [0, 1]) \setminus P_0$, and $U_1 \setminus D_1$. There are $c(A) = E(p, q)$ vertices in the interior part of P_0. Further, there are 3 vertices on T_0^2, which correspond to two vertices of L_0 and the intersection point of L_0 and ∂D_0. Similarly, there are 3 vertices of P_1 on T_1^2. It remains to show that P_1 is a special polyhedron. This can be proven by analogy with Lemma 3. □

Below we show that one can decrease the number of vertices "inside of P_0" by one and the number of vertices "near each U_i" by four. This gives a spine with $E(p, q) + 6 - 1 - 4 - 4 = E(p, q) - 3$ vertices.

Recall that the parallels σ_i are defined only modulo meridians μ_i. Thus, the θ-curves L_i are defined only up to powers of the Dehn twists along the meridians, that is, up to transformations $\sigma_i \mapsto \sigma_i + n_i\mu_i$, $n_i \in \mathbb{Z}$. By varying n_0 and n_1, one can decrease the distance in Γ between $B^{n_0}W_0$ and $C^{n_1}AW_0$ and thus decrease the number of the vertices inside of P_0; here W_0 is the standard hexagon, $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ is the matrix representing the Dehn twist along μ_0, and $C = ABA^{-1}$ is the matrix corresponding to the Dehn twist along μ_1.

Lemma 5. $\min\limits_{n_0,n_1\in\mathbb{Z}} d(B^{n_0}W_0, C^{n_1}AW_0) = E(p, q) - 1.$

Proof. Note that the graph Γ contains edges $B^uW_0\, B^{u+1}W_0$ and $C^vAW_0\, C^{v+1}AW_0$ for all $u, v \in \mathbb{Z}$. Since Γ is a tree, there are $m_0, m_1 \in \mathbb{Z}$ such that the path from $B^{n_0}W_0$ to $C^{n_1}AW_0$ for all $n_0, n_1 \in \mathbb{Z}$ consists of the following three legs: $B^{n_0}W_0\, B^{m_0}W_0$, $B^{m_0}W_0\, C^{m_1}AW_0$, and $C^{m_1}AW_0\, C^{n_1}AW_0$. Now it is obvious that $\min\limits_{n_0,n_1\in\mathbb{Z}} d(B^{n_0}W_0, C^{n_1}AW_0) = d(B^{m_0}W_0, C^{m_1}AW_0)$. By considering the three legs of the path from W_0 to AW_0, one can see that $m_0 = 1$ (because $p > q > 0$), $m_1 = 0$ (because both positive and negative Dehn twists along μ_1 do not affect the leading vertex (p, q) of AW_0 and thus increase the distance to W_0), and the length of the middle leg of this path is $d(BW_0, AW_0) = d(W_0, AW_0) - |m_0| - |m_1| = E(p,q) - 1$. $\square$

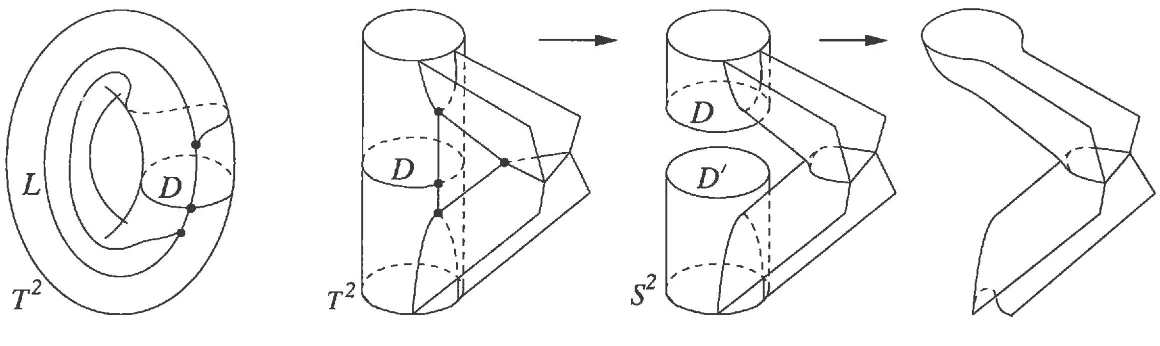

FIGURE 14. Simplification of P_0 near T_i^2

By virtue of Lemma 5, we can decrease by one the number of the vertices inside of P_0 by another choice of a θ-curve L_0. Now we have a special spine of $L_{p,q}$ with three punctures having $E(p,q)+5$ vertices. The disks D_0 and D_1 are components with short boundaries. By τ_i denote the edge of L_i that intersects ∂D_i. Note that two other edges of L_i form the meridian of T_i, the first flip in the sequence connecting L_0 with L_1 is performed along τ_0 while the last flip in this sequence is performed along τ_1 (flips along other edges are equivalent to meridional Dehn twists and thus do not lead out of the mainstreams $\gamma(B) = \{B^{n_0}W_0 \mid n_0 \in \mathbb{Z}\}$ and $\gamma(C) = \{C^{n_1}AW_0 \mid n_1 \in \mathbb{Z}\}$, while the path between L_0 and L_1 is the shortest path that connects these mainstreams). We can apply the following simplification move in the neighborhood of D_i. First, add a parallel copy D_i' of D_i. Second, delete the lateral surface of the cylinder bounded by D_i, D_i', and a thin strip of T_i^2. Finally, delete the cell of P_1 adjacent to τ_i; this cell is triangular, because τ_i is the edge involved in the flip in P_0 closest to T_i^2, see Fig. 14. So, the first step adds one vertex on each T_i^2, the second step kills two vertices on each T_i^2, and the last step kills three vertices near each of T_i^2. By P_1 denote the polyhedron obtained by the construction above. Obviously, it has $E(p,q)-3$ vertices. Further, one can see that two remaining edges of L_i (which differ from τ_i) form a closed triple line S_i^1 and the complement $L_{p,q} \setminus P_1$ still consists of three cells, two of which are bounded by the spheres S_i^2 obtained from the torus T_i^2 and their meridional disks D_i, D_i' by deleting the thin strip bounded by ∂D_i and $\partial D_i'$ from T_i^2. The circles S_i^1 divide the spheres S_i^2 into two disks each; one of the disks contains D_i, the other contains D_i'. Delete from P_1 the disks of the S_i^2 that contain D_i'. This yields a polyhedron P with $E(p,q)-3$ vertices such that $L_{p,q} \setminus P$ is a cell.

Theorem 10. *The polyhedron P is a pseudominimal special spine of $L_{p,q}$ with $E(p,q)-3$ vertices. It coincides with the spine of $L_{p,q}$ presented in* [9]. $\square$

It was shown in [9] that the spines constructed in that paper are pseudominimal. So it is sufficient to prove only the second statement; we leave it to the reader.

REFERENCES

1. S. Anisov, *Flip equivalence of triangulations of surfaces*, (in Russian; English transl. in Moscow Univ. Math. Bull. **49** (1994) no. 2, 55–60), Vestnik Moskov. Univ. Ser. I Mat. Mekh. (1994), no. 2, 61–67.

2. S. Anisov and S. Lando, *Topological complexity of torus bundles over S^1*, in: Topics in Quantum Groups and Finite-type Invariants (Mathematics at the Independent University of Moscow) AMS Translations ser. 2, Vol. 185; Advances in the Mathematical Sciences (also available at `http://www.botik.ru/~duzhin/iumvol1/anisov.ps.gz`) (B. Feigin, V. Vassiliev, eds.), 1998, pp. 129–135.
3. B. Casler, *An embedding theorem for connected 3-manifolds with boundary*, Proc. Amer. Math. Soc. **16** (1965), 559–566.
4. A. Casson and S. Bleiler, *Automorphisms of Surfaces After Nielsen and Thurston*, London Mathematical Society Student Texts, 9, Cambridge University Press, Cambridge – New York, 1988, pp. iv+105.
5. H. Davenport, *The Higher Arithmetic. An Introduction to the Theory of Numbers*, Cambridge University Press, Cambridge, UK, 1992, pp. 217.
6. S. Matveev, *Transformations of special spines, and the Zeeman conjecture*, (in Russian; English transl. in Math. USSR-Izv. 31 (1988), no. 2, 423–434), Izv. Akad. Nauk SSSR Ser. Mat. **51** (1987), no. 5, 1104–1116, 1119.
7. S. Matveev, *Complexity theory of three-dimensional manifolds*, Acta Appl. Math. **19** (1990), 101–130.
8. S. Matveev, *Computer recognition of three-manifolds*, Experimental Mathematics **7** (1998), no. 2, 153–161.
9. S. Matveev, *Tables of 3-manifolds up to complexity* 6, (.dvi and .ps files are available through `http://www.mpim-bonn.mpg.de/html/preprints/preprints.html` ; the .ps file exceeds 60 Mbytes), Max Planck Institute preprint MPI 1998-67, 1–50.
10. V. Nikulin and I. Shafarevich, *Geometries and Groups*, (in Russian; English transl.: *Geometries and Groups*, Universitext. Springer Series in Soviet Mathematics, Springer–Verlag, Berlin–New York, 1987, pp. viii+251), Nauka, Moscow, 1983, pp. 240.
11. M. Ovchinnikov, *A table of closed orientable irreducible 3-manifolds of complexity* 7, (in Russian), Chelyabinsk University preprint (1997).
12. U. Pachner, *Bistellare Äquivalenz kombinatorischer Mannigfaltigkeiten*, (in German), Arch. Math. (Basel) **30** (1978), no. **1**, 89–98.
13. P. Scott, *The geometries of 3-manifolds*, Bull. London Math. Soc. **15** (1983), no. 5, 401–487.
14. J.-P. Serre, *Cours d'arithmétique*, (in French; English transl.: *A course in arithmetic.* Graduate Texts in Mathematics, No. 7, Springer–Verlag, New York–Heidelberg, 1973, pp. viii+115), Collection SUP: "Le Mathématicien", 2, Presses Universitaires de France, Paris, 1977, pp. 188.
15. J.-P. Serre, *Arbres, amalgames,* SL_2, (in French; English transl.: *Trees*, Springer–Verlag, Berlin–New York, 1980, pp. ix+142), Astérisque, No. 46, Société Mathématique de France, Paris, 1977, pp. 189.
16. V. Turaev and O. Viro, *State sum invariants of 3-manifolds and quantum 6j-symbols*, Topology **31** (1992), no. 4, 865–902.
17. I. Vinogradov, *Foundations of Number Theory*, (in Russian; English transl.: *An Introduction to the Theory of Numbers.* Pergamon Press, London–New York, 1955, pp. vi+155), Nauka, Moscow, 1981.

INSTITUTE FOR SYSTEM STUDIES, RUSSIAN ACADEMY OF SCIENCES

Current address: I. Newton Institute for Mathematical Sciences, 20 Clarkson Road, Cambridge, CB3 0EH, U.K.

E-mail address: `anisov@mccme.ru`