Classifying Spaces of Singularities and Thom Polynomials

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Abstract.
We discuss the theory of characteristic classes of manifolds, Poincaré dual to singular sets of smooth maps. We review the proof of the existence of Thom polynomials and present some methods of computation of these polynomials. Our approach is based on the study of classifying spaces of singularities and their geometric properties.

Keywords: Singularity, Thom polynomial, characteristic class, classifying space, characteristic spectral sequence

1. Thom polynomials

Theorems of global singularity theory express global topological invariants (of manifolds, bundles, etc.) in terms of the geometry of singularities of some differential geometry structures. A classical example is the Hopf theorem expressing the Euler characteristic of a manifold via singular points of a vector field on it. Another example is the Maslov class of a Lagrange submanifold in the cotangent bundle defined as the cohomology class Poincaré dual to the (properly co-oriented) critical set of the projection to the base of the bundle, see e.g. [1]. Many results in this theory are formulated as theorems on existence and computation of so called *Thom polynomials*. In these notes we explain the definition of these polynomials based on the notion of the classifying space of singularities. This approach makes the 'existence theorem' trivial and also gives some ideas on computing these polynomials.

1.1. Theorem on the existence of Thom polynomial

Many classification problems in singularity theory can be formulated as the problem of classification of orbits for some Lie group action. In the case of singularities of maps consider the space

$$V = J_0^K(\mathbb{R}^m, \mathbb{R}^n) \tag{1}$$

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of K-jets at the origin of map germs $(\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$. We consider this space together with the action of the Lie group

$$G = J_0^K \mathrm{Diff}(\mathbb{R}^m) \times J_0^K \mathrm{Diff}(\mathbb{R}^n) \qquad (2)$$

of K-jets of left-right changes at the origin.

Definition. A *singularity class* is any G-invariant subset $\Sigma \subset V$.

If $f : M^m \to N^n$ is a generic smooth map then the singularity locus $\Sigma(f) \subset M$ consisting of points with the given local singularity type Σ has the same codimension as $\Sigma \subset V$. Moreover if the singularity class $\Sigma \subset V$ is an algebraic subvariety then the singularity locus $\Sigma(f)$ carries a fundamental $\mathbb{Z}_2$-homology class and its Poincaré dual cohomology class is well defined.

THEOREM 1 (Thom, [21, 9]). *For any algebraic singularity class $\Sigma \subset V$ there exists a universal polynomial P_Σ (called* Thom polynomial*) in Stiefel-Whitney classes $\omega_1(M), \ldots, \omega_m(M)$, $f^*\omega_1(N), \ldots, f^*\omega_n(N)$, such that for any generic map $f : M \to N$ the cohomology class Poincaré dual to the singularity locus $\Sigma(f)$ is given by this polynomial,*

$$[\Sigma(f)] = P_\Sigma(\omega(M), f^*\omega(N)) \in H^*(M, \mathbb{Z}_2).$$

1.2. Genericity condition

The genericity condition in the theorem above is formulated as follows. Consider the smooth fiber bundle $E \to M$ whose fiber E_x over a point x is isomorphic to V and consists of all K-jets of map germs $(M, x) \to (N, f(x))$. The structure group of this bundle can be reduced to G, and up to an isomorphism of G-bundles, it is independent of the representative f in the same homotopy class of smooth maps $M \to N$. The singularity class Σ gives rise to a well-defined subvariety $\Sigma(E) \subset E$ in the jet bundle E. If Σ is locally algebraic then so $\Sigma(E)$ is and the codimension of Σ in V is equal to that of $\Sigma(E)$ in E. Therefore its dual cohomology class $[\Sigma(E)] \in H^*(E, \mathbb{Z}_2)$ is always well defined. The map f defines a natural section s_f of the bundle E whose value at a point $x \in M$ is the K-jet of the map f itself at this point. A reformulation of Theorem 1 claims that for any (generic or not) map f there is an equality

$$P_\Sigma(\omega(M), f^*\omega(N)) = s_f^*[\Sigma(E)] \in H^*(M, \mathbb{Z}_2).$$

If Σ is algebraic then the space E admits a Whitney stratification such that $\Sigma(E)$ is the union of several strata.

Definition. The map $f : M \to N$ is called *transversal* if the jet extension section $s_f : M \to E$ is transversal to each stratum of the stratification of E.

The generic maps of Theorem 1 are those which are transversal. By the transversality theorem, any generic section $s : M \to E$ is transversal. Moreover by the strong version of this theorem due to Thom the transversality condition for the section s_f can be achieved by some small perturbation of the map f, i.e. within the class of 'integrable' sections.

1.3. Complex version

There is a complex version of Theorem 1 where smooth maps of smooth manifolds are replaced by holomorphic maps of complex analytic manifolds, $\mathbb{Z}_2$-cohomology by integer cohomology, Stiefel-Whitney classes by the corresponding Chern classes, etc. In the complex case the transversality condition is open but not necessarily dense. For any map f the characteristic class represented by the corresponding Thom polynomial can be defined as

$$P_\Sigma(c(M), f^*c(N)) = s_f^*[\Sigma(E)]. \qquad (3)$$

It can be interpreted as follows.

- If the section s_f is transversal then $P_\Sigma(c(M), f^*c(N)) = [\Sigma(f)]$.
- The equality (3) can be applied if the singularity locus $\Sigma(f)$ has the 'expected codimension'. In this case the components of $\Sigma(f)$ should be taken with multiplicities prescribed by the scheme structure of $\Sigma(f) = s_f^{-1}(\Sigma(E))$.
- If the codimension of $\Sigma(f)$ is less than that expected, then the Poincaré dual of $P_\Sigma(c(M), f^*c(N))$ can be represented by some closed singular chain in $\Sigma(f)$. It follows, in particular, that $P_\Sigma(c(M), f^*c(N)) \neq 0$ implies $\Sigma(f) \neq \varnothing$.
- In any case one may neglect the holomorphic structure on M and consider a generic C^∞-perturbation $s : M \to E$ of the section s_f. Then $P_\Sigma(c(M), f^*c(N))$ is Poincaré dual to the singularity locus $\Sigma(s)$ (which is a real locally analytic co-oriented subvariety in M).

In many problems there is a correspondence between the classifications of real and complex singularities: every complex singularity has a real representative and the real codimension of a real singularity class is equal to the complex codimension of its complexification. There is no *à priori* proof of this statement. Moreover there are counterexamples that show that this is not always the case. All known counterexamples are very degenerate and have a very large codimension, see e.g. [24]. So we can formulate a general *complexification principle* ([4]) which should be proved in each particular case independently: *the Thom polynomial of a real singularity can be obtained from the Thom polynomial of the corresponding complex singularity by replacing the Chern classes by the corresponding Stiefel-Whitney classes and reducing all coefficients modulo* 2.

1.4. Classifying space of singularities and determination of Thom polynomials

The characteristic classes dual to singularity loci of a smooth map $f : M \to N$ are defined using an auxiliary (V, G)-bundle $E \to M$, see (1), (2). This bundle can be induced from the universal classifying (V, G)-bundle $\mathbf{B}V \to \mathbf{B}G$. The construction of the classifying space $\mathbf{B}V$ presented below is used also in Borel's definition of equivariant cohomology for the G-space V.

Consider the classifying principal G-bundle $\mathbf{E}G \to \mathbf{B}G$, i.e. a contractible space $\mathbf{E}G$ with a free action of the group G. This action extends to the diagonal action on the product space $V \times \mathbf{E}G$.

Definition. ([11, 12]). The *classifying space of singularities* $\mathbf{B}V$ is the total space of the (V, G)-bundle associated with the classifying principal bundle $\mathbf{E}G \to \mathbf{B}G$,

$$\mathbf{B}V = V \times_G \mathbf{E}G = (V \times \mathbf{E}G)/G.$$

The projection to the second factor $\mathbf{B}V \to \mathbf{E}G/G = \mathbf{B}G$ is a bundle with fiber isomorphic to V and structure group G. Since the space V is contractible the projection $\mathbf{B}V \to \mathbf{B}G$ induces an isomorphism of (co)homology groups,

$$H^*(\mathbf{B}V, \mathbb{Z}_2) \cong H^*(\mathbf{B}G, \mathbb{Z}_2).$$

On the other hand each singularity class $\Sigma \subset V$ defines a subspace $\mathbf{B}\Sigma = \Sigma \times_G \mathbf{E}G \subset \mathbf{B}V$. If Σ is an algebraic subvariety then $\operatorname{codim}_{\mathbf{B}V} \mathbf{B}\Sigma = \operatorname{codim}_V \Sigma$ and the cohomology class dual to $\mathbf{B}\Sigma$ is well defined.

Definition. The *Thom polynomial* of the singularity class Σ is the cohomology class $P_\Sigma \in H^*(\mathbf{B}V, \mathbb{Z}_2) = H^*(\mathbf{B}G, \mathbb{Z}_2)$ dual to the locus $\mathbf{B}\Sigma \subset \mathbf{B}V$.

The Lie group G of jets of left-right changes is contractible to its subgroup $\mathrm{GL}(m) \times \mathrm{GL}(n)$ of linear changes and hence to its maximal compact subgroup $\mathrm{O}(m) \times \mathrm{O}(n)$. Therefore the Thom polynomial is an element of the ring

$$H^*(\mathbf{B}G, \mathbb{Z}_2) \cong H^*(\mathbf{B}\mathrm{O}(m) \times \mathbf{B}\mathrm{O}(n), \mathbb{Z}_2) \cong \mathbb{Z}_2[\omega_1, \dots, \omega_m, \omega'_1, \dots, \omega'_n]$$

of polynomials in Stiefel-Whitney classes.

Remark. The classifying spaces $\mathbf{B}G$, $\mathbf{B}V$, etc. have infinite dimensions and thus the definition above should be clarified. The simplest way to overcome this difficulty is to replace the classifying principal bundle $\mathbf{E}G \to \mathbf{B}G$ by a finite dimensional smooth principal bundle $\mathbf{E}G_N \to \mathbf{B}G_N$ with N-connected total space $\mathbf{E}G_N$. Then we get isomorphisms $H^p(\mathbf{B}G_N, \mathbb{Z}_2) \cong H^p(\mathbf{B}G, \mathbb{Z}_2)$ for all $p < N$ and can set

$$P_\Sigma = [\Sigma \times_G \mathbf{E}G_N] \in H^c(V \times_G \mathbf{E}G_N, \mathbb{Z}_2) \cong H^c(\mathbf{B}G_N, \mathbb{Z}_2) \cong \\ \cong H^c(\mathbf{B}G, \mathbb{Z}_2),$$

where $c = \operatorname{codim} \Sigma$. This cohomology class is independent of the choice of the finite dimensional approximation $\mathbf{E}G_N$ provided that $N > \operatorname{codim} \Sigma$.

Remark. In [19] Szücs and Rimányi used an alternative approach to the definition of the classifying space of singularities based on Szücs's idea of gluing the classifying spaces of symmetry groups of singularities. They considered only simple singularities, and the very clear topology of the classifying space does not follow from their construction. It should be noticed nevertheless that their construction works as well for the case of multisingularities, see [17, 18, 20] for some applications. It is an interesting problem to find an *à priori* construction for the classifying space of multisingularities and to describe its topology (the work [19] implies that it should be related to cobordism theory).

1.5. Proof of Theorem 1

Let $f : M \to N$ be a smooth map and $E \to M$ be the associated (V, G)-bundle whose fiber over a point $x \in M$ consists of all K-jets of map germs $(M, x) \to (N, f(x))$. This bundle, like any other G-bundle, can be induced from the classifying space $\mathbf{B}G$ by some continuous map $\kappa : M \to \mathbf{B}G$. This map extends to the map $\tilde{\kappa} : E \to \mathbf{B}V$ of total spaces of (V, G)-bundles. Thus we get the diagram of maps

$$M \xrightarrow{s_f} E \xrightarrow{\tilde{\kappa}} \mathbf{B}V.$$

The maps in this diagram induce both characteristic classes and partitions by singularity classes. Hence the induced homomorphism of cohomology groups

$$H^*(\mathbf{B}G, \mathbb{Z}_2) \xrightarrow{\bar{\kappa}^*} H^*(E, \mathbb{Z}_2) \xrightarrow{s_f^*} H^*(M, \mathbb{Z}_2)$$

sends the Thom polynomial of Σ to the corresponding polynomial in Stiefel-Whitney classes of M and N, and the cohomology class dual to the singularity locus $\mathbf{B}\Sigma$ to the cohomology class dual to the singularity locus $\Sigma(f)$. □

1.6. Stabilization

One of the most important invariants of a map germ $y = f(x)$, $f : (\mathbb{R}^m, 0) \to (\mathbb{R}^{m+k}, 0)$ is the *local algebra* $Q_f = \mathfrak{m}_x / f^* \mathfrak{m}_y$. This is the quotient algebra of the algebra of function germs $(\mathbb{R}^m, 0) \to (\mathbb{R}, 0)$ over the ideal generated by the components f_i of the germ f.

Definition. Two map germs $f : (\mathbb{R}^m, 0) \to (\mathbb{R}^{m+k}, 0)$ and $f' : (\mathbb{R}^{m'}, 0) \to (\mathbb{R}^{m'+k}, 0)$ (with the same k and possibly different m, m') are called *stably equivalent* if they have isomorphic local algebras, $Q_f \cong Q_{f'}$. A class Σ of singularities is called *stable* if it contains together with each map germ f any map germ stably equivalent to it. (Do not confuse with the notion of a stable singularity!)

The stabilization allows us to compare singularities of map germs of manifolds of different dimensions. The codimension of a stable singularity class of map germs $(\mathbb{R}^m, 0) \to (\mathbb{R}^{m+k}, 0)$ given by some collection of local algebras does not depend on m (but it *does* depend on k). Therefore for each k we get an independent problem of stable classification of map germs $(\mathbb{R}^m, 0) \to (\mathbb{R}^{m+k}, 0)$ for all m.

THEOREM 2 (cf. [5]). *The Thom polynomial of any stable class of singularities can be expressed as a polynomial in the relative Stiefel-Whitney classes $\omega_i(f^*TN - TM)$ defined as the homogeneous components of the expression*

$$\omega(f^*TN - TM) = \frac{1 + f^*\omega_1(N) + f^*\omega_2(N) + \dots}{1 + \omega_1(M) + \omega_2(M) + \dots}.$$

Proof. First note that the proof of Theorem 1 implies that the statement can be extended to a generic section of any (V, G)-bundle over M, not necessary related to a map $M \to N$. A (V, G)-bundle and a section may be given by a family of germs of manifolds E_x,

F_x, and of map germs $E_x \to F_x$ depending on the point $x \in M$. In particular, to any map $f : M \to N$ we associate the family of map germs $f_x : (M, x) \to (N, f(x))$, $x \in M$. The Stiefel-Whitney classes in this extended version of Theorem 1 are those of the vector bundles $\bigcup_x T_0 E_x$ and $\bigcup_x T_0 F_x$ respectively.

Now a map germ $x \mapsto f(x)$ is stably equivalent to the map germ $(x, z) \mapsto (f(x), z)$ where $z = (z_1, \dots, z_l)$ is any number of additional variables. This observation may be globalized as follows. The jet extension of a map $f : M \to N$ may be considered as the family of map germs $f_x : (M, x) \to (N, f(x))$ depending on the point $x \in M$. Consider an arbitrary vector bundle $U \to M$. Then we may construct a new family of map germs

$$f_x \times \mathrm{id} : (M \times U_x, x \times 0) \to (N \times U_x, f(x) \times 0).$$

The map germs of the new family are stably equivalent to those of the original one. Therefore the singularity loci and their cohomology classes coincide for both families. By the extended version of Theorem 1 mentioned above the cohomology class dual to the locus of singularity Σ is given by $P_\Sigma(\omega(TM \oplus U), \omega(f^*TN \oplus U))$. We may chose the bundle U arbitrary. For example, we can chose it in such a way that the bundle $TM \oplus U$ is trivial. Then $\omega(f^*TN \oplus U) = \omega(f^*TN - TM)$ and we get

$$P_\Sigma(\omega(TM), \omega(f^*TN)) = P_\Sigma(1, \omega(f^*TN - TM)). \qquad \square$$

Theorem 2 implies that for computing Thom polynomials of a stable class of singularities it is sufficient to consider the case when the target space is a fixed Euclidean space $N = \mathbb{R}^n$.

Remark. The formal definition of the classifying space given above admits a very simple geometrical realization of this space. We present a construction for a version of the classifying space which takes into account the stabilization used in Theorem 2. For fixed k, K choose large integers $m, N \gg 0$ and consider the Euclidean space $\mathbb{R}^{N+n} = \mathbb{R}^N \times \mathbb{R}^n$, $n = m + k$. Denote by $\mathcal{G}_m(\mathbb{R}^{N+n})$ *the manifold of all K-jets of germs at 0 of m-dimensional submanifolds in* $(\mathbb{R}^{N+n}, 0)$. This manifold is homotopy equivalent to the Grassmannian $G_m(\mathbb{R}^{N+n})$ (since the space of germs of submanifolds with a fixed tangent plane is contractible) and the $\mathbb{Z}_2$-cohomology ring of this space is generated by Stiefel-Whitney classes. The points of this space are classified according to the singularities of the projection to the coordinate plane $\mathbb{R}^n \subset \mathbb{R}^N \times \mathbb{R}^n$. Denote by $\tilde{\Sigma} \subset \mathcal{G}_m(\mathbb{R}^{N+n})$ the collection of points for which this projection belongs to the given stable class Σ of singularities. The cohomology class dual to this cycle

$$[\widetilde{\Sigma}] \in H^*(\mathcal{G}_m(\mathbb{R}^{N+n}), \mathbb{Z}_2)$$

(or, more exactly, the expression of this class in terms of the multiplicative generators of the cohomology ring of the Grassmannian) may be taken as an independent definition of the Thom polynomial. This construction reduces the problem of finding Thom polynomials to the study of the geometry of Grassmannians.

Remark. Similar stabilizations allow one to compare singularities in the spaces of different dimensions exist for other problems of singularity theory. For instance, the stable classification of critical points of functions is related to Lagrange singularities and leads to Lagrange characteristic classes, see [22, 14] and Section 3.4 below.

2. Computing Thom polynomials

A large number of Thom polynomials for various kinds of singularities were found by different authors, see [2, 17] and references therein. As an example of computation of Thom polynomials, we present several proofs of the classical formulas for the Thom-Porteous classes. These proofs illustrate different methods which can be used for other classes. In this section by cohomology we mean cohomology with $\mathbb{Z}_2$-coefficients.

2.1. Thom-Porteous classes

Let E, F be two vector bundles of ranks $m, n = m + k$ respectively over a smooth manifold M and $f : E \to F$ be a generic morphism of vector bundles. Denote by $\Sigma_d \subset M$ the set of points $x \in M$ where the rank of the linear map $f_x : E_x \to F_x$ is at most $m - d$ (i.e. the dimension of the kernel of f_x is at least d; we assume that $d \geq \max(0, -k)$). The classes dual to the loci Σ_d are called *Thom-Porteous classes*.

THEOREM 3 ([16]). *Generically Σ_d is a subvariety of codimension $d(d+k)$ and its Thom polynomial is given by $[\Sigma_d] = \Delta_{d,d+k}(\omega(F - E))$, where for a formal series $a = 1 + a_1 + a_2 + \ldots$ we denote*

$$\Delta_{p,q}(a) = \begin{vmatrix} a_q & a_{q+1} & \cdots & a_{q+p-1} \\ a_{q-1} & a_q & \cdots & a_{q+p-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{q-p+1} & a_{q-p+2} & \cdots & a_q \end{vmatrix}.$$

In particular, let $f : M \to N$ be a generic smooth map. Set $E = TM$, $F = f^*TN$. Then the formula of Theorem 3 expresses the Thom polynomial for the locus $\Sigma_d(f)$ consisting of points $x \in M$ where the rank of the differential $df_x : T_xM \to T_{f(x)}N$ is at most $m - d$.

In the case of Thom-Porteous classes the singularity type is determined by 1-jet of the map and it is sufficient to set $K = 1$. The classifying space is the Grassmann manifold $G_m(\mathbb{R}^{N+n})$, where $N, m \gg 0$, $n = m + k$. The cycle $\widetilde{\Sigma}_d$ is formed by the m-planes whose projections to the fixed subspace $\mathbb{R}^n$ have rank at most $m - d$. Equivalently, it is formed by planes whose intersections with the fixed subspace $\mathbb{R}^N$ are at least d-dimensional. Theorem 3 is equivalent therefore to the equality

$$[\widetilde{\Sigma}_d] = \Delta_{d,d+k}(\omega(-E)) \in H^*(G_m(\mathbb{R}^{N+n})), \tag{4}$$

where E denotes the m-dimensional tautological vector bundle on the Grassmannian.

2.2. Schubert calculus

The cohomology group of the Grassmannian has two natural bases. The first is given by monomials in Stiefel-Whitney classes and the second by classes of Schubert cells. The Schubert basis is more geometric and in many cases it is possible to express the cohomology classes given by particular cycles on the Grassmannian via Schubert cells. The passage between the two bases is described by the Giambelli formula ([8]).

In the case of Theorem 3, the cycle $\widetilde{\Sigma}_d \subset G_m(\mathbb{R}^{N+n})$ is the closure of the Schubert cell denoted by $(d + k, \dots, d + k)$ (d entries) in the Schubert calculus. The formula (4) is a particular case of the Giambelli formula. □

2.3. Resolutions of singularities

Another direct method of proving Theorem 3 uses resolutions of singularities. This method may be used for proving Giambelli formulas as well as for finding Thom polynomials for other singularities. Consider the product space $G_d(\mathbb{R}^N) \times G_m(\mathbb{R}^{N+n})$, where we identify $\mathbb{R}^N$ with a fixed N-subspace in $\mathbb{R}^{N+n}$. Consider the submanifold $Z \subset G_d(\mathbb{R}^N) \times G_m(\mathbb{R}^{N+n})$ formed by pairs $(K, L) \in G_d(\mathbb{R}^N) \times G_m(\mathbb{R}^{N+n})$ such that $K \subset L$. Then $\widetilde{\Sigma}_d$ is the image of Z under the projection

$$\pi : G_d(\mathbb{R}^N) \times G_m(\mathbb{R}^{N+n}) \to G_m(\mathbb{R}^{N+n})$$

to the second factor. Moreover the restriction $\pi|_Z : Z \to \Sigma_d$ is one-to-one over an open dense set, so that $[\widetilde{\Sigma}_d] = \pi_*[Z]$, where π_* :

$H^*(G_d(\mathbb{R}^N) \times G_m(\mathbb{R}^{N+n})) \to H^*(G_m(\mathbb{R}^{N+n}))$ is the Gysin homomorphism. So the problem is split into two: computing the class $[Z]$ and computing the homomorphism π_*.

For the first problem we note that the cycle Z may be identified with the zero section of the bundle $\mathrm{Hom}(K, \mathbb{R}^{N+n}/L)$, where we denote by K, L the tautological bundles on the two Grassmannians. Therefore $[Z]$ is the top Stiefel-Whitney class of this bundle,

$$[Z] = \omega_{d(N+k)}(\mathrm{Hom}(K, \mathbb{R}^{N+n}/L)) = \Delta_{d,N+k}(\omega(-K-L)).$$

The last equality can by proved purely algebraically using, for example, the splitting principle.

Now we compute the homomorphism π_*. Denote by $Q = \mathbb{R}^N/K$ the universal $(N-d)$-dimensional quotient bundle on $G_d(\mathbb{R}^N)$. Then

$$\omega_s(-K-L) = \omega_s(Q-L) = \sum_{i=0}^{N-d} \omega_{N-d-i}(Q)\omega_{s-(N-d)+i}(-L).$$

Substituting this in the determinant $\Delta_{d,N+k}$ we obtain

$$[Z] = \Delta_{d,N+k}(\omega(Q-L)) = (\omega_{N-d}(Q))^d \Delta_{d,d+k}(\omega(-L)) + \dots,$$

where the dots denote terms whose degree in $\omega_i(Q)$ is strictly less than $d(N-d) = \dim G_d(\mathbb{R}^N)$. The Gysin homomorphism vanishes on these terms for dimensional reasons and thus

$$[\widetilde{\Sigma}_d] = \pi_*[Z] = \pi_*((\omega_{N-d}(Q))^d)\, \Delta_{d,d+k}(\omega(-L)).$$

It remains to note that the equality $\pi_*((\omega_{N-d}(Q))^d) = 1$ reflects the fact that given d generic lines in $\mathbb{R}^N$, there exists a unique d-plane containing them. □

2.4. Symmetries of singularities

Recently R. Rimányi [17] invented a new method for finding Thom polynomials. His method reduces this problem to the linear algebra problem of inverting a large matrix. In general it requires less computations for computing particular Thom polynomials, though it usually does not give closed formulas for series of singularity classes. The main idea is very simple. Since the Thom polynomials are universal, every example where we may compute both the Stiefel-Whitney classes and the class dual to the singularity locus gives linear relations on the coefficients of the Thom polynomial. If the number of examples is large enough then these relations should be sufficient to determine the polynomial completely.

Many examples may be produced in the following way. Let Σ be an orbit of the action of the equivalence group G on the jet space V. Then as a test manifold we can take a tubular neighborhood U of the submanifold $\mathbf{B}\Sigma$ in the classifying space $\mathbf{B}V$ (it can be identified with the total space of a normal bundle of $\mathbf{B}\Sigma$). The test manifold in this case is homotopy equivalent to $\mathbf{B}\Sigma = (\Sigma \times \mathbf{E}G)/G = (\mathrm{pt} \times \mathbf{E}G)/G_\Sigma \cong \mathbf{B}G_\Sigma$, where G_Σ is the 'symmetry group' of the singularity Σ, the stationary group of any point pt $\in \Sigma$ (or a maximal compact subgroup in it). Moreover the normal bundle of $\mathbf{B}\Sigma$ may be identified with the space of the universal bundle over $\mathbf{B}G_\Sigma$ associated with the action of G_Σ on any G_Σ-invariant transversal slice to Σ. The locus of the singularity Σ for this test manifold is the zero section of the normal bundle and hence its dual coincides with the Euler class of the bundle. It is usually not difficult to describe explicitly the homomorphism $H^*(\mathbf{B}G) \to H^*(\mathbf{B}G_\Sigma)$ and to compute the corresponding Euler class.

In the particular case of Thom-Porteous singularities the arguments above can be reduced to the following. Consider vector bundles K, L of ranks d, $d+k$ respectively with Stiefel-Whitney classes $a_i = \omega_i(K)$, $b_j = \omega_j(L)$ over some smooth base B. Let the test manifold M be the total space of the bundle $\mathrm{Hom}(K, L)$ over B. The singularity locus $\Sigma_d(M)$ in this case is the zero section of the bundle $M \to B$. Therefore

$$\begin{aligned}[\Sigma_d(M)] &= \omega_{d(d+k)}(\mathrm{Hom}(K, L)) \\ &= \Delta_{d,d+k}(\omega(L-K)) \in H^*(B) \cong H^*(M).\end{aligned}$$

(The last equality is an algebraic exercise on the application of the splitting principle.) The base B can be chosen arbitrary. For example we can chose B to be (a finite dimensional approximation of) the product space $\mathbf{BO}(d) \times \mathbf{BO}(d+k)$ of the classifying spaces for d- and $(d+k)$-dimensional vector bundles respectively, and K, L to be the corresponding canonical bundles. We see that the Thom polynomial $P_{\Sigma_d}(\omega_1, \omega_2, \ldots)$ has the following property: *after the substitution* $\omega = \frac{1+a_1+\ldots+a_d}{1+b_1+\ldots+b_{d+k}}$ *it coincides with* $\Delta_{d,d+k}(\omega)$. One can verify that the homomorphism $H^*(\mathbf{BO}) \to H^*(\mathbf{BO}(d) \times \mathbf{BO}(d+k))$ given by $1 + \omega_1 + \ldots \mapsto \frac{1+a_1+\ldots+a_d}{1+b_1+\ldots+b_{d+k}}$ is injective up to degree $d(d+k)$ so the relation above determines the polynomial completely. □

Remark. There are many results in global singularity theory that involve classes dual to cycles of multisingularities (see, for example, [2] and references therein). The method of Rimányi may be effectively applied to this kind of problem as well, see [17, 18].

3. Universal complex of singularity classes and characteristic spectral sequence

The singularity classes in *real* classification problems usually form semi-algebraic rather than algebraic subvarieties in the jet space. In order for the cohomology class dual to some union of singularity classes to be well defined, the (formal) boundary of this union must vanish. A similar problem appears when one tries to define an integer characteristic class dual to some combination of singularity classes. In this case all singularity classes of this combination must be co-oriented and the (formal) co-oriented boundary of the combination must vanish. These observations are formalized in the notions of the universal complex of singularity classes [22] and the characteristic spectral sequence [11, 12, 15].

3.1. Classifications

Consider a classification problem of singularity theory formulated as the classification of orbits of an equivalence Lie group G acting on a contractible jet space V. A *finite G-classification* ([22]) is a finite G-invariant Whitney stratification of V. If the group G is not connected then its elements may permute some strata. Unions of strata containing points of one orbit are called *classes*.

Some classes may consist of only one orbit. For other classes orbits may form families (modules). If, for each class, the moduli space of orbits is smooth and contractible then the G-classification is called *cellular*. In this case, a maximal compact subgroup in the stationary group is independent (up to an isomorphism) of a point of the given class Σ. This group is called the *symmetry group* of the class. The existence of cellular G-classifications for any algebraic action is proved in [22].

A singularity class $\Sigma \subset V$ is called *co-orientable* if it admits a G-invariant co-orientation in V. For cellular classifications this is equivalent to the condition that the symmetry group preserves the orientation of the normal space to the class.

3.2. Characteristic spectral sequence

For a given G-classification on V consider the filtration formed by *open* subspaces

$$F_0(V) \subset F_1(V) \subset \cdots \subset V,$$

where F_i is the union of classes of codimension less than or equal to i. This filtration defines an *equivariant* spectral sequence $E_*^{*,*}$ called the *characteristic spectral sequence*. This sequence converges to the *equivariant* cohomology $H^*_G(V)$ of V. Since V is contractible, one has $H^*_G(V) \cong H^*_G(\mathrm{pt}) \cong H^*(\mathbf{B}G)$.

The reformulation of this definition in the language of classical cohomology groups is as follows. The filtration on V induces a corresponding filtration on the classifying space $\mathbf{B}V = V \times_G \mathbf{E}G$,

$$F_0(\mathbf{B}V) \subset F_1(\mathbf{B}V) \subset \cdots \subset \mathbf{B}V, \qquad F_p(\mathbf{B}V) = F_p(V) \times_G \mathbf{E}G.$$

The characteristic spectral sequence $E_*^{*,*}$ defined by this filtration converges to $H^*(\mathbf{B}V) \cong H^*(\mathbf{B}G)$.

This spectral sequence contains all cohomological information on adjacencies of singularities and their symmetry groups. Its initial term $E_1^{p,*} \cong H^*_G(F_p(V), F_{p-1}(V)) \cong H^*(F_p(\mathbf{B}V), F_{p-1}(\mathbf{B}V))$ is isomorphic to the cohomology group of the Thom space of the normal bundle of the codimension p smooth manifold $F_p(\mathbf{B}V) \setminus F_{p-1}(\mathbf{B}V)$. It is the direct sum of the cohomology groups of the corresponding Thom spaces over all classes of codimension p. In the case of cellular classifications both the submanifold $\mathbf{B}\Sigma \cong \mathbf{B}G_\Sigma$ and its normal bundle (the universal bundle over $\mathbf{B}G_\Sigma$ corresponding to the action of G_Σ on a G_Σ-invariant transversal slice to Σ) may be determined intrinsically in terms of the singularity class Σ and its symmetry group G_Σ. The first differential δ_1 is given by adjacencies of singularities of neighboring codimensions; the higher differentials δ_r correspond to adjacencies of singularities whose codimensions differ by r; for details see [11, 12, 15] and the recent preprint [7].

3.3. Universal complex of singularity classes

The cohomology classes corresponding to the fundamental cycles of singularity loci are described by the row $E_*^{*,0}$ of the spectral sequence.

Definition. The row $(E_1^{*,0}, \delta_1)$ is called the ***universal complex of singularity classes.***

The cohomology classes of this complex give rise to well defined characteristic classes via the canonical homomorphism

$$E_2^{*,0} \to E_\infty^{*,0} \subset H^*(\mathbf{B}G).$$

Among these are, for example, the cohomology classes dual to algebraic singularity loci. Below we give an abstract geometric-algebraic definition of this complex as it appeared in [22].

In the case of cohomology with $\mathbb{Z}_2$-coefficients, the free generators of this complex in degree p correspond to the singularity classes of codimension p. The differential is given by

$$\delta\Sigma = \sum_{\operatorname{codim}\Omega = \operatorname{codim}\Sigma + 1} [\Sigma, \Omega]\, \Omega,$$

where the *incidence coefficients* $[\Sigma, \Omega] \in \mathbb{Z}_2$ are defined as follows. Consider a germ of some $(\operatorname{codim}\Omega)$-dimensional transversal T to the class $\Omega \subset V$. The points of singularity type Σ form a collection of curves in T going out of the origin. The coefficient $[\Sigma, \Omega]$ is equal to the parity of the number of these curves.

In the case of integer coefficients, the term of degree p in the universal complex is freely generated by *co-orientable* classes (with some fixed choice of the co-orientations). The coboundary operator is defined in a similar way, but now the incidence coefficient $[\Sigma, \Omega]$ is an integer. It is defined as the algebraic number of curves (together with their signs) of singularity type Σ in the transversal T to the singularity class Ω. The *sign* of every such curve (positive or negative) is defined as follows. Consider a small sphere in T centered at the origin. This sphere is oriented as the boundary of a small ball oriented by the chosen co-orientation of Ω. In a neighborhood of the intersection point with a curve of singularity Σ the sphere has an additional orientation as the germ of a transversal to the singularity class Σ. The sign is positive (negative) if the two orientations on the sphere coincide (respectively, are opposite).

3.4. Example: fiber singularities of functions

A number of applications of the notions introduced in this section to different problems of singularity theory are considered in [22], [23] and [12]–[15]. In this section we discuss characteristic classes related to the classification of critical points of functions. Consider the following diagram of holomorphic maps of complex analytic manifolds:

$$\begin{array}{ccc} W & \xrightarrow{f} & C \\ \downarrow{\scriptstyle\pi} & & \\ B & & \end{array} \tag{5}$$

We assume that the differential of π is surjective at every point, so the fibers of π form locally a smooth fibration; C is a complex curve. (The case when π is the trivial bundle and $C = \mathbb{C}P^1$ is already interesting enough.) We study singularities of the restrictions of f to the fibers. Let $M \subset W$ be the subset of all critical points of such restrictions.

Generically M is smooth and has codimension $n = \dim W - \dim B$. It can be identified with the zero locus of the section $df|_V$ of the bundle $\mathrm{Hom}(V, I)$, where $V \subset TW$ is the subbundle of vectors tangent to the fibers of π and I is the complex line bundle $I = f^*TC$.

Let Ω be any class of singularities of functions (an algebraic subvariety in some jet space of function germs $\mathbb{C}^n, 0 \to \mathbb{C}, 0$ which is invariant with respect to the group of left-right changes of co-ordinates). We shall use the same letter Ω to denote the class of function germs $\mathbb{C}^{n'}, 0 \to \mathbb{C}, 0$, $n' \neq n$, stably equivalent to the functions from Ω. Recall that two germs of functions on spaces of possibly different dimensions are called *stably equivalent* if after adding suitable non-degenerate quadratic forms in new variables they can be reduced to each other by a left-right change of variables.

Define $\Omega(f) \subset M$ as the locus of points at which the restriction of f to the fiber belongs to the given singularity class Ω. According to the general principle of Thom the cohomology class Poincaré dual to the locus $\Omega(f)$ is independent of f (provided that genericity conditions for f analogous to those of Section 1.2 are satisfied) and can be expressed as a universal polynomial in Chern classes of W, B, C. We claim that this polynomial can be expressed in terms of some particular combinations of these classes. Namely, denote $u = c_1(I) = f^*c_1(TC)$, $c_i = c_i(V) = c_i(TW - \pi^*TB)$, and define the classes $a_i = c_i(V^* \otimes I - V)$ as the homogeneous components in the expansion of

$$1 + a_1 + a_2 + \ldots = \frac{(1+u)^n - (1+u)^{n-1}c_1 + (1+u)^{n-2}c_2 - \ldots \pm c_n}{1 + c_1 + c_2 + \ldots + c_n}. \quad (6)$$

These classes satisfy the relations

$$(1 + a_1 + a_2 + \ldots)\Big(1 - \frac{a_1}{1+u} + \frac{a_2}{(1+u)^2} - \ldots\Big) = 1, \quad (7)$$

following from the identity $U + U^* \otimes I = 0$, where U is the formal difference $U = V^* \otimes I - V$. These relations allow us to expand the squares of the classes a_i, and hence any polynomial in $u, a_1, a_2, \ldots$ can be expressed as a linear combination of monomials $u^{i_0} a_1^{i_1} a_2^{i_2} \ldots$, $i_0 \geq 0$, $i_k \in \{0, 1\}$ $(k > 0)$.

THEOREM 4 ([14]). *For any singularity class Ω, the class in $H^*(M)$ Poincaré dual to the locus $\Omega(f)$ can be expressed as a universal polynomial P_Ω in $u, a_1, a_2, \ldots$. This polynomial (called the* Thom polynomial*) is independent of n.*

The Poincaré dual of the locus $\Omega(f)$ considered as a locus in W is equal to $(u^n - u^{n-1}c_1 + \ldots \pm c_n)\, P_\Omega(u, a_1, a_2, \ldots) \in H^*(W)$.

For the singularity classes of codimension not greater than 6 *the Thom polynomials are given in Table* I.

Table I. Thom polynomials of singularities of functions of codim ≤ 6

$$
\begin{aligned}
A_2 &= a_1 \\
A_3 &= 3a_2 \quad + ua_1 \\
A_4 &= 3a_1a_2+6a_3 \quad + 4ua_2 + u^2a_1 \\
D_4 &= a_1a_2-2a_3 \quad - ua_2 \\
A_5 &= 27a_1a_3+6a_4 \quad + u(16a_1a_2-12a_3) - 4u^2a_2 + u^3a_1 \\
D_5 &= 6a_1a_3-12a_4 \quad + u(4a_1a_2-14a_43) - 4u^2a_2 \\
A_6 &= 87a_2a_3+54a_1a_4+78a_5 \quad + \\
&\qquad u(127a_1a_3-53a_4) + u^2(59a_1a_2-126a_3) - 41u^3a_2 + u^4a_1 \\
D_6 &= 12a_2a_3-24a_5 \quad + u(14a_1a_3-40a_4) + u^2(8a_1a_2-30a_3) - 8u^3a_2 \\
E_6 &= 9a_2a_3-12a_1a_4+6a_5 \quad + 3ua_4 + u^2(3a_1a_2-6a_3) - 3u^3a_2 \\
A_7 &= 135a_1a_2a_3+465a_2a_4+264a_1a_5+522a_6 \quad + u(516a_2a_3-16a_1a_4+485a_5) + \\
&\qquad u^2(305a_1a_3-70a_4) + u^3(190a_1a_2-440a_3) - 165u^4a_2 + u^5a_1 \\
D_7 &= 24a_1a_2a_3-24a_2a_4+48a_1a_5-144a_6 \quad + u(8a_2a_3+44a_1a_4-224a_5) + \\
&\qquad u^2(48a_1a_3-172a_4) + u^3(20a_1a_2-88a_3) - 20u^4a_2 \\
E_7 &= 9a_1a_2a_3+6a_2a_4-42a_1a_5+36a_6 \quad + u(21a_2a_3-61a_1a_4+80a_5) + \\
&\qquad u^2(43a_4-6a_1a_3) + u^3(7a_1a_2-8a_3) - 7u^4a_2 \\
P_8 &= a_1a_2a_3-6a_2a_4+6a_1a_5-4a_6 \quad + \\
&\qquad u(7a_1a_4-4a_2a_3-10a_5) + u^2(2a_1a_3-8a_4) - 2u^3a_3
\end{aligned}
$$

This theorem can be formally applied to the case when the manifolds W, B, C are real and the function f is smooth (with Chern classes replaced by the corresponding Stiefel-Whitney classes). But in the real case the $\mathbb{Z}_2$-classes u, a_i vanish and so all cohomology classes of singularities are trivial. Moreover ([12, 15]), *for any locally trivial fiber bundle with compact fibers there exists a real-valued function on the total space whose restrictions to the fibers have no singularities more complicated than* A_2.

The $\mathbb{Z}_2$-reductions of the polynomials listed in Table I are non-trivial when they are considered in the context of the *theory of Lagrange and Legendre singularities*, see [22, 23, 11, 14]. Non-trivial classes appear also if we consider the *global* singularities of the restriction of f to the fibers. Assume that in diagram (5) π is a smooth locally trivial bundle with oriented fibers diffeomorphic to S^1. Then the Chern-Euler class $e = c_1(\pi) \in H^2(B)$ of the bundle π can be interpreted as follows in terms of the fiber singularities of a generic smooth function $f : W \to \mathbb{R}$ on the total space of the bundle.

We study the global minima of the restrictions $f_b : W_b \cong S^1 \to \mathbb{R}$, $b \in B$. Denote by $(a_1, \dots, a_l) = (a_1, \dots, a_l)_f \subset B$ the locus of points $b \in B$ such that the function f_b attains its global minimum at l consecutive points $x_1, \dots, x_l$ on the circle W_b, and has a critical point of multiplicity a_i at x_i (i.e. f_b is equivalent to $(x - x_i)^{a_1+1}$ near x_i). The numbers a_i are odd positive integers; their order is defined up to a cyclic permutation. Generically the locus $(a_1, \dots, a_l)$ is smooth and has codimension $(\sum a_i) - 1$.

THEOREM 5 ([13]). *Every singularity class $(a_1, \dots, a_l) \subset B$ of even codimension has a natural co-orientation. For any integer $r > 0$ there is a universal (independent of f) linear combination with rational coefficients of the classes $(a_1, \dots, a_l)$ of codimension $2r$ such that the cohomology class dual to this combination is well defined and equals the characteristic class e^r of the bundle π. For $r \leq 4$ these combinations are given in Table* II.

Table II. Characteristic classes of $\mathrm{codim}_{\mathbb{R}} \leq 8$ singularities of the global minimum.

$$
\begin{aligned}
-2e &= (1^3) - (3)\,, \\
12e^2 &= (1^5) - (3,1^2) + 2(5)\,, \\
-120e^3 &= (1^7) - (3,1^4) + (3^2,1) + 2(5,1^2) - 5(7)\,, \\
1680e^4 &= (1^9) - (3,1^6) + 2(5,1^4) + \tfrac{31}{15}(3^2,1^3) - \tfrac{1}{15}(3,1^2,3,1) - \\
&\quad \tfrac{21}{5}(3^3) - \tfrac{14}{15}(5,3,1) - \tfrac{14}{15}(5,1,3) - \tfrac{91}{15}(7,1^2) + 14(9)\,.
\end{aligned}
$$

In Table II (m^l) stands for $(m, \dots, m)$ (l times). The universal complex of singularity classes responsible for this problem is closely related to cyclic homology theory. This relation is studied in [13].

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