HOMOLOGY OF SPACES OF KNOTS IN ANY DIMENSIONS

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I shall describe the recent progress in the study of cohomology rings of spaces of knots in $\mathbb{R}^n$, $H^*(\{\text{knots in } \mathbb{R}^n\})$, with arbitrary $n \geq 3$. "Any dimensions" in the title can be read as dimensions n of spaces $\mathbb{R}^n$, as dimensions i of the cohomology groups H^i, and also as a parameter for different generalizations of the notion of a knot.

An important subproblem is the study of *knot invariants*; in our context they appear as 0-dimensional cohomology classes of the space of knots in $\mathbb{R}^3$. It turns out that our more general problem is never less beautiful. In particular, nice algebraic structures arising in the related homological calculations have equally (or maybe even more) compact description, of which the classical "zero-dimensional" part can be obtained by easy factorization; see especially §2.5.

There are many good expositions of the theory of related knot invariants (for some references see [11]); therefore I shall deal almost completely with results in higher (or arbitrary) dimensions.

1. Main construction

We consider both the standard *compact* knots, i.e. smooth embeddings $S^1 \to \mathbb{R}^n$, and the *long knots*, i.e. embeddings $\mathbb{R}^1 \to \mathbb{R}^n$ coinciding with a standard linear embedding outside some compact subset in $\mathbb{R}^1$, see Fig. 1.

The study of the latter space is more essential, because the algebraic structure of the cohomology ring of the space of standard knots is built of that of the similar ring for long knots (which plays here the role of the "coefficient ring") and the topological nontriviality of the circle S^1 and certain its configuration spaces.

Let us denote by $\mathcal{K}$ the space of all smooth maps $S^1 \to \mathbb{R}^n$ (respectively, of maps $\mathbb{R}^1 \to \mathbb{R}^n$ with such boundary conditions). This is a linear (respectively, an affine)

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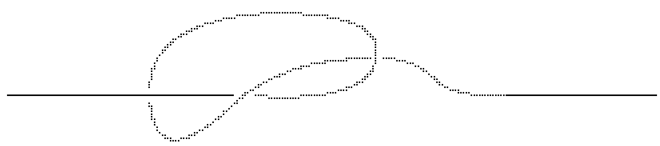

Figure 1. A long knot

space. The *discriminant* $\Sigma \subset \mathcal{K}$ is the set of all maps which are not smooth embeddings, i.e. have either self-intersections or singular points. The space of knots is the difference $\mathcal{K} \setminus \Sigma$.

1.1. **Arnold's reduction.** It is convenient to study the cohomology group of the space of knots by a sort of the Alexander duality,

$$\tilde{H}^i(\mathcal{K} \setminus \Sigma) \simeq \bar{H}_{n\infty - i - 1}(\Sigma). \tag{1}$$

The bar in the notation $\bar{H}_*$ means that we consider *Borel–Moore* homology, i.e. the homology group of the one point compactification, and $n\infty$ is the notation for the dimension of $\mathcal{K}$. Of course, the whole right-hand part in (1) is, strictly speaking, senseless. However it can be given some strict sense by means of appropriate finite-dimensional approximations to the space $\mathcal{K}$, see §1.8: roughly speaking, the elements of this group are the semialgebraic cycles of *codimension* $i + 1$ in $\mathcal{K}$. A reduction like (1) was used first by V.I. Arnold [4] (in the finitedimensional situation of the standard discriminant varieties in the space of polynomials in $\mathbb{C}^1$) and is very useful in the whole theory of discriminants. Indeed, the discriminant sets of singular maps are singular varieties, stratified in the correspondence with the classifications of (multi)singularities, and (as we shall see in our special case) a lot of their topological properties can be expressed in the terms of these stratifications.

1.2. **Simplicial resolutions.** Further, it is convenient to study the topology of discriminants by means of the *simplicial* (or, more generally, *conical*) resolutions. These resolutions provide topological spaces homotopy equivalent to initial ones (in particular having the same homology groups), but having more transparent homological structure which is easier to calculate. An important illustration of this method comes from the theory of *plane arrangements.*

Let us consider a finite collection of affine planes (of arbitrary dimensions) in $\mathbb{R}^m$,

$$L = \bigcup_{i=1}^{N} L_i, \tag{2}$$

and suppose that we need to calculate the cohomology group of its complement $\mathbb{R}^m \setminus L$ (or, equivalently, the Alexander dual group $\bar{H}_*(\Sigma)$).

The resolutions of three *line* arrangements shown in the lower part of Fig. 2 are given in its upper part. On the first step we take these lines separately, and then add some furniture spanning the points of separated lines arising from one and the same point below. For two left pictures all standard constructions of simplicial resolutions give essentially one and the same space. Namely, if we have a double intersection point of the arrangement then we mark the corresponding two points on separated lines and join them by a segment. However, there are two main different ways to resolve the right-hand arrangement.

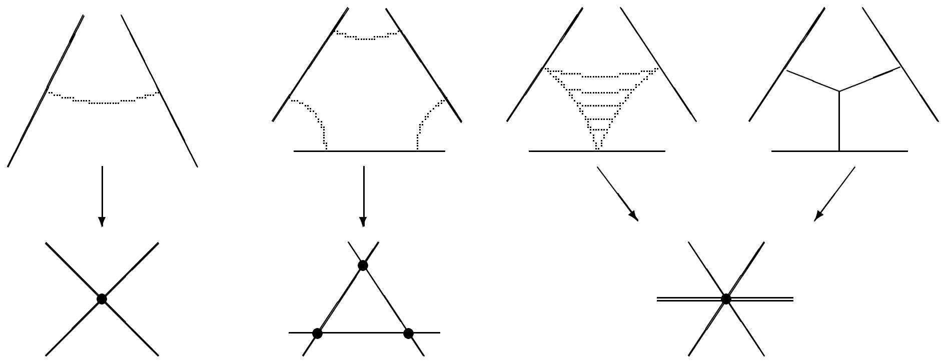

FIGURE 2. Simplicial resolutions of line arrangements

One of them (modelling the combinatorial formula of inclusions and exclusions) will first join by segments all pairs of intersection points of any two planes (independently on whether these points belong to some planes more or not). Then over all triple intersection points of the arrangement we obtain a triple of segments forming a triangle "without interior part", which will be filled on the next step; on the next step the preimages of quadruple points (if any) will be filled by tetrahedra, etc. We shall call this resolution (and its generalizations) the *naive* resolution, in contrast with the *economical* one, which provides the upper right-hand picture in Fig. 2 and is based on the notion of the *order complex* of a partially ordered set (=*poset*), cf. [24].

Definition 1. *Given a poset $(A, >)$, the corresponding order complex $P(A)$ is the simplicial complex, whose vertices are the points of the set A, and the simplices span all the sequences of such points monotone with respect to the partial order.*

An important family of posets is provided by the theory of plane arrangements. Given such an arrangement (2), for any subset I of the set of indices $\{1, \dots, N\}$ denote by L_I the plane $\bigcap_{i \in I} L_i$. All planes of the form L_I are (not canonically) called the *strata* of the arrangement L. The set of all strata is a poset (by inclusion); let $P(L)$ be the corresponding order complex. The order complexes of three arrangements of Fig. 2 are shown in Fig. 3; here the vertex labeled by (12) denotes the intersection plane (point) of the first and the second planes (lines) labeled by (1) and (2) respectively.

The economical simplicial resolution of the arrangement L will be defined as a subset of the direct product $P(L) \times \mathbb{R}^m$. For any nonempty stratum L_I, let $\Delta(I) \subset P(L)$ be the *order subcomplex subordinate to* L_I, i.e. the subcomplex of $P(L)$ consisting of only those simplices all whose vertices correspond to planes containing L_I (or coinciding with L_I). This is a compact contractible space: indeed, all its maximal simplices have the common vertex $\{L_I\}$. Then the resolved arrangement $\widetilde{L}$ is defined by the

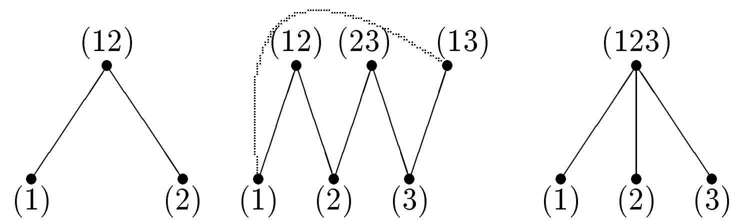

FIGURE 3. Order complexes for line arrangements

formula

$$\tilde{L} = \bigcup (\Delta(I) \times L_I) \subset P(L) \times \mathbb{R}^m, \tag{3}$$

union over all geometrically distinct strata L_I.

For two left arrangements in the bottom row of Fig. 2, the corresponding graphs drawn above them can be considered as the pictures of such resolutions as well, especially if we distinguish the middle points of the inserted segments. These segments can be considered as products $\Delta(I) \times L_I$ for two-element sets I, so that L_I are the intersection points of some two lines, and the order subcomplex $\Delta(I)$ is the union of two segments joining the corresponding vertex to two vertices corresponding to these two lines.

The obvious projection $P(L) \times \mathbb{R}^m \to \mathbb{R}^m$ defines a map $p : \tilde{L} \to L$. This map is proper and semialgebraic, and all its fibers are different spaces of the form $\Delta(I)$, therefore it is a homotopy equivalence. Moreover, its extension to the map of one-point compactifications $\overline{\tilde{L}} \to \overline{L}$ is also a homotopy equivalence, in particular defines an isomorphism of Borel–Moore homology groups. But why is the resolved space $\tilde{L}$ better than the initial one?

1.3. **The filtration.** There is a natural increasing filtration

$$F_1 \subset \cdots \subset F_{n-1} = \tilde{L} \tag{4}$$

on the resolved space $\tilde{L}$: its term F_p equals the union like (3) but over the strata of codimension $\leq p$ only. The difference $F_p \setminus F_{p-1}$ is the union of products $\check{\Delta}(I) \times L_I$ over all strata L_I of codimension exactly p, where $\check{\Delta}(I)$ is equal to $\Delta(I)$ less the *link* $\partial\Delta(I)$ of $\Delta(I)$, i.e. the union of simplices not containing the minimal vertex $\{L_I\}$. Indeed, the set $\partial\Delta(I) \times L_I$ belongs to the lower term F_{p-1} of the filtration. This filtration can be extended to a filtration $\{\bar{F}_0 \subset \bar{F}_1 \subset \ldots\}$ of the compactification $\overline{\tilde{L}}$: its term $\bar{F}_0$ consists of the added point, and other terms $\bar{F}_p$ are just the closures of the similar terms of the filtration on $\tilde{L}$.

Theorem 1. *This filtration homotopically splits into the wedge of corresponding quotient spaces: there is a homotopy equivalence*

$$\bar{\tilde{L}} \sim \bar{F}_1 \vee (\bar{F}_2/\bar{F}_1) \vee \ldots \vee (\bar{F}_{N-1}/\bar{F}_{N-2}). \tag{5}$$

This theorem was proved in [71]; for an equivalent (and obtained simultaneously) result in the terms of the "naive" resolution see [61].

In particular, we have the splitting of the Borel–Moore homology group of $\bar{L}$ (or, which is the same by the Alexander duality, of the cohomology group of $\mathbb{R}^m \setminus L$):

$$\begin{aligned} H^{m-i-1}(\mathbb{R}^m \setminus L) &\simeq \bar{H}_i(\overline{L}) \equiv \bar{H}_i(\bar{\tilde{L}}) \simeq \\ \simeq \oplus \bar{H}_i(\check{\Delta}(I) \times L_I) &\equiv \oplus H_{i-\dim\, L_I}(\Delta(I), \partial\Delta(I)); \end{aligned} \tag{6}$$

here $\tilde{H}_*$ denotes the homology group reduced modulo a point, and summation is over all strata L_I of the arrangement.

This expression was obtained first by Goresky and Macpherson [32] by a different method. It implies that the homology groups of $\mathbb{R}^m \setminus L$ are completely determined by dimensions of spaces L_I.

The splitting (5) implies that even the *stable* homotopy type of this complementary space depends on these data only.

1.4. **Geometrical interpretation.** The formula (6) has the following direct realization (see [71], [43]). Suppose an Euclidean metric is fixed in $\mathbb{R}^m$. Consider a constant vector field V ("power") in $\mathbb{R}^m$ in general position with respect to L. For any k-dimensional simplex of the order subcomplex $\Delta(I)/\partial\Delta(I)$ (i.e. for a decreasing sequence of $k+1$ strata $L_{I_1} \supset L_{I_2} \supset \ldots \supset L_{I_k} \supset L$) and for any point $x \in L_I$ consider the sequence of $k+1$ rays in $\mathbb{R}^m$ issuing from x, namely the trajectories of x in the planes $\mathbb{R}^m, L_{I_1}, \ldots, L_{I_k}$ under the action of this power. (We can realize V as the gradient field of a generic linear function $\theta : \mathbb{R}^m \to \mathbb{R}$, then these rays will be the trajectories of gradients of restrictions of θ to these planes.) As V is in general position, these rays are linearly independent, and their convex hull is linearly homeomorphic to an $(k+1)$-dimensional octant with origin at x. Such octants over all $x \in L_I$ sweep out an $(i+1+\dim L_I)$-dimensional wedge in $\mathbb{R}^m$.

If we have a cycle α of the complex $\Delta(I)/\partial\Delta(I)$, then the sum of (uniformly oriented) corresponding wedges is a relative cycle in $\mathbb{R}^m (\bmod L)$, and the relative homology class $\nabla\alpha$ of the latter cycle depends on the class of α in $H_*(\Delta(I), \partial\Delta(I))$ only.

Finally we take the class in $H^*(\mathbb{R}^m \setminus L)$ Poincaré–Lefschetz dual to $\nabla\alpha$ in $\mathbb{R}^m \setminus L$, i.e. defined by intersection indices with the relative cycle $\nabla\alpha$.

This realization depends on the choice of the direction V, but not very much. Two elements in $\bar{H}_*(\mathbb{R}^m, L)$, corresponding in this way to one and the same class $\alpha \in$

$H_*(\Delta(I), \partial\Delta(I))$ via different generic functions can differ by elements of lower filtration only, i.e. by a sum of similar classes coming from the summands $H_*(\Delta(J), \partial\Delta(J))$ corresponding to planes L_J *strictly containing* L_I.

Moreover, if all strata L_I have codimensions ≥ 2 in all greater strata L_J, then the isomorphism (6) is canonical: in this case the space of generic (in the desired sense) vectors V is path-connected.

By analogy with the knot theory, such realizations of elements of $H^*(\mathbb{R}^m \setminus L)$ can be called their *combinatorial expressions.*

1.5. **Multiplication in cohomology.** Unfortunately the usual homotopy type of the complement of an arrangement cannot be determined by the dimensional data.

The most developed case is that of *complex hyperplane* arrangements. In this case the multiplicative structure of the integral cohomology ring of the complement is determined by the dimensional data: the direct expression was obtained by Orlik and Solomon [45] with the help of some ideas from pioneering works of Arnold and Brieskorn [3], [16]. However even in this case (and even for *central*, i.e. passing through the origin, arrangements in $\mathbb{C}^3$) the fundamental group of the complement is not determined by these data: there exist pairs of arrangements with equal dimensions of all strata but with different fundamental groups, see [49].

For arbitrary (not hyperplane) complex arrangements the cohomology ring of the complement is also defined by the dimensional data: in the case of rational coefficients this was proved in [23], and in the more complicated integral case in [26] with the help of some ideas from [69].

Still, something good can be said even in the most general case of an arbitrary arrangement of real affine planes of arbitrary (may be different) dimensions in $\mathbb{R}^m$: the graded ring associated with the filtered ring $H^*(\mathbb{R}^m \setminus L)$ also is defined by dimensional data (and some information on mutual orientations of all planes L_I).

Indeed, the splitting (6) is not canonical: the summands in the second line of (6) related to some stratum L_I define well some elements of the first line only up to lower terms of the filtration (more precisely, only up to elements of similar terms $L_{\tilde{I}}$ with $L_{\tilde{I}} \stackrel{\neq}{\supset} L_I$). However, let us rewrite the equation (6) as that for associated graded groups:

$$GrH^*(\mathbb{R}^m \setminus L) \cong \oplus H_{m-*-1-\dim L_I}(\Delta(I), \partial\Delta(I)) \tag{7}$$

The splitting in this formula is already canonical (up to the choice of orientations of planes L_I), and the multiplication in the associated graded *ring* is as follows.

Let us consider two strata $L_I, L_J \subset L$ and two cycles A, B of the quotient complexes $\Delta(I)/\partial\Delta(I)$ and $\Delta(J)/\partial\Delta(J)$, dim $A = u$, dim $B = v$, represented by linear combinations of simplices of subcomplexes $\Delta(I), \Delta(J)$ with boundaries in $\partial\Delta(I)$ and $\partial\Delta(J)$ only. The *shuffle product* $A \odot B$ of these cycles is defined as follows (see [69]).

If L_I and L_J are not transversal (i.e. belong to some proper plane in $\mathbb{R}^m$) or have no intersection points, then $A \odot B = 0$. Now suppose that L_I and L_J are transversal and $L_K = L_I \cap L_J \neq \emptyset$ (we can take $K = I \cup J$). Let $a \subset A$ and $b \subset B$ be some two simplices with $u+1$ and $v+1$ vertices respectively, i.e. some decreasing sequences of strata of L having $\{L_I\}$ and $\{L_J\}$ as their last elements. Consider all $\binom{u+v+2}{u+1}$ possible *shuffles* of these sequences, i.e. all (nonmonotone) sequences of $u+v+2$ strata in which all elements of a and b appear preserving their orders in the sequences a and b. To any such shuffle a monotone sequence corresponds: any element λ of the shuffle coming from the sequence a (respectively, b) should be replaced by the intersection of the corresponding stratum with the last stratum coming from the sequence b (respectively, a) and staying before λ in the shuffle. The obtained monotone sequence is by definition an $(u+v+1)$-dimensional simplex of the order complex L_K. The shuffle product of our simplices a and b is defined as the sum of all such simplices taken with signs equal to parities of the corresponding shuffles (i.e. the numbers of transpositions reducing them to the simple concatenation of sequences a and b) multiplied by one sign more, which depends on multi-indices I, J and K only and is defined by the comparison of the fixed coorientation of the plane L_K in $\mathbb{R}^m$ with the ordered pair of coorientations of L_I and L_J. The shuffle product of cycles A and B is defined by linearity. It is a relative cycle defining an element of the summand in the right-hand part of (7) corresponding to the stratum L_K; this element depends only on homology classes of A and B in the summands corresponding to L_I and L_J.

Theorem 2 (cf. [69], [29], [25], [26]). *The isomorphism (7) commutes the shuffle product in its right-hand part and the multiplication in its left part obtained from the usual cohomological multiplication. If all strata L_I have codimensions ≥ 2 in all greater strata L_J, then the same is true for the isomorphism (6) and the multiplication in the ring $H^*(\mathbb{R}^m \setminus L)$ itself, and not in its graded ring only.*

This is a corollary of the explicit construction described in §1.4. Given two strata L_I, L_J and classes $\alpha \in H_*(\Delta(I), \partial\Delta(I))$, $\beta \in H_*(\Delta(J), \partial\Delta(J))$, we can realize corresponding elements in the left part of (6) with the help of directions V_I, V_J in $\mathbb{R}^m$ that are in general position if L_I and L_J have nonempty transversal intersection; if not then these directions should be opposite to one another and transversal to a plane separating or containing these strata.

1.6. **All the same in the space of curves.** The discriminant in the space of curves $\mathcal{K}$ also is a union of planes: for any pair of points a, b in $\mathbb{R}^1$ we consider the plane $L(a,b) \subset \mathcal{K}$ consisting of all maps $f : \mathbb{R}^1 \to \mathbb{R}^n$ such that $f(a) = f(b)$ if $a \neq b$ or $f'(a) = 0$ if $a = b$. Any point of the discriminant belongs to at least one such plane. Then we take the order complex of all possible intersections

$$L(a_1, b_1) \cap L(a_2, b_2) \cap \ldots \tag{8}$$

and limit positions of such intersections (all of them are affine planes in $\mathcal{K}$ whose codimensions are multiples of n), *supply it with a natural topology*, and define the simplicial resolution in exactly the same way as previously, i.e. as a subset of the direct product of this order complex and the space $\mathcal{K}$. Then we define the filtration on this resolution by the codimensions (divided by n) of these planes and consider the arising spectral sequence.

The unique serious difficulty here appears from the fact that some points of Σ belong to infinitely many planes $L(a, b)$: for instance a map f sending a segment of $\mathbb{R}^1$ into one point or sending two segments of $\mathbb{R}^1$ into one and the same arc in $\mathbb{R}^n$. It is impossible to carry out the standard construction of the order complex counting such infinite objects. (There is a more refined construction of *conical resolutions*, which helps us in some troubles of this kind, see e.g. [67], [65] and §3.1 below, but in the case of knots this difficulty remains very serious.)

Therefore we restrict ourselves to the case of finite intersections: for any d we consider only the poset Δ_d of planes (8) of codimension $\leq nd$ in $\mathcal{K}$, construct the corresponding order complex, and define the general order complex Δ as the direct limit of such complexes over $d \to \infty$; the numbers d define a natural increasing filtration on them. Any term ϕ_d of this filtration is finite-dimensional, and any difference $\phi_d \setminus \phi_{d-1}$ is naturally divided in a finite family of finite-dimensional cells, so that its one-point compactification is a finite cell complex.

The homological study of this filtered complex is a major problem in the theory of finite type cohomology groups of knot spaces (and the theory of finite type invariants is its part considering only the cells of two upper dimensions in any term of the filtration).

Indeed, the resolved discriminant $\sigma \subset \Delta \times \mathcal{K}$ can be naturally projected to both Δ and $\mathcal{K}$. The first projection induces a natural filtration $F_1 \subset F_2 \subset \dots$ on it from the filtration $\{\phi_d\}$ on Δ. The restriction of this projection to the difference $F_d \setminus F_{d-1}$ is a locally trivial bundle over $\phi_d \setminus \phi_{d-1}$ whose fibers are subspaces of codimension nd in $\mathcal{K}$. Thus the "Borel–Moore homology group of finite codimension" of $F_d \setminus F_{d-1}$ is reduced (via some sort of the Thom isomorphism) to the usual (finite-dimensional) homology group of the base (in particular is finitely generated). This allows us to calculate in principle all the (finite codimension) homology groups of spaces F_d. The finite type homology classes of Σ are nothing else than direct images of their elements under the second projection $\sigma \to \Sigma$, and the finite type cohomology classes of the space of knots are their Alexander duals. The "order" (i.e. the filtration) of these classes is defined by our filtration of the resolved discriminant.

The cellular structure of terms $\phi_d \setminus \phi_{d-1}$ (and hence also of $F_d \setminus F_{d-1}$) together with incidence coefficients of cells is explicitly described in [59], [57]. It consists of the

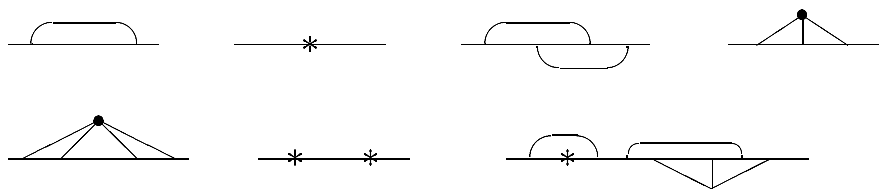

FIGURE 4. Examples of generalized chord diagrams

enumeration of different families of planes (8) and the simplicial structure of inserted order complexes.[1]

The families of planes (8) are classified and depicted in the terms of (generalized) *chord diagrams*, see Fig. 4: any particular plane $L(a,b)$ is depicted by an arc (chord) connecting the points a, b of the line $\mathbb{R}^1$ or the circle S^1, and finite collections of such planes (giving planes (8) as their intersections) by collections of such chords or more complicated objects. For instance, seven pictures of Fig. 4 denote the following planes respectively: a plane $L(a,b)$, $a \neq b$; a plane $L(a,a)$; a plane $L(a,c) \cap L(b,d)$ where $a < b < c < d \in \mathbb{R}^1$; a plane $L(a,b) \cap L(b,c) \equiv L(b,c) \cap L(c,a) \equiv L(c,a) \cap L(a,b)$ where $a < b < c$; a plane of codimension $3n$ consisting of maps gluing together some fixed four points of $\mathbb{R}^1$; a plane $L(a,a) \cap L(b,b)$; a certain plane of codimension $5n$.

The order complex arising over the plane of third type is just a segment (or, more precisely, the union of two segments joining the vertex corresponding to the plane $L(a,c) \cap L(b,d)$ with two vertices corresponding to planes $L(a,c)$ and $L(b,d)$, cf. the left picture of Fig. 2). The order complex arising over the fourth picture coincides with that shown in the upper right part of Fig. 2. The order complex over the fifth picture is two-dimensional and is equal to the cone over the graph given in the lower part of Fig. 5 (not containing the segments with endpoint (1234)), the whole this picture presents the corresponding poset (more precisely, only its primitive edges).

The theory of these resolutions is related very much to the graph theory. For instance, let us resolve in the *naive* way the stratum of Σ consisting of maps $\mathbb{R}^1 \to \mathbb{R}^n$ with unique k-fold selfintersection point. This stratum consists of intersections of $\binom{k}{2}$ planes $L(a_i, a_j)$, $1 \leq i < j \leq k$. These planes correspond to the vertices of the inserted simplex. They are conveniently described by the edges connecting some pairs (i,j) of k numbered points, while the faces of this simplex are the collections of such edges, i.e. just the graphs on these k vertices (without double edges or loops). Some of these faces belong to the lower term of our filtration of the discriminant: they are exactly the faces corresponding to the non-connected graphs. Therefore we obtain naturally the *complex of connected graphs*, cf. [61]. This complex arises

[1]in [59], I have used the "naive" resolution like the second from the right upper picture in Fig. 2: in this (equivalent) approach the study of inserted order complexes is replaced by the study of inserted simplices reduced modulo their subcomplexes lying in the lower terms of the filtration, cf. [61]

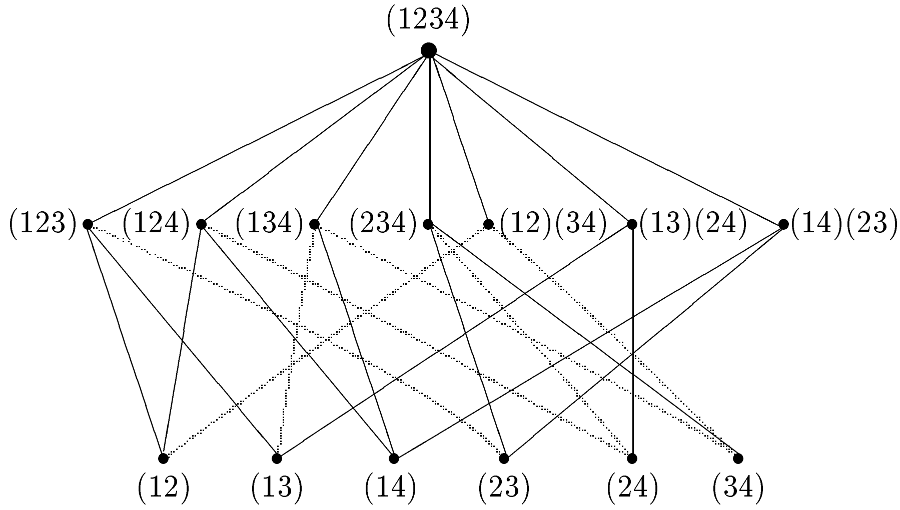

FIGURE 5. Poset and order complex for a quadruple point

also in the naive resolution of the "diagonal" plane arrangement in $\mathbb{R}^{kn}$ consisting of all ordered collections of k points in $\mathbb{R}^n$ at least two of which coincide. Another important related complex is that of *two-connected graphs*, see [63], [10], [51], [52]. On the other hand, the order complex arising from the economical resolution of the same stratum leads (after combining together some simplices) to the *graph-complex of trees* due to Kontsevich.

For the study of knot invariants in $\mathbb{R}^3$ it is enough to consider only the simplest chord diagrams like the ones in the first and the third pictures of Fig. 4, i.e. with all different endpoints. More precisely, such chord diagrams (and the corresponding cells) generate the homology groups responsible for knot invariants, while the relations between them are described in the terms of similar diagrams allowing either one asterisk as in the second picture, or one triple point as in the fourth one.

However, for the calculation of higher cohomology groups of spaces of knots the consideration of more complicated diagrams like the fifth and the last ones is absolutely necessary.

1.7. **The spectral sequence and its convergence.** The calculation of homology groups (of finite filtration) of the resolved discriminant (or of the cohomology classes in $\mathcal{K} \setminus \Sigma$ Alexander dual to them) can be presented by a cohomological spectral sequence with the support in the second quadrant, see Fig. 6. Its initial term E_1 is given by

$$E_1^{p,q} \simeq \bar{H}_{n\infty-p-q-1}(F_{-p} \setminus F_{-p-1}) \simeq \bar{H}_{p(n+1)-q-1}(\phi_{-p} \setminus \phi_{-p-1}, A). \tag{9}$$

Here A is the orientation sheaf of the $n(\infty + p)$-dimensional affine bundle $(F_{-p} \setminus F_{-p-1}) \to (\phi_{-p} \setminus \phi_{-p-1})$. If n is even then this sheaf is isomorphic to $\mathbb{Z}$ (as the bundle

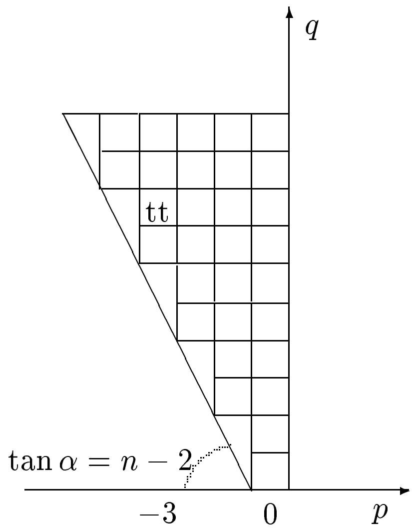

FIGURE 6. The spectral sequence

is orientable) but for n odd is, generally, not. The order complexes ϕ_d do not depend on n, thus for different numbers n of the same parities the columns $E_1^{p,*}$ of spectral sequences calculating the cohomology of spaces of knots in $\mathbb{R}^n$ coincide canonically up to a shift along the q-axis; in the case of $\mathbb{Z}_2$-coefficients the same is true also for n of different parities.

If n is greater than 3 then there are only finitely many nonzero cells on any diagonal $\{p+q = const\}$. Using the machinery of finite dimensional approximations, it is easy to prove that in this case the infinitely degenerate strata of Σ do not contribute to the calculation of cohomology classes, therefore *if $n \geq 4$ then our spectral sequence converges exactly to the cohomology group $H^*(\mathcal{K} \setminus \Sigma)$ of the space of knots in $\mathbb{R}^n$.*

For the most intriguing case $n = 3$ this is not the case (or at least is not proved). Something good can be a priori said on the lower diagonal $\{p + q = 0\}$ responsible for the knot invariants: any nonzero element of the group $E_\infty^{-i,i}$ actually defines a nontrivial knot invariant of filtration i (modulo the group of invariants of smaller filtration). This filtration has a transparent geometrical description in the terms of finite differences, see [13], [12] or §0.2 in [59]. However for the elements of terms $E_\infty^{p,q}$ on higher diagonals it is not known whether the infinitely degenerate strata will not spoil them. Any such element defines a $(p+q)$-dimensional cohomology class of $\mathcal{K} \setminus \Sigma$ (again, modulo the elements of lower filtration), but we cannot be sure a priori that this class is not trivial, i.e. that the corresponding cycle in the discriminant is not a boundary.

1.8. **Justifications and approximations.** Using the Weierstrass approximation theorem, we can choose a perfect (in some sense) system of finite dimensional affine approximating subspaces $\{\mathcal{K}_\nu\}$, $\nu \to \infty$, of the space of curves $\mathcal{K}$. The corresponding rings $H^*(\mathcal{K}_\nu \setminus \Sigma)$ converge to the ring $H^*(\mathcal{K} \setminus \Sigma)$. Also, we can assume that all planes $\mathcal{K}_\nu$ are *in general position*, in particular transversal to the natural stratification of Σ. Then for any particular ν the cohomological spectral sequence calculating $H^*(\mathcal{K}_\nu \setminus \Sigma)$ and constructed from the simplicial resolution of $\Sigma \cap \mathcal{K}_\nu$ also looks as in Fig. 6 (although it will have only finitely many nontrivial columns). The stabilization of spectral sequences means the following. For any natural s there exists ν such that terms $E_1^{p,q}$, $p \geq -s$, of all our spectral sequences calculating cohomology of $\mathcal{K}_\nu \setminus \Sigma$, $\mathcal{K}_{\nu+1} \setminus \Sigma$, etc. are canonically isomorphic, and the images of differentials $d^r : E_r^{p,q} \to E_r^{p+r,q-r+1}$, $r \leq s$, acting from these cells to the right also coincide. Thus the limit spectral sequence $E_r^{p,q} \equiv \lim_{\nu\to\infty} E_r^{p,q}(\nu)$ is well defined.

A very important role in the birth of this theory was played by the V. Arnold's problem on the *stable cohomology ring of complements of discriminants of complex hypersurface singularities*, see [5]. Being still of finite dimensional nature, this problem forced me to look for the homology classes arising uniformly in "very highdimensional" discriminant varieties, and also to think on the nature of their stabilization, see [58].

2. Further results and problems

2.1. Kontsevich integral.

Theorem 3. *For any $n \geq 3$, our spectral sequence with complex coefficients stabilizes at the first term:*

$$E_\infty^{p,q}{}_{/\mathbb{C}} \simeq E_1^{p,q}{}_{/\mathbb{C}}. \tag{10}$$

This theorem was proved by Kontsevich about 1994 and is surely true. Its published part proves the stabilization of the diagonal responsible for knot invariants, i.e. the equality (10) for $n = 3$ and $p+q = 0$, see [37], [20]. For an arbitrary $n \geq 3$, almost the same integral proves the identity (10) on the lower boundary of the spectral sequence, i.e. for cells $E^{p,q}$ with $q + (n-2)p = 0$, but for the upper cells the proof uses some extra efforts.

A great problem is whether the same is true over the integers.

I conjecture that *in the case of long knots* this is true, and moreover *the homotopy splitting* (*5*) *of any finite term $\bar{F}_d$ of the filtration of the one-point compactification of our resolved discriminant holds in some precise sense*, see e.g. [64].

Another great achievement coming from the Kontsevich's works is an integral representation for the cohomology classes.

A spectral sequence similar to (but easier than) the one outlined in §§1.4–1.5 allows us to calculate the cohomology groups of spaces Y^X of continuous maps $X \to Y$ where X is an m-dimensional finite cell complex and Y an m-connected one. Indeed, Y is homotopy equivalent to the space $\mathbb{R}^N \setminus \Lambda$ where N is sufficiently large and Λ is a closed conical subset of codimension $\geq m+2$. Then we consider the vector space of all continuous maps $X \to \mathbb{R}^N$, define the discriminant in it as the space of all

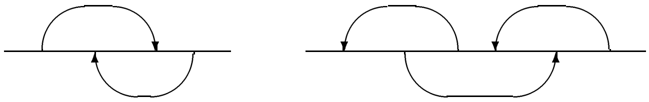

FIGURE 7. Arrow diagrams

maps whose images intersect Λ, resolve this discriminant as previously and obtain a spectral sequence converging to the cohomology group of the complement of this discriminant, i.e. of the space $(\mathbb{R}^N \setminus \Lambda)^X \sim Y^X$, see [56], [57]. This spectral sequence extends the Anderson's spectral sequence [2] to the case when X is not a smooth manifold, and is isomorphic to it if X is. It provides some information also if Y is only $(m-1)$-connected, but in this case we cannot be sure that it calculates all the cohomology groups of Y^X, in full analogy with the discussion at the end of §1.7. If X is a circle, then it is covered by the Adams–Eilenberg–Moore spectral sequence [1], [27] (calculating in particular the cohomology of loop spaces). The "deRhamization" of the latter spectral sequence is known as the theory of iterated path integrals, see [19], [35]. The Kontsevich's integral (and possibly also its more smart versions proving Theorem 3 in full generality) can be considered as its extension to the problems "of second order", see §2.4 below.

2.2. **Combinatorial expressions.** The most well-studied part of this theory is, of course, that of knot invariants. Shortly after its appearance, different *combinatorial formulas* for these invariants were developed. They express the values of invariants in the terms of the geometrical disposition of the knot, see e.g. [39], [17] and [48]. The most convenient formulas of this kind were announced and partly proved by M. Polyak and O. Viro in [46], [47], see also [54], [55].

These formulas are described in terms of *arrow diagrams*, i.e. pictures like the ones shown in Fig. 7.

Let us fix a direction in $\mathbb{R}^3$ transverse to the common direction "at infinity" of our long knots. Given a generic long knot $f : \mathbb{R}^1 \to \mathbb{R}^3$ (see Fig. 1), the value of the left picture in Fig. 7 on it is equal to the number of 4-configurations $(a < b < c < d) \subset \mathbb{R}^1$ (counted with appropriate signs) such that the point $f(a)$ lies above $f(c)$ with respect to the chosen direction, and $f(d)$ lies above $f(b)$. It turns out (see [46]) that this value actually is a knot invariant, namely it coincides with the unique invariant of filtration 2.

M. Goussarov has proved a wonderful theorem:

Theorem 4 (see [33]). *Any invariant of finite filtration of long knots in $\mathbb{R}^3$ can be represented by a finite linear combination of arrow diagrams.*

Formally speaking, any cohomology class of finite filtration of the space of knots in $\mathbb{R}^n$, $n \geq 3$, also should have combinatorial representations (although maybe formulated in terms of more complicated conditions whose total complexity it is difficult to estimate); the strength of the previous theorem consists in the fact that in the case of invariants we can use only conditions of a very special kind. To find the combinatorial formulas for other cohomology classes $\alpha \in H^*(\mathcal{K} \setminus \Sigma)$ effectively, *it is convenient to consider such a combinatorial formula as a semialgebraic relative cycle in $\mathcal{K}$ (mod Σ), such that α equals the linking number with the boundary of this cycle in Σ.*

It is natural to construct such cycles by induction over our spectral sequence. For an illustration, let us consider again the theory of plane arrangements and their complements. In the case of the line arrangement shown in Fig. 2 left, the entire group $E^1_{2,*}$ appears from the unique crossing point $L_{(12)}$. This group is nontrivial only for $* = -1$, is isomorphic to $\mathbb{Z}$ and generated by the homology class of the segment $\Delta(1,2)$ modulo its endpoints (lying in F_1). The splitting formula (5) means that we *can* extend this relative cycle in $\bar{F}_2$ (mod $\bar{F}_1$) to a (Borel–Moore) cycle in entire $\tilde{L}$. However, to be able to define the value of this generating element on any 0-dimensional cycle in $\mathbb{R}^2 \setminus L$ we need to *choose* such an extension explicitly. Then we project it to L and get a cycle there. Finally, we need to choose a relative cycle in $\mathbb{R}^2$ (mod L) whose boundary coincides with this cycle. Then we call this relative cycle "a combinatorial formula": its value on a point in $\mathbb{R}^2 \setminus L$ is equal to the multiplicity of this cycle in the neighborhood of this point.

If we have a more complicated plane arrangement, then we can construct this extension step by step over our filtration. Our starting element $\gamma \in E^1_{p,q}$ is represented by a cycle with closed supports in $F_p \setminus F_{p-1}$ (or, equivalently, by a relative cycle in $\bar{F}_p/\bar{F}_{p-1}$). We take its *first boundary* $d_1(\gamma)$, which is a cycle in $F_{p-1} \setminus F_{p-2}$. Then we *span* it, i.e. construct a chain $\tilde{\gamma}_1 \subset F_{p-1} \setminus F_{p-2}$ such that $\partial\tilde{\gamma}_1 = d_1(\gamma)$ there. Then we take the boundary of $\gamma + \tilde{\gamma}_1$ in the space $F_{p-2} \setminus F_{p-3}$ and span it there by a chain $\tilde{\gamma}_2$, etc. The splitting formula (5) ensures that all this sequence of choices can be accomplished. Moreover, a precise final result of this sequence is known since [71]: see §1.4. It appears if we span our cycles in the most obvious way: by the trajectories of generic flows.

The case of knots (say, of long knots) is very similar to that of plane arrangements. For instance, here is a heuristic interpretation of the Polyak–Viro arrow diagram formulas. A knot invariant can be considered as a relative cycle of full dimension $n\infty$ in the space of curves $\mathcal{K}$ (mod Σ): its value at a knot f equals the multiplicity of the cycle in a neighborhood of f in $\mathcal{K}$. All strata of the discriminant which can generate (finite-type) homology classes of this dimension are defined by ordinary chord diagrams only: all points a_i, b_i in (8) should be different. At them, the corresponding planes $L(a_i, b_i)$ meet normally, so that the corresponding order subcomplexes $\Delta(\cdot)$ are simplices (or, more precisely, their first barycentric subdivisions). I do not know a suitable analog of a globally defined vector field V from §1.4 on the space $\mathcal{K}$. However, in the construction of §1.4 we could use not the one vector V but just a generic family of such vectors, one for each stratum L_I, whose trajectories span them in greater strata. In the case of knots, when the planes $L(a_i, b_i)$ in (8) are defined by conditions

$f(a_i) = f(b_i)$, it is natural to take a vector field preserving the projection of our knot to $\mathbb{R}^2$ but increasing all the differences $z(b_i) - z(a_i)$, where z is some coordinate in $\mathbb{R}^3$, say the one normal to the "blackboard" plane $\mathbb{R}^2$. To make this formula correct we need to order the endpoints of any chord, i.e. to call one of them a_i and the other b_i. Thus the arrow diagrams appear. The union of wedges emanating from the point f as in §1.4 will then consist of curves with the same projection to $\mathbb{R}^2$ but with $f(b_i)$ "above" $f(a_i)$. The knot theory is very nonlinear (in contrast to the theory of plane arrangements), in particular such wedges corresponding to chord diagrams of the same topological type but with different configurations of points a_i, b_i can have intersections in $\mathcal{K}$. The algebraic multiplicity of such an intersection at some point $\phi \in \mathcal{K} \setminus \Sigma$ is exactly the value (in the Polyak-Viro sense) of the arrow diagram on the corresponding knot.

Of course, everything is not so easy. Indeed, the strata corresponding to different chord diagrams have common boundaries as the endpoints of different chords tend to one another. Some additional trouble comes from singular maps with nongeneric projections to $\mathbb{R}^2$. Therefore the wedges constructed as above have some extra boundary components. Constructing the combinatorial formulas we need to span these boundaries by some other chains in $\mathcal{K}$ or try to choose the orientations of arrows in such a way that these boundaries of different wedges annihilate. The Goussarov's theorem means (in our terms) that it is always possible to choose the orientations of arrows in such a way that for the spanning chain we can take sums of similar wedges emanating from the strata (8) of lower complexity.

The above heuristic speculations are helpful also in the case of higher dimensions (in any of senses indicated in the preface), i.e. in constructing the combinatorial expressions of higher-dimensional cohomology classes of spaces of knots in $\mathbb{R}^n$, $n \geq 3$.

In [68], natural classes of semialgebraic subvarieties in $\mathcal{K}$ and in different terms $F_i \setminus F_{i-1}$ of the filtration were introduced, of which (some of) these spanning chains can be built.

2.2.1. *Example: Teiblum–Turchin cocycle and its realization.* The first positive dimensional cohomology class of finite filtration of the space of long knots in $\mathbb{R}^3$ was calculated by my students, D. M. Teiblum and V. E. Turchin, about 1995. It is a class of dimension 1 and filtration 3. (Accordingly to [59], there are no cohomology classes of filtration ≤ 2 other than the simplest knot invariant.) However, this calculation was quite implicit: they have calculated just the corresponding group $E_1^{-3,4} \sim \mathbb{Z}$ of the spectral sequence in the terms of generalized chord diagrams. It is clear from the shape of the spectral sequence that this group survives and the final group $E_\infty^{-3,4}$ also is isomorphic to $\mathbb{Z}$, so that its generator can be extended to a well defined 1-dimensional cohomology class of the space of knots.

However, the fact that this class is nontrivial does not follow from the general considerations, cf. the discussion in §1.7. This fact was proved in [68] by means of an explicit combinatorial formula, see Fig. 8 and the following theorem.

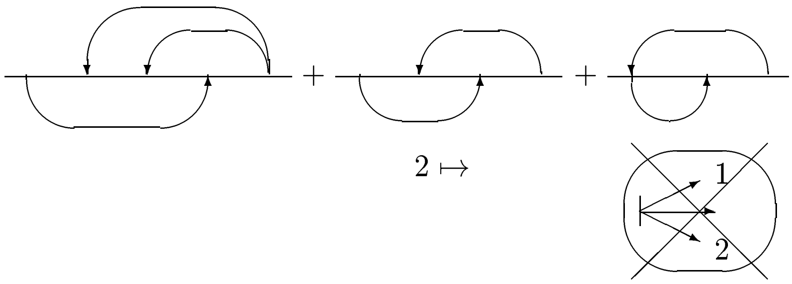

FIGURE 8. Combinatorial formula for Teiblum–Turchin cocycle

Let us choose a direction "to the right" in the "blackboard" plane $\mathbb{R}^2$ (i.e. in the quotient of the spaces $\mathbb{R}^3$ by the direction chosen previously).

Theorem 5 (see [68]). *The value of the Teiblum–Turchin cocycle on any* generic *loop in the space $\mathcal{K}$ of long knots (i.e. on a closed 1-parametric family of such knots) is equal mod 2 to the number of points of this loop such that one of three holds (cf. Fig. 8):*

a) there are five points $a < b < c < d < e$ in $\mathbb{R}^1$ such that $f(a)$ is above $f(d)$, and $f(e)$ is above $f(c)$ and $f(b)$;

b) there are four points $a < b < c < d$ in $\mathbb{R}^1$ such that $f(a)$ is above $f(c)$, $f(b)$ is below $f(d)$, and the projection of the derivative $f'(b)$ to $\mathbb{R}^2$ is directed to the right;

c) there are three points $a < b < c$ in $\mathbb{R}^1$ such that $f(a)$ is above $f(b)$ but below $f(c)$, and the "exterior" angle in $\mathbb{R}^2$ formed by projections of $f'(a)$ and $f'(b)$ contains the direction "to the right" (i.e. this direction is equal to a linear combination of these projections, and at least one of coefficients in this combination is nonpositive).

(These points of the loop in $\mathcal{K}$ should be counted with multiplicities equal to the numbers of different point configurations in $\mathbb{R}^1$ for which the corresponding condition a), b) or c) is satisfied.)

This statement remains true if we replace $\mathbb{R}^3$ by any $\mathbb{R}^n$, $n \geq 3$, $\mathbb{R}^2$ by $\mathbb{R}^{n-1}$, a generic loop in the space of knots by a generic $(3n-8)$-dimensional cycle, and the 1-dimensional Teiblum–Turchin cocycle by its $(3n-8)$-dimensional stabilization, see discussion in §1.7.

Further, let us consider the connected sum of two equal (long) trefoil knots in $\mathbb{R}^3$ and a path in the space of knots connecting this knot with itself as in the proof of the commutativity of the knot semigroup: we shrink the first summand, move it "through" the second, and then blow up again.

Proposition 1. *This closed path in the space of long knots has exactly seven intersection points (counted with multiplicities) with the union of three varieties indicated in items a, b and c of the previous theorem.*

But the Teiblum–Turchin cocycle is a well-defined integral cohomology class, thus its value on (the integral homology class of) this loop is not equal to zero, and the group generated by this cocycle is free.

Remark 1. I cannot yet reprove the Goussarov's theorem in this way: the combinatorial formulas for knot invariants obtained by the straightforward application of our algorithm can include some varieties in $\mathcal{K}$ more complicated than just the varieties given by arrow diagrams, cf. Theorem 5. The construction of spanning cycles participating in this algorithm leaves many choices, e.g. how to order the endpoints a_i, b_i of a chord. The Goussarov's theorem implies that it is possible to choose these possibilities in such a way that all the awkward varieties will be cancelled. I hope that a deeper understanding of its proof will help to formulate the exact rule for this.

Also, in all situations more complicated than that of invariants I do have, strictly speaking, not an algorithm (i.e. something definitely converging to an answer), but just a collection of tricks which succeed to give such answers in particular problems like that of the Teiblum-Turchin cocycle or the one considered in the next subsection.

Remark 2. The *virtual knots* introduced by L. Kauffman in 1997 and applied in [33] to the construction of combinatorial formulas can be identified as another (extremely big) class of subvarieties of the space of curves $\mathcal{K}$.

2.3. **Cohomology of spaces of compact knots.** A similar theory exists for the space of compact knots $S^1 \to \mathbb{R}^n$. There is a one-to-one correspondence between invariants of compact and long knots in $\mathbb{R}^3$, but in higher dimensions many extra cohomology classes of spaces of compact knots arise from the topological nontriviality of the circle. For instance, already in filtration 1 we have two such classes of dimensions $n-2$ and $n-1$ (with coefficients in $\mathbb{Z}_2$, and if n is even then also with integer coefficients). The combinatorial formulas for all such classes of filtrations 1 and 2 were found in [68]. E.g. the $(n-2)$-dimensional class of filtration 1 is Alexander dual to the variety in $\mathcal{K}$ formed by all maps $f : S^1 \to \mathbb{R}^N$ gluing together some two *opposite* points of S^1, see [63].

The corresponding combinatorial formula consists of two varieties distinguished by the following conditions (referring to a circular coordinate $S^1 \sim \mathbb{R}/2\pi\mathbb{Z}$ in S^1):

a) there is a point $\alpha \in [0, \pi)$ such that $f(\alpha)$ is above $f(\alpha+\pi)$ with respect to the chosen direction;

b) the projection of the point $f(0)$ to $\mathbb{R}^{n-1}$ lies "to the right" from the projection of $f(\pi)$.

As usual, all of this theory can be literally extended to the spaces of links, i.e. embeddings of a disjoint union of finitely many circles.

2.4. **Theories of further orders.** The knot theory is a theory of the second degree of complexity in the same way as the problem mentioned in the end of §2.1 is of degree one: the forbidden discriminant set in the knot theory is defined by a condition on

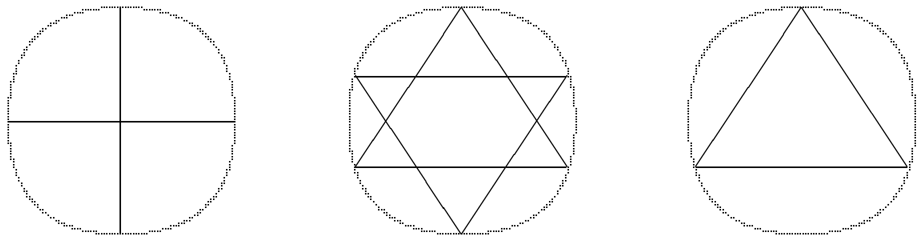

FIGURE 9. Simplest invariants of knots and doodles

the simultaneous behavior of our map $\mathbb{R}^1 \to \mathbb{R}^n$ at some two points, while in the theory of generalized loop spaces any point is responsible for its own behavior only.

The typical example of a problem of order 3 is the study of invariants of plane immersed curves $S^1 \to \mathbb{R}^2$ without triple self-intersection points.

This problem was raised by V.I. Arnold [8], [9], who indicated also the simplest such invariant distinguishing homotopic immersions. This is the *strangeness* Alexander dual to the fundamental cycle of the whole discriminant variety of curves having forbidden triple points.

Similarly to the case of knots, this variety is swept out by the 3-parametric family (parametrized by three-point configurations in S^1) of flat manifolds of codimension 4 in entire space of plane curves (these manifolds form open dense subsets in the planes also parametrized by triples of points and distinguished by the condition that the images of these three points should coincide). It follows easily that this discriminant variety is the image of a smooth orientable manifold, in particular carries a fundamental cycle.

A similar problem formulated in [61], [62] and studied in [36], [42], [43], [66] a.o., concerns the classification of *all* smooth plane curves $S^1 \to \mathbb{R}^2$ (not necessarily immersions) without triple points or singularities obtained as their degenerations. (Since [36], they are called *doodles.*)

These problems have lead to the calculus of *triangular diagrams* (see [66]) in the same way as the knot theory leads to the chord algebra. E.g., the Arnold's "strangeness" is an invariant of filtration 2 and can be depicted by a single triangle, see Fig. 9 right. However, it is not an invariant of doodles. The simplest invariant of doodles (discovered first by A. Merkov [42] by different methods) can be naturally depicted by the simplest triangular diagram, whose triangles have no neighboring points in the circle (see Fig. 9 center) in the same way as the first knot invariant corresponds to the simplest chord diagram with the same property (see Fig. 9 left or the third picture of Fig. 4).

The relation with the graph theory (see page 9) is almost literally replaced by that with the theory of 3-hypergraphs, and the analogy with the "diagonal" plane arrangement by the analogy with the "k-equal" arrangement of planes in $\mathbb{R}^{nm}$ =

$(\mathbb{R}^n)^m$ consisting of such collections $(x_1, \ldots, x_m)$, $x_i \in \mathbb{R}^n$, that $x_{i_1} = \cdots = x_{i_k}$ for some set of indices $1 \le i_1 < \cdots < i_k \le m$, see [15], [61].

2.5. **The V. Turchin's calculation.** The theory of finite type invariants of knots has born many beautiful algebraic objects, such as the Hopf algebra of chord diagrams and graph-complex of trees, see e.g. [38], [12].

It was shown recently by V. Turchin [53] that these structures are nonseparable parts of more general theories, related with entire cohomology rings of spaces of knots and formulated in terms of generalized chord diagrams. The corresponding multiplicative structures resemble the multiplication discussed in §1.5 (although are, of course, much more complicated). It was proved in [53] that the first term of the main spectral sequence calculating the rational homology of the space of long knots in $\mathbb{R}^n$, $n \ge 3$, is described in terms of the Hochschild homology of the Poisson algebras operad if n is odd (respectively, of the Gerstenhaber algebras operad if n is even). Namely, the Hochschild homology of these operads is in both cases some polynomial algebra in infinitely many even and odd variables. To obtain the first term of the spectral sequence in the case of even n we need to factorize the corresponding polynomial algebra by one odd generator $[x_1, x_2]$. In the case of odd n we need to factorize by two generators: one even (equal to $[x_1, x_2]$) and one odd (equal to $[[x_1, x_3], x_2]$).

In particular, the standard bialgebra of chord diagrams factorized through the 4-term relations (see [37], [12]) is some subspace in the Hochschild homology of the Poisson algebras operad. To obtain the algebra of finite order invariants (i.e. cohomology of degree zero in the case $n = 3$) we should factorize this bialgebra by one generator $[x_1, x_2]$.

3. Discussion

3.1. **Which method of resolution is better:** the naive or economical one (see Fig. 2 right)?[2]

In the case of knot spaces they are more or less equal: the constructions are equivalent and the complexities of related calculations are comparable. But generally the "economical" method (or rather its suitable generalization) is stronger. Indeed, sometimes we need to resolve discriminant spaces swept out by families of planes, infinitely many of which pass through one and the same point. The classical example is the *determinant variety* of all degenerate linear operators (whose Borel–Moore homology group is Alexander dual to the cohomology group of GL_n), or the space of singular algebraic projective hypersurfaces of a given degree (that arises in the study of the complementary space of nonsingular varieties), etc., see [67], [65]. The natural extension of the "economical" construction based on the notions of *conical resolutions*

[2]Asked by the chairman of the session, Prof. N. Hitchin FRS

and *continuous order complexes* allows us to overcome this difficulty, while the naive construction does not work.

On the other hand, sometimes *both methods are better*. This means that the very fact that both constructions are homotopy and homology equivalent provides interesting combinatorial relations. For instance, the resolutions of diagonal arrangements like in [3] (arising also in the study of some discriminant strata of the space of knots) is related very much to the graph theory, and we obtain many natural problems and comparison theorems in its homological theory, see e.g. [15], [61], [63], [10], [51], [52], etc.

3.2. **Caution.** The initial part of the described theory, i.e. the study of knot invariants of finite filtration, became very popular (see e.g. [11]) owing, in particular, to the fact that its basic definitions can be formulated in very elementary terms of finite differences, see [13], [12] and §0.2, 0.4 in [59].

However the literal translation of these definitions to such problems as e.g. theories of higher orders in the sense of §2.4 or the study of higher dimensional cohomology classes of spaces of knots will not help us to guess adequate geometrical constructions or equally beautiful algebraic structures. For instance, the group of "order k" (in this sense) invariants of triple points free plane curves will be not finitely generated for any k. The reason for this consists in the fact that in these theories *the singularities of discriminant sets essential for the calculation of cohomology classes and invariants are more complicated than just the normal crossings*. Moreover, the *indices* which invariants and cohomology classes define at all such essential strata (by some far analogues of conditions of type

$$f\,(-|-) + f\,(\,\dashv\!\vdash\,) = f\,(-\!\!\bullet\!\!-)$$

from the theory of knot invariants) in most situations are not scalar: they take values in certain homology groups associated with these strata, cf. [63]. Therefore *the elementary interpretation of the filtration, as well as the very definition of objects of finite type, should be modified adequately in any particular theory of this sort*, cf. [62], [40], [66].

Notice however the beautiful theory of finite type invariants of 3-manifolds started by T. Ohtsuki and extended by S. Garoufalidis, M. Goussarov and others, see [44], [31], [34]. In this theory some comparatively close modification of the basic geometrical characterization of knot invariants of finite filtration is very important. Maybe this can be explained by the fact that the classification of 3-manifolds is a (very nontrivial) quotient of the link theory by the Kirby relations.

Unfortunately I cannot include this theory in the general framework of the discriminant theory.

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References

[1] Adams, J.F. (1957) On the cobar construction. In: Colloque de topologie algebrique (Louvain, 1956). Paris: George Thone, 81–87.

[2] Anderson, D. W. (1972) A generalization of the Eilenberg–Moore spectral sequence, *Bull. AMS* **78**, 784–788.

[3] Arnold, V.I. (1969) Cohomology ring of the group of dyed braids. *Mat. Zametki* **5**:2, 227–231.

[4] Arnold, V. I. (1970) On some topological invariants of algebraic functions. *Transact. (Trudy) Mosc. Mat. Soc.* **21**, 27–46.

[5] Arnold, V. I. (1976) Some unsolved problems of the singularity theory. In: S. V. Uspenskii (ed.), *Proc. of the S. L. Sobolev Seminar*, Math. Inst. of the Siberian branch of Russian Ac. Sci., Novosibirsk, 5–15. (in Russian; extended Engl. transl. in [6].)

[6] Arnold, V. I. (1983) Some unsolved problems in the theory of singularities, in: P. Orlik (ed.) *Singularities. Part 1 (Arcata 1981),* AMS, Providence RI, 57–69.

[7] Arnold, V. I. (1989) Spaces of functions with mild singularities. *Funct. Anal. and its Appl.* **23**:3, 1–10.

[8] Arnold, V. I. (1994) Plane curves, their invariants, perestroikas and classifications, in V. I. Arnold (ed.), *Singularities and Bifurcations,* Adv. in Soviet Math. **21**, AMS, Providence RI, 33–91.

[9] Arnold, V. I. (1995) Invariants and perestroikas of fronts in the plane, *Proc. Steklov Math. Inst.* **209**, 14–64.

[10] Babson E., Björner A., Linusson S., Shareshian J., and Welker V. (1999), The complexes of not i-connected graphs, *Topology*, **38**:2, 271–299.

[11] Bar-Natan, D. (1994–) *Bibliography of Vassiliev Invariants.* Web publication `http://www.ma.huji.ac.il/~drorbn/VasBib/VasBib.html`

[12] Bar-Natan, D. (1995) On the Vassiliev knot invariants, *Topology*, **34**, 423–472.

[13] Birman, J. S. and Lin, X.-S, (1993) Knot polynomials and Vassiliev's invariants, *Invent. Math.* **111**(2), 225–270.

[14] Birman, J. (1993) New points of view in knot theory. *Bull. AMS (N.S.)*, **28**:3, 253–287.

[15] Björner, A. and Welker, V. (1995) The homology of "k-equal" manifolds and related partition lattices, *Adv. Math.* **110**:2, 277–313.

[16] Brieskorn, E. (1973) Sur les groupes de tresses (d'aprés V.I. Arnol'd), in *Seminaire Bourbaki*, 1971/72, No. 401. Berlin: Springer, 21–44. (Lect. Notes Math, 317).

[17] Cartier, P. (1993) Construction combinatoire des invariants de Vassiliev, *C.R.Acad. Sci. Paris, Série I,* **316**, 1205–1210.

[18] A. S. Cattaneo, A. S., Cotta-Ramusino, P., and Longoni, R. (1999-2000) *Configuration spaces and Vassiliev classes in any dimension,* preprint `math.GT/9910139`

[19] Chen, K.T. (1977) Iterated path integrals. *Bull. AMS,* 1977, **83**(5), 831–879.

[20] Chmutov, S.V. and Duzhin, S.V. (1998) Kontsevich integral. To appear in *Acta Applicandae Math.*, available at `ftp://ftp.botik.ru/pub/local/zmr/ki.ps.gz`

[21] Chmutov, S. V., Duzhin, S.V., and Kaishev, A.I. (1998) The algebra of 3-graphs, *Proc. Steklov Math. Inst.*, **221**, 168–196.

[22] Chmutov, S. V., Duzhin, S. V., and Lando, S. V. (1994) Vassiliev knot invariants II. Intersection graph conjecture for trees, in V. I. Arnold, ed. *Singularities and curves*, Adv. in Soviet Math., **21**, AMS, 127-134.

[23] De Concini, C. and Procesi, C. (1995) Wonderful models of subspace arrangements, *Selecta Math., New Ser.*, **1**:3, 459–494.

[24] Deligne, P. (1972) Immeubles des groupes de tresses généraisés. *Invent Math.* **17**, 273–302.

[25] Deligne, P., Goresky, M., and MacPherson, R. (2000) L'algebre de cohomologie du complément, dans un espace affine, d'une famille finie de sous-espaces affines, *Michigan J. Math.* **48**

[26] de Longueville, M., and Schultz, C. (2000) The cohomology rings of complements of subspace arrangements, Math. Annalen (to appear). Preprint version available at: `http://www.math.fu-berlin.de/~cschultz/arrangements.html`

[27] Eilenberg, S. and Moore, J.C. (1966) Homology and fibrations. I, *Comment. Math. Helv.* **40**, 199–236.

[28] Fenn, R. and Taylor, P. (1977) Introducing doodles, in R. Fenn (ed.), *Topology of low-dimensional manifolds*, Lect. Notes Math., Springer, Berlin a.o., **722**, 37–43.

[29] Feichtner, E. M., and Ziegler, G. M. (2000) On cohomology algebras of complex subspace arrangements, *Trans. AMS*, in print.

[30] Fock, V. V., Nekrasov, N. A., Rosly, A. A., and Selivanov, K. G. (1992) What we think about the higher-dimensional Chern-Simons theories, in *Sakharov Memorial Lectures in Physics.* V.1. Commack, NY: Nova Science Publishers, Inc., 465–471.

[31] Garoufalidis, S. (1996) On finite type 3-manifold invariants, *J. of Knot Theory and its Ramifications,* **5**, 441–462.

[32] Goresky, M. and MacPherson, R. (1988) *Stratified Morse Theory,* Springer, Berlin a.o.

[33] Goussarov, M., Polyak, M., and Viro, O. (2000) Finite type invariants of classical and virtual knots, *Topology* **39**:5, 1045–1068.

[34] Goussarov, M. (1999) Finite type invariants and n-equivalence of 3-manifolds, *C. R. Ac. Sci. Paris,* **329** Sér. **1**, 517–522.

[35] Hain, R. (1984) Iterated integrals and homotopy periods, *Mem. AMS*, **47**:291, 98 pp.

[36] Khovanov, M. (1996) Doodle groups, *Trans. AMS* **349**, 2297–2315.

[37] Kontsevich, M. (1993) Vassiliev's knot invariants, in *Adv. in Sov. Math.,* **16:2,** AMS, Providence RI, 137–150.

[38] Kontsevich, M. (1993) Formal (non-)commutative symplectic geometry. In: L. Corvin, I. Gel'fand, J. Lepovsky (eds.), *The I. M. Gel'fand's mathematical seminars 1990–1992,* Birkhäuser, Basel, 173–187.

[39] Lannes, J. (1993) Sur les invariants de Vassiliev de degreé inferieur ou égal à 3. *L'Enseignement Mathématique* **39**, 295–316.

[40] Merkov, A. B. (1998) Finite order invariants of ornaments, *J. of Math. Sciences* **90**:4, 2215–2273.

[41] Merkov, A. B. (1999) Vassiliev invariants classify flat braids, in S. L. Tabachnikov (ed.), *Differential and Symplectic Topology of Knots and Curves,* AMS Translations Ser. 2, **190**, AMS, Providence RI, 83–102.

[42] Merkov, A. B. (1998) Vassiliev invariants classify plane curves and doodles, Preprint, `http://www.botik.ru/~duzhin/as-papers/finv-dvi.zip`

[43] Merkov, A. B. (2000) Segment–arrow diagrams and invariants of ornaments, *Mat. Sbornik*, to appear.

[44] Ohtsuki, T. (1996) Finite type invariants of integral homology 3-spheres, *J. of Knot Theory and its Ramifications,* **5**, 101–115.

[45] Orlik, P. and Solomon, L. (1980) Combinatorics and topology of complements of hyperplanes, *Invent. Math.*, **56**(2), 167–189.

[46] Polyak, M. and Viro, O. (1994) Gauss diagram formulas for Vassiliev invariants, *Internat. Math. Res. Notes* **11**, 445–453.

[47] Polyak, M. and Viro, O. (1996) On the Casson knot invariant, *Tel Aviv University and Uppsala University preprint.* `math.GT/9903158`

[48] Piunikhin, S.A. (1995) Combinatorial expression for universal Vassiliev link invariant, *Comm. Math. Phys.* **168**:1, 1–22.

[49] Rybnikov, G.L. (1998) On the fundamental group of the complex hyperplane arrangement, `math.AG/9805056`.

[50] Shumakovich, A. (1996) Explicit formulas for strangeness of plane curves, *St. Petersburg Math. J.* **7**:3, 445–472.

[51] V. E. Turchin (1997) Homology groups of complexes of two-connected graphs, *Uspekhi Math. Nauk*, **52**:2, 189–190. English Transl.: *Russian Math. Surveys*, **52**:2, 426–427.

[52] V. E. Turchin, (1998) Homology isomorphism of the complex of 2-connected graphs and the graph complex of trees, in: B. Feigin and V. Vassiliev, eds., *Topics in Quantum Groups and Finite-Type Invariants. Mathematics at the Independent University of Moscow.* AMS Translations. Ser. 2. Vol. 185. Advances in the Mathematical Sciences. AMS, Providence RI, 145–153.

[53] Tourtchine, V. (2000) Sur l'homologie des espaces des nœuds non-compacts, arXiv:`math.QA/0010017`

[54] Tyurina, S. D. (1999) On the Lannes and Viro-Polyak type formulas for finite type invarians, *Matem. Zametki*, **66:**4, 635–640. English transl.: *Math. Notes*, **66**, No.3–4, 525–530.

[55] Tyurina, S. D. (1999) Diagrammatic formulas of the Viro-Polyak type for finite degree invariants, *Russian Math. Surv. (Uspekhi)* **54**:3, 658–659.

[56] Vassiliev, V. A. (1994) *Complements of discriminants of smooth maps: topology and applications, Revised ed.*, Translations of Math. Monographs, AMS, Providence RI.

[57] Vassiliev, V. A. (1997) *Topology of complements of discriminants,* Phasis, Moscow (in Russian).

[58] Vassiliev, V. A. (1987-88) Stable cohomology of complements of discriminant manifolds of singularities of holomorphic functions, *Russian Math. Surveys* **42**:2, 307–308 and *J. Soviet Math.* **52**:4, 3217–3230.

[59] Vassiliev, V.A. (1990) Cohomology of knot spaces, in: *Theory of Singularities and its Applications* (V. I. Arnold, ed.), Advances in Soviet Math. Vol. 1, AMS, Providence, RI, 23–69.

[60] Vassiliev, V. A. (1991) A geometric realization of the homology of classical Lie groups, and complexes, S-dual to the flag manifolds, *Algebra i Analiz* **3**:4, 113–120. English transl.: *St.-Petersburg Math. J.*, **3**:4, 809–815.

[61] Vassiliev, V. A. (1993) Complexes of connected graphs, in L. Corvin, I. Gel'fand, J. Lepovsky (eds.), *The I. M. Gel'fand's mathematical seminars 1990–1992,* Birkhäuser, Basel, 223–235.

[62] Vassiliev, V. A. (1994) Invariants of ornaments, in V. I. Arnold (ed.), *Singularities and Bifurcations, Advances in Soviet Math.,* **21**, AMS, Providence RI, 225–262.

[63] Vassiliev, V. A. (1999) Topology of two-connected graphs and homology of spaces of knots, in S. L. Tabachnikov (ed.), *Differential and Symplectic Topology of Knots and Curves,* AMS Transl., Ser. 2, **190**, AMS, Providence RI, 253–286.

[64] Vassiliev, V. A. (1999) Homology of i-connected graphs and invariants of knots, plane arrangements, etc., in *Proc. of the Arnoldfest Conference, Fields Inst. Communications,* **24**, AMS, Providence RI, 451–469.

[65] Vassiliev, V. A. (1999) How to calculate homology groups of spaces of nonsingular algebraic projective hypersurfaces. *Proc. Steklov Math. Inst.,* **225**, 121–140.

[66] Vassiliev, V. A. (1999) On finite-order invariants of triple points free plane curves, in A. Astashkevich and S. Tabachnikov (eds.) *D. B. Fuchs Anniversary Volume,* AMS Transl., Ser. 2, **194**, AMS, Providence RI, 275–300.

[67] Vassiliev, V. A. (1999) Topological order complexes and resolutions of discriminant sets, *Publications de l'Institut Mathématique Belgrade, Nouvelle série* **66(80)**, 165–185.

[68] Vassiliev, V. A. (2001) On combinatorial formulas for cohomology of spaces of knots, *Moscow Math. J.*, to appear

[69] Yuzvinsky, S. (1998) Small rational model of subspace complement, preprint `http://xxx.lanl.gov/abs/math.CO/9806143`, to appear in Trans. AMS.

[70] Yuzvinsky, S. (1999) Rational model of subspace complement on atomic complex, *Publications de l'Institut Mathématique Belgrade, Nouvelle série* **66(80)**, 157–164.

[71] Ziegler, G. M. and Živaljević, R. T. (1993) Homotopy type of arrangements via diagrams of spaces, *Math. Ann.* **295**, 527–548.

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