Germs of integral curves in contact 3-space, plane and space curves

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1. Introduction and main results.

Consider the contact space $\mathbb{R}^{2n+1}$. Two germs at $0 \in \mathbb{R}$ of curves $\gamma_1, \gamma_2 : (\mathbb{R}, 0) \to (\mathbb{R}^{2n+1}, 0)$ are called equivalent (resp. contact equivalent) if there exists a local diffeomorphism $\Phi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ and a local diffeomorphism (resp. contactomorphism, i.e. a diffeomorphism preserving the contact structure) $\Psi : (\mathbb{R}^{2n+1}, 0) \to (\mathbb{R}^{2n+1}, 0)$ such that $\gamma_2 = \Psi \circ \gamma_1 \circ \Phi^{-1}$.

A curve $\gamma : \mathbb{R} \to \mathbb{R}^{2n+1}$ is called integral if $\gamma^* \omega = 0$, where ω is a 1-form describing the contact structure. This means that the image of γ is tangent to the contact structure at any nonsingular point. By Darboux-Givental' theorem [1] any two germs of immersed integral curves are contact equivalent.

Assume now that γ_1 and γ_2 are germs of integral curves at the singular point $0 \in \mathbb{R}$. Is it true that their equivalence implies contact equivalence? Recently V.I. Arnol'd showed [2] that if $n \geq 2$ then the answer is negative even for the simplest A-singularities: for any $k \geq 1$ there exist integral curves $\gamma_1, \ldots, \gamma_{2k+1}$ such that each of them is equivalent to the curve $t \to (t^2, t^{2k+1}, 0, \ldots, 0)$ and γ_i is not contact equivalent to γ_j if $i \neq j$.

The case $n = 1$, i.e. the 3-dimensional case, is simpler. In the present paper we prove several results showing that the contact classification of integral curves in contact 3-space is very close to the classification of space curves as well as to the classification of plane curves.

Convention. We work in a fixed category which is either C^∞ or real-analytic. Throughout the paper by a curve γ in $\mathbb{R}^n$ we mean *the germ* at $0 \in \mathbb{R}$ of a mapping $(\mathbb{R}, 0) \to (\mathbb{R}^n, 0)$. A curve is singular if $0 \in \mathbb{R}$ is not an immersed point, i.e. $\gamma'(0) = 0$.

Theorem 1. *In the set of integral curves in a contact 3-space there exists a subset E of infinite codimension such that away from this subset the equivalence of two integral curves implies their contact equivalence.*

The set E does not depend on the contact structure. It consists of integral curves γ whose formal series $\hat{\gamma}$ satisfies one of the following conditions:

(i) $\hat{\gamma} = 0$;
(ii) $\hat{\gamma}$ is equivalent to $(t^q, 0, 0)$, $q \geq 2$;
(iii) $\hat{\gamma}$ is equivalent to $(t^q, t^p, 0)$, where $2 \leq q < p$ and the numbers q, p have a common factor > 1.

Theorem 1 is proved in sect.3. The proof is based on Proposition 1 slightly extending the 3-dimensional case of one of A.B. Givental's theorems on singular Legendrian submanifolds [3] and Proposition 2 on integral curves defining the

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orientation of the space.

The classification of space curves was started in [4], see also [5]. Note that Theorem 1 does not reduce the contact classification of integral curves in contact 3-space to the classification of space curves because the set of space curves that are equivalent to integral ones is rather complicated subset of the set of all space curves, see Appendix 2.

On the other hand, the contact classification of *simple singularities* of integral curves reduces to the classification of *plane curves.* Recall that a curve γ in $\mathbb{R}^n$ is called simple (in the set of all curves in $\mathbb{R}^n$) if for some finite k all curves whose k-jet is sufficiently close to the k-jet of γ belong to a finite set of equivalence classes.

Definition. *An integral curve in a contact space is called contact simple in the set of integral curves (or just contact simple) if for some finite k all integral curves whose k-jet is sufficiently close to the k-jet of γ belong to a finite set of contact equivalence classes.*

Throughout the paper we will use the following notation.

Notation. Given an integral curve γ in the contact space $(\mathbb{R}^3, dz-ydx)$ we denote by $\pi(\gamma)$ its projection to the (x,y)-plane. Given a plane curve $\mu : (x(t), y(t))$ we denote by $\pi^{-1}(\mu) = \pi^{-1}(x(t), y(t))$ the integral curve γ such that $\pi(\gamma) = \mu$ and $\gamma(0) = 0$.

Theorem 2.
1. Let γ be an integral curve in the contact space $(\mathbb{R}^3, dz - ydx)$. The following statements are equivalent:

(i) γ is contact simple (in the set of integral curves);
(ii) γ is simple (in the set of all space curves);
(iii) $\pi(\gamma)$ is simple (in the set of all plane curves).

2. Two contact simple integral curves γ_1, γ_2 in the contact space $(\mathbb{R}^3, dz-ydx)$ are contact equivalent if and only if the curves $\pi(\gamma_1), \pi(\gamma_2)$ are equivalent.

Remarks. The equivalence of (i) and (ii) in Theorem 2 does not follow from Theorem 1 (Theorem 1 implies only $(ii) \Rightarrow (i)$). The second statement in Theorem 2 does not hold for sufficiently deep (but of finite codimension) singularities: there are contact equivalent integral curves γ_1, γ_2 in $(\mathbb{R}^3, dz-ydx)$ such that the plane curves $\pi(\gamma_1), \pi(\gamma_2)$ are not equivalent, and there are equivalent plane curves μ_1, μ_2 such that the integral curves $\pi^{-1}(\mu_1), \pi^{-1}(\mu_2)$ are not contact equivalent, see Appendix 3.

Theorem 2 and the known classification of simple plane curves [6] lead to the classification of contact simple integral curves. The classification of singular simple curves $(\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ in [6] easily implies the classification of singular simple curves $(\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$. The list of normal forms includes four series

$$(t^2,\ t^{2k+1}), \tag{1_k}$$

$$(t^3,\ t^{3k+1} + (\pm 1)^i t^{3k+2+3i}), \tag{$2_{k,i}$}$$

$$(t^3,\ t^{3k+2} + (\pm 1)^{i+1} t^{3k+4+3i}), \qquad (3_{k,i})$$

$$(t^4,\ t^6 + t^{2k+5}), \qquad (4_k)$$

where $k \geq 1,\ i \in \{0, 1, \ldots, k-1\}$, and 8 sporadic curves

$$(t^4, t^5 \pm t^7),\ (t^4, t^5),\ (t^4, t^7 \pm t^9),\ (t^4, t^7 \pm t^{13}),\ (t^4, t^7). \qquad (5)$$

The normal forms $(2_{k,k-1})$ and $(3_{k,k-1})$ can be simplified - they are equivalent to (t^3, t^{3k+1}) and (t^3, t^{3k+2}) respectively.

Theorem 3.
1. A singular integral curve γ in a contact 3-space is contact simple if and only if one of the following holds:

(i) the 3-jet of γ is different from zero and γ is not infinitely degenerated;
(ii) the 5-jet of γ is equivalent to $(t^4, t^5, 0)$;
(iii) the 6-jet of γ is equivalent to $(t^4, t^6, 0)$ and γ is not infinitely degenerated;
(iv) the 7-jet of γ is equivalent to $(t^4, t^7, 0)$.

2. Any contact simple singular integral curve in $(\mathbb{R}^3, dz - ydx)$ is contact equivalent to one and only one curve of the form $\pi^{-1}(\mu)$, where μ is a plane curve of the list (1-5).

The infinitely degenerated curves in Theorem 3 are those whose formal series is equivalent to $(t^2, 0, 0)$ or to $(t^3, 0, 0)$ or to $(t^4, t^6, 0)$.

The set of integral curves which are not contact simple has codimension 8 in the space of all integral curves, see Appendix 1. Consequently, the germ at any point of any curve of a generic l-parameter family of globally defined integral curves is contact simple provided that $l \leq 6$.

Remark. The conditions (i)-(iv) in Theorem 3 do not involve the contact structure. This corresponds to Theorem 2 which implies that if γ is an integral curve with respect to two contact structures then the property of γ of being contact simple does not depend on the choice of contact structure.

The proof of Theorems 2 and 3 is based on Theorem 4 below having an independent significance for the contact classification of integral curves, not necessarily simple. We need the following notation.

Notation. Given two integers $1 < q < p$ denote by $T(q, p)$ the set of all integers $s > p$ which cannot be expressed in the form $s = \alpha_1 q + \alpha_2 p$ with integers $\alpha_1 \geq -1$ and $\alpha_2 \geq 0$.

It is easy to see that the set $T(q, p)$ is empty if and only if either $(q, p) = (2, 2r+1), r \geq 1$, or $(q, p) = (3, 4)$, or $(q, p) = (3, 5)$. The set $T(q, p)$ is finite if and only if the numbers q and p are mutually prime (have no common factor > 1). For example, the set $T(3, 7)$ consists of the only number 8, the set $T(4, 5)$ consists of the only number 7, the set $T(5, 6)$ consists of two numbers $8, 9$, and the set $T(6, 7)$ consists of 6 numbers $9, 10, 11, 16, 17, 23$.

Theorem 4. *Let γ be a singular integral curve in the contact space $(\mathbb{R}^3, dz - ydx)$. Assume that the formal series of γ is not zero and is not equivalent to $(t^q, 0, 0), q \geq 2$. Then*

1. γ is contact equivalent to an integral curve of the form $\pi^{-1}(t^q, f(t))$, where $f(t) = t^p + o(t^p))$, $2 \leq q < p$, and p is not divisible over q;

2. If $s > p$ and $s \notin T(q,p)$ then γ is contact equivalent to an integral curve of the form $\pi^{-1}(t^q, g(t))$, where $g(t) = j^{s-1}f(t) + o(t^s)$.

3. The formal series of γ is contact equivalent to the formal series

$$\pi^{-1}\left(t^q,\ t^p + \sum_{i\in T(q,p)} a_i t^i\right). \tag{6}$$

4. If q and p are mutually prime then the set $T(q,p)$ is finite and the normal form (6) holds in analytic and smooth categories.

Remark. The normal form (6) is, in general, preliminary - it contains contact equivalent integral curves unless the set $T(q,p)$ is empty.

In sect.2 we prove the first three statements of Theorem 4. This allows us to prove Theorem 1 in sect.3. The fourth statement of Theorem 4 and Theorems 2 and 3 are proved in sect.4. The Appendix 1 is devoted to adjacencies and codimension of singularities. In Appendix 2 we analyze how the space curves equivalent to integral ones are placed in the set of all space curves. In Appendix 3 we show that for sufficiently deep singularities neither the contact equivalence of integral curves in $(\mathbb{R}^3, dz - ydx)$ implies the equivalence of their projections to the (x,y)-plane, nor the equivalence of the projections implies the contact equivalence of the integral curves.

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2. Proof of Theorems 4 (statements 1-3).

The first statement easily follows from the existence of the following local contactomorphisms:

$$x \to y,\ y \to -x,\ z \to z - xy,$$

$$x \to x,\ y \to y + ax^m,\ z \to z + \frac{ax^{m+1}}{(m+1)},\ m \geq 1,\ a \in \mathbb{R}.$$

The third statement is a corollary of the second one, so we will concentrate on the proof of the second statement.

The assumption $s \notin T(q,p)$ means that s can be expressed in the form

$$s = (\beta_1 - 1)q + \beta_2 p,\quad \beta_1, \beta_2 \in \{0, 1, 2, \dots\}.$$

Set

$$H = cx^{\beta_1}y^{\beta_2}, \ c \in \mathbb{R},$$

$$X = -\frac{\partial H}{\partial y}\frac{\partial}{\partial x} + \frac{\partial H}{\partial x}\frac{\partial}{\partial y} + \left(H - y\frac{\partial H}{\partial y}\right)\frac{\partial}{\partial z}.$$

The vector field X vanishes at the origin, therefore it generates a flow Ψ^t of local diffeomorphisms. A simple calculation shows that the Lie derivative of the 1-form $dz - ydx$ along X is equal to zero. Therefore X is a contact vector field - the diffeomorphisms Ψ^t are contactomorphisms (moreover, they preserve the form $dz - ydx$).

Denote $\Psi = \Psi^1$. We will show that the contactomorphism Ψ and a suitable reparametrization $t \to \Phi(t)$ brings the curve $\gamma = \pi^{-1}(t^q, f(t)),\ f(t) = t^p + o(t^p))$ to the required form $\pi^{-1}(t^q,\ j^{s-1}f(t) + o(t^s))$. The curve $\pi(\Psi \circ \gamma)$ has the form

$$x(t) = t^q - c\beta_2 t^{s+q-p} + o(t^{s+q-p}), \quad y(t) = f(t) + c\beta_1 t^s + o(t^s).$$

Make a reparametrization of t bringing $x(t)$ back to the form t^q. This reparametrization has the form

$$t \to t + (c\beta_2/q)t^{s-p+1} + o(t^{s-p+1}).$$

It brings $y(t)$ to the form $\hat{y}(t) = f(t) + (c/q)(q\beta_1 + p\beta_2)t^s + o(t^s)$. Since $q\beta_1 + p\beta_2 \neq 0$ then for a suitable c we have $\hat{y}(t) = j^{s-1}f(t) + o(t^s)$. ■

3. Proof of Theorem 1. Integral curves orienting $\mathbb{R}^3$.

At first we will reformulate Theorem 1. We will denote by (ω) the contact structure described by a contact 1-form ω. By a symmetry of a curve $\gamma : (\mathbb{R}, 0) \to (\mathbb{R}^3, 0)$ we mean a local diffeomorphism $\Psi : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ preserving the image of γ. Theorem 1 is equivalent to the following statement.

Theorem 1,a. *Any singular curve γ in $\mathbb{R}^3$ which is integral with respect to a local contact structure (ω_0) has the following property provided that γ does not belong to the set E of infinite codimension in Theorem 1: if γ is also integral with respect to a local contact structure (ω_1) then there exists a local symmetry of γ bringing (ω_1) to (ω_0).*

The first step in proving Theorem 1,a is the following statement slightly extending the 3-dimensional case of one of A.B. Givental's theorem on singular Legendrian manifolds. Recall that any contact structure (ω) on $\mathbb{R}^3$ defines a canonical orientation on $\mathbb{R}^3$. Fix a volume form Ω on $\mathbb{R}^3$. Two local contact structures (ω_0) and (ω_1) define the same orientation if the numbers $(\omega_0 \wedge d\omega_0)/\Omega(0)$ and $(\omega_1 \wedge d\omega_1)/\Omega(0)$ have the same sign.

Proposition 1 (cf. [3], Theorem 1′.) *The conclusion of Theorem 1,a holds provided that $\gamma \notin E$ and the local contact structures (ω_0) and (ω_1) define the same orientation of $\mathbb{R}^3$.*

Theorem 1,a is a direct corollary of Proposition 1 and the following statement.

Proposition 2. *Any singular integral curve $\gamma \notin E$ has at least one of the following properties:*

(a) γ admits an orientation reversing symmetry - a local symmetry Ψ such that $det\Psi'(0) < 0$;

(b) γ defines an orientation of $\mathbb{R}^3$ in the following sense: if γ is an integral curve with respect to two contact structures then these contact structures define the same orientation of $\mathbb{R}^3$.

The property (a) (resp. (b)) holds if the formal series of γ is equivalent (resp. not equivalent) to $(t^q, t^p, 0)$.

In terms of normal form (6) the property (b) holds if the set $T(q,p)$ is not empty and at least one of the coefficients a_i is not zero. The first occurring singularity of integral curves defining orientation of the space is the singularity $\pi^{-1}(t^3, t^7 + t^8)$ in the contact space $(\mathbb{R}^3, dz - ydx)$. This singularity has codimension 6 in the set of all germs of integral curves.

Proof of Proposition 1. We will say that two contact structures can be joined by a segment of contact structures if there exists 1-forms ω_0 and ω_1 describing these contact structures such that the form $\omega_s = \omega_0 + s(\omega_1 - \omega_0)$ is contact for all $s \in [0,1]$.

Lemma 1. *Two local contact structures on $\mathbb{R}^3$ can be joined by a segment of contact structures if and only if they define the same orientation of $\mathbb{R}^3$.*

Let ω_0 and ω_1 be the 1-forms in Proposition 1, $\omega_s = \omega_0 + s(\omega_1 - \omega_0)$. Lemma 1 allows to assume without loss of generality that the 3-form $\omega_s \wedge d\omega_s$ does not vanish as $s \in [0,1]$. Define a family X_s of vector fields by the relation

$$X_s \rfloor (\omega_s \wedge d\omega_s) = \omega_s \wedge (\omega_1 - \omega_0) = \omega_0 \wedge \omega_1, \ \ s \in [0,1]. \tag{7}$$

The assumption $\gamma \notin E$ implies that the image of γ is smooth at all points except the origin. Since ω_0 and ω_1 annihilate any vector tangent to the image of γ then by (7) the vector field X_s is tangent to the image of γ at any point except the origin. Using again that $\gamma \notin E$ we obtain that $X_s(0) = 0$ for all $s \in [0,1]$. In fact, if $X_s(0) \neq 0$ for some s then the tangency of X_s to the image of γ implies that the image of γ belongs to a smooth curve which is possible only in the case $\gamma \in E$.

The given properties of X_s allow to construct a path of local diffeomorphisms Ψ_s by the system of ODE's $\frac{d\Psi_s}{ds} = X_s(\Psi_s)$, $\Psi_0 = id$ and to conclude that Ψ_s preserves the image of γ for any $s \in [0,1]$. Now we show that $\Psi_s^*(\omega_s) = (\omega_0)$. Denote by L_{X_s} the Lie derivative along the vector field X_s.

Lemma 2. $L_{X_s}\omega_s + \omega_1 - \omega_0 = h_s\omega_s$ *for some family h_s of functions.*

Denote $A(s) = \Psi_s^*\omega_s$. Differentiating by s and using Lemma 2 we obtain

$$A'(s) = \Psi_s^*(L_{X_s}\omega_s + \omega_1 - \omega_0) = Q_s A(s), \ \ Q_s = h_s(\Psi_s).$$

It follows that

$$A(s) = R_s A(0), \ \ R_s = e^{\int_0^s Q_\nu d\nu}.$$

Therefore $\Psi_s^*\omega_s = R_s\omega_0$ which means that $\Psi_s^*(\omega_s) = (\omega_0)$. To prove Proposition 1 it remains to prove Lemmas 1 and 2.

Proof of Lemma 2. The relation (7) implies that $(X_s\rfloor\omega_s)d\omega_s = \lambda_s \wedge \omega_s$ for some 1-form λ_s. It follows that $(X_s\rfloor\omega_s)\omega_s \wedge d\omega_s = 0$ and consequently $X_s\rfloor\omega_s = 0$ for all s. Then (7) leads to the relation $(L_{X_s}\omega_s + \omega_1 - \omega_0)\wedge\omega_s = 0$. Since $\omega_s(0) \neq 0$ we obtain the required relation. ■

Proof of Lemma 1. It is clear that if the orientations defined by the local contact structures are different then these contact structures cannot be joined by any continuous path of contact structures. Assume that the orientations are the same. Fix a volume form Ω such that $(\omega_0 \wedge d\omega_0)/\Omega(0) > 0$ and $(\omega_1 \wedge d\omega_1)/\Omega(0) > 0$. Consider the function

$$F(s,r) = (\omega_{s,r} \wedge d\omega_{s,r})/\Omega(0),$$

$$\omega_{s,r} = \omega_0 + s(r\omega_1 - \omega_0).$$

To prove the Lemma we will show that there exists $r^* \neq 0$ such that $F(s,r^*) > 0$ for all $s \in [0,1]$. We will use the relation

$$\frac{\partial F}{\partial s}(0,r) = \left(-2\omega_0 \wedge d\omega_0 + r(\omega_0 \wedge d\omega_1 + \omega_1 \wedge d\omega_0)\right)/\Omega(0).$$

Let $\theta = (\omega_0 \wedge d\omega_1 + \omega_1 \wedge d\omega_0)(0)$. If $\theta \neq 0$ then there exists $r = r^*$ such that $\frac{\partial F}{\partial s}(0,r^*) > 0$. Since the function $F(s,r^*)$ is quadratic with respect to s and takes positive values as $s = 0$ and $s = 1$ then $F(s,r^*) > 0$ for all $s \in [0,1]$. If $\theta = 0$ then $F(s,r) = \left((1-s)^2\omega_0 \wedge d\omega_0 + s^2r^2\omega_1 \wedge d\omega_1\right)/\Omega(0)$, therefore $F(s,r) > 0$ for all s and all $r \neq 0$. ■

Proof of Proposition 2. At first assume that $\gamma \notin E$ and the formal series of γ is equivalent to $(t^q, t^p, 0)$. In this case, by the definition of the set E, the numbers q and p are mutually prime and by Weierstrass-Malgrange preparation theorem γ is equivalent to $(t^q, t^p, 0)$ in the analytic or smooth category. The latter curve has an orientation reversing symmetry $(x,y,z) \to (x,y,-z)$.

Assume now that $\gamma \notin E$ and the formal series of γ is not equivalent to $(t^q, t^p, 0)$. We will show that in this case any two contact structures (ω_0) and (ω_1) such that $\gamma^*\omega_0 = \gamma^*\omega_1 = 0$ define the same orientation of $\mathbb{R}^3$.

There is no loss of generality to assume that $\omega_0 = dz - ydx$. By Theorem 4 there exist integers $1 < q < p < s$ such that p is not divisible over q and $s \in T(q,p)$ and the projection of γ to the (x,y)-plane has the form $x = t^q, y = t^p + at^s + o(t^s)$, $a \neq 0$. Then γ has the form

$$x = x(t) = t^q,\ \ y = y(t) = t^p + at^s + o(t^s),\ \ z = z(t) = bt^{p+q} + ct^{s+q} + o(t^{s+q})$$

with $b = q/(p+q)$ and $c = aq/(s+q)$. Let

$$\omega_1 = A(x,y,z)dx + B(x,y,z)dy + C(x,y,z)dz.$$

Then

$$F(t) = A(x(t),y(t),z(t))x'(t) + B(x(t),y(t),z(t))y'(t) + C(x(t),y(t),z(t))z'(t) \equiv 0.$$

We will express some coefficients of the formal series $\hat{F}(t) = f_1 t + f_2 t^2 + \cdots$ of $F(t)$ in terms of the coefficients of the formal series of A, B, C. Using the inequalities $1 < q < p < s$, the condition that p is not divisible over q, and the condition that s belongs to the set $T(q,p)$ we obtain

$$f_{q-1} = qA(0), \quad f_{p-1} = pB(0),$$
$$f_{p+q-1} = q\frac{\partial A}{\partial y}(0) + p\frac{\partial B}{\partial x}(0) + qC(0),$$
$$f_{s+q-1} = aq\frac{\partial A}{\partial y}(0) + as\frac{\partial B}{\partial x}(0) + aqC(0).$$

It follows that $A(0) = B(0) = 0$. The relations $f_{p+q-1} = f_{s+q-1} = 0$ give a system of linear equations with respect to $\frac{\partial A}{\partial y}(0)$ and $\frac{\partial B}{\partial x}(0)$. The determinant of the matrix of this system is equal to $qa(s-p) \neq 0$, and this system has unique solution $\frac{\partial A}{\partial y}(0) = -C(0)$, $\frac{\partial B}{\partial x}(0) = 0$. Therefore $(\omega_1 \wedge d\omega_1)(0) = C^2(0)dxdydz$ which means that the contact structures (ω_0) and (ω_1) define the same orientation of $\mathbb{R}^3$. ■

4. Proof of Theorem 4, statement 4, and Theorems 2 and 3.

The fourth statement of Theorem 4 is a corollary of Theorem 1 and the following well-known fact (see, for example, [4]): if q and p are mutually simple numbers then a space curve of the form $(t^q, t^p + o(t^p), f(t))$ is finitely determined (in analytic and smooth categories) for any function $f(t)$.

Theorem 2 is a corollary of Theorem 3, it can be obtained by comparing the classification of simple integral curves with the classification of simple plane and space curves in [4,6].

Now we start to prove Theorem 3. Theorem 1 implies that if γ is an integral curve with respect to two contact structures (ω_0) and (ω_1) then it is contact simple with respect to (ω_0) if and only if it is contact simple with respect to (ω_1). Therefore proving the first statement of Theorem 3 we can assume that $\omega_0 = dz - ydx$.

Assume that γ satisfies one of the conditions (i)-(iv) of Theorem 3. Then by Theorem 4 γ is contact equivalent to an integral curve $\pi^{-1}(t^q, t^p + o(t^p))$, where (q,p) is one of the pairs $(2, 2k+1), (3, 3k+1), (3, 3k+2), (4,5), (4,6), (4,7)$.

In the case $(q,p) = (2, 2k+1)$ the set $T(q,p)$ is empty and by Theorem 4 γ is contact equivalent to $\pi^{-1}(t^2, t^{2k+1})$.

If $(q,p) = (3,4)$ or $(q,p) = (3,5)$ then the set $T(q,p)$ is empty and by Theorem 4 γ is contact equivalent to $\pi^{-1}(t^3, t^4)$ or $\pi^{-1}(t^3, t^5)$.

In the case $(q,p) = (3, 3k+1), k \geq 2$ the set $T(q,p)$ consists of $k-1$ numbers $3k+2+3i, i \in \{0, 1, \ldots, k-2\}$. By Theorem 4 γ is contact equivalent either to the curve $\pi^{-1}(t^3, t^{3k+1})$ or to an integral curve of the form

$$x = t^3,\ y = t^{3k+1} + at^{3k+2+3i} + o(t^{3k+2+3i}),\ z = bt^{3k+4} + ct^{3k+5+3i} + o(t^{3k+5+3i}),$$

where $i \in \{0, 1, \ldots, k-2\}$, $a \neq 0$, $b = 3/(3k+4)$, $c = 3a/(3k+5+3i)$. It is clear that this curve is equivalent to

$$x = t^3,\ y = t^{3k+1} + \kappa t^{3k+2+3i} + o(t^{3k+2+3i}),\ z = t^{3k+5+3i} + o(t^{3k+5+3i}), \qquad (8)$$

where $\kappa = 1$ if i is even and $\kappa = \pm 1$ if i is odd. Note that any integer $> 3k+2+3i$ can be expressed in the form $\beta_1 \cdot 3 + \beta_2 \cdot (3k+1) + \beta_3 \cdot (3k+5+3i)$. Therefore all curves of the form (8) are equivalent. By Theorem 1 we obtain that the integral curve γ is contact equivalent to $\pi^{-1}\mu$, where μ has the normal form (2).

The case $(q,p) = (3, 3k+2), k \geq 2$ is similar. The set $T(q,p)$ consists of $k-1$ numbers $3k+4+3i, i \in \{0, 1, \ldots, k-2\}$. By Theorem 4 γ is contact equivalent either to the $\pi^{-1}(t^3, t^{3k+2})$ or to an integral curve of the form

$$x = t^3,\ y = t^{3k+2} + at^{3k+4+3i} + o(t^{3k+4+3i}),\ z = bt^{3k+5} + ct^{3k+7+3i} + o(t^{3k+7+3i}),$$

where $i \in \{0, 1, \ldots, k-2\}$, $a \neq 0$, $b = 3/(3k+5)$, $c = 3a/(3k+7+3i)$. This curve is equivalent to

$$x = t^3,\ y = t^{3k+2} + \kappa t^{3k+4+3i} + o(t^{3k+4+3i}),\ z = t^{3k+7+3i} + o(t^{3k+7+3i}), \tag{9}$$

where $\kappa = 1$ if i is odd and $\kappa = \pm 1$ if i is even. Since any integer $> 3k+4+3i$ can be expressed in the form $\beta_1 \cdot 3 + \beta_2 \cdot (3k+2) + \beta_3 \cdot (3k+7+3i)$ then all curves of the form (9) are equivalent. By Theorem 1 we obtain that the integral curve γ is contact equivalent to $\pi^{-1}\mu$, where μ has the normal form (3).

Consider now the case $(q,p) = (4,5)$. In this case the set $T(q,p)$ consists of the single number 7, and by Theorem 4 the integral curve γ is contact equivalent to $\pi^{-1}(t^4, t^5 + at^7)$. A contactomorphism $(x,y,z) \to (k_1 x, k_2 y, k_1 k_2 z)$ and reparametrization $t \to k_3 t$ with suitable k_1, k_2, k_3 reduce the parameter a to ± 1 unless $a = 0$. Therefore γ is contact equivalent to one of the curves $\pi^{-1}(t^4, t^5 \pm t^7)$, $\pi^{-1}(t^4, t^5)$.

The next case is $(q,p) = (4,6)$. By Theorem 4 γ is contact equivalent to an integral curve of the form

$$(t^4,\ t^6 + at^{2k+1} + o(t^{2k+1}),\ t^{10} + bt^{2k+5} + o(t^{2k+5})), \tag{10}$$

where $k \geq 3, a \neq 0, b = 4/(2k+5)$. The contact equivalence of γ to the curve $\pi^{-1}(t^4, t^{2k+1})$ follows from Theorem 1 and the equivalence of all space curves of the form (10). The latter can be proved as follows. Any plane curve of the form $(t^4, t^6 + at^{2k+1}), a \neq 0$ is equivalent to $(t^4, t^6 + t^{2k+1})$, see [6]. Therefore any space curve (10) is equivalent to

$$(t^4,\ t^6 + t^{2k+1},\ t^{2k+5} + b_1 t^{2k+6} + b_2 t^{2k+7} + o(t^{2k+7})).$$

The parameter b_1 can be reduced to zero by a change of coordinates $z \to z - b_1 x^m y$ if $k = 2m$ or $z \to z - b_1 x^{m+2}$ if $k = 2m+1$. The parameter b_2 also can be reduced to zero by a change of coordinates $z \to z - (b_2/2)y^2 + (b_2/2)x^3$ (it is essential that $k \geq 3$). It remains to note that the function $o(t^{2k+7})$ also reduces to zero since any integer $\geq 2k+8$ can be expressed in the form $\beta_1 \cdot 4 + \beta_2 \cdot 6 + \beta_3 \cdot (2k+5)$ with nonnegative integers $\beta_1, \beta_2, \beta_3$.

Finally, consider the case $(q,p) = (4,7)$. In this case $T(q,p) = \{9, 13\}$ and by Theorem 4 γ is contact equivalent to an integral curve of the form

$$(t^4,\ t^7 + a_1 t^9 + a_2 t^{13},\ b_0 t^{11} + b_1 t^{13} + b_2 t^{17}) \tag{11}$$

where $b_0 = 4/11, b_1 = 4a_1/13, b_2 = 4a_2/17$. It is clear that if $a_1 \neq 0$ then the space curve (11) is equivalent to the curve $(t^4, t^7 \pm t^9, t^{13})$, and if $a_1 = 0, a_2 \neq 0$ then (11) is equivalent to $(t^4, t^7 \pm t^{13}, t^{17})$. By Theorem 1 the integral curve γ is contact equivalent to one of the curves $\pi^{-1}(t^4, t^7 \pm t^9), \pi^{-1}(t^4, t^7 \pm t^{13}), \pi^{-1}(t^4, t^7)$.

Denote by B the set of integral curves that do not satisfy any of the conditions (i)-(iv) of Theorem 3. To prove Theorem 3 it remains to show that the set B contains no contact simple curves.

Denote by B_1 (resp. B_2) the set of integral curves that are contact equivalent to $\pi^{-1}(t^4, t^9 + o(t^9))$ (resp. $\pi^{-1}(t^5, t^6 + o(t^6))$). By Theorem 4 for any integral curve γ of the set B there exists an integral curve $\tilde{\gamma}$ of the set $B_1 \cup B_2$ such that $j^9\tilde{\gamma}$ is arbitrary close to $j^9\gamma$. Therefore to prove that B contains no contact simple curves it suffices to prove that the sets B_1 and B_2 contain no contact simple curves.

By Theorem 4 an integral curve of the set B_1 is contact equivalent to an integral curve of the form $(t^4, t^9 + a_1 t^{10} + a_2 t^{11}, 0) + o(t^{11})$, and an integral curve of the set B_2 is contact equivalent to an integral curve of the form $(t^5, t^6 + a_1 t^8 + a_2 t^9, 0) + o(t^9)$. Generic curves of this form are equivalent to $(t^4, t^9 + t^{10} + at^{11}, 0) + o(t^{11})$ and $(t^5, t^6 \pm t^8 + at^9, 0) + o(t^9)$ respectively. In these normal forms the parameter a is the modulus with respect to the RL-equivalence of space curves (see [6]), therefore the sets B_1 and B_2 do not contain contact simple curves. ■

Appendix 1. Codimension of simple singularities. Adjacencies.

By Theorems 3 and 4 the codimension of contact simple singularities of integral curves and adjacencies between such singularities are the same as those for plane curves.

Given two integers $1 < q < p$ such that p is not divisible over q, denote by $[q, p]$ the singularity class consisting of integral curves which are contact equivalent to the integral curves of the form $\pi^{-1}(t^q, t^p + o(t^p))$. By Theorem 4 the codimension of this singularity class in the space of all germs of integral curves is equal to $2(q-1) + t$, where t is the number of integers from q to $p-1$ (including $p-1$) that are not divisible over q.

In particular, the singularity classes $[5, 6]$ and $[4, 9]$ have codimensions 8 and 9 respectively. Therefore the set B defined at the end of sect.4 has codimension 8 and consequently *the set of integral curves which are not contact simple has codimension 8*. Here and below by codimension we mean the codimension in the space of all germs of integral curves.

The codimensions of the singularity classes $[q, p]$ containing contact simple integral curves and the adjacencies between these singularity classes are as follows:

$$codim[2, 2k+1] = k-1, \;\; codim[3, 3k+1] = 2k+2, \;\; codim[3, 3k+2] = 2k+3,$$

$$codim[4, 5] = 6, \;\; codim[4, 6] = 7, \;\; codim[4, 7] = 8,$$

$$[2, 2k+1] \longleftarrow [2, 2k+3], \;\;\; [3, 3k+1] \longleftarrow [3, 3k+2] \longleftarrow [3, 3k+4], \;\;\; k \geq 1,$$

$$[4, 5] \longleftarrow [4, 6] \longleftarrow [4, 7],$$

$$[3, m] \longleftarrow [2, m+1], \;\;\; m = 4, 8, 10, 14, 16, \ldots,$$

$$[3, m] \longleftarrow [2, m], \;\;\; m = 5, 7, 11, 13, 17, \ldots,$$

$$[4,5] \longleftarrow [3,5], \quad [4,6] \longleftarrow [3,7].$$

Each of the singularity classes $[2,2k+1]$, $k \geq 1$, $[3,4]$, and $[3,5]$ consists of one orbit with respect to the contact equivalence.

The singularity classes $[3,3k+1]$ and $[3,3k+2], k \geq 2$ consist of k orbits corresponding to the normal forms $(2)_{k,i}$ and $(3)_{k,i}$, $i = 0,\dots,k-1$. The codimensions of these singularities are equal to $2k+2+i$ and $2k+3+i$ respectively.

The singularity class $[4,5]$ consists of 3 orbits described by the normal forms $\pi^{-1}(t^4, t^5+t^7)$, $\pi^{-1}(t^4, t^5-t^7)$ and $\pi^{-1}(t^4, t^5)$. The codimension of these singularities is equal to $6,6,7$ respectively.

The singularity class $[4,6]$ consists, modulo infinitely degenerated curves, of infinite number of singularities described by the normal forms $(4)_k$. The codimension of these singularities is equal to $k+6$.

The singularity class $[4,7]$ consists of 5 orbits described by the normal forms $\pi^{-1}(t^4, t^7+t^9), \pi^{-1}(t^4, t^7-t^9)$, $\pi^{-1}(t^4, t^7+t^{13})$, $\pi^{-1}(t^4, t^7-t^{13})$, $\pi^{-1}(t^4, t^7)$. The codimension of these singularities is equal to 8,8,9,9,10 respectively.

The adjacencies within each of the singularity classes $[3,2k+1], [3,3k+2], [4,5]$, $[4,6], [4,7]$ correspond to the codimension of singularities: the singularity α of codimension a is adjacent to a singularity β of codimension b if and only if $a < b$.

Appendix 2. Space curves diffeomorphic to integral curves.

This Appendix is devoted to the following question: which curves in a contact 3-space are equivalent to integral curves? By the Darboux theorem on the local equivalence of all contact structures, this question is equivalent to the following one: for which space curves $\gamma : (\mathbb{R},0) \to (\mathbb{R}^3,0)$ there exists a *contact* 1-form ω on $\mathbb{R}^3$ such that $\gamma^*\omega = 0$?

It is easy to see that all A-singularities - the space curves diffeomorphic to one of the curves $A_{2k} : x = t^2, y = t^{2k+1}, z = 0$ - are equivalent to integral curves. In fact, the curve A_{2k} is integral with respect to the contact structure described by the 1-form $dz + (2k+1)ydx - 2xdy$.

The normal forms A_{2k} serve for all space curves whose Taylor series starts with quadratic terms except certain infinitely degenerated curves. The first occurring singularity with zero 2-jet is represented by the curve $\gamma : x = t^3, y = t^4, z = t^5$. Already this curve is not equivalent to any integral curve. To see this, assume that $\gamma^*\omega = 0$, where $\omega = A(x,y,z)dx + B(x,y,z)dy + C(x,y,z)dz$. Then $A(t^3,t^4,t^5) \cdot 3t^2 + B(t^3,t^4,t^5) \cdot 4t^3 + C(t^3,t^4,t^5) \cdot 5t^4 \equiv 0$, and this relation easily implies that $A(0) = B(0) = C(0) = 0$. This means that $\omega(0) = 0$ and consequently the relation $\gamma^*\omega = 0$ holds for no contact 1-form ω.

Nevertheless, there are deeper qubical singularities which are equivalent to integral curves. This is the case, for example, for the space curves $x = t^3, y = t^4, z = 0$ or $x = t^3, y = t^5, z = 0$. These curves are integral with respect to the contact structures described by 1-forms $dz + 4ydx - 3xdy$ and $dz + 5ydx - 3xdy$ respectively.

Analyzing the normal forms for qubical singularities (space curves with nonzero 3-jet and zero 2-jet) given in [4,5] one can obtain the following result. We will use the notations from the paper [5]. Any qubical singularity, except infinitely degenerated ones, can be described by one of the normal forms

$$E_{6k,p,i} : \ x = t^3, \ y = t^{3k+1} + t^{3k+2+3i}, \ z = t^{3k+2+3p},$$

$$E_{6k+2,p,i}:\ x=t^3,\ y=t^{3k+2}+t^{3k+4+3i},\ z=t^{3k+4+3p},$$

where $k\geq 1$, $0\leq p\leq k$, $0\leq i\leq k-1$.

Theorem. *The curve $E_{6k,p,i}$ or $E_{6k+2,p,i}$ is RL-equivalent to an integral curve if and only if $p=i+1$.*

For example, the curve $E_{12,1,0}:(t^3,t^7+t^8,t^{11})$ is equivalent to an integral curve, and the curves (t^3,t^7,t^8) and $(t^3,t^7+t^8,0)$, which are equivalent to the normal forms $E_{12,0,0}$ and $E_{12,2,0}$ respectively, are not equivalent to any integral curve.

We end this Appendix by a similar analysis for the space curves whose 5-jet is equivalent to $(t^4,t^5,0)$. Any curve with such 5-jet is RL-equivalent to one of the curves

$$\begin{array}{ccccccc} (t^4,t^5,t^6) & \longleftarrow & (t^4,t^5,t^7) & \longleftarrow & (t^4,t^5+t^7,t^{11})^* & \longleftarrow & (t^4,t^5,t^{11}) \\ & & & & \uparrow & & \uparrow \\ & & & & (t^4,t^5+t^7,0) & \longleftarrow & (t^4,t^5,0)^*. \end{array}$$

Here the arrows mean the adjacencies, and by * are marked those and only those curves which are equivalent to integral curves.

Appendix 3. Contact equivalence of integral curves and equivalence of their projections.

In this Appendix we give two examples showing that

(a) there are contact equivalent integral curves γ_1,γ_2 in the contact space $(\mathbb{R}^3,dz-ydx)$ whose projections $\pi(\gamma_1)$, $\pi(\gamma_2)$ to the (x,y)-plane are not equivalent;

(b) there are equivalent plane curves μ_1,μ_2 such that the integral curves $\pi^{-1}(\mu_1),\pi^{-1}(\mu_2)$ in the contact space $(\mathbb{R}^3,dz-ydx)$ are not contact equivalent.

By Theorem 2 such examples are impossible if γ_1 and γ_2 are contact simple curves or if μ_1 and μ_2 are simple curves. The examples below involve nonsimple singularities of big (but finite) codimension.

Example 1. Consider the integral curves

$$\gamma_0:\ (x(t),y(t),z(t))=\pi^{-1}(t^7,\ t^{10}+t^{11}+t^{12}).$$

Let

$$\tilde{x}(t)=x(t),\ \tilde{z}(t)=z(t)(1+ax(t)),\ \tilde{y}(t)=\tilde{z}'(t)/\tilde{x}'(t),$$

where a is a parameter. The curve

$$\gamma_a:\ (\tilde{x}(t),\tilde{y}(t),\tilde{z}(t))$$

is integral and contact equivalent to the curve γ_0 via the contactomorphism

$$\tilde{x}=x,\quad \tilde{z}=z(1+ax),\quad \tilde{y}=\frac{d\tilde{z}}{d\tilde{x}}=y(1+ax)+az.$$

One can calculate that the 19-jet of the plane curve

$$\pi(\gamma_a):\ \tilde{x}(t)=t^7,\ \tilde{y}(t)=t^{10}+t^{11}+t^{12}+a\left(\frac{24}{17}t^{17}+\frac{25}{18}t^{18}+\frac{26}{19}t^{19}\right)$$

is equivalent to the 19-jet

$$\eta_a:\ \left(t^7,\ t^{10}+t^{11}+t^{12}+\frac{14}{17\cdot 18\cdot 19}at^{19}\right)$$

and that the 19-jets η_a and η_0 are not equivalent unless $a=0$. Therefore the curves $\pi(\gamma_a)$ and $\pi(\gamma_0)$ are not equivalent if $a\neq 0$.

Example 2. Consider the plane curves

$$\mu_1:\ x=t^{10},\ y=t^{15}+t^{16}+t^{17},$$

$$\mu_2:\ x=t^{10},\ y=t^{15}+t^{16}+t^{17}+t^{10}(t^{15}+t^{16}+t^{17}).$$

It is clear that these curves are equivalent. One can calculate that the 37-jets of the integral curves $\pi^{-1}(\mu_1)$ and $\pi^{-1}(\mu_2)$ are not equivalent. Therefore the integral curves $\pi^{-1}(\mu_1)$ and $\pi^{-1}(\mu_2)$ are not contact equivalent.

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