Generic geometry of symplectic relations

by

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[1] [2] [3]

Abstract

Stratification of Lagrangian Grassmannian in the product symplectic space $(N \times M, \pi_M^* \omega_M - \pi_N^* \omega_N)$ is constructed and the global homological properties of the strata are investigated. Basing on the symplectic Gauss map, which prescribes to each point of a Lagrangian submanifold (symplectic relation) in the product symplectic space the tangent linear Lagrangian subspace, the generic properties of symplectic relations are described and the corresponding local symplectic invariants are derived. Classification of local models for generic symplectic relations is reduced to the classification of conjugacy classes of smooth generic matrices.

1 Introduction.

Symplectic structure on a manifold M is a 2-form ω, which is closed and nondegenerate. Symplectic structures appear naturally in various branches of physics, e.g. mechanics, optics, thermodynamics, etc. In most of these cases the symplectic space is a cotangent bundle to a manifold with a symplectic form being the differential of the canonical Liouville one-form. For a symplectic manifold (M, ω), say the cotangent bundle with its Liouville form, we can consider submanifolds L which are isotropic with respect to the symplectic form, $\omega \mid_L = 0$. If L has a maximal possible dimension equals half of the dimension of the symplectic manifold, then it is called Lagrangian. The Lagrangian submanifolds (possibly with singularities) are interesting objects since for instance in optics they represent the systems of rays producing an evolving wavefront, in mechanics they correspond to Hamiltonian systems and in thermodynamics they exactely describe the space of states of the system being in thermodynamical equilibrium.

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There are an extensive local and global studies of Lagrangian submanifolds and their singularities (cf. [19, 3, 8, 11]). An important object in global investigations of Lagrangian submanifolds is the Lagrangian Grassmannian Λ_n; the manifold of linear Lagrangian subspaces in $2n$-dimensional linear symplectic space. The canonical stratification $\Lambda_n = \cup_{k=0}^{n} \Lambda_{n,k}^{\alpha}$, $\Lambda_{n,k}^{\alpha} = \{\beta \in \Lambda_n : dim(\beta \cap \alpha) = k\}$, where α is a fixed element of Λ_n, allows us to indicate the geometry of Lagrangian submanifolds and their singularities. The set $\Lambda_n^{(1)} = \cup_{k=1}^{n} \Lambda_{n,k}^{\alpha}$ is orientable and its singular part has codimension strictly greater than 2 in Λ_n. Thus $\Lambda_n^{(1)}$ determines the singular cycle which is Poincaré dual to the universal Maslov class $\mu \in H^1(\Lambda_n, \mathbb{Z})$ (cf. [2]). Investigations of Lagrangian submanifolds in product symplectic manifold $(M_1 \times M_2, \pi_2^* \omega_2 - \pi_1^* \omega_1)$, called also the symplectic relations, as it was shown in [9], need to use the another natural stratification of the Lagrangian Grassmannian Λ_{n+m} $(dim M_1 = n, dim M_2 = m)$ determined by the imposed product structure properties.

The main goal of this paper is to explore the global geometry of the product stratification of the Lagrangian Grassmannian and indicate the local symplectic invariants by means of the normal forms for generic symplectic relations. This type of problems are natural and turn out to be intimately related to the generalization of the symplectic transformation group and investigations of splitting of the potential systems (cf. [16, 19, 18, 9]).

In Section 2 of this paper we develop the local description of Lagrangian submanifolds in product symplectic manifold. We show that by the general symplectomorphisms acting in both components of the product any Lagrangian germ can be generated by a function without Morse parameters. To proceed, further on, the investigations of symplectic relations, in Section 3 we construct the stratification of Lagrangian Grassmannian in product linear symplectic space. This stratification distinguish the symplectically nonequivalent vertical positions of symplectic relations and measures their difference to canonical relations formed by the graphs of symplectomorphisms. In Section 4 we show that a first singular stratum of the Grassmannian Λ_{2n} is coorientable, we calculate the first homology group of the strata and find the cycle in this group realizing the class dual to the universal Maslov class of the Grassmannian Λ_{2n}. For a smooth symplectic relation, in Section 5, using the tangential "Gauss" map into Grassmannian Λ_{n+m} and transversality of this map we investigate the generic properties of symplectic relations with respect to their position

according to the Cartesian product projections.

2 Product symplectic space

Let $(M_1,\omega_1),(M_2,\omega_2)$ be two symplectic manifolds, $dimM_1 = 2n$, $dimM_2 = 2m$. The product symplectic space is defined as

$$\mathcal{M} = (M_1 \times M_2, \omega_2 \ominus \omega_1),$$

where $\pi_i, i = 1,2$, are the canonical projections, $\pi_i : M_1 \times M_2 \to M_i$, $\omega_2 \ominus \omega_1 = \pi_2^*\omega_2 - \pi_1^*\omega_1$.

By C, L, I, we denote the coisotropic, Lagrangian and isotropic submanifolds of $\mathcal{M}$ with respect to the symplectic structure $\Omega = \omega_2 \ominus \omega_1$ (cf. [19, 9]). There is a natural question concerning the typical positions of the submanifolds with respect to the canonical projections. In this paper we will consider the generic properties of Lagrangian submanifolds known also as the symplectic relations or correspondences (cf. [18, 8]).

By (L,p) we denote a Lagrangian submanifold germ in $\mathcal{M}$. Now we introduce the equivalence relation acting in the space of germs of Lagrangian submanifolds.

DEFINITION 2.1 *The two Lagrangian germs $(L_1,p_1),(L_2,p_2) \subset (\mathcal{M},\Omega)$ are called equivalent if there exist two symplectomorphism germs $B_1 : (M_1,\pi_1(p_1)) \to (M_1,\pi_1(p_2))$ and $B_2 : (M_2,\pi_2(p_1)) \to (M_2,\pi_2(p_2))$ such that the symplectomorphism $B_1 \times B_2 : \mathcal{M} \to \mathcal{M}$ sends L_1 into L_2 and p_1 into p_2.*

Now we have the preliminary

LEMMA 2.1 *If (L,p) is a Lagrangian germ in $\mathcal{M}$, then there are local cotangent bundle structures around $\pi_1(p)$, say T^*X and around $\pi_2(p)$, say T^*Y, such that (L,p) is generated in the product space*

$$\mathcal{M} \cong (T^*(X \times Y), \omega_Y \ominus \omega_X)$$

by a germ of a generating function $F : (X \times Y, \pi_{X\times Y}(p)) \to \mathbb{R}$ such that, in local coordinates on $(X \times Y, \pi_{X\times Y}(p))$,

$$F(x,y) = \sum_{i=1}^{n}\sum_{j=1}^{m} x_i y_j \phi_{ij}(x,y), \tag{1}$$

where ω_Y, ω_X are the corresponding Liouville forms on T^*Y and T^*X respectively, $\pi_{X\times Y} : T^*(X \times Y) \to X \times Y$ is a canonical cotangent bundle projection, $dimX = n$, $dimY = m$.

Proof. Let $((p,q),(\bar{p},\bar{q}))$ be Darboux coordinates on $T^*(X \times Y)$, then

$$\Omega = \sum_{i=1}^{m} d\bar{p}_i \wedge d\bar{q}_i - \sum_{i=1}^{n} dp_i \wedge dq_i.$$

By [3] (Section III 19.3) we can find the partition $I \cup J = \{1,\ldots,n\}$, $I \cap J = \emptyset$, $\bar{I} \cup \bar{J} = \{1,\ldots,m\}$, $\bar{I} \cap \bar{J} = \emptyset$ such that there exists a smooth function

$$(p_I, q_J, \bar{p}_{\bar{I}}, \bar{q}_{\bar{J}}) \to S(p_I, q_J, \bar{p}_{\bar{I}}, \bar{q}_{\bar{J}}),$$

which is a generating function for (L,p) (cf. [2, 3]). By the symplectomorphism Φ of $\mathcal{M}$,

$$\Phi(p,q,\bar{p},\bar{q}) = (-q_I, p_J, p_I, q_J, -\bar{q}_{\bar{I}}, \bar{p}_{\bar{J}}, \bar{p}_{\bar{I}}, \bar{q}_{\bar{J}}) = (\xi, x, \eta, y),$$

which preserves the product structure of $\mathcal{M}$, we get the generating function $(x,y) \to F(x,y)$ for (L,p) in the canonical cotangent bundle symplectic structure $T^*X \times T^*Y$ on $\mathcal{M}$. We can write

$$F(x,y) = F_1(x) + \sum_{i=1}^{n}\sum_{j=1}^{m} x_i y_j \phi_{ij}(x,y) + F_2(y)$$

and then, taking the equivalence $B_1 \times B_2$,

$$B_1(\xi,x) = (\xi - gradF_1(x), x), \quad B_2(\eta,y) = (\eta - gradF_2(y), y)$$

we get the reduced form (1) . □

Now we have the first simple equivalency class distinguished by the property

$$rank T\pi_2 \mid_{T_pL} = 2m,$$

where $m \leq n$.

PROPOSITION 2.1 *If (L,p) projects onto $(M_2, \pi_2(p))$, $dim M_2 = 2m$, $m \leq n$, then (L,p) is parallelizable, i.e. it is equivalent to its tangent space T_pL with the generating function*

$$F(x,y) = \sum_{i=1}^{m} x_i y_i$$

Proof. By assumption (L,p) may be parameterized by the following mapping

$$\Phi(\bar{p},\bar{q},q_I) = (\phi_I(\bar{p},\bar{q},q_I), \phi_J(\bar{p},\bar{q},q_I), q_I, \psi_J(\bar{p},\bar{q},q_I); \bar{p},\bar{q})$$

i.e. $\Phi^*\Omega = 0$, where $I \cup J = \{1, \dots n\}$, $I \cap J = \emptyset$, and ϕ_I, ϕ_J, ψ_J are smooth function-germs. Let us construct the symplectomorphism

$$\Xi : M_1 \to M_1$$

such that

$$\Xi(\xi, x) = (\xi_I + \phi_I(\xi_J, x_J, x_I), \phi_J(\xi_J, x_J, x_I), x_I, \psi_J(\xi_J, x_J, x_I), \xi_J, x_J).$$

It is really a symplectomorphism, in fact

$$\Xi^*(dp_I \wedge dq_I + dp_J \wedge dq_J) = d\xi_I \wedge dx_I + d\phi_I(...) \wedge dx_I + d\phi_J(...) \wedge d\psi_J(...).$$

But because

$$d\bar{p} \wedge d\bar{q} - d\phi_I(\bar{p}, \bar{q}, q_I) \wedge dq_I - d\phi_J(...) \wedge d\psi_J(...) = 0$$

we get

$$\Xi^*(dp_I \wedge dq_I + dp_J \wedge dq_J) = d\xi_I \wedge dx_I + d\xi_J \wedge dx_J.$$

Taking the equivalency (Ξ^{-1}, id) we get a germ, which is equivalent to (L, p) written in the form

$$\xi_J = -y_J, \quad \eta_J = x_J, \quad \xi_I = 0,$$

where $J = \{1, \dots, m\}$, and $\Omega = d\eta_J \wedge dy_J - d\xi \wedge dx$. But this germ is generated by the following generating function

$$F(x, y) = \sum_{i=1}^{m} x_i y_i.$$

□

REMARK 2.1 *In the case of Proposition 2.1, the image of Φ (see Proof of the Prop. 2.1) is a coisotropic submanifold of (M_1, ω_1) and moreover (M_2, ω_2) is isomorphic to the reduced symplectic manifold. For generic germ (L, p) this is the typical situation excluding strata of codimension ≥ 0 in (L, p) where the $rank T\pi_2 \mid_{T_pL}$ is not maximal.*

3 Lagrangian Grassmannian

Now we assume that the space $\mathcal{M}$ is linear and build with the linear symplectic spaces (N, ω_1) and (M, ω_2), $\mathcal{M} = (N \times M, \omega_2 \ominus \omega_1)$, $dim N = 2n$, $dim M = 2m$, $m \leq n$. By Λ_{n+m}

we denote the Lagrangian Grassmannian of linear $n+m$-dimensional Lagrangian subspaces in $\mathcal{M}$. Let N and M be canonically placed in the product. If $L \in \Lambda_{n+m}$ then there are two possibilities; L is transversal to N (i.e. $rank\pi_2 \mid_L$ is maximal), or L is not transversal to N. The set of those $L \in \Lambda_{n+m}$, which are not transversal to N we denote by $C\Lambda_{n+m}$ and call it the critical subset of Λ_{n+m}. Naturally if L is transversal to N then L is a linear reduction relation. The subspace of Λ_{n+m} consisting such L we will denote by RSp_{n+m}. The case of $n = m$ was studied in [9] and in the first two sections we generalize these results to $n \neq m$. If $n = m$, RSp_{n+m} is equal to the space of graphs of symplectic isomorphisms GSp_{2n}. Analogously, the most singular set of elements in Λ_{n+m} is called supercritical. It is denoted by $S\Lambda_{n+m}$, and defined as $S\Lambda_{n+m} = \Lambda_n \times \Lambda_m$, where Λ_n and Λ_m are the Lagrangian Grassmannians in N and M respectively. We easily calculate the codimension of the most critical stratum

$$codim S\Lambda_{n+m} = dim\Lambda_{n+m} - dim\Lambda_n - dim\Lambda_m = nm.$$

Now we introduce the composition of symplectic linear relations. Let R_1 and R_2 be the Lagrangian subspaces in the product spaces, say $(M_1 \times M_2, \omega_2 \ominus \omega_1)$ and $(M_2 \times M_3, \omega_3 \ominus \omega_2)$ respectively. Then we define the composition of R_1 and R_2, denoted by $R_2 \circ R_1$ as the following Lagrangian subspace

$$R_2 \circ R_1 = \{(v_1, v_3) \in M_1 \times M_3 : \exists_{v_2 \in M_2}, (v_1, v_2) \in R_1, (v_2, v_3) \in R_2\}$$

in the product symplectic space $(M_1 \times M_3, \omega_3 \ominus \omega_1)$. The transposed relation to $R \subset (M_1 \times M_2, \omega_2 \ominus \omega_1)$ is defined to be the Lagrangian subspace of $(M_2 \times M_1, \omega_1 \ominus \omega_2)$ having the form

$$R^t = \{(v_2, v_1) \in M_2 \times M_1 : (v_1, v_2) \in R\}.$$

Using the description method of the Grassmannian by compositions of symplectic relations we have the following result.

LEMMA 3.1 *If $L \in RSp_{n+m}$, then L has a following unique decomposition*

$$L = \tilde{L} \circ R,$$

where $\tilde{L}$, R *are symplectic relations,* $\tilde{L} \subset (\tilde{N} \times M, \omega_M \ominus \omega_{\tilde{N}})$, $R \subset (N \times \tilde{N}, \omega_{\tilde{N}} \ominus \omega_N)$ *and* R *is a graph of the coisotropic projection* ρ *onto* $(\tilde{N}, \omega_{\tilde{N}})$ *and* $\omega_{\tilde{N}}-$ *is defined by the formula*

$$\rho^* \omega_{\tilde{N}} = \omega_N \mid_{\pi_1(L)},$$

and $\tilde{L}$ *is an element of the Grassmannian* Λ_{2m} *in the symplectic space* $(\tilde{N} \times M, \omega_M \ominus \omega_{\tilde{N}})$, $dim\tilde{N} = dimM = 2m$, *being the graph of a symplectic isomorphism.*

On the basis of this unique decomposition we immediately have

$$dimRSp_{n+m} = dimI^{2n}_{n-m} + dim\Lambda_{2m},$$

where $dimI^{2n}_{n-m}$ is dimension of an isotropic Grassmannian of $n-m-$spaces in $2n-$dimensional symplectic space. This dimension is equal to the dimension of the coisotropic Grassmannian of coisotropic spaces of codimension $n-m$ denoted by C^{2n}_{n-m}. Then using the formula

$$dimC^{2n}_{n-m} = 2n(n-m) - \frac{1}{2}(n-m)(3(n-m)-1)$$

we get

$$\begin{aligned} dimRSp_{n+m} &= 2n(n-m) - \frac{1}{2}(n-m)(3(n-m)-1) + 2m^2 + m \\ &= \frac{1}{2}(n+m)(n+m+1) = dim\Lambda_{n+m} \end{aligned}$$

Finally we have the resulting decomposition of $C\Lambda_{n+m}$.

THEOREM 3.1 *The critical set of* Λ_{n+m} *has a canonical partition into smooth strata*

$$C\Lambda_{n+m} = \bigcup_{k=1}^{m} C_k\Lambda_{n+m},$$

where the elements of $C_k\Lambda_{n+m}$ *are determined by the pairs of two coisotropic subspaces* V_1 *in* N *and* V_2 *in* M *of codimension* $n-m+k$ *and* k *respectively, and the symplectic linear automorphism of the* $2n-2k-$*dimensional symplectic space.*

Proof. By π_N (π_M resp.) we denote the canonical projections of $\mathcal{M} = N \times M$ onto N (M resp.). The simple geometric argument shows that if $L \in C\Lambda_{n+m}$ then $\pi_N(L) \subset N$ and $\pi_M(L) \subset M$ are the coisotropic subspaces with the equally dimensional uniquely defined reduced symplectic spaces. This property defines the corresponding strata distinguished

implicitly in Lemma (3.1) and the decomposition of the Theorem follows immediately.

□

As a result of this theorem we obtain

COROLLARY 3.1 *The strata $C_k\Lambda_{n+m}$ are smooth submanifolds of Λ_{n+m} and*

$$codim C_k\Lambda_{n+m} = k^2 + k(n-m). \tag{2}$$

Proof. It is pure calculational result.

$$codim C_k\Lambda_{n-m} = dim\Lambda_{n+m} - dim I^{2n}_{n-m+k} - dim I^{2m}_k - dim\Lambda_{2m-2k}.$$

Using the standard formulas for the respective dimensions we get the result. □

DEFINITION 3.1 *If $L \in C_k\Lambda_{n+m}$ then L is called the $k-$vertical element of the Grassmannian Λ_{n+m}.*

Hence Λ_{n+m} is decomposed into strata of $k-$vertical elements with $k = 0, \ldots, m \leq n$.

4 The geometry of Λ_{2n}

The purpose of this section is to explore the geometry of the Lagrangian Grassmannian in the product of two symplectic spaces of the same dimension. Let (M^{2n}, ω) be the symplectic linear space and $(\mathcal{M}, \Omega) = (M \times M, \omega \ominus \omega)$.

We consider the partition of the singular set $C\Lambda_{2n}$ into the smooth submanifolds, (cf. [9])

$$C\Lambda_{2n} = \bigcup_{k=1}^{n} C_k\Lambda_{2n}$$

every stratum $C_k\Lambda_{2n}$, for $k = 1, \ldots, n-1$, is fibered in the following way

$$\begin{array}{ccc} Sp(2n-2k) & \xrightarrow{\quad r \quad} & C_k\Lambda_{2n} \\ & & \Big\downarrow p \\ & & (\mathcal{I}^{2n}_k)^2 \end{array}$$

where p is a canonical projection into the symplectic polars (isotropic spaces) to coisotropic spaces prescribed to the elements of $C_k\Lambda_{2n}$ and r is a fiber inclusion. The symbol $\mathcal{I}_k^{2n}$ denotes the isotropic Grassmannian of $k-$dimensional isotropic subspaces (or $(2n-k)-$dimensional coisotropic subspaces) in $2n-$dimensional symplectic space, the symbol $Sp(2n-2k)$ denotes the group of symplectic linear automorphisms of the $(2n-2k)-$dimensional symplectic space.

PROPOSITION 4.1 *The first homology group of the set $C_k\Lambda_{2n}$, for $k = 1, \ldots, n-1$, with the real coefficients is equal to* $\mathbf{R}$.

Proof. We take the exact homotopy sequence for the fibration

$$\ldots \to \pi_1(Sp(2n-2k)) \to \pi_1(C_k\Lambda_{2n}) \to \pi_1((\mathcal{I}_k^{2n})^2) \to \ldots$$

Since $\pi_1(Sp(2n-2k)) \simeq \mathbf{Z}$ and $\pi_1((\mathcal{I}_k^{2n})^2) \simeq \mathbf{Z}_2 \oplus \mathbf{Z}_2$ ([13, 1]) is a torsion group, $Sp(2n-2k)$ and $C_k\Lambda_{2n}$ are connected, so we obtain that $H_1(C_k\Lambda_{2n}, \mathbf{R})$ is equal to $\mathbf{R}$ or is trivial. We will examine the generator of the group $H_1(C_k\Lambda_{2n}, \mathbf{R})$ coming from $Sp(2n-2k)$. We fix the point in the base $I_0 = \mathbf{R}^k \times \mathbf{R}^k \in (\mathcal{I}_k^{2n})^2$, so above I_0 we have the inclusion of the fibre $r : Sp(2n-2k) \hookrightarrow C_k\Lambda_{2n}$. As a generator of $H_1(Sp(2n-2k), \mathbf{R})$ we take the class $[\gamma_k(t)]$ of the matrix cycle in $Sp(2n-2k)$

$$\gamma_k(t) = \begin{pmatrix} e^{it} & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix}$$

for $t \in [0, 2\pi]$.

Let $(\varepsilon_1, \gamma_k(t)\varepsilon_1), (\varepsilon_2, \gamma_k(t)\varepsilon_2), \ldots, (\varepsilon_{n-k}, \gamma_k(t)\varepsilon_{n-k})$ be the complex basis of $graph\gamma_k(t) \subset \mathbf{C}^{n-k} \oplus \mathbf{C}^{n-k}$, where $\varepsilon_1, \ldots, \varepsilon_{n-k}$ is the standard basis in $\mathbf{C}^{n-k}$. We recall that the symplectic structure in the product $\mathcal{M}$ is given by the form $\Omega = \pi_2^*\omega - \pi_1^*\omega = \omega \ominus \omega$, so we conjugate the first variable due to the sign minus in the form Ω. The appropriate real basis is the following:

$(\varepsilon_1, \gamma_k(t)\varepsilon_1) = (1, 0, \ldots, 0, e^{it}, 0, \ldots, 0)$

$(\overline{i\varepsilon_1}, i\gamma_k(t)\varepsilon_1) = (-i, 0, \ldots, 0, ie^{it}, 0, \ldots, 0)$

$(\varepsilon_l, \gamma_k(t)\varepsilon_l) = (\varepsilon_l, \varepsilon_l)$

$(\overline{i\varepsilon_l}, i\gamma_k(t)\varepsilon_l) = (-i\varepsilon_l, i\varepsilon_l)\,, \quad for \quad l = 2, \ldots, n-k\,.$

We denote by $[\widetilde{\gamma_k(t)}]$ the image of the generator $[\gamma_k(t)]$ under the mapping

$$r_* : H_1(Sp(2n-2k), \mathbf{R}) \to H_1(\,C_k\Lambda_{2n}, \mathbf{R}).$$

For every Lagrangian subspace $L \subset (\mathcal{M}, \Omega)$ we have

$$L = \{\begin{pmatrix} A \\ B \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \ldots \\ \alpha_{2n} \end{pmatrix}, \alpha_i \in \mathbf{R}\}$$

for some $\begin{pmatrix} A \\ B \end{pmatrix} \in Sp(2n)$ where A and B are matrices of dimensions $n \times 2n$ (**cf.** [14]). We will represent $[\widetilde{\gamma_k(t)}]$ using the matrix $\begin{pmatrix} A \\ B \end{pmatrix}$.

The matrix A has the following form

$$A = \begin{pmatrix} X_A & 0 \\ 0 & Q_A \end{pmatrix},$$

where

1. $X_A = (\varepsilon_1, \overline{i\varepsilon_1}, \ldots, \varepsilon_{n-k}, \overline{i\varepsilon_{n-k}})$ is the matrix of dimension $(n-k) \times (2n-2k)$.
2. The matrix $Q_A = (E_k, 0)$ has the dimension $k \times 2k$ and consists of two parts: the identity matrix E_k and the zero matrix.

We have the analogous form of the matrix B:

$$B = \begin{pmatrix} X_B & 0 \\ 0 & Q_B \end{pmatrix},$$

where dimensions of the appropriate parts are the same as in the matrix A and

1. $X_B = (\gamma_k(t)\varepsilon_1, i\gamma_k(t)\varepsilon_1, \ldots, \gamma_k(t)\varepsilon_{n-k}, i\gamma_k(t)\varepsilon_{n-k})$
2. $Q_B = (0, E_k)$.

We will show that the class of the cycle $\widetilde{\gamma_k(t)}$ does not vanish in homology. We calculate, that

$$\det(\widetilde{\gamma_k(t)}) = \det \begin{pmatrix} A \\ B \end{pmatrix} = \pm e^{it}(2i)^{n-k},$$

where $t \in < 0, 2\pi >$.

Now we consider the mapping

$$\det{}^2 : \Lambda_{2n} \to S^1 , \quad [D] \mapsto (\det D)^2 ,$$

where the element $[D] \in \Lambda_{2n} \simeq U(2n)/O(2n)$ represents a Lagrangian linear subspace in $4n-$ dimensional symplectic space. We recall, that the universal Maslov class of the Lagrangian Grassmannian is defined as the element μ_{2n} of the first cohomology group $H^1(\Lambda_{2n}, \mathbf{Z}) \simeq \mathbf{Z}$ such, that its evaluation on a $1-$dimensional cycle γ in Λ_{2n} is the intersection number of γ and a certain hypersurface. The class μ_{2n} is the generator of $H^1(\Lambda_{2n}, \mathbf{Z})$ and it is the image of the generator of $H^1(S^1, \mathbf{Z})$ under the mapping $\det{}^{2*} : H^1(S^1, \mathbf{Z}) \to H^1(\Lambda_{2n}, \mathbf{Z})$ ([2],[17]). Dually the map of homology $\det{}^2_* : H_1(\Lambda_{2n}, \mathbf{Z}) \simeq \mathbf{Z} \to H_1(S^1, \mathbf{Z}) \simeq \mathbf{Z}$ is an isomorphism and the image of the cycle $[\widetilde{\gamma_k(t)}]$ under the mapping $\det{}^2_*$ is equal to $2g$, where g is the generator of $H_1(S^1, \mathbf{Z})$. We conclude that $[\widetilde{\gamma_k(t)}]$ is nonzero. The mapping r_* is an epimorphism, so consequently r_* is an isomorphism and we have, that $H_1(\, C_k\Lambda_{2n}, \mathbf{R}) \simeq \mathbf{R}$. $\square$

We denote by μ^*_{2n} an element of $H_1(\Lambda_{2n}, \mathbf{R}) \simeq \mathbf{R}$ dual to the universal Maslov class μ_{2n}, i.e. the evaluation of the class μ_{2n} on the class μ^*_{2n} is equal to one. So we get

$$[\widetilde{\gamma_k(t)}] = [\deg \det{}^2(\widetilde{\gamma_k(t)})]\, \mu^*_{2n} = 2\mu^*_{2n}.$$

For the inclusion $j : C_k\Lambda_{2n} \hookrightarrow \Lambda_{2n}$, $k = 1, \ldots, n-1$, we have $j_*([\widetilde{\gamma_k(t)}]) = 2\mu^*_{2n}$. Thus we proved the following theorem:

THEOREM 4.1 *For every stratum $C_k\Lambda_{2n}$, $k = 1, \ldots, n-1$, we can find the cycle in $H_1(\, C_k\Lambda_{2n}, \mathbf{R}) \simeq \mathbf{R}$ realizing the class $\mu^*_{2n} \in H_1(\Lambda_{2n}, \mathbf{R})$ dual to the universal Maslov class for the Grassmannian Λ_{2n}.*

REMARK 4.1 *We observe that in every stratum $C_k\Lambda_{2n}$, $k = 1, \ldots, n-1$ we can find another cycle which represents precisely the cycle μ^*_{2n}, for example in $C_1\Lambda_4$ we have the following cycle*

$$\begin{pmatrix} e^{it} & 0 & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & i \end{pmatrix} \qquad t \in (0, \pi).$$

Let l_0 be a fixed element of Λ_n and L be an arbitrary element of Λ_{2n}. We define the image of l_0 by L:

$$L(l_0) = \{p \in M : \exists p' \in l_0 \quad (p', p) \in L\}.$$

We consider the mapping ([9]) $\rho : \Lambda_{2n} \to \Lambda_n$; $\rho(L) = L(l_0)$. Let ρ_k be the mapping ρ reduced to the stratum $C_k\Lambda_{2n}$ for $k = 1, \ldots, n-1$. Although ρ is not continuous on the strata $C_k\Lambda_{2n}$ for $k = 1, \ldots, n-1$ but the loop $\widetilde{\gamma_k(t)} \subset C_k\Lambda_{2n}$ is transformed to a continuous loop in Λ_n. We can easily calculate, that for each $t \in (0, 2\pi)$ the Lagrangian subspace $\rho_k(\widetilde{\gamma_k(t)})$ in (M^{2n}, ω) is represented by the unitary matrix from $U(n)$

$$\begin{pmatrix} e^{it} & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix}$$

Thus the cycle $[\rho_k(\widetilde{\gamma_k(t)})]$ is equal to $2\mu_n^*$, where μ_n^* is the generator in $H_1(\Lambda_n, \mathbf{Z})$ and simultaneously it is a class dual to the universal Maslov class of the Grassmannian Λ_n.

We described the strata $C_k\Lambda_{2n}$, for $k = 1, \ldots, n-1$. Now we are interested in two extreme strata: supercritical set $C_n\Lambda_{2n} = S\Lambda_{2n} \simeq \Lambda_n \times \Lambda_n$ and in $GSp_{2n} = Sp(2n)$ consisting of the graphs of the linear symplectomorphisms. In these two cases we have no problem with continuity. As a generator of $H_1(Sp(2n), \mathbf{R}) \simeq \mathbf{R}$ we take the class of the matrix cycle $\widetilde{\gamma_n(t)}$, $t \in < 0, 2\pi >$, which were described above for the cases $k = 1, \ldots, n-1$.

In the case $k = 0$ every element of the cycle $\widetilde{\gamma_n(t)}$ represents the basis of the Lagrangian linear subspace which is a graph of a symplectomorphism. Let $\rho_0 = \rho\,|_{Sp(2n)}$. Considering the mappings

$$Sp(2n) \overset{j}{\hookrightarrow} \Lambda_{2n} \overset{\det^2}{\to} S^1$$

$$Sp(2n) \overset{\rho_0}{\to} \Lambda_n \overset{\det^2}{\to} S^1$$

we obtain the same results as for the strata $C_k\Lambda_{2n}$, where $k = 1 \ldots, n-1$. The mapping $\rho : \Lambda_{2n} \to \Lambda_n$ reduced to the supercritical set $S\Lambda_{2n} \simeq \Lambda_n \times \Lambda_n$ (we denote it by ρ_n) is

simply the projection π_2 on the second component of the product $M^{2n} \times M^{2n}$. Thus, the analogous arguments for the mappings

$$\begin{array}{ccc} & \overset{j}{} & \\ \Lambda_{2n} & \hookleftarrow & S\Lambda_{2n} \\ & & \downarrow \quad \pi_2 = \rho_n \\ & & \Lambda_n \end{array}$$

leads us to the conclusion:

COROLLARY 4.1

1. *For the mapping* $j_* : H_1(S\Lambda_{2n}, \mathbf{R}) \simeq \mathbf{R} \oplus \mathbf{R} \to H_1(\Lambda_{2n}, \mathbf{R}) \simeq \mathbf{R}$ *we obtain*

$$j_*(a\mu_n^*, b\mu_n^*) = (-a+b)\mu_{2n}^*,$$

where the classes μ_n^* *and* μ_{2n}^* *are dual to the universal Maslov classes for the Grassmannians* Λ_n *and* Λ_{2n}, $a, b \in \mathbf{R}$. *In particular we have* $j_*(-\mu_n^*, \mu_n^*) = 2\mu_{2n}^*$.

2. *The map* $\rho_{n*} : H_1(S\Lambda_{2n}, \mathbf{R}) \to H_1(\Lambda_n, \mathbf{R})$ *is the projection on the second factor, so* $\rho_{n*}(v, \mu_n^*) = \mu_n^*$ *for an arbitrary* v.

The Lagrangian Grasssmannian is an orientable manifold if and only if n is an odd integer (cf. [6]). So in our case Grassmannian Λ_{2n} is not orientable for an arbitrary $n \in \mathbf{N}$.

PROPOSITION 4.2 *The first singular set* $C_1\Lambda_{2n}$ *is coorientable in* Λ_{2n}.

Proof. We will construct the normal direction in an arbitrary point $L \in C_1\Lambda_{2n}$, $L \subset (\mathcal{M}, \Omega)$. Let $f_1 \in \pi_1(L)^\perp$, $f_2 \in \pi_2(L)^\perp$, where the symbol $\perp$ denotes the orthogonal complement in M with respect to the scalar product $<,>$. Observe that $(if_1, 0)$, $(0, if_2)$ belong to the subspace L, because

$$\Omega((if_1, 0), (v, w)) = -\omega(if_1, v) = -ReH(f_1, v) = - < f_1, v >= 0,$$

where H is the Hermitian product in M and analogously $\Omega((0, if_2), (v, w)) = 0$. The normal direction in the point $L \in C_1\Lambda_{2n}$ is defined as follows:

$$L(t) = span\{(tf_1, if_2), (if_1, -tf_2)\} \oplus L^\natural, \quad t \in (0, 1),$$

where the subspace $L^\natural$ is an orthogonal complement in the Lagrangian subspace L (with respect to the scalar product in $\mathcal{M}$) to the space $span\{(0, if_2), (if_1, 0)\}$. We check, that $span\{(f_1, 0), (0, f_2)\}$ are skew-orthogonal to $L^\natural$ and to each other i.e.

$$\Omega((tf_1, if_2), (if_1, -tf_2)) = \omega(if_2, -tf_2) - \omega(tf_1, if_1) = (-t) - (-t) = 0.$$

□

5 Local properties of symplectic relations

Let $L \subset \mathcal{M}$ be a symplectic relation. We can associate with L the symplectic "Gauss" map

$$G_L : L \ni p \to T_pL \in \Lambda_{n+m},$$

where the tangent space T_pL is identified with the linear subspace of $\mathcal{M}$.

DEFINITION 5.1 *We call L to be in general position (or it is generic) if G_L is transversal to the stratification $C\Lambda_{n+m} = \bigcup_{k=1}^{m} C_k\Lambda_{n+m}$. We say that L has $k-$vertical position at $p \in L$ if $G_L(p) \in C_k\Lambda_{n+m}$. The index k is called the rank of the vertical position.*

We see that the 0-vertical position at p corresponds to the case when $T_pL \in \Lambda_{n+m} - C\Lambda_{n+m}$. Following Theorem 3.1 and Lemma 2.1 we obtain the following result.

PROPOSITION 5.1 *Let $p \in L$, and L has $k-$vertical position at p, then the germ (L, p) is equivalent to one in $(T^*X \times T^*Y, \omega_Y \ominus \omega_X)$ generated by the generating function*

$$F(x, y) = \sum_{i=1}^{n}\sum_{j=1}^{m} x_i y_j \phi_{ij}(x, y),$$

where the rank of the vertical position of L at p is equal to the corank of the matrix $(\frac{\partial^2 F}{\partial x_i \partial y_j}(0, 0))$, $k = corank(\phi_{ij}(0, 0))$ at $(0, 0) = \pi_{X\times Y}(p)$.

Let $M_{n\times m}$ $(n \geq m)$ denote the space of $n \times m$ matrices of real numbers. For each natural r, $0 \leq r \leq m$ let Σ_r denote the subset of $M_{n\times m}$ consisting of matrices of corank r. Σ_r is a submanifold of $M_{n\times m}$ of codimension $r^2 + r(n - m)$ (cf. [7]). Let $E_{n\times m}$ denote the set of $n \times m$ matrices of smooth function-germs at 0 on $X \times Y$, i.e. a representant of a germ is a smooth mapping of some open neighborhood of $0 \in X \times Y$ into $M_{n\times m}$. $\Phi \in E_{n\times m}$ is

called generic if it is transverse to all Σ_r, $r = 0, 1, \ldots, m$. By Lemma 2.1 and Proposition 5.1 to a neighborhood U of $p \in L$, for some choice of the cotangent bundle structures on $\mathcal{M}$, we associate the generating function $\pi_{X\times Y}(U) = V \ni (x, y) \to F(x, y)$, where L is transversal to the fibering. We can treat coordinates $(x, y) \in U$ as a parameterization of L. To each $(\bar{x}, \bar{y}) \in U$ we associate the reduced, with respect to terms depending only on $x - \bar{x}$ and $y - \bar{y}$ separately, the two-jet

$$j^2_{(\bar{x},\bar{y})}F = \sum_{i=1}^{n}\sum_{j=1}^{m}(x_i - \bar{x}_i)(y_j - \bar{y}_j)a_{ij}(\bar{x}, \bar{y}).$$

The smooth mapping $V \ni (x, y) \to a_{ij} \in M_{n\times m}$ we will call the one-jet extension of the parametrization of L. Now we have.

PROPOSITION 5.2 *Let $L \subset \mathcal{M}$ be a symplectic relation in $\mathcal{M}$. Then the following conditions are equivalent*

1. *G_L is transversal to the stratification of the critical set $C\Lambda_{n+m} = \bigcup_{k=1}^{m} C_k\Lambda_{n+m}$.*
2. *For any germ of a symplectic relation (L, p) the corresponding one-jet extension of the local parameterization of L is generic.*

Proof. We know, by Proposition 5.1 that the corresponding stratifications of Λ_{n+m} and $M_{n\times m}$ coincide. Thus the one-jet extension $(a_{ij}(x, y))$ reconstructs the Gauss map locally. So if $V \ni (x, y) \to (a_{ij}(x, y))$ is generic, then G_L is transversal to the stratification of Λ_{n+m} and opposite if G_L is transverse to the stratification of Λ_{n+m}, then by the extension and reduction method (see also [4, 5]) the corresponding local one-jet extensions are generic matrices. □

REMARK 5.1

A. *If $a_{ij}(\bar{x}, \bar{y})$ is generic then $\phi_{ij}(x, y)$ is generic in some open neighbourhood of 0. And because of $a_{ij}(0, 0) = \phi_{ij}(0, 0)$, $rank(a_{ij}(0, 0)) = rank(\phi_{ij}(0, 0))$, we have the local equivalency of generic matrices $(a_{ij}(\bar{x}, \bar{y}))$ and $(\phi_{ij}(x, y))$ (cf. [12]). In fact*

$$(a_{ij} \circ \psi)(x, y) = \sum_{kl} \alpha_{ik}(x, y)\phi_{kl}(x, y)\beta_{lj}(x, y),$$

where ψ is a local diffeomorphism, $\phi(0) = 0$, and α, β are local invertible matrices; $\alpha(0) = I_{n\times n}$, $\beta(0) = I_{m\times m}$.

B. *On the basis of Proposition 5.2, the k-vertical points of generic* $L \subset (N \times M, \omega_2 \ominus \omega_1)$, $dimN = 2n, dimM = 2m$ *can not be removed by a small perturbation of* L *if*

$$k^2 + k(n-m) \leq n + m, \quad (n \geq m, k \leq m). \tag{3}$$

For generic L, *the isolated points of k-vertical position appear only if* $m \in \mathbb{N}$ *fulfills the following condition*

$$m = \frac{1}{8}(4 + h^2 - s^2),$$

for some natural numbers $h \in \mathbb{N}$ *and* $s \in \mathbb{N}$, $s \geq 2$. *In this case the relation* L *has an isolated* $k-$*vertical point, not removable by a small perturbation of* L, *for* $k = \frac{1}{2}(2 + h - s)$. *If* $n = m$ *the* $k-$*vertical points generically appearing in* L *are given by the inequality* $k^2 \leq 2n$. *The isolated points of the* $k-$*vertical position appear only if* $n = 2h^2$, $h \in \mathbb{N}$ *and these are* $2h-$*vertical points.*

C. *If* $n = m = 2$, *then the supercritical points, i.e. points* $p \in L$ *such that* $G_L(p) \in C_2\Lambda_4$, *appear in generic* L *as the isolated points. At such points* L *is generated locally by the following generating function*

$$F(x, y) = \sum_{i,j=1}^{2} x_i y_j \phi_{ij}(x, y),$$

where $\phi_{ij}(0, 0) = 0$, *and the transversality of* G_L *to* $C_2\Lambda_4$ *is equivalent to the maximal rank property*

$$rank D\Phi(0, 0) = 4,$$

where $\Phi(x, y) = (\phi_{ij}(x, y)) \in M_{2\times 2}$.

If $n = m = 1$, *the supercritical points for generic* L *are not isolated. In this case the generating function has the form,*

$$F(x, y) = xyf(x, y),$$

and the transversality condition means that f *has no critical point at* 0. *Moreover the transversality condition ensures the infinitesimal symplectic stability of such supercritical points.*

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