Combinatorial Models for Weyl Characters

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0. Introduction

In this paper, we define an "admissible system" as a set-with-operators satisfying a certain list of axioms (see (A0)–(A4) in Section 2). Our goal is to show that these axioms abstract a minimal set of properties for understanding the combinatorics of the Weyl character formula for representations of semisimple Lie groups or algebras, and more generally for symmetrizable Kac-Moody algebras.

Axioms (A0)–(A3) can be recognized as defining, although with slightly different notation, what is known as a "crystal" in the theory that Kashiwara has developed for bases of representations of quantized universal enveloping algebras [**K2**]. The remaining axiom (A4) postulates the existence of what we call a "coherent timing pattern," and is designed precisely to force a certain natural operation to be a sign-reversing involution.

We would like to emphasize that all proofs in this paper are self-contained, aside from a few standard facts about root systems and Weyl groups. Furthermore, it uses nothing from representation theory, and (if one is willing to drop all considerations of motivation) can be understood at a combinatorial level as a collection of theorems about generating functions expressible as ratios of alternating sums over Weyl groups.

In Section 2, after introducing the defining axioms, we prove simultaneously that (1) every admissible system has a "character" (i.e., a generating series) that is a nonnegative sum of Weyl characters, and (2) there is a simple product decomposition rule for multiplying the character by any Weyl character (see Theorem 2.4). As a corollary, one also obtains a branching rule for decomposing the character of any admissible system relative to Weyl characters for root subsystems. We remark that these results are false for general crystals; i.e., systems that fail to satisfy axiom (A4).

At this point the main issue is existence; i.e., for each Weyl character $\chi(\lambda)$, does there exist (and if so, how can one construct) an admissible system whose character is $\chi(\lambda)$? Indeed, once we have such constructions, the previous results immediately yield tensor product and branching rules for Weyl characters. We give two solutions to the existence problem, the first (for finite root systems only) is close to the philosophy of crystal bases, and the second is provided by Littelmann's path model [**L1–4**].

In the first approach, we define a product construction for admissible systems (see Section 3). This coincides exactly with the usual definition of the product of crystals, except that we have the added (light) burden of proving that axiom (A4) is respected.

In Section 4, we study "thin" admissible systems; these are the systems for which the most trivial timing pattern—a constant function—suffices. We prove that the only significant thin systems are those arising from minuscule and quasi-minuscule representations (Theorem 4.3).

In Section 5, we digress to discuss the example of semistandard tableaux for the root system $\mathcal{A}_n$; these can be viewed as forming an admissible subsystem of a product of thin systems. Although tableaux are well understood from many points of view, including that of crystal bases and the path model, it is nonetheless worth emphasizing how easy it is to deduce in this way (1) the equivalence of the bi-alternant and tableaux definitions of the Schur functions, and (2) the Littlewood-Richardson rule (see Proposition 5.1).

In Section 6, we confront (the lack of) complete reducibility. Although there is a canonical decomposition of any admissible system into irreducible subsystems, there exist irreducible systems whose characters are sums of more than one Weyl character; we call these "tangled" systems. Entanglement is not an issue for the crystals corresponding to irreducible (integrable, highest weight) modules, since the latter belong to a category that enjoys complete reducibility. We prove (Theorem 6.4) that products of minuscule and quasi-minuscule systems suffer no entanglements, and this allows us to deduce (Theorem 7.1 and Corollary 7.2) that for every Weyl character $\chi(\lambda)$ for every finite root system, there is an admissible subsystem with character $\chi(\lambda)$ in such a product.

The crystals arising from representations are known to be unique and well-behaved under tensor product (see Theorems 4.1 and 4.2 of [**K2**]). Since the admissible systems arising from minuscule and quasi-minuscule representations are easily seen to be the crystals of these representations, it follows that the admissible systems whose existence we prove in Sections 6 and 7 must be identical to the crystals of representations in Kashiwara's theory.

It would be interesting to identify a stronger set of axioms whose only models are the crystals of representations. It is conceivable that the axioms for a "strongly untangled" admissible system (see Section 6) have this property. For simply-laced root systems, a more specific proposal would be the axioms for an admissible system, together with the "combinatorial Serre relations" described in Lemmas 6.5 and 6.6 (see also Section 7.3 of [**K1**] and Remark 6.7(d) below).

In Section 8, we return to the general case, and show that Littelmann's path model fits the axioms for an admissible system. More specifically, we treat the instance of the path model corresponding to Lakshmibai-Seshadri paths in [**L1**], although the derivation we give is closer in spirit to the approaches based on the Weyl character formula in [**L2–3**]. In this context, the timing pattern records the time of deepest penetration of a path through each wall of the fundamental chamber, and axiom (A4) can be viewed as extracting the essential combinatorial properties of these penetration times.

One reason for using the Lakshmibai-Seshadri path model is that the objects can be explicitly and uniformly described—in this case, as weighted chains in the Bruhat ordering of the Weyl group (hence we refer to the objects as "Lakshmibai-Seshadri chains"). However, this causes no real loss of generality, since the crystal defined by any path model depends only on the highest weight (see [**L2**] or Theorem 2 of [**L3**]). Thus all path models yield admissible systems, although this should not be hard to prove directly.

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1. Preliminaries

Let V be a finite-dimensional real vector space with a non-degenerate symmetric bilinear form $\langle\ ,\ \rangle$, and let $\Phi \subset V$ be a crystallographic root system with simple roots $\{\alpha_i : i \in I\}$. By this we mean that Φ is the set of real roots of some symmetrizable Kac-Moody algebra. The finite root systems of this type are the root systems of semisimple Lie algebras.

For each root $\alpha \in \Phi$, we let $\alpha^\vee := 2\alpha/\langle\alpha,\alpha\rangle$ denote the co-root and $\sigma_\alpha \in GL(V)$ the reflection corresponding to α, so that $\sigma_\alpha(\lambda) = \lambda - \langle\lambda,\alpha^\vee\rangle\alpha$. For each $i \in I$, we let s_i denote the reflection corresponding to the simple root α_i. The Weyl group W is the subgroup of $GL(V)$ generated by $\{s_i : i \in I\}$ (or equivalently, $\{\sigma_\alpha : \alpha \in \Phi\}$); it is finite if and only if Φ is finite.

We remark that Φ can be characterized by the following axioms.

(R1) $\{\alpha_i : i \in I\}$ is a linearly independent set.

(R2) $\langle \alpha_i, \alpha_i \rangle > 0$ for all $i \in I$.

(R3) $\langle \alpha_i, \alpha_j^\vee \rangle \in \mathbf{Z}^{\leqslant 0}$ for all $i, j \in I$ such that $i \neq j$.

(R4) $\Phi = \bigcup_{i \in I} W\alpha_i$.

We let Φ^+ denote the set of positive roots; i.e., the roots in the nonnegative linear span of the simple roots. One knows that Φ is the disjoint union of Φ^+ and $-\Phi^+$.

We let $\Lambda := \{\lambda \in V : \langle \lambda, \alpha_i^\vee \rangle \in \mathbf{Z},\ \text{all } i \in I\}$ denote the lattice of (integral) weights. This is slightly misleading terminology, since Λ is not a lattice in V, but rather a lattice in V/Z, where $Z = \{\lambda \in V : \langle \lambda, \alpha_i^\vee \rangle = 0,\ \text{all } i \in I\}$.[1] Those $\lambda \in V$ such that $\langle \lambda, \alpha^\vee \rangle \geqslant 0$ for all $\alpha \in \Phi^+$ (or equivalently, $\langle \lambda, \alpha_i^\vee \rangle \geqslant 0$ for all $i \in I$) are said to be dominant. One knows that every W-orbit in V has at most one dominant member. We let Λ^+ denote the semigroup of dominant integral weights.

The (integral) Tits cone Λ_c is defined to be the union of all W-orbits of dominant integral weights, or equivalently,

$$\Lambda_c = \{\lambda \in \Lambda : \langle \lambda, \alpha^\vee \rangle < 0 \text{ for finitely many } \alpha \in \Phi^+\}.$$

We have $\Lambda = \Lambda_c$ in the finite case, but not otherwise.

We now need to define a suitable ring R that contains the characters of all integrable highest weight modules for the corresponding Kac-Moody algebra. In the finite case, one may simply take R to be the group ring of Λ, but in general more care is required.

First, choose a height function $\mathrm{ht}: V \to \mathbf{R}$; i.e., a linear map such that $\mathrm{ht}(\alpha_i) = 1$ for all $i \in I$. Second, for each $\lambda \in \Lambda$, let e^λ denote a formal exponential subject to the rules $e^\mu \cdot e^\nu = e^{\mu+\nu}$ for all $\mu, \nu \in \Lambda$. These given, we define R to be the ring consisting of all formal sums $\sum_{\lambda \in \Lambda} c_\lambda e^\lambda$ ($c_\lambda \in \mathbf{Z}$) satisfying the condition that for all $h \in \mathbf{R}$, there are only finitely many weights λ such that $\mathrm{ht}(\lambda) > h$ and $c_\lambda \neq 0$. A subtle complication is the fact that the W-action $e^\lambda \mapsto e^{w\lambda}$ on exponentials does not extend to the full ring R.[2]

Note that R includes the formal power series ring $R_0 = \mathbf{Z}[[e^{-\alpha_i} : i \in I]]$. In particular, if $f \in R_0$ has constant term 1, then $e^\lambda f$ has a multiplicative inverse in R.

For each $\lambda \in \Lambda^+$ with a finite W-stabilizer, we define

$$\Delta(\lambda) := \sum_{w \in W} \mathrm{sgn}(w) e^{w\lambda},$$

where $\mathrm{sgn}(w) = \det(w)$ denotes the sign character. It is not hard to show that $\Delta(\lambda)$ is a well-defined member of R, and more generally, the same holds if we extend the definition to any $\lambda \in \Lambda_c$ with a finite W-stabilizer. Moreover,

$$\Delta(w\lambda) = \mathrm{sgn}(w)\Delta(\lambda) \qquad (w \in W),$$

and $\Delta(\lambda) \neq 0$ if and only if λ has a trivial stabilizer. If λ is dominant, then $\Delta(\lambda) \neq 0$ if and only if λ is strongly dominant (i.e., $\langle \lambda, \alpha_i^\vee \rangle > 0$ for all $i \in I$). In that case, $e^{-\lambda}\Delta(\lambda) \in R_0$ has constant term 1 and $\Delta(\lambda)$ is invertible in R.

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[1]In general, we have to allow for the possibility that the simple roots span a proper subspace of V. Indeed, it can happen that the bilinear form is degenerate on the span of the simple roots.

[2]For example, consider how s_i should act on $\sum_{k \geqslant 0} e^{-k\alpha_i}$.

Since $\langle\ ,\ \rangle$ is non-degenerate, we may select $\rho \in \Lambda^+$ so that $\langle \rho, \alpha_i^\vee \rangle = 1$ for all $i \in I$. This given, for each $\lambda \in \Lambda^+$ we define

$$\chi(\lambda) := \frac{\Delta(\lambda+\rho)}{\Delta(\rho)} = \frac{\sum_{w \in W} \operatorname{sgn}(w) e^{w(\lambda+\rho)-\rho}}{\sum_{w \in W} \operatorname{sgn}(w) e^{w\rho-\rho}} \in R.$$

It is easy to show that $w\rho - \rho$, and hence $\chi(\lambda)$, does not depend on the choice of ρ. By the Kac-Weyl character formula **[Ka]**, these are the characters of the irreducible integrable highest weight modules for the corresponding Kac-Moody algebra.

2. Admissible Systems

Fix a crystallographic root system Φ with simple roots $\{\alpha_i : i \in I\}$ and weight lattice Λ.

DEFINITION 2.1. An *admissible system* is a 4-tuple $(X, \mu, \delta, \{F_i : i \in I\})$, where

- X is a set whose members are called *objects*,
- μ and δ are maps $X \to \Lambda$ (i.e., assignments of integral weights to objects), and
- for each $i \in I$, F_i is a bijection between two subsets of X,

subject to axioms (A0)–(A4) below.

By abuse of notation, we identify the system with the set X.

Our first requirement is

(A0) For all $h \in \mathbf{R}$, there are only finitely many objects x such that $\operatorname{ht}(\mu(x)) > h$.

In other words, the generating series $G_X := \sum_{x \in X} e^{\mu(x)}$ is a well-defined member of R. In case Φ is finite, it is reasonable to use the stronger hypothesis that X is finite.

For each $x \in X$, we call $\mu(x)$, $\delta(x)$ and $\varepsilon(x) := \mu(x) - \delta(x)$ the *weight*, *depth* and *rise* of x. We also require

(A1) $\delta(x) \in -\Lambda^+,\ \ \varepsilon(x) = \mu(x) - \delta(x) \in \Lambda^+$.

It is convenient to introduce the notations $\mu(x,i) = \langle \mu(x), \alpha_i^\vee \rangle$, $\delta(x,i) = \langle \delta(x), \alpha_i^\vee \rangle$, and $\varepsilon(x,i) = \langle \varepsilon(x), \alpha_i^\vee \rangle$ for all $i \in I$. We call $\delta(x,i)$ and $\varepsilon(x,i)$ the depth and rise of x in the direction of α_i. In these terms, we have

$$\delta(x,i) \leqslant \min(0, \mu(x,i)) \quad (i \in I),$$
$$\varepsilon(x,i) \geqslant \max(0, \mu(x,i)) \quad (i \in I),$$

and either one of these is equivalent to (A1).

Next, we require the domain and co-domain of F_i to be the set of objects with nonzero rise and depth in the direction of α_i; i.e.,

(A2) F_i is a bijection $\{x \in X : \varepsilon(x,i) > 0\} \to \{x \in X : \delta(x,i) < 0\}$.

We further impose the conditions

(A3) $\mu(F_i(x)) = \mu(x) - \alpha_i,\ \ \delta(F_i(x), i) = \delta(x,i) - 1$.

Hence also $\varepsilon(F_i(x), i) = \varepsilon(x,i) - 1$. We let $E_i = F_i^{-1}$ denote the inverse map.

It is easy to see that (A1)–(A3) imply

$$F_i^k(x) = E_i^{-k}(x) \text{ is defined} \Leftrightarrow \delta(x,i) \leqslant k \leqslant \varepsilon(x,i). \tag{2.1}$$

The maps E_i and F_i act as raising and lowering operators that provide a partition of the objects into α_i-strings that are closed under the action of E_i and F_i. For example, the *α_i-string through x* is (by definition)

$$F_i^\varepsilon(x),\ \ldots,\ F_i(x),\ x,\ E_i(x),\ \ldots,\ E_i^{-\delta}(x),$$

where $\delta = \delta(x,i)$ and $\varepsilon = \varepsilon(x,i)$. The top member of the string, $E_i^{-\delta}(x) = F_i^\delta(x)$ has a depth of 0 and a rise of $\varepsilon - \delta$ in the direction of α_i; the bottom member $F_i^\varepsilon(x)$ has a rise of 0 and a depth of $\delta - \varepsilon$.

LEMMA 2.2. *If X is a system satisfying (A0)–(A3), then*

(a) *G_X is W-invariant, and*

(b) *$\mu(x) \in \Lambda_c$ for all $x \in X$.*

Proof. (a) We seek to prove that for all $\mu \in \Lambda$ and all $i \in I$, the coefficients of e^μ and $e^{s_i\mu}$ in G_X are the same. Given an object $x \in X$ of weight $\mu = \mu(x)$, let $\delta = \delta(x,i)$ and $\varepsilon = \varepsilon(x,i)$. Since $\delta \leqslant 0$ and $\varepsilon \geqslant 0$, it follows that $\delta \leqslant \delta + \varepsilon \leqslant \varepsilon$, whence $x' := F_i^{\delta+\varepsilon}(x)$ is a valid object of weight $\mu - (\delta+\varepsilon)\alpha_i = s_i\mu$ (cf. (2.1)). Since there can be at most one member of the α_i-string through x' whose weight is $s_i\mu(x')$ (namely, x), it follows that the map $x \mapsto x'$ is an involution.

(b) It follows from (A0) that $\{\mathrm{ht}(\mu(x)) : x \in X\}$ has a maximum. Assume toward a contradiction that $\mu(x)$ has maximum height among all $x \in X$ such that $\mu(x) \notin \Lambda_c$. Since $\mu(x)$ cannot be dominant, we have $\mu(x,i) < 0$ for some $i \in I$, hence the element $x' \in X$ in the α_i-string through x of weight $s_i\mu(x)$ constructed in (a) has greater height. The maximality of $\mathrm{ht}(\mu(x))$ then implies $s_i\mu(x) \in \Lambda_c$, a contradiction. □

In the following, it will be convenient to define

$$x \preccurlyeq_i y \quad \text{if } x = F_i^k(y) \text{ for some } k \geqslant 0.$$

Any assignment of real numbers $t(x,i) \in \mathbf{R}$ for all pairs (x,i) with $\delta(x,i) < 0$ is called a *timing pattern* for X. Our final requirement is

(A4) There exists a *coherent* timing pattern $t(\ ,\ )$ for X; i.e.,

for all pairs (x,i) such that $\delta(x,i) < 0$ and $F_i(x)$ is defined (i.e., $\varepsilon(x,i) > 0$), we have

(i) $t(x,i) \leqslant t(F_i(x),i)$,

and for all $j \neq i$, all integers $\delta < 0$, and all $t \leqslant t(x,i)$,

(ii) there is an object $y \succcurlyeq_j x$ such that $\delta(y,j) = \delta$ and $t(y,j) = t$ if and only if there is an object $y' \succcurlyeq_j F_i(x)$ such that $\delta(y',j) = \delta$ and $t(y',j) = t$.

Note that if $F_i^2(x)$ also exists, then for all $t \leqslant t(F_i(x),i)$ we can apply (ii) with $F_i(x)$ replacing x. Bearing in mind (i), it follows that for $t \leqslant t(x,i)$, there is an object $y \succcurlyeq_j x$ such that $\delta(y,j) = \delta$ and $t(y,j) = t$ if and only if there is an object $y' \succcurlyeq_j F_i^2(x)$ such that $\delta(y',j) = \delta$ and $t(y',j) = t$. By iteration, we obtain more generally

LEMMA 2.3. *For all distinct pairs $i, j \in I$, all objects $x' \preccurlyeq_i x$ such that $\delta(x,i) < 0$, all integers $\delta < 0$, and all $t \leqslant t(x,i)$, there is an object $y \succcurlyeq_j x$ such that $\delta(y,j) = \delta$ and $t(y,j) = t$ if and only if there is an object $y' \succcurlyeq_j x'$ such that $\delta(y',j) = \delta$ and $t(y',j) = t$.*

For some basic examples of admissible systems, see Sections 4 and 5.

THEOREM 2.4. *If X is an admissible system and $\nu \in \Lambda^+$, then*

$$\chi(\nu) \cdot G_X = \sum_{x \in X:\, \nu+\delta(x) \in \Lambda^+} \chi(\nu + \mu(x)).$$

In particular (taking $\nu = 0$),

$$G_X = \sum_{\delta(x) \in \Lambda^+} \chi(\mu(x)).$$

It should be noted that $\nu + \mu(x) = (\nu + \delta(x)) + (\mu(x) - \delta(x))$, so (A1) implies that each term $\nu + \mu(x)$ appearing in the above expansion is dominant. Note also that $\delta(x)$ is dominant if and only if $\delta(x,i) = 0$ for all $i \in I$. In that case, we say that x is *maximal*.

Proof. Since G_X is W-invariant by Lemma 2.2(a), we have

$$\begin{aligned}\Delta(\rho+\nu)\cdot G_X &= \sum_{w\in W, x\in X} \operatorname{sgn}(w) e^{w(\rho+\nu)+\mu(x)} \\ &= \sum_{w\in W, x\in X} \operatorname{sgn}(w) e^{w(\rho+\nu+\mu(x))} = \sum_{x\in X} \Delta(\rho+\nu+\mu(x)). \end{aligned} \tag{2.2}$$

The fact that the summands $\Delta(\rho + \nu + \mu(x))$ are well-defined members of R follows from Lemma 2.2(b). (In particular, $\mu \in \Lambda_c$ implies that $\rho + \nu + \mu$ has a finite W-stabilizer.)

Now let $k_i = k_i(\nu) = -\langle \rho + \nu, \alpha_i^\vee \rangle < 0$. For all $x \in X$, we define

$$J(x) = \{i \in I : \delta(x,i) \leqslant k_i\}.$$

For each $i \in J(x)$, there is a unique object $x_i \succcurlyeq_i x$ such that $\delta(x_i, i) = k_i$. Assuming that $J(x)$ is nonempty, we say that x is "bad" with respect to ν.

By (A4), there is a coherent timing pattern $t(\ ,\ )$ for X.

Given that x is bad, choose $i \in J(x)$ so that $t(x_i, i)$ is minimized. If there is more than one minimizing choice, select i to be the first one relative to some fixed ordering of I. Setting $\delta = \delta(x,i)$, $\varepsilon = \varepsilon(x,i)$ and $l = \varepsilon + \delta - k_i$, we have $\delta \leqslant l \leqslant \varepsilon$, so it follows by (2.1) that $x' := F_i^l(x)$ is a valid object. Moreover by (A3),

$$\delta(x', i) = \delta - l = k_i - \varepsilon \leqslant k_i,$$

so $i \in J(x')$, x' is bad with respect to ν, and

$$\mu(x') = \mu(x) - l\alpha_i = \mu(x) - \langle \mu(x), \alpha_i^\vee \rangle \alpha_i + k_i \alpha_i,$$

whence x' is the unique member of the α_i-string through x such that

$$s_i(\rho + \nu + \mu(x)) = \rho + \nu + \mu(x'). \tag{2.3}$$

We claim that the map $x \mapsto x'$ is an involution on the set of all members of X that are bad with respect to ν. To see this, for each $j \in J(x')$ let x'_j denote the unique object such that $x'_j \succcurlyeq_j x'$ and $\delta(x'_j, j) = k_j$. Since x and x' are on the same α_i-string, we have $x'_i = x_i$. Choose $j \in J(x')$ so as to minimize $t(x'_j, j)$, and if there is more than one choice, select j to be first in the ordering of I. We have $t(x'_j, j) \leqslant t(x'_i, i) = t(x_i, i)$ and $x' \preccurlyeq_i x$ or $x \preccurlyeq_i x'$, so if we assume $j \neq i$, then Lemma 2.3 implies that there is an object $y \succcurlyeq_j x$ such that $\delta(y, j) = k_j$ and $t(y, j) = t(x'_j, j)$. Hence, $j \in J(x)$ and y is the unique object on the α_j-string through x such that $\delta(y, j) = k_j$ (i.e., $y = x_j$), so $t(x_j, j) \leqslant t(x_i, i)$, contradicting our choice of i. Therefore $j = i$ and the object x'' such that $x' \mapsto x''$ is (by (2.3)) the unique member of the α_i-string through x' (or x) such that

$$s_i(\rho + \nu + \mu(x')) = \rho + \nu + \mu(x'').$$

Thus $x = x''$ and the claim follows.

Since (2.3) implies $\Delta(\rho + \nu + \mu(x)) = -\Delta(\rho + \nu + \mu(x'))$, the net contribution of bad objects to (2.2) is zero. The remaining "good" objects are characterized by the property that $\langle \rho + \nu, \alpha_i^\vee \rangle + \delta(x, i) > 0$ for all $i \in I$, or equivalently, $\nu + \delta(x) \in \Lambda^+$, whence

$$\Delta(\rho + \nu) \cdot G_X = \sum_{x \in X : \nu + \delta(x) \in \Lambda^+} \Delta(\rho + \nu + \mu(x)).$$

Now divide by $\Delta(\rho)$. $\square$

Given $J \subseteq I$, let Φ_J denote the root subsystem of Φ with simple roots $\{\alpha_j : j \in J\}$. We let $W_J \subseteq W$, $\Lambda_J \supseteq \Lambda$, and R_J denote the corresponding Weyl group, weight lattice, and character ring. Provided that we use the height function inherited from Φ (in which case $R_J \supseteq R$), it is easy to see that any admissible system X can also be viewed as an admissible system relative to Φ_J using only the operators E_j and F_j for $j \in J$.

An immediate consequence of the second part of Theorem 2.4 is the following "branching rule" for decomposing G_X as a sum of Weyl characters relative to Φ_J.

Corollary 2.5. *If X is an admissible system and $J \subseteq I$, then*

$$G_X = \sum_{\delta(x) \in \Lambda_J^+} \chi(\mu(x); J),$$

where $\chi(\lambda; J) \in R_J$ denotes the Weyl character (relative to Φ_J) corresponding to $\lambda \in \Lambda_J^+$.

Note that $\delta(x) \in \Lambda_J^+$ if and only if $\delta(x, j) = 0$ for all $j \in J$.

3. The Product Construction

Let X and Y be admissible systems. In the following, we will construct an admissible system XY whose set of objects is the Cartesian product of X and Y. For brevity, we will represent the objects of XY as concatenations xy with $x \in X, y \in Y$.

The weight, depth[3] and rise of the object xy are as follows:

$$\begin{aligned} \mu(xy) &= \mu(x) + \mu(y), \\ \delta(xy, i) &= \min(\delta(x,i), \mu(x,i) + \delta(y,i)) = \delta(x,i) + \min(0, \varepsilon(x,i) + \delta(y,i)), \\ \varepsilon(xy, i) &= \max(\varepsilon(y,i), \varepsilon(x,i) + \mu(y,i)) = \varepsilon(y,i) + \max(0, \varepsilon(x,i) + \delta(y,i)). \end{aligned}$$

Since $\delta(x,i) \leqslant 0$ and $\delta(y,i) \leqslant \mu(y,i)$, it follows that $\delta(xy,i) \leqslant \min(0, \mu(xy,i))$ and hence (A1) holds. It is also clear that $G_{XY} = G_X G_Y$, so (A0) is immediate.

The raising and lowering operators E_i and F_i are defined by

$$F_i(xy) = \begin{cases} F_i(x)y & \text{if } \varepsilon(x,i) + \delta(y,i) > 0, \\ xF_i(y) & \text{if } \varepsilon(x,i) + \delta(y,i) \leqslant 0, \end{cases}$$

$$E_i(xy) = \begin{cases} E_i(x)y & \text{if } \varepsilon(x,i) + \delta(y,i) \geqslant 0, \\ xE_i(y) & \text{if } \varepsilon(x,i) + \delta(y,i) < 0. \end{cases}$$

We claim that these maps are well-defined inverse pairs satisfying (A2)–(A3). Indeed, consider an object xy with $\varepsilon(xy,i) > 0$. If $\varepsilon(x,i) + \delta(y,i) > 0$, then $\varepsilon(x,i) > 0$, whence $F_i(x)$ is defined, $F_i(xy) = F_i(x)y$, and F_i decreases both the depth and rise in the direction of α_i by 1. Furthermore, we now have $\varepsilon(F_i(x),i) + \delta(y,i) \geqslant 0$, whence $E_i(F_i(x)y) = xy$. Otherwise, if $\varepsilon(x,i) + \delta(y,i) \leqslant 0$, then $\varepsilon(y,i) = \varepsilon(xy,i) > 0$, whence $F_i(y)$ is defined, $F_i(xy) = xF_i(y)$, and again F_i decreases both the depth and rise in the direction of α_i by 1. Furthermore, we have $\varepsilon(x,i) + \delta(F_i(y),i) < 0$, and hence $E_i(xF_i(y)) = xy$. Also, in both cases it is clear that $\mu(F_i(xy)) = \mu(xy) - \alpha_i$. Conversely, a similar argument shows that E_i is well-defined and inverted by F_i whenever $\delta(xy,i) < 0$, so the claim follows.

LEMMA 3.1. *Let $x, x' \in X$, $y, y' \in Y$, and $j \in I$.*

(a) *If $x'y' \succcurlyeq_j xy$, then $x' \succcurlyeq_j x$ and $y' \succcurlyeq_j y$.*
(b) *If $x' \succcurlyeq_j x$, then there exists $y'' \in Y$ such that $x'y'' \succcurlyeq_j xy$ and $\varepsilon(x',j) + \delta(y'',j) \geqslant 0$.*
(c) *If $\varepsilon(x',j) + \delta(y',j) \leqslant 0$, then $x'y' \succcurlyeq_j xy$ if and only if $x' = x$ and $y' \succcurlyeq_j y$.*

Proof. The action of F_j on xy decreases $\varepsilon(x,j) + \delta(y,j)$ by 1. Furthermore, whenever $\varepsilon(x,j) + \delta(y,j)$ is positive, $F_j(xy)$ is defined and acts on x; similarly, $\varepsilon(x,j) + \delta(y,j) < 0$ implies that $E_j(xy)$ is defined and acts on y. It follows that every α_j-string in XY has a unique object xy for which $\varepsilon(x,j) + \delta(y,j) = 0$, and the string itself takes the form

$$E_j^{-\delta}(x)y \succcurlyeq_j \cdots \succcurlyeq_j E_j(x)y \succcurlyeq_j xy \succcurlyeq_j xF_j(y) \succcurlyeq_j \cdots \succcurlyeq_j xF_j^{\varepsilon}(y), \tag{3.1}$$

where $\delta = \delta(x,j)$ and $\varepsilon = \varepsilon(y,j)$.

All three assertions are easy consequences of this observation. □

[3]Strictly speaking, we have not defined $\delta(xy)$ here, but rather $\delta(xy,i)$ for all $i \in I$. However the properties of admissible systems depend only on the latter, so it is a moot point.

PROPOSITION 3.2. *If X and Y are admissible systems, then XY is also admissible.*

Proof. All that remains is to verify (A4); i.e., the existence of a coherent timing pattern. By applying order-preserving injections $\mathbf{R} \to \mathbf{R}^{<0}$ and $\mathbf{R} \to \mathbf{R}^{>0}$, we may select coherent timing patterns for X and Y so that $t(x,i) < 0$ and $t(y,i) > 0$ for all $x \in X, y \in Y, i \in I$ where these times are defined (i.e., $\delta(x,i) < 0$ and $\delta(y,i) < 0$). These given, we define

$$t(xy,i) = \begin{cases} t(x,i) & \text{if } \varepsilon(x,i) + \delta(y,i) \geqslant 0, \\ t(y,i) & \text{if } \varepsilon(x,i) + \delta(y,i) < 0 \end{cases}$$

whenever $\delta(xy,i) < 0$ and claim that this is a coherent timing pattern for XY.

From (3.1), one can see that the values of $t(\cdot\,,i)$ along a given α_i-string in XY consist of the negative, nondecreasing values of $t(\cdot\,,i)$ along the upper portion of an α_i-string in X, followed by the positive, nondecreasing values of $t(\cdot\,,i)$ along the bottom portion of an α_i-string in Y. This confirms part (i) of (A4).

For part (ii), choose a distinct pair $i,j \in I$, let xy be an object such that $\delta(xy,i) < 0$ and $\varepsilon(xy,i) > 0$, and consider an object $x'y' \succcurlyeq_j xy$ such that $\delta = \delta(x'y',j) < 0$ and $t = t(x'y',j) \leqslant t(xy,i)$. We need to establish the existence of an object $x''y'' \succcurlyeq_j F_i(xy)$ such that $\delta(x''y'',j) = \delta$ and $t(x''y'',j) = t$.

Case 1: $\varepsilon(x,i) + \delta(y,i) > 0$ (i.e., $F_i(xy) = F_i(x)y$). We have $t(xy,i) = t(x,i) < 0$, so $t < 0$, whence $t = t(x',j)$, $\varepsilon(x',j) + \delta(y',j) \geqslant 0$, and $\delta = \delta(x',j)$. Since $t(\,,)$ is coherent on X and $x' \succcurlyeq_j x$ (Lemma 3.1(a)), there must be an object $x'' \succcurlyeq_j F_i(x)$ with $t(x'',j) = t(x',j) = t$ and $\delta(x'',j) = \delta(x',j) = \delta$. By Lemma 3.1(b), it follows that there is an object $y'' \in Y$ such that $x''y'' \succcurlyeq_j F_i(x)y$ and $\varepsilon(x'',j) + \delta(y'',j) \geqslant 0$, so we have $\delta(x''y'',j) = \delta(x'',j) = \delta$ and $t(x''y'',j) = t(x'',j) = t$.

Case 2: $\varepsilon(x,i) + \delta(y,i) \leqslant 0$ (i.e., $F_i(xy) = xF_i(y)$) and $t < 0$. The latter of these forces $t = t(x',j)$, whence $\varepsilon(x',j) + \delta(y',j) \geqslant 0$ and $\delta = \delta(x',j)$. Applying Lemma 3.1(b), there is an object $y'' \in Y$ such that $x'y'' \succcurlyeq_j xF_i(y)$ and $\varepsilon(x',j) + \delta(y'',j) \geqslant 0$, and the latter implies $\delta(x'y'',j) = \delta(x',j) = \delta$ and $t(x'y'',j) = t(x',j) = t$.

Case 3: $\varepsilon(x,i) + \delta(y,i) \leqslant 0$ (i.e., $F_i(xy) = xF_i(y)$) and $t > 0$. Here, $t(xy,i) \geqslant t > 0$, which forces $t(xy,i) = t(y,i)$ and $t = t(x'y',j) = t(y',j)$, whence $\varepsilon(x',j) + \delta(y',j) < 0$, $\delta(y',j) < 0$ (since $\varepsilon(x',j) \geqslant 0$), and $\delta = \mu(x',j) + \delta(y',j)$. By Lemma 3.1(c), we must have $x' = x$ and $y' \succcurlyeq_j y$, and since $t(\,,)$ is coherent on Y, there is an object $y'' \succcurlyeq_j F_i(y)$ such that $t(y'',j) = t(y',j) = t$ and $\delta(y'',j) = \delta(y',j)$. Applying Lemma 3.1(c) again, we obtain $xy'' \succcurlyeq_j xF_i(y)$, $t(xy'',j) = t(y'',j) = t$, and $\delta(xy'',j) = \mu(x,j) + \delta(y'',j) = \delta$.

Almost identical reasoning applies if we start with an object $x''y'' \succcurlyeq_j F_i(xy)$ such that $\delta(x''y'',j) < 0$ and $t(x''y'',j) \leqslant t(xy,i)$, so this confirms part (ii) of (A4). □

REMARK 3.3. In an l-fold product $X_1 \cdots X_l$, iteration of the above construction yields objects that are l-tuples $x_1 \cdots x_l$ ($x_i \in X_i$), with

$$\mu(x_1 \cdots x_l) = \mu(x_1) + \cdots + \mu(x_l),$$

$$\delta(x_1 \cdots x_l, i) = \min_{1 \leqslant j \leqslant l} \mu(x_1, i) + \cdots + \mu(x_{j-1}, i) + \delta(x_j, i), \tag{3.2}$$

$$\varepsilon(x_1 \cdots x_l, i) = \max_{1 \leqslant k \leqslant l} \varepsilon(x_k, i) + \mu(x_{k+1}, i) + \cdots + \mu(x_l, i). \tag{3.3}$$

Furthermore, the raising and lowering operators are given by

$$E_i(x_1 \cdots x_l) = x_1 \cdots E_i(x_j) \cdots x_l,$$
$$F_i(x_1 \cdots x_l) = x_1 \cdots F_i(x_k) \cdots x_l,$$

where j and k denote the smallest and largest indices for which the minimum and maximum are achieved in (3.2) and (3.3), respectively.

4. Thin Systems

The simplest admissible systems are those that admit a coherent timing pattern that is constant. We say that such systems are *thin*. For example, it can happen that all strings of objects have length at most one, so conditions (i) and (ii) of (A4) hold vacuously; in this case *every* timing pattern is coherent.

In general, if a constant timing pattern is to be coherent, then condition (ii) implies that for all distinct $i, j \in I$, all $\delta < 0$, and all objects x such that $\delta(x,i) < 0$ and $\varepsilon(x,i) > 0$, there is an object $y \succcurlyeq_j x$ such that $\delta(y,j) = \delta$ if and only if there is an object $y' \succcurlyeq_j F_i(x)$ such that $\delta(y',j) = \delta$. Conversely, it is easily seen that this forces any constant timing pattern to be coherent. To simplify even further, observe that the existence of an object $y \succcurlyeq_j x$ such that $\delta(y,j) = \delta$ is equivalent to the condition $\delta \geqslant \delta(x,j)$. This yields the following characterization of thinness.

PROPOSITION 4.1. *A 4-tuple $(X, \mu, \delta, \{F_i : i \in I\})$ satisfying (A0)–(A3) is a thin admissible system if and only if $\delta(F_i(x), j) = \delta(x,j)$ for all $x \in X$ and distinct $i, j \in I$ such that $\delta(x,i) < 0$ and $\varepsilon(x,i) > 0$.*

REMARK 4.2. If $X = X_1 \cdots X_l$ is a product of thin admissible systems, it follows from Remark 3.3 and the proof of Proposition 3.2 that a coherent timing pattern for X can be devised by setting $t(x_1 \cdots x_l, i) = j$ when E_i acts on $x_1 \cdots x_l$ at x_j. Equivalently, if $\delta(x_1 \cdots x_l, i) < 0$ and j is the least index for which

$$\delta(x_1 \cdots x_l, i) = \mu(x_1, i) + \cdots + \mu(x_{j-1}, i) + \delta(x_j, i),$$

then one may set $t(x_1 \cdots x_l, i) = j$ (cf. (3.2)).

A. Trivial systems.

A weight $\theta \in \Lambda$ is said to be *trivial* if $\langle \theta, \alpha_i^\vee \rangle = 0$ for all $i \in I$. Since V need not be spanned by the simple roots, there may be nonzero trivial weights. It is easy to see that one may construct a (thin) admissible system consisting of a single object with a trivial weight. Moreover, if X is any admissible system, one can create a new admissible system by adding a fixed trivial weight θ to the weight of each $x \in X$; this can be seen as a special case of the product construction.

B. Minuscule systems.

A weight $\mu \in \Lambda$ is said to be *minuscule* if $\langle \mu, \alpha^\vee \rangle \in \{0, \pm 1\}$ for all $\alpha \in \Phi$. This includes the possibility that μ is trivial. Since W permutes Φ, every element in the W-orbit of a minuscule weight is also minuscule.

For any minuscule $\lambda \in \Lambda^+$, it is easy to construct a thin admissible system $X(\lambda)$ whose object set is the W-orbit of λ. Naturally, the weight of the object μ is μ itself, and if we define the depth and rise of μ so that

$$\delta(\mu,i) = \min(0, \langle \mu, \alpha_i^\vee \rangle) = \begin{cases} -1 & \text{if } \langle \mu, \alpha_i^\vee \rangle = -1, \\ 0 & \text{otherwise}, \end{cases}$$
$$\varepsilon(\mu,i) = \max(0, \langle \mu, \alpha_i^\vee \rangle) = \begin{cases} 1 & \text{if } \langle \mu, \alpha_i^\vee \rangle = 1, \\ 0 & \text{otherwise}, \end{cases}$$

then it is easy to see that (A1) holds. In particular, $\langle \mu, \alpha_i^\vee \rangle = \delta(\mu,i) + \varepsilon(\mu,i)$ for $i \in I$.

For the raising and lowering operators, we define

$$F_i(\mu) = s_i\mu = \mu - \alpha_i \quad \text{if } \langle \mu, \alpha_i^\vee \rangle = 1 \quad (\text{i.e., } \varepsilon(\mu,i) > 0),$$
$$E_i(\mu) = s_i\mu = \mu + \alpha_i \quad \text{if } \langle \mu, \alpha_i^\vee \rangle = -1 \ (\text{i.e., } \delta(\mu,i) < 0).$$

It is clear that these maps are inverse pairs satisfying (A2)–(A3). Since there are no objects μ in $X(\lambda)$ that satisfy $\delta(\mu,i) < 0$ and $\varepsilon(\mu,i) > 0$, we are in precisely the situation mentioned at the beginning of this section: every timing pattern is coherent. In particular, $X(\lambda)$ is a thin admissible system; we call these *minuscule systems.*

Dominant minuscule weights are fairly rare and have a well-known classification (e.g., see Exercise VI.4.15 of [**B**]). In particular, no (irreducible) infinite root system has a nontrivial minuscule weight (this follows from Lemma 4.5 below), and in the finite case, there is one dominant minuscule weight in each coset of Λ modulo $\mathbf{Z}\Phi$ (see Lemma 7.3).

C. Quasi-minuscule systems.

We say that a weight $\mu \in \Lambda$ is *quasi-minuscule* if $\langle \mu, \alpha^\vee \rangle \in \{0, \pm 1, \pm 2\}$ for all $\alpha \in \Phi$, and there is a unique $\alpha \in \Phi$ such that $\langle \mu, \alpha^\vee \rangle = 2$. Note that every element in the W-orbit of a quasi-minuscule weight is also quasi-minuscule.

For simplicity, assume now that Φ is irreducible. In that case, we should also assume that Φ is finite; otherwise, there are no dominant quasi-minuscule weights (Lemma 4.5). Under these circumstances, it is well-known that Φ must have either one or two orbits of roots. In the two-orbit case, the roots in each orbit have different lengths ("long" and "short"); in the one-orbit case, we can agree that all roots are short by convention. In either case, we let Φ_s denote the set of short roots, and let $I_s = \{i \in I : \alpha_i \in \Phi_s\}$ denote the indices of the short simple roots.

A fundamental property of every short root β is that $\langle \beta, \alpha^\vee \rangle \leqslant 2$ for all $\alpha \in \Phi$, and equality occurs if and only if $\beta = \alpha$. (This can be proved by examining the rank two root systems; e.g., see [**B**, §VI.1.3].) In other words, short roots are quasi-minuscule. In fact, the short dominant root, denoted $\bar{\alpha}$, is the unique dominant quasi-minuscule weight, aside from trivial translations (see Lemma 4.6 below).

One may construct a thin admissible system $X(\bar{\alpha})$ whose object set is $\Phi_s \cup \{0_i : i \in I_s\}$, where 0_i denotes an object of weight 0, and each root $\beta \in \Phi_s$ is defined to have weight β. The depth and rise are defined by setting

$$\delta(\beta,j) = \min(0, \langle \beta, \alpha_j^\vee \rangle), \quad \delta(0_i,j) = -\delta_{ij},$$
$$\varepsilon(\beta,j) = \max(0, \langle \beta, \alpha_j^\vee \rangle), \quad \varepsilon(0_i,j) = \delta_{ij}$$

for all $\beta \in \Phi_s$, $i \in I_s$, and $j \in I$. Thus $\langle \beta, \alpha_j^\vee \rangle = \delta(\beta, j) + \varepsilon(\beta, j)$ and $\delta(0_i, j) + \varepsilon(0_i, j) = 0$, so (A1) holds. Note that $\delta(\beta, j) \geqslant -2$ for all $j \in I$, with equality if and only if $\beta = -\alpha_j$ and $j \in I_s$. Similarly, $\varepsilon(\beta, j) \leqslant 2$, with equality if and only if $\beta = \alpha_j$ and $j \in I_s$.

For the raising and lowering operators, we define

$$F_i(\alpha_i) = 0_i,\ F_i(0_i) = -\alpha_i,\quad F_j(\beta) = s_j\beta = \beta - \alpha_j \text{ (if } \langle \beta, \alpha_j^\vee \rangle = 1),$$
$$E_i(-\alpha_i) = 0_i,\ E_i(0_i) = \alpha_i,\quad E_j(\beta) = s_j\beta = \beta + \alpha_j \text{ (if } \langle \beta, \alpha_j^\vee \rangle = -1)$$

for $\beta \in \Phi_s$, $i \in I_s$, and $j \in I$. As in the minuscule case, it is easy to check that these are inverse pairs satisfying (A2)–(A3). However, here there do exist objects x such that $\delta(x, i) < 0$ and $\varepsilon(x, i) > 0$; namely, the objects 0_i ($i \in I_s$). For these, we have $\delta(0_i, j) = \delta(-\alpha_i, j) = 0$ for $j \neq i$, so $X(\bar{\alpha})$ is a thin admissible system by Proposition 4.1.

Even if Φ is not irreducible, the same construction can be used if $\bar{\alpha}$ is the short dominant root of some (finite) irreducible component of Φ. Even more generally, if $\lambda = \theta + \bar{\alpha}$ for some trivial $\theta \in \Lambda$, we define $X(\lambda)$ to be the (thin) system obtained by shifting the weights of $X(\bar{\alpha})$ by θ; thus $X(\lambda) \cong X(\theta)X(\bar{\alpha})$. We call these *quasi-minuscule systems.*

D. Classification.

An explicit classification of all thin admissible systems would be unreasonably complicated, given the possibility of tangled systems (see Section 6). Nevertheless, the following result shows that the only "interesting" thin systems are the minuscule and quasi-minuscule systems (see also Remark 6.7(c)).

THEOREM 4.3. *Assume that Φ is irreducible of rank > 1. If X is a thin admissible system, then the weight of every object is minuscule or quasi-minuscule. In particular, G_X is a sum of Weyl characters $\chi(\lambda)$ such that $\lambda \in \Lambda^+$ is minuscule or quasi-minuscule. Furthermore, nontrivial weights can occur only if Φ is finite.*

Our proof will require a series of lemmas.

LEMMA 4.4. *If X is an admissible system with an object x such that $\delta(x, i) \leqslant -2$ and $\delta(x, j) \leqslant -1$ for some pair $i, j \in I$ satisfying $\langle \alpha_i, \alpha_j \rangle < 0$, then X cannot be thin.*

Proof. Assume toward a contradiction that X is thin. If $\varepsilon(x, j) > 0$, we can replace x with $F_j(x)$; indeed, Proposition 4.1 implies that $\delta(F_j(x), i) = \delta(x, i) \leqslant -2$. Thus by iteration, we may assume that $\varepsilon(x, j) = 0$. Now consider the object $y = E_i(x)$. We have $\delta(y, i) < 0$ and $\varepsilon(y, i) > 0$, so Proposition 4.1 implies $\delta(x, j) = \delta(y, j)$. However, $\mu(y) = \mu(x) + \alpha_i$, so we have $\varepsilon(y, j) = \varepsilon(x, j) + \langle \alpha_i, \alpha_j^\vee \rangle = \langle \alpha_i, \alpha_j^\vee \rangle < 0$, contradicting (A1). □

LEMMA 4.5. *If Φ is infinite and irreducible, then for all nontrivial $\lambda \in \Lambda^+$, the set $\{\langle \lambda, \beta^\vee \rangle : \beta \in \Phi^+\}$ is unbounded.*

Proof. We may write $\lambda = \sum_{i \in I} \langle \lambda, \alpha_i^\vee \rangle \omega_i$ for suitable $\omega_i \in \Lambda$ such that $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$. Since λ is dominant and nontrivial, the coefficients $\langle \lambda, \alpha_i^\vee \rangle$ are nonnegative, and at least one is positive. It thus suffices to show that $\{\langle \omega_i, \beta^\vee \rangle : \beta \in \Phi^+\}$ is unbounded for all $i \in I$.

For each $\beta \in \Phi^+$, we have $\beta^\vee = \sum_{i \in I} \langle \omega_i, \beta^\vee \rangle \alpha_i^\vee$. Bearing in mind that $\langle \alpha_i, \alpha_j \rangle \leqslant 0$ for all $j \neq i$, it follows that for any particular such j,

$$\langle \alpha_i, \beta^\vee \rangle \leqslant 2\langle \omega_i, \beta^\vee \rangle + \langle \alpha_i, \alpha_j^\vee \rangle \langle \omega_j, \beta^\vee \rangle. \tag{4.1}$$

Now let $J \subset I$ denote the set of indices $j \in I$ such that $\{\langle \omega_j, \beta^\vee \rangle : \beta \in \Phi^+\}$ is unbounded. Clearly J is nonempty, since $\sum_{i \in I} \langle \omega_i, \beta^\vee \rangle$ is the height of $\beta^\vee$ (as a co-root), and there can be at most finitely many positive co-roots of bounded height. If $J \neq I$, then by the irreducibility of Φ, there must be a pair $j \in J$, $i \in I - J$ such that $\langle \alpha_i, \alpha_j \rangle < 0$. In that case, $\langle \omega_i, \beta^\vee \rangle$ is bounded above, whereas $\langle \omega_j, \beta^\vee \rangle$ is not, so (4.1) implies that $\langle \alpha_i, \beta^\vee \rangle$ cannot be bounded below. However, $\langle \omega_i, s_i \beta^\vee \rangle = \langle \omega_i, \beta^\vee \rangle - \langle \alpha_i, \beta^\vee \rangle$, so this contradicts the fact that $\langle \omega_i, \beta^\vee \rangle$ is bounded. □

LEMMA 4.6. *Assume that Φ is finite and irreducible. If $\lambda \in \Lambda$ is quasi-minuscule and α is the unique root for which $\langle \lambda, \alpha^\vee \rangle = 2$, then $\lambda - \alpha$ is trivial. Moreover, if λ is dominant, then $\lambda - \bar{\alpha}$ is trivial.*

Proof. We may assume that λ is dominant. In that case, we must have $\alpha = \bar{\alpha}$, since $\bar{\alpha}^\vee$ is the highest co-root. Now if $\lambda - \bar{\alpha}$ fails to be dominant, say $\langle \lambda - \bar{\alpha}, \alpha_i^\vee \rangle < 0$, then $0 \leqslant \langle \lambda, \alpha_i^\vee \rangle < \langle \bar{\alpha}, \alpha_i^\vee \rangle$. We cannot have $\bar{\alpha} = \alpha_i$; otherwise $\langle \lambda, \alpha_i^\vee \rangle = \langle \bar{\alpha}, \alpha_i^\vee \rangle = 2$. Thus since $\bar{\alpha}$ is quasi-minuscule, we must have $\langle \bar{\alpha}, \alpha_i^\vee \rangle = 1$ and $\langle \lambda, \alpha_i^\vee \rangle = 0$. In that case, $\beta = s_i \bar{\alpha}$ is a root distinct from $\bar{\alpha}$ such that $\langle \lambda, \beta^\vee \rangle = \langle \lambda, \bar{\alpha}^\vee \rangle = 2$, a contradiction.

Hence $\lambda - \bar{\alpha}$ is dominant, and if it is also nontrivial, we must have $\langle \lambda - \bar{\alpha}, \alpha_i^\vee \rangle > 0$ for some $i \in I$. Moreover, since $\bar{\alpha}$ is a dominant root, the coefficient of $\alpha_i^\vee$ in $\bar{\alpha}^\vee$ must be positive, so we obtain $\langle \lambda - \bar{\alpha}, \bar{\alpha}^\vee \rangle > 0$, whence $\langle \lambda, \bar{\alpha}^\vee \rangle > 2$, a contradiction. □

LEMMA 4.7. *Assume that Φ is finite, irreducible, and of rank > 1. If $\lambda \in \Lambda^+$ is neither minuscule nor quasi-minuscule, then there is a weight $\mu \in W\lambda$ that is neither minuscule nor quasi-minuscule relative to some irreducible subsystem Φ_J of rank 2.*

Proof. Proceeding by induction, we assume that the rank is > 2. Since $\bar{\alpha}^\vee$ is the highest co-root and λ is not minuscule, we must have $\langle \lambda, \bar{\alpha}^\vee \rangle \geqslant 2$. Every orbit of roots includes a simple root, so one may select $w \in W$ so that $w\bar{\alpha} = \alpha_i$ for some $i \in I$. Taking $\mu = w\lambda$, we have $\langle \mu, \alpha_i^\vee \rangle \geqslant 2$, so μ is not minuscule relative to Φ_J for all $J \subset I$ such that $i \in J$.

A spanning tree with at least two vertices has at least two end nodes, so there must be at least one $j \in I - \{i\}$ such that the subsystem indexed by $J = I - \{j\}$ is irreducible. If μ is not quasi-minuscule with respect to Φ_J, we continue the induction. Otherwise, α_i must necessarily be the unique root in Φ_J such that $\langle \mu, \alpha_i^\vee \rangle = 2$, and $\mu - \alpha_i$ must be trivial relative to Φ_J (Lemma 4.6); i.e., $\langle \mu - \alpha_i, \alpha_k^\vee \rangle = 0$ for all $k \neq j$. On the other hand, μ is not quasi-minuscule relative to Φ, so we must have $\langle \mu - \alpha_i, \alpha_j^\vee \rangle \neq 0$.

We may therefore assume that i and j are the *unique* indices whose deletion leaves an irreducible diagram; if j' were a third (or second, if i failed to have this property), then μ could not be quasi-minuscule relative to the subsystem indexed by $I - \{j'\}$, so again we could continue the induction.

It follows in particular that i and j are non-adjacent (i.e., $\langle \alpha_i, \alpha_j \rangle = 0$); a spanning tree with only two end nodes must be a path, and (since Φ has rank > 2) the end nodes of the path are non-adjacent. Hence $\langle \mu, \alpha_j^\vee \rangle \neq 0$. Setting $\mu' = s_j \mu$, we claim that μ' cannot be quasi-minuscule relative to Φ_J. We have $s_j \alpha_i = \alpha_i$ and $\langle \mu', \alpha_i^\vee \rangle \geqslant 2$, so if μ' were quasi-minuscule, then $\mu' - \alpha_i$, and hence also $\mu' - \mu$, would be trivial relative to Φ_J. However, $\mu' - \mu$ is a nonzero multiple of α_j and Φ is irreducible, so this is a contradiction. □

Proof of Theorem 4.3. Let λ be the weight of some object in X. Since $\lambda \in \Lambda_c$ and the weights are W-stable (Lemma 2.2), we may assume that λ is dominant.

Case 1: Φ is infinite. If λ is nontrivial, then there exists $\beta \in \Phi^+$ so that $\langle \lambda, \beta^\vee \rangle \geqslant p+2$, where $-p < 0$ denotes the smallest entry in the Cartan matrix $[\langle \alpha_i, \alpha_j^\vee \rangle]$ (Lemma 4.5). Now select $w \in W$ so that $-w\beta$ is simple, say $w\beta = -\alpha_i$. Since Φ is irreducible, we can also find $j \in I$ so that $\langle \alpha_i, \alpha_j \rangle < 0$.

There must be an object $x \in X$ with $\mu(x) = w\lambda$, whence $\delta(x,i) \leqslant \mu(x,i) \leqslant -2-p < -2$. If $\mu(x,j) < 0$, then $\delta(x,j) \leqslant \mu(x,j) < 0$ and we contradict Lemma 4.4, so it must be the case that $\mu(x,j) \geqslant 0$. If $\mu(x,j) = 0$, then consider the object $y = E_i(x)$. We have $\delta(y,i) \leqslant -1-p \leqslant -2$ and $\mu(y,j) = \mu(x,j) + \langle \alpha_i, \alpha_j^\vee \rangle < 0$, so y is an object whose existence contradicts Lemma 4.4. The remaining possibility is that $\mu(x,j) > 0$. In that case, $\varepsilon(x,j) > 0$, so there is an object $z = F_j(x)$. This object has $\delta(z,j) < 0$ and $\mu(z,i) = \mu(x,i) - \langle \alpha_i, \alpha_j^\vee \rangle \leqslant \mu(x,i) + p \leqslant -2$, so again we contradict Lemma 4.4.

Case 2: Φ is finite. Assume toward a contradiction that λ is neither minuscule nor quasi-minuscule. In that case, Lemma 4.7 shows that we may assume Φ has rank 2.

Since λ is not minuscule, there is a root β such that $\langle \lambda, \beta^\vee \rangle \geqslant 2$, so there is an object x whose weight is not quasi-minuscule and satisfies $\delta(x,i) \leqslant -2$ for some $i \in I$. The lowest object x' on the α_i-string through x is either x, or satisfies $\mu(x',i) = \delta(x',i) \leqslant -3$, in which case $\mu(x')$ is also not quasi-minuscule. Replacing x with x', we may thus assume $\varepsilon(x,i) = 0$. We may further require that x has minimal height among all the objects at the bottom of some string of length $\geqslant 2$ whose weight is in the same (finite) W-orbit.

Since Φ has rank 2, there is only one other simple root, say α_j, and since Φ is irreducible, we have $\langle \alpha_i, \alpha_j \rangle < 0$. Since X is thin, it must be the case that $\delta(x,j) = 0$; otherwise, we contradict Lemma 4.4. We also claim that $\varepsilon(x,j) \leqslant 1$. Indeed, if $\varepsilon(x,j) \geqslant 2$, then the lowest object x' on the α_j-string through x satisfies $\delta(x',j) \leqslant -2$ and $\mu(x',j) = s_j\mu(x)$, contradicting our choice of x. Now since $\delta(E_i(x),j) = \delta(x,j) = 0$ (Proposition 4.1), it follows that $0 \leqslant \varepsilon(E_i(x),j) = \varepsilon(x,j) + \langle \alpha_i, \alpha_j^\vee \rangle$. But we have $\varepsilon(x,j) \leqslant 1$, so this leaves only the possibility that $\varepsilon(x,j) = 1$ and $\langle \alpha_i, \alpha_j^\vee \rangle = -1$.

Finally, we claim that $\delta(x,i) = -2$. If not, we have $\delta(x,i) \leqslant -3$ and $\delta(E_i^2(x),j) = 0$ (Proposition 4.1), so $0 \leqslant \varepsilon(E^2(x),j) = \varepsilon(x,j) + 2\langle \alpha_i, \alpha_j^\vee \rangle < 0$, a contradiction.

To summarize, we have shown that $\varepsilon(x,i) = \delta(x,j) = 0$, $\delta(x,i) = -2$, $\varepsilon(x,j) = 1$, and $\langle \alpha_i, \alpha_j^\vee \rangle = -1$. Hence, $\mu(x,i) = -2$, $\mu(x,j) = 1$, and $\mu(E_i(x),i) = \mu(E_i(x),j) = 0$. In other words, $\mu(E_i(x)) = \mu(x) + \alpha_i$ is a trivial weight. However, $\langle \alpha_i, \alpha_j^\vee \rangle = -1$ implies that α_i is short, so $-\alpha_i$ (and hence $\mu(x)$) is quasi-minuscule, a contradiction. □

5. Semistandard Tableaux

Consider the root system $\Phi = A_{n-1} = \{\varepsilon_i - \varepsilon_j : 1 \leqslant i \neq j \leqslant n\}$, where $\varepsilon_1, \dots, \varepsilon_n$ denote an orthonormal basis of $V = \mathbf{R}^n$. As a set of simple roots, we take $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i = 1, \dots, n-1$. The Weyl group acts as the group of permutations of $\varepsilon_1, \dots, \varepsilon_n$.

It is not hard to see that $\omega_k := \varepsilon_1 + \cdots + \varepsilon_k$ $(0 \leqslant k \leqslant n)$ is a dominant minuscule weight, although $\omega_0 = 0$ and ω_n are trivial. The W-orbit of ω_k can be naturally identified with the k-element subsets of $[n] := \{1, \dots, n\}$, so the construction of Section 4B yields a thin admissible system X_k whose objects are the k-subsets of $[n]$.

The product of l copies of $X = X_0 \cup \cdots \cup X_n$ is (by Proposition 3.2) an admissible system whose objects consist of all l-tuples of subsets of $[n]$. Furthermore, the weight, depth, and rise of the l-tuple $T = (T_1, \ldots, T_l)$ are as follows (cf. Remark 3.3):

$$\mu(T) = N_1(T)\varepsilon_1 + \cdots + N_n(T)\varepsilon_n,$$

$$\delta(T, i) = \min_{0 \leqslant j \leqslant l} N_i(T_{\leqslant j}) - N_{i+1}(T_{\leqslant j}), \tag{5.1}$$

$$\varepsilon(T, i) = \max_{0 \leqslant j \leqslant l} N_i(T_{>j}) - N_{i+1}(T_{>j}), \tag{5.2}$$

where $T_{\leqslant j} = (T_1, \ldots, T_j)$, $T_{>j} = (T_{j+1}, \ldots, T_l)$, and $N_i(T)$ denotes the number of occurrences of i among $T_1, \ldots, T_l$. Thus (for example), the depth of T in the direction of α_i can be computed by scanning the subsets T_j from left to right, finding the smallest cumulative difference between the number of i's and $i+1$'s. The depth is 0 only if the number of i's accumulated is at least the number of $i+1$'s at all stages of the scanning process.

As a particular instance of the product construction, the operator E_i acts as follows. Assuming $\delta(T, i) < 0$, locate the least index j such that $\delta(T, i) = N_i(T_{\leqslant j}) - N_{i+1}(T_{\leqslant j})$. Under these circumstances, T_j must include $i+1$ but not i. One then obtains $T' = E_i(T)$ by replacing $i+1$ with i in T_j. Consequently, j is now the greatest index for which $\delta(T', i) = N_i(T'_{<j}) - N_{i+1}(T'_{<j})$, or equivalently, $\varepsilon(T', i) = N_i(T'_{\geqslant j}) - N_{i+1}(T'_{\geqslant j})$, and F_i inverts E_i by changing the i in T'_j back to $i+1$.

Now choose integers $l = \lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0$, and consider the dominant (integral) weight $\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n$. This weight is clearly in the nonnegative integral span of the ω_k's; indeed, the coefficient of ω_k in λ is $\lambda_k - \lambda_{k+1}$. Thus we may uniquely write

$$\lambda = \omega_{k_1} + \cdots + \omega_{k_l},$$

where $1 \leqslant k_1 \leqslant \cdots \leqslant k_l \leqslant n$.

Given an object $T = (T_1, \ldots, T_l)$, one may treat each subset T_j as a column whose members are listed in increasing order from top to bottom, with the top and subsequent entries of each column aligned in rows. If the j-th column has k_j entries for $1 \leqslant j \leqslant l$, we say that T forms a *tableau* of *shape* λ.[4] In addition, if the rows are non-increasing from left to right, we say that the tableau is *semistandard*. For example, if $l = 5$, $n = 6$, and $\lambda = 5\varepsilon_1 + 4\varepsilon_2 + 3\varepsilon_3$, then the array

$$\begin{matrix} 3 & 2 & 2 & 1 & 1 \\ & 4 & 3 & 3 & 2 \\ & & 6 & 4 & 3 \end{matrix}$$

represents a semistandard tableau T of shape λ and weight $2\varepsilon_1 + 3\varepsilon_2 + 4\varepsilon_3 + 2\varepsilon_4 + \varepsilon_6$. The reader can also check that $\delta(T, i) = (-2, -1, 0, 0, -1)$ for $i = 1, \ldots, 5$.

[4]The columns of these tableaux are listed in reverse order from the usual English tradition.

Proposition 5.1. *The set of semistandard tableaux of shape λ forms an admissible subsystem $Y(\lambda)$ of X^l. Furthermore, $Y(\lambda)$ has a unique maximal object, and the weight of this object is λ. Therefore, $G_{Y(\lambda)} = \chi(\lambda)$, and for all $\nu = \nu_1\varepsilon_1 + \cdots + \nu_n\varepsilon_n \in \Lambda^+$,*

$$\chi(\lambda) = \sum_{\mu\in\Lambda} K_{\lambda,\mu}e^{\mu} \quad \text{and} \quad \chi(\lambda)\chi(\nu) = \sum_{\mu\in\Lambda^+} c_{\lambda,\mu,\nu}\chi(\mu),$$

where $K_{\lambda,\mu}$ denotes the number of semistandard tableaux of shape λ and weight μ, and $c_{\lambda,\mu,\nu}$ denotes the number of semistandard tableaux T of shape λ and weight $\mu - \nu$ such that $\nu_i + N_i(T_{\leqslant j}) \geqslant \nu_{i+1} + N_{i+1}(T_{\leqslant j})$ for all $i < n$ and $j \leqslant l$.

Proof. To show that semistandard tableaux (or any subset of X^l) form an admissible system, it suffices to show that the operators E_i and F_i preserve semistandardness. Thus consider a pair (T, i) such that T is semistandard and $\delta(T, i) < 0$. As we noted above, E_i operates on T by changing a single $i + 1$ to an i in the leftmost column T_j such that $\delta(T, i) = N_i(T_{\leqslant j}) - N_{i+1}(T_{\leqslant j})$. Since $i \notin T_j$, the only way $E_i(T)$ could fail to be semistandard would be if the entry in the same row of T_{j+1} were equal to $i + 1$. However in that case, there must also be an i directly above the $i + 1$ in T_{j+1} (or else we violate the defining property of j), and hence there must be an entry $\geqslant i$ directly above the $i+1$ in T_j, a contradiction. Similar reasoning shows that F_i preserves semistandardness.

If T is semistandard and maximal (i.e., $\delta(T, i) = 0$ for all i), then the top entry in the leftmost column must be a 1. By semistandardness, it follows that the top entry in every column is a 1. Iterating this reasoning, the leftmost entry (and hence all entries) in the second row must be a 2, and so on. Hence, there is a unique maximal semistandard tableau of shape λ, and it has weight λ.

The remaining assertions are now immediate consequences of Theorem 2.4. □

6. Untangled Systems

Let X be an admissible system. We say that $Y \subseteq X$ is *F-saturated* if for all $i \in I$, we have $F_i(y) \in Y$ for all $y \in Y$ in the domain of F_i. We define E-saturated subsets of X similarly, and say that Y is *saturated* if it is both E- and F-saturated. It is easy to see that Y is an admissible subsystem of X if and only if it is saturated.

An admissible system is *irreducible* if it is nonempty and contains no proper subsystems. Intersections of saturated sets are saturated, so for every $Y \subset X$ there is a smallest saturated subset containing Y. Since the complement of a saturated subset is saturated, it follows that every admissible system is a disjoint (countable[5]) union of irreducible subsystems. Furthermore, the decomposition is unique up to order.

Recall that $x \in X$ is said to be *maximal* if $\delta(x, i) = 0$ for all $i \in I$; i.e., $\delta(x) \in \Lambda^+$. Every nonempty admissible system has at least one maximal object; this follows from the second part of Theorem 2.4, or one can argue that an object of maximum height is maximal (cf. the proof of Lemma 2.2(b)).

An irreducible system need not have a unique maximal object.

For example, let X be any (irreducible) admissible system, and let X' denote an isomorphic copy of X, disjoint from X, with $x \mapsto x'$ an isomorphism. It is clear that $X \cup X'$ also

[5]The countability of an admissible system follows from (A0).

forms an admissible system. However, one may create a new admissible system with object set $X \cup X'$ by redefining $F_i(x) := y'$ and $F_i(x') := y$ for an arbitrary set of triples (x, y, i) satisfying $y = F_i(x)$. If X is sufficiently connected (e.g., there exists a pair $x, y \in X$ such that there are two ways to write $y = F_{i_1} \cdots F_{i_l}(x)$), then this can be done in a way that yields a system with only one irreducible component. At the same time, this system has twice as many maximal objects as X.

We say that an admissible system X is *untangled* if every irreducible component of X has a unique maximal object; otherwise, X is *tangled*. We say that X is *strongly untangled* if it is untangled as an admissible system relative to Φ_J for all $J \subseteq I$.

PROPOSITION 6.1. *Minuscule and quasi-minuscule systems are strongly untangled. Moreover, all irreducible components of such systems relative to Φ_J (for all $J \subseteq I$) are minuscule or quasi-minuscule.*

Proof. First consider a minuscule system $X(\lambda)$ as in Section 4B. The objects (and weights) are the members of the W-orbit of λ, and it is clear that each such weight is also minuscule relative to Φ_J. Each W_J-orbit of weights has a unique member $\mu \in \Lambda_J^+$, and this orbit forms a Φ_J-subsystem of $X(\lambda)$ that is isomorphic to the minuscule Φ_J-system of highest weight μ.

Now consider a quasi-minuscule system $X(\lambda)$ as in Section 4C. There is no loss of generality in assuming that Φ is finite and irreducible, and that $\lambda = \tilde{\alpha}$, the short dominant root. For $i \in I_s$, the object 0_i is maximal relative to Φ_J if and only if $i \notin J$, in which case the singleton $\{0_i\}$ is an irreducible (minuscule) component of $X(\tilde{\alpha})$ relative to Φ_J. All other objects are short roots β; if β is maximal relative to Φ_J, then it is either minuscule relative to Φ_J, or it must be a short dominant root of Φ_J. In the former case, the W_J-orbit of β forms a minuscule subsystem of $X(\tilde{\alpha})$ relative to Φ_J; in the latter case, β generates a quasi-minuscule subsystem. □

Given an admissible system X and a subset $J \subseteq I$, define a partial order $\preccurlyeq_J$ on X by taking the transitive closure of the relations

$$x \prec_J y \quad \text{if } x = F_i(y) \text{ for some } i \in J.$$

This extends the notation '$\preccurlyeq_i$' introduced in Section 2. Note that the maximal objects of X relative to Φ_J are the maximal elements of this partial order.

The following is a type of "diamond" criterion for strong disentanglement.

LEMMA 6.2. *An admissible system X is strongly untangled if and only if for all distinct pairs $i, j \in I$ and all $x \in X$ such that $\delta(x, i) < 0$ and $\delta(x, j) < 0$, there is an object y such that $E_i(x) \preccurlyeq_{\{i,j\}} y$ and $E_j(x) \preccurlyeq_{\{i,j\}} y$.*

Proof. If X is strongly untangled, then there is a unique maximal object in every irreducible component of X relative to $\Phi_{\{i,j\}}$. In particular, since $E_i(x)$ and $E_j(x)$ belong to the same component, there is an object y satisfying the stated condition.

Conversely, given an admissible system X satisfying the stated condition, consider an object $y_0 \in X$ that is maximal relative to some Φ_J, and let Y denote the F_J-saturated (i.e., $\{F_i : i \in J\}$) subset of X generated by y_0. It suffices to show that Y is E_J-saturated, since it then follows that Y is an admissible Φ_J-subsystem of X, which by construction has y_0 as the unique maximal object.

To show that Y is E_J-saturated, we proceed by induction with respect to height. Given a pair $x \in Y$, $i \in J$ such that $\delta(x,i) < 0$, it must be the case that $x = F_j(x')$ for some $x' \in Y$ and $j \in J$ (by definition of Y). If $i = j$, we immediately obtain $E_i(x) \in Y$. Otherwise, $i \neq j$ and the stated hypothesis provides for the existence of an object y such that $E_i(x) \preccurlyeq_{\{i,j\}} y$ and $x' \preccurlyeq_{\{i,j\}} y$. We must have $y \in Y$, since by the induction hypothesis, any object $\succcurlyeq_{\{i,j\}} x'$ belongs to Y. It follows that $E_i(x) \in Y$, since $E_i(x) \preccurlyeq_{\{i,j\}} y \in Y$ shows that $E_i(x)$ is in the F_J-saturated set generated by y_0. □

QUESTION 6.3. *Are products of (strongly) untangled systems (strongly) untangled?*

The following special case is particularly useful, since it will lead to constructions of admissible systems for all Weyl characters of all finite root systems.

THEOREM 6.4. *Products of any number of minuscule and quasi-minuscule systems are strongly untangled.*

Proof. Let $X = X_1 \cdots X_l$ be a product of minuscule and quasi-minuscule systems. To prove that X is strongly untangled, it suffices by Lemma 6.2 to show that X is untangled relative to Φ_J for all doubletons $J \subseteq I$. However by Proposition 6.1, the irreducible components of each X_i relative to Φ_J are still minuscule or quasi-minuscule. Since products distribute naturally with respect to disjoint union, X is therefore a disjoint union of products of minuscule and quasi-minuscule systems relative to Φ_J. In other words, we may assume that Φ has rank 2; say $I = \{1,2\}$. We may also assume that Φ is finite, since otherwise X would be a product of singletons (cf. Theorem 4.3).

Thus there are four cases: $\Phi = \mathcal{A}_1 \times \mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{B}_2$, or $\mathcal{G}_2$.

LEMMA 6.5. *Assume $\langle \alpha_1, \alpha_2 \rangle = 0$ (i.e., $\Phi \cong \mathcal{A}_1 \times \mathcal{A}_1$).*

(a) *For all $x \in X$ such that $\delta(x,1) < 0$, we have $\delta(E_1(x),2) = \delta(x,2)$.*

(b) *If $\delta(x,1) < 0$ and $\delta(x,2) < 0$, then $E_1E_2(x) = E_2E_1(x)$.*

Proof. A quasi-minuscule system for $\Phi = \mathcal{A}_1 \times \mathcal{A}_1$ consists of an α_1-string or α_2-string of length 2; a (nontrivial) minuscule system consists of an α_1-string or α_2-string of length 1, or a product of both. Thus we may assume that each factor X_i is a single string.

Now consider an object $x = x_1 \cdots x_l \in X$ such that $\delta(x,1) < 0$, and recall that the action of E_1 is such that $E_1(x) = x_1 \cdots E_1(x_i) \cdots x_l$ for some i (see Remark 3.3). Since X_i must be an α_1-string, $x_i \mapsto E_1(x_i)$ has no effect on $\delta(x_i,2)$. Furthermore, since $\langle \alpha_1, \alpha_2 \rangle = 0$, this replacement also has no effect on $\mu(x_i,2)$, and hence no effect on $\delta(x,2)$ (see (3.2)), proving (a).

It follows that if $\delta(x,2) < 0$ and E_2 acts on position j of x, then it must also act on position j of $E_1(x)$ (again, see Remark 3.3). We must also have $j \neq i$, since E_2 does not act on any object of X_i. By the symmetry of the roles of α_1 and α_2, we conclude that the actions of E_1 and E_2 commute. □

LEMMA 6.6. *Assume* $\langle \alpha_1, \alpha_2^\vee \rangle = \langle \alpha_2, \alpha_1^\vee \rangle = -1$ *(i.e.,* $\Phi \cong A_2$*).*

(a) *For all* $x \in X$ *such that* $\delta(x,1) < 0$*, we have* $\delta(E_1(x),2) - \delta(x,2) \in \{0,-1\}$.

(b) *If* $\delta(x,1) < 0$ *and* $\delta(x,2) < 0$*, then exactly one of the following holds:*

(i) $\delta(E_1(x),2) = \delta(x,2)$ *and* $E_1E_2(x) = E_2E_1(x)$,

(ii) $\delta(E_2(x),1) = \delta(x,1)$ *and* $E_1E_2(x) = E_2E_1(x)$, *or*

(iii) $\delta(E_1(x),2) < \delta(x,2)$, $\delta(E_2(x),1) < \delta(x,1)$ *and* $E_1E_2^2E_1(x) = E_2E_1^2E_2(x)$ *(and both expressions are defined).*

Proof. In the notation of Section 5, the root system $\Phi = A_2$ has two nontrivial dominant minuscule weights (up to trivial shifts): ε_1 and $\varepsilon_1 + \varepsilon_2$. Furthermore, it is easy to check that $X(\varepsilon_1 + \varepsilon_2)$ occurs as a subsystem of $X(\varepsilon_1)X(\varepsilon_1)$, and the quasi-minuscule system $X(2\varepsilon_1 + \varepsilon_2)$ occurs as a subsystem of $X(\varepsilon_1)X(\varepsilon_1 + \varepsilon_2)$. Thus we may reduce to the case where each factor X_i is isomorphic to $X(\varepsilon_1)$; i.e., $X = X(\varepsilon_1)^l$.

As in Section 5, we can identify $X(\varepsilon_1)$ with a system of three objects $1, 2, 3$, with corresponding weights $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and the string structure $1 \succ_1 2 \succ_2 3$. In this way, the objects of X are words of length l over the alphabet $\{1,2,3\}$, and the depth and rise are given as very special cases of (5.1) and (5.2).

Given a word $x = x_1 \cdots x_l \in X$ such that $\delta(x,1) < 0$ and E_1 acts on x at x_i, it must be the case that $x_i = 2$ and $E_1(x_i) = 1$. This action of E_1 has the effect of decreasing $N_2(x_{\leqslant j}) - N_3(x_{\leqslant j})$ by 1 for $j \geqslant i$ and no effect for $j < i$. Since $\delta(x,2)$ is the minimum value of this expression over all j, this yields (a).

Now assume $\delta(x,2) < 0$ as well, and consider the case in which $\delta(E_1(x),2) = \delta(x,2)$. By the previous analysis, E_2 must act on x at a position $j < i$. Since E_1 preserves $x_{\leqslant j}$, it follows that E_2 acts at position j in both x and $E_1(x)$. Meanwhile, in $E_2(x)$ there is an additional 2 prior to position i, the first position of x that achieves the minimum value for $N_1(x_{\leqslant i}) - N_2(x_{\leqslant i})$. Hence, this minimum decreases by 1 (and $\delta(E_2(x),1) < \delta(x,1)$), but the position where this minimum first occurs (namely, i) does not. That is, E_1 acts at position i in both x and $E_2(x)$, so the actions of E_1 and E_2 on x commute.

By interchanging α_1 and α_2, we reach a similar conclusion if $\delta(E_2(x),1) = \delta(x,1)$, so the only remaining possibility is that $\delta(E_1(x),2) - \delta(x,2) = \delta(E_2(x),1) - \delta(x,1) = -1$. Assuming that E_1 and E_2 act on x at a pair of respective positions i and j with $j < i$, it must be the case (in order for the action of E_1 to decrease $\delta(x,2)$) that there is a first position $j' > i$ where $N_2(x_{\leqslant j'}) - N_3(x_{\leqslant j'}) = \delta(x,2)$, and E_2 acts at position j' on $E_1(x)$. In $E_2E_1(x)$, the positions $\leqslant j$ are unchanged and $\delta(E_2E_1(x),2) = \delta(x,2)$, so E_2 acts on $E_2E_1(x)$ at position j. We thus have

$$x = \cdots 3 \cdots 2 \cdots 3 \cdots,$$
$$E_2^2E_1(x) = \cdots 2 \cdots 1 \cdots 2 \cdots,$$

with the changed positions being j, i, j' (in order). Since the number of 1's and 2's in the first $i-1$ positions of $E_2^2E_1(x)$ is the same as in $x_{\leqslant i}$, we have $\delta(E_2^2E_1(x),1) = \delta(x,1) < 0$, and E_1 acts on $E_2^2E_1(x)$ at some position i' such that $j \leqslant i' < i$.

On the other hand, E_2 acts on x at position j, decreasing $\delta(x,1)$ by 1, but the position where E_1 acts on $E_2(x)$ remains i. The first $j'-1$ positions of $E_1E_2(x)$ and $E_2^2E_1(x)$ are now identical, so E_1 acts on $E_1E_2(x)$ at position i'. Comparing the cumulative number of 2's and 3's among the first j' positions of $E_1^2E_2(x)$ and x, the only difference is that $E_1^2E_2(x)$ has one less 3 (starting at j), one more 2 (from j to $i'-1$), and one less 2 (starting at j). It follows that $\delta(E_1^2E_2(x),2) = \delta(x,2) < 0$, E_2 acts on $E_1^2E_2(x)$ at position j', and thus $E_1E_2^2E_1(x) = E_2E_1^2E_2(x)$.

This proves the conclusion of (iii) under the added condition that $j < i$; i.e., E_2 acts to the left of E_1 on x. However, the hypotheses of the lemma are invariant under switching 1 and 2, so the conclusion must also be valid when E_2 acts to the right of E_1. □

Lemmas 6.5(b) and 6.6(b) establish the existence of a "diamond" fitting the conditions of Lemma 6.2, and thus prove Theorem 6.4 for all simply-laced root systems (i.e., root systems whose rank two subsystems are all of type $\mathcal{A}_1 \times \mathcal{A}_1$ or $\mathcal{A}_2$).

The case $\Phi = \mathcal{B}_2$. One way to realize $\mathcal{B}_2$ is through an automorphism of $\mathcal{A}_3$. Letting $\{\alpha_1', \alpha_2', \alpha_3'\}$ denote the simple roots of $\mathcal{A}_3$, there is a linear automorphism σ that fixes α_2' and interchanges α_1' and α_3'. In this way, the short roots of $\mathcal{B}_2$ can be identified with the roots of $\mathcal{A}_3$ fixed by σ, and the long roots can be identified with the sums $\alpha + \sigma(\alpha)$ for $\alpha \in \mathcal{A}_3$ such that $\sigma(\alpha) \neq \alpha$. In particular, the simple roots of $\mathcal{B}_2$ can be chosen so that $\alpha_1 = \alpha_2'$ and $\alpha_2 = \alpha_1' + \alpha_3'$. Using the coordinates from Section 5, we have $\alpha_1 = \varepsilon_2 - \varepsilon_3$, $\alpha_2 = \varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4$, and σ acts by interchanging ε_1 with $-\varepsilon_4$ and ε_2 with $-\varepsilon_3$.

The nontrivial $\mathcal{B}_2$-weight $\omega = (1/2)(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4)$ is dominant minuscule, and is the unique such weight up to trivial shifts. The corresponding minuscule system $X(\omega)$ consists of four objects with the string structure $1 \succ_1 2 \succ_2 3 \succ_1 4$, and their weights are the four weights fixed by σ that can be obtained from ω by permuting $\varepsilon_1, \dots, \varepsilon_4$. From this it is easy to check that the quasi-minuscule system $X(\bar{\alpha})$ is a subsystem of $X(\omega)^2$, so we may reduce to the case where each factor X_i is isomorphic to $X(\omega)$; i.e., $X = X(\omega)^l$.

The weight ω is also dominant minuscule relative to $\mathcal{A}_3$, and one can view $X(\omega)$ as the σ-invariant part of the minuscule $\mathcal{A}_3$-system $X'(\omega)$ in the following way. Since ω is fixed by σ, the $\mathcal{A}_3$-orbit of ω is σ-stable, so there is an induced automorphism σ of $X'(\omega)$. This automorphism fixes four of the six objects, and the weights of the fixed objects are the weights in $X(\omega)$. Letting E_i', F_i' $(i = 1,2,3)$ denote the raising and lowering operators of $X'(\omega)$, one finds that the four fixed objects are x, $F_2'(x)$, $F_1'F_3'F_2'(x) = F_3'F_1'F_2'(x)$, and $F_2'F_1'F_3'F_2'(x) = F_2'F_3'F_1'F_2'(x)$, where x denotes the maximal object of $X'(\omega)$. Identifying these objects with those of $X(\omega)$, we have $F_1 = F_2'$ and $F_2 = F_1'F_3' = F_3'F_1'$.

The automorphism σ extends naturally to the l-fold product $X'(\omega)^l$, and the fixed points of this automorphism can be identified with the objects of $X(\omega)^l$. Furthermore, under this identification, the raising operators E_1 and E_2 act on $X(\omega)^l$ in the same way that E_2' and $E_1'E_3'$ act on the σ-fixed objects of $X'(\omega)^l$.

Now consider an object $x = x_1 \cdots x_l \in X(\omega)^l$ such that $\delta(x,1) < 0$ and $\delta(x,2) < 0$. Since $\mathcal{A}_3$ is simply-laced, we know that $X'(\omega)^l$ is strongly untangled, so there is a unique maximal object y in the irreducible component of $X'(\omega)^l$ that contains x, $E_1(x) = E_2'(x)$, and $E_2(x) = E_1'E_3'(x)$. We claim that y is fixed by σ, and hence an object of $X(\omega)^l$. If not, then let y' be a σ-fixed object of maximum height in this component. As a non-maximal object of $X'(\omega)^l$, it must be the case that $\delta(y',i) < 0$ for some i. However if $\delta(y',2) < 0$, then $E_2'(y') = E_1(y')$ would be a higher counterexample. On the other hand, since y' is σ-fixed, the only remaining possibility is that $\delta(y',1) = \delta(y',3) < 0$, in which case Lemma 6.5(a) shows that $E_1'E_3'(y') = E_2(y')$ would be a higher counterexample, a contradiction. This same reasoning also shows E_1 or E_2 can be applied to any non-maximal σ-fixed object of $X'(\omega)^l$, so it follows that $E_1(x) \preccurlyeq_{\{1,2\}} y$ and $E_2(x) \preccurlyeq_{\{1,2\}} y$, whence by Lemma 6.2, $X = X(\omega)^l$ is strongly untangled.

The case $\Phi = \mathcal{G}_2$. Choose an order-three linear automorphism σ of the root system $\mathcal{D}_4$. One can label the simple roots α'_i $(i = 0, 1, 2, 3)$ so that σ fixes α'_0 and cyclically permutes $\alpha'_1, \alpha'_2, \alpha'_3$, which are mutually orthogonal. The short roots of $\mathcal{G}_2$ can then be realized as the σ-fixed roots of $\mathcal{D}_4$, and the long roots are sums of nontrivial σ-orbits in $\mathcal{D}_4$. In particular, the simple roots of $\mathcal{G}_2$ can be chosen so that $\alpha_1 = \alpha'_0$ and $\alpha_2 = \alpha'_1 + \alpha'_2 + \alpha'_3$.

In this case, there are no (nontrivial) minuscule weights, so we may assume that each factor X_i is isomorphic to the quasi-minuscule system $X(\bar{\alpha})$; i.e., $X = X(\bar{\alpha})^l$. There are six short roots and an object of weight 0 in $X(\bar{\alpha})$; these objects have the string structure

$$\bar{\alpha} \succ_1 \alpha_1 + \alpha_2 \succ_2 \alpha_1 \succ_1 0 \succ_1 -\alpha_1 \succ_2 -\alpha_1 - \alpha_2 \succ_1 -\bar{\alpha}.$$

Furthermore, $\bar{\alpha}$ is also the dominant root of $\mathcal{D}_4$, and we can extend the action of σ to an automorphism of the quasi-minuscule $\mathcal{D}_4$-system $X'(\bar{\alpha})$ by permuting the weight 0 objects 0_i in the same way that σ permutes the simple roots α'_i $(0 \leqslant i \leqslant 3)$. Thus $X(\bar{\alpha})$ can be viewed as the σ-invariant part of $X'(\bar{\alpha})$. Letting E'_i, F'_i $(0 \leqslant i \leqslant 3)$ denote the raising and lowering operators of $X'(\bar{\alpha})$, we have $E_1 = E'_0$ and $E_2 = E'_1 E'_2 E'_3$ under this identification.

The automorphism σ extends naturally from $X'(\bar{\alpha})$ to $X'(\bar{\alpha})^l$, and the σ-invariant part can be identified with $X(\bar{\alpha})^l$. Using Lemma 6.5 and the fact that $\alpha'_1, \alpha'_2, \alpha'_3$ are mutually orthogonal, it follows that E_1 and E_2 act on $X(\bar{\alpha})^l$ in the same way that E'_0 and $E'_1 E'_2 E'_3$ act on the σ-fixed part of $X'(\bar{\alpha})^l$. Since $X'(\bar{\alpha})^l$ is strongly untangled ($\mathcal{D}_4$ is simply-laced), the same reasoning used for $\mathcal{B}_2$ shows that $X(\bar{\alpha})^l$ is also strongly untangled. □

REMARK 6.7. (a) It follows that if X and Y are any admissible systems that occur as subsystems of products of (quasi-)minuscule systems, then XY is strongly untangled.

(b) In an untangled system, the irreducible component containing a given maximal object x is simply the F-saturated subset generated by x.

(c) Assuming Φ is irreducible of rank > 1, an untangled thin system must be a disjoint union of (thin) systems whose generating series are single Weyl characters $\chi(\lambda)$ with λ minuscule or quasi-minuscule (Theorem 4.3). Moreover, one can show that any such system must be isomorphic to one of the minuscule or quasi-minuscule systems $X(\lambda)$ of Section 4. Thus in this context, products of untangled thin systems are strongly untangled.

(d) The relations $E_1 E_2 = E_2 E_1$ and $E_1 E_2^2 E_1 = E_2 E_1^2 E_2$ in Lemmas 6.5(b) and 6.6(b) can be viewed as combinatorial analogues of the Serre relations. In general, for each object x in a strongly untangled system, one can ask for a minimal object y such that $E_i(x) \preccurlyeq_{\{i,j\}} y$ and $E_j(x) \preccurlyeq_{\{i,j\}} y$ (assuming that $E_i(x)$ and $E_j(x)$ are defined). There must be a sequence of raising operators starting with E_i that takes x to y, and another starting with E_j; their equality is a *combinatorial Serre relation.* In $\mathcal{B}_2$, there are at least four of these relations; namely,

$$E_1 E_2 E_1 E_2 E_1 = E_2 E_1^3 E_2, \quad E_2 E_1^3 E_2^2 E_1 = E_1 E_2^2 E_1^3 E_2,$$

and the two that occur in $\mathcal{A}_2$. In $\mathcal{G}_2$, there are at least 15 such relations.

7. Generation of Finite Systems

Throughout this section, we assume that Φ is finite. It follows that there is no harm in further assuming that V is spanned by the simple roots. Under these circumstances, there are no nonzero trivial weights, Λ is a true lattice, and $\Lambda/\mathbf{Z}\Phi$ is a finite abelian group.

The following result implies (and is roughly equivalent to) the fact that every Weyl character occurs as a summand of a product of minuscule or quasi-minuscule characters. This fact does not seem to be well-known; in any case, we have not seen it in the literature.

THEOREM 7.1. *For every $\lambda \in \Lambda^+$, there is a product of minuscule and quasi-minuscule systems that includes a maximal object of weight λ.*

Combining this with Theorem 6.4, we obtain

COROLLARY 7.2. *For every $\lambda \in \Lambda^+$, there is an admissible subsystem of a product of minuscule and quasi-minuscule systems whose generating series is $\chi(\lambda)$.*

Our proof of Theorem 7.1 (or rather, the stronger result in Theorem 7.6 below) is constructive in that for each λ, we show how to identify a suitable sequence of minuscule and quasi-minuscule systems and a maximal object of weight λ in their product. In order to explicitly realize the irreducible component containing this object, one still needs to identify the F-saturated set that it generates (cf. Remark 6.7(b)).

LEMMA 7.3.

(a) *Each coset of Λ modulo $\mathbf{Z}\Phi$ contains a unique dominant minuscule weight.*

(b) *If Φ is irreducible, then the lattice generated by any nonzero W-orbit of minuscule weights includes $\mathbf{Z}\Phi$.*

Proof. (a) This is well-known and easy to prove; e.g., see Corollary 1.13 of [**St**].

(b) Given a nonzero minuscule orbit Ω, it suffices to show that $\mathbf{Z}\Omega$ includes every simple root. However if $\alpha_i \notin \mathbf{Z}\Omega$, then α_i must be orthogonal to the linear span of Ω. Otherwise, we would have $\langle \mu, \alpha_i^\vee \rangle \neq 0$ for some $\mu \in \Omega$, and therefore $\langle \mu, \alpha_i^\vee \rangle = \pm 1$ (since μ is minuscule), whence $\alpha_i = \pm(\mu - s_i\mu) \in \mathbf{Z}\Omega$. Thus the simple roots in $\mathbf{Z}\Omega$ are orthogonal to those not in $\mathbf{Z}\Omega$. Given that $\Omega \neq \{0\}$, this contradicts the irreducibility of Φ. $\square$

LEMMA 7.4. *If Φ is irreducible and Ω is a W-stable set of minuscule and quasi-minuscule weights, then for every dominant $\lambda \in \mathbf{Z}\Omega$, there is a decomposition*

$$\lambda = \mu_1 + \cdots + \mu_l \quad (\mu_i \in \Omega)$$

with $\mu_1 + \cdots + \mu_i$ dominant for $1 \leqslant i \leqslant l$.

Proof. The sum of all members of a W-orbit in Λ is W-invariant, hence orthogonal to all simple roots, and hence zero. It follows that each element of $-\Omega$ is in the *nonnegative* integer span of Ω, and thus there exist decompositions $\lambda = \mu_1 + \cdots + \mu_l$ with $\mu_i \in \Omega$.

If $l \leqslant 1$, there is nothing further to prove. Proceeding by induction, fix $l \geqslant 2$ and choose a decomposition that minimizes the height of μ_l within its W-orbit. If $\lambda - \mu_l$ is dominant, then the induction hypothesis provides a decomposition of $\lambda - \mu_l$ with dominant partial sums, and we are done. Otherwise, we have $\langle \lambda - \mu_l, \alpha_i^\vee \rangle < 0$ for some $i \in I$ and hence $\langle \mu_j, \alpha_i^\vee \rangle < 0$ for some $j < l$ and $\langle \mu_l, \alpha_i^\vee \rangle > 0$, since λ is dominant.

Since μ_l is minuscule or quasi-minuscule, it must be the case that $\langle \mu_l, \alpha_i^\vee \rangle = 1$ or 2. In the former case, $\mu_l - \alpha_i = s_i \mu_l$ is in the same W-orbit; in the latter case, μ_l must be quasi-minuscule and $\mu_l - \alpha_i = 0$ (Lemma 4.6). Similarly, $\mu_j + \alpha_i$ is either zero or in the W-orbit of μ_j, so we may replace μ_l with $\mu_l - \alpha_i$ and μ_j with $\mu_j + \alpha_i$ in our decomposition, suitably reducing l if either of these replacements vanishes. However, if this new decomposition of λ still has length l, we contradict our original choice. Hence it must be shorter, and we may appeal to the induction hypothesis. □

REMARK 7.5. This result is false if we drop the hypothesis that V is spanned by the simple roots. For example, using the coordinates for $\Phi = \mathcal{A}_{n-1}$ from Section 5, there is no suitable decomposition of the dominant weight $\lambda = -\varepsilon_n$ for the minuscule orbit $\Omega = \{\varepsilon_i : 1 \leqslant i \leqslant n\}$. Instead one can argue that the result remains valid in the general case if we are allowed to shift λ by a trivial weight.

Now consider any lattice Λ' such that $\mathbf{Z}\Phi \subseteq \Lambda' \subseteq \Lambda$. By Lemma 7.3(a), we know that $\Lambda'/\mathbf{Z}\Phi$ is a finite group generated by a set of dominant minuscule weights; let Ω^+ denote any such set of generators.

THEOREM 7.6. *Assume Φ is irreducible, and let Λ' and Ω^+ be as above.*

(a) *If $\Lambda' \neq \mathbf{Z}\Phi$, then for all dominant $\lambda \in \Lambda'$, there is a sequence $\omega_1, \dots, \omega_l \in \Omega^+$ such that the minuscule product $X(\omega_1) \cdots X(\omega_l)$ has a maximal object of weight λ.*

(b) *If $\Lambda' = \mathbf{Z}\Phi$ (i.e., $\Omega^+ \subseteq \{0\}$), then for all dominant $\lambda \in \Lambda'$, there is an $l \geqslant 0$ such that the quasi-minuscule product $X(\bar{\alpha})^l$ has a maximal object of weight λ.*

Proof. (a) Let Ω denote the W-stable set generated by Ω^+. Since $\mathbf{Z}\Omega$ includes members of all cosets of Λ' modulo $\mathbf{Z}\Phi$, it follows via Lemma 7.3(b) that $\Lambda' = \mathbf{Z}\Omega$. Applying Lemma 7.4, we deduce that every dominant $\lambda \in \Lambda'$ has a decomposition $\lambda = \mu_1 + \cdots + \mu_l$ with $\mu_i \in \Omega$ and $\mu_1 + \cdots + \mu_i$ dominant for all i. Letting ω_i denote the dominant member of the W-orbit of μ_i, this means precisely that the l-tuple $(\mu_1, \dots, \mu_l)$ is a maximal object of $X(\omega_1) \cdots X(\omega_l)$ of weight λ (see (3.2)).

(b) The short roots Φ_s generate $\mathbf{Z}\Phi$. If all roots are short this is vacuous; if there are long roots, then there is a pair $i, j \in I$ such that α_i is short, α_j is long, and $\langle \alpha_i, \alpha_j^\vee \rangle = -1$. In that case, $\alpha_j = s_j \alpha_i - \alpha_i \in \mathbf{Z}\Phi_s$, and hence $\mathbf{Z}\Phi_s$ includes the entire orbit of long roots. Applying Lemma 7.4, we deduce that every dominant $\lambda \in \mathbf{Z}\Phi$ has a decomposition $\lambda = \beta_1 + \cdots + \beta_l$ with $\beta_i \in \Phi_s$ and $\beta_1 + \cdots + \beta_i$ dominant for all i. This implies (via (3.2)) that the l-tuple $(\beta_1, \dots, \beta_l)$ is a maximal object of $X(\bar{\alpha})^l$ of weight λ. □

REMARK 7.7. (a) The lattices Λ' such that $\mathbf{Z}\Phi \subseteq \Lambda' \subseteq \Lambda$ index the various classes of (connected, semisimple, complex) Lie groups with root system Φ, and the dominant weights in Λ' index the (Weyl) characters of their irreducible representations.

(b) For irreducible root systems, the groups $\Lambda'/\mathbf{Z}\Phi$ are almost always cyclic. In such cases one can take Ω^+ to be a singleton and deduce that there are maximal objects of all possible dominant weights in Λ' among the powers of a single minuscule or quasi-minuscule system. The only exceptions occur when $\Phi = \mathcal{D}_n$, $\Lambda' = \Lambda$, and n is even.

8. Lakshmibai-Seshadri Chains

Given a fixed $\lambda \in \Lambda^+$, we let '$<$' denote the usual Bruhat ordering of the W-orbit of λ; i.e., the transitive closure of the relations

$$\sigma_\alpha \mu < \mu \quad \text{if } \langle \mu, \alpha^\vee \rangle > 0 \quad (\mu \in W\lambda,\ \alpha \in \Phi^+).$$

Similarly, if λ is anti-dominant (i.e., $-\lambda \in \Lambda^+$), we define the Bruhat ordering of $W\lambda$ in exactly the same way. We write $\nu \lessdot \mu$ to indicate that μ covers ν. This happens only if $\nu = \sigma_\alpha \mu$ for some $\alpha \in \Phi^+$ as above, but not conversely.

The Bruhat orderings of $W\lambda$ and $-W\lambda$ are dual-isomorphic (in fact, $\mu < \nu$ if and only if $-\nu < -\mu$), so by employing arguments that simultaneously apply to orbits generated by dominant or anti-dominant weights, one may instantly dualize any property of the Bruhat order. In the finite case such distinctions are unnecessary, since every finite W-orbit is generated by a dominant member.

Given $\pm\lambda \in \Lambda^+$ and a fixed $b \in \mathbf{R}$, we define the *b-Bruhat ordering* '$<_b$' by taking the transitive closure of the relations

$$\sigma_\alpha \mu <_b \mu \quad \text{if } \sigma_\alpha \mu \lessdot \mu \text{ and } b\langle \mu, \alpha^\vee \rangle \in \mathbf{Z} \quad (\mu \in W\lambda,\ \alpha \in \Phi^+).$$

Thus μ covers ν in the b-Bruhat order if and only if μ covers ν in the normal Bruhat order and $b(\mu - \nu)$ is an integer multiple of a root. In particular,

$$\nu \leqslant_b \mu \ \Rightarrow\ b(\mu - \nu) \in \mathbf{Z}\Phi, \tag{8.1}$$

and the only nontrivial values of b are rational with a denominator that divides $\langle \lambda, \alpha^\vee \rangle$ for some root α. (Otherwise, $\nu \leqslant_b \mu$ only if $\nu = \mu$.)

Note that the 1-Bruhat ordering is the normal Bruhat ordering.

LEMMA 8.1. *If $\langle \mu, \alpha_i^\vee \rangle > 0$ and $\nu \leqslant_b \mu$, then either*

(a) *$\langle \nu, \alpha_i^\vee \rangle \leqslant 0$ and $\nu \leqslant_b s_i\mu <_b \mu$, or*
(b) *$\langle \nu, \alpha_i^\vee \rangle > 0$ and $s_i\nu \leqslant_b s_i\mu$.*

Proof. Proceed by induction, the base being the (trivial) case $\nu = \mu$. In all other cases, we have $\nu <_b \mu$ and there is a positive root α such that $\nu \leqslant_b \sigma_\alpha\mu <_b \mu$ and $\sigma_\alpha\mu \lessdot \mu$.

Case 1: $\alpha = \alpha_i$. We may assume $\langle \nu, \alpha_i^\vee \rangle > 0$; otherwise, (a) holds and we are done. Now since $s_i\mu <_b \mu$ is necessarily a covering relation, it follows that $b\langle \mu, \alpha_i^\vee \rangle \in \mathbf{Z}$. Furthermore, since $b(\mu - \nu) \in \mathbf{Z}\Phi$ by (8.1), we thus have $b\langle \nu, \alpha_i^\vee \rangle \in \mathbf{Z}$, whence $s_i\nu <_b \nu \leqslant_b s_i\mu <_b \mu$ and (b) holds.

Case 2: $\alpha \neq \alpha_i$. In this case, $\beta := s_i\alpha$ must be positive and $\langle s_i\mu, \beta^\vee \rangle = \langle \mu, \alpha^\vee \rangle > 0$, so we have $s_i\sigma_\alpha\mu = \sigma_\beta s_i\mu < s_i\mu$. Hence $\langle \sigma_\alpha\mu, \alpha_i^\vee \rangle > 0$; otherwise, $\sigma_\alpha\mu \leqslant s_i\sigma_\alpha\mu < s_i\mu < \mu$, contradicting the fact that $\sigma_\alpha\mu \lessdot \mu$.

Claim: $s_i\sigma_\alpha\mu <_b s_i\mu$. Since $\sigma_\alpha\mu <_b \mu$, we have $b\langle s_i\mu, \beta^\vee \rangle = b\langle \mu, \alpha^\vee \rangle \in \mathbf{Z}$, so to prove the claim it suffices to show that $s_i\sigma_\alpha\mu \lessdot s_i\mu$. If this were false, then there would be a chain of length $\geqslant 3$ from $s_i\sigma_\alpha\mu$ to μ. However the chain $s_i\sigma_\alpha\mu < \sigma_\alpha\mu < \mu$ is unrefinable, contradicting the fact that the Bruhat order is graded (e.g., see [**D**]).

Now since $\langle \sigma_\alpha\mu, \alpha_i^\vee \rangle > 0$ and $\nu \leqslant_b \sigma_\alpha\mu$, it follows by induction that

(a) $\langle \nu, \alpha_i^\vee \rangle \leqslant 0$ and $\nu \leqslant_b s_i\sigma_\alpha\mu <_b \sigma_\alpha\mu$, or
(b) $\langle \nu, \alpha_i^\vee \rangle > 0$ and $s_i\nu \leqslant_b s_i\sigma_\alpha\mu$.

In the latter case, the claim yields $s_i\nu \leqslant_b s_i\sigma_\alpha\mu <_b s_i\mu$. In the former case, the claim yields $\nu \leqslant_b s_i\sigma_\alpha\mu <_b s_i\mu$. We also obtain $s_i\mu <_b \mu$ in this case, since the relations $s_i\mu >_b s_i\sigma_\alpha\mu <_b \sigma_\alpha\mu <_b \mu$ imply $b(\mu - s_i\mu) \in \mathbf{Z}\Phi$ via (8.1). In either case, this completes the induction. □

Given $\pm\lambda \in \Lambda^+$, we say that a pair consisting of a Bruhat chain $\mu_0 < \mu_1 < \cdots < \mu_l$ in the W-orbit of λ and an increasing sequence of rationals $0 < b_1 < \cdots < b_l < 1$ is a *Lakshmibai-Seshadri chain* (or *LS chain*) if

$$\mu_0 <_{b_1} \mu_1 <_{b_2} \cdots <_{b_l} \mu_l.$$

To simplify the description of certain operators, it will be convenient to identify this object with the map $x : (0,1] \to W\lambda$, where

$$x(t) = \begin{cases} \mu_0 & \text{if } 0 < t \leqslant b_1, \\ \mu_k & \text{if } b_k < t \leqslant b_{k+1}, \\ \mu_l & \text{if } b_l < t \leqslant 1. \end{cases}$$

Note that the piecewise-constant left-continuous maps x that arise in this fashion (i.e., from LS chains) can be characterized by the property

$$x(t) \leqslant_t x(t^+) \qquad (0 < t < 1), \tag{8.2}$$

where $x(t^+)$ denotes limiting value of x approaching t through values $> t$.

We claim that these maps form the objects of an admissible system whose generating series is $\chi(\lambda)$ (assuming $\lambda \in \Lambda^+$), although for this to be a precise statement we first need to assign weights and depths, and construct lowering operators.

Given x as above, we define the weight of x to be

$$\mu(x) = \int_0^1 x(t)\,dt \;=\; \mu_l - \sum_{1\leqslant k\leqslant l} b_k(\mu_k - \mu_{k-1}). \tag{8.3}$$

The fact that this is an integral weight is a consequence of (8.1). The depth (and resulting rise) of x in the direction of α_i is defined (respectively, given) by

$$\delta(x,i) = \min_{0\leqslant t\leqslant 1} \int_0^t \langle x(s), \alpha_i^\vee\rangle\,ds, \tag{8.4}$$

$$\varepsilon(x,i) = \max_{0\leqslant t\leqslant 1} \int_t^1 \langle x(s), \alpha_i^\vee\rangle\,ds. \tag{8.5}$$

We remark that the piecewise-linear map $t \mapsto \int_0^t x(s)\,ds$ is a "Lakshmibai-Seshadri path" in the sense of Littelmann [**L1**].

To see that the depth and rise are integral, note that

$$\int_0^t \langle x(s), \alpha_i^\vee\rangle\,ds = t\langle x(t), \alpha_i^\vee\rangle - \sum_{1\leqslant k\leqslant j} b_k\langle \mu_k - \mu_{k-1}, \alpha_i^\vee\rangle,$$

where j is the largest index such that $b_j < t$. Hence by (8.1),

$$\int_0^t \langle x(s), \alpha_i^\vee \rangle \, ds \in \mathbf{Z} \;\Leftrightarrow\; t\langle x(t), \alpha_i^\vee \rangle \in \mathbf{Z}. \tag{8.6}$$

In particular, since $s_i\mu <_b \mu$ implies $b\langle \mu, \alpha_i^\vee \rangle \in \mathbf{Z}$, Lemma 8.1(a) yields

$$\langle x(t), \alpha_i^\vee \rangle \leqslant 0,\ \langle x(t^+), \alpha_i^\vee \rangle > 0 \;\Rightarrow\; \int_0^t \langle x(s), \alpha_i^\vee \rangle \, ds \in \mathbf{Z}. \tag{8.7}$$

Assuming that the minimum in (8.4) does not occur at an endpoint (otherwise $\delta(x,i) = 0$ and $\varepsilon(x,i) = \mu(x,i)$ or vice versa), it must occur at some t for which $\langle x(t), \alpha_i^\vee \rangle \leqslant 0$ and $\langle x(t^+), \alpha_i^\vee \rangle > 0$, in which case (8.7) implies $\delta(x,i) \in \mathbf{Z}$.

To construct a lowering operator $x \mapsto F_i(x)$, assume $\varepsilon(x,i) > 0$, let t_1 be the largest value of t for which equality occurs in (8.4) and (8.5), and let t_2 be the smallest value of $t > t_1$ such that $\int_t^1 \langle x(s), \alpha_i^\vee \rangle \, ds = \varepsilon(x,i) - 1$. We then define

$$F_i(x)(t) = \begin{cases} s_i x(t) & \text{if } t_1 < t \leqslant t_2, \\ x(t) & \text{otherwise.} \end{cases}$$

Clearly $F_i(x)$ is a piecewise-constant left-continuous map $(0,1] \to W\lambda$.

LEMMA 8.2. *Let x, t_1, t_2 be as above.*

(a) *$F_i(x)$ is the map corresponding to some LS chain.*
(b) *$\mu(F_i(x)) = \mu(x) - \alpha_i$ and $\delta(F_i(x), i) = \delta(x,i) - 1$.*
(c) *t_2 is the smallest value of t such that $\int_0^t \langle F_i(x)(s), \alpha_i^\vee \rangle \, ds = \delta(x,i) - 1$.*
(d) *t_1 is the largest value of $t < t_2$ such that $\int_0^t \langle F_i(x)(s), \alpha_i^\vee \rangle \, ds = \delta(x,i)$.*

Proof. (a) Let $\hat{x}(t) = \int_0^t \langle x(s), \alpha_i^\vee \rangle \, ds$. Bearing in mind (8.2), we need to show that

(i) $x(t_1) \leqslant_{t_1} s_i x(t_1^+)$ (assuming $t_1 > 0$),
(ii) $s_i x(t) \leqslant_t s_i x(t^+)$ for $t_1 < t < t_2$, and
(iii) $s_i x(t_2) \leqslant_{t_2} x(t_2^+)$ (assuming $t_2 < 1$).

For (i), note that having the minimum value of $\hat{x}(t)$ at $t = t_1 > 0$ forces $\langle x(t_1), \alpha_i^\vee \rangle \leqslant 0$ and $\langle x(t_1^+), \alpha_i^\vee \rangle \geqslant 0$. The fact that t_1 is the largest value with this property forces $\langle x(t_1^+), \alpha_i^\vee \rangle > 0$, and hence (i) follows from Lemma 8.1(a).

For (iii), it suffices to show that $s_i x(t_2) \leqslant_{t_2} x(t_2)$, or equivalently, $t_2\langle x(t_2), \alpha_i^\vee \rangle \in \mathbf{Z}^{>0}$. The integrality follows from (8.6) and the fact that $\hat{x}(t_2) = \delta(x,i) + 1 \in \mathbf{Z}$, and the positivity follows from t_2 being the smallest value of $t > t_1$ such that $\hat{x}(t) = \delta(x,i) + 1$.

For (ii), we claim that $\langle x(t), \alpha_i^\vee \rangle > 0$ for $t_1 < t \leqslant t_2$. Since the previous argument shows that $\langle x(t_2), \alpha_i^\vee \rangle > 0$, this could fail only if $\langle x(t), \alpha_i^\vee \rangle \leqslant 0$ and $\langle x(t^+), \alpha_i^\vee \rangle > 0$ for some $t \in (t_1, t_2)$. In that case, (8.7) implies $\hat{x}(t) \in \mathbf{Z}$. However, the definitions of t_1 and t_2 force $\delta(x,i) < \hat{x}(t) < \delta(x,i) + 1$ for $t_1 < t < t_2$, so this is impossible. Given the claim, (ii) now follows from Lemma 8.1(b).

(b)–(d) Let $y = F_i(x)$. We have

$$\mu(x) - \mu(y) = \int_{t_1}^{t_2} (x(t) - s_i x(t)) \, dt = \int_{t_1}^{t_2} \langle x(t), \alpha_i^\vee \rangle \alpha_i \, dt = \alpha_i,$$

yielding the first part of (b). Since $x(t) = y(t)$ for $t \leqslant t_1$, we have $\hat{y}(t) \geqslant \delta(x,i)$ for all such t, with equality at $t = t_1$. Since $\langle y(t), \alpha_i^\vee \rangle = -\langle x(t), \alpha_i^\vee \rangle < 0$ for $t_1 < t \leqslant t_2$, it follows that $\hat{y}(t)$ strictly decreases from t_1 to t_2 by the amount $\hat{x}(t_2) - \hat{x}(t_1) = 1$, so $\hat{y}(t) \geqslant \delta(x,i) - 1$ for $t \leqslant t_2$, with equality if and only if $t = t_2$. Finally, we have $y(t) = x(t)$ for $t > t_2$, so $\hat{y}(t) = \hat{x}(t) - \hat{x}(t_2) + \hat{y}(t_2) = \hat{x}(t) - 2$ for such t, and to complete the proof of (b), (c) and (d), it suffices to show that $\hat{x}(t) \geqslant \delta(x,i) + 1$ for $t > t_2$. If this were false, there would be a local minimum for $\hat{x}(t)$ in this region strictly below $\delta(x,i) + 1$. However, (8.7) shows that such a minimum would have to be integer-valued, and hence equal to the global minimum $\delta(x,i)$, contradicting the definition of t_1. □

The following result is essentially due to Littelmann [**L1**], the main difference being that we have formulated it in the setting of admissible systems.

THEOREM 8.3. *For all $\lambda \in \Lambda^+$, the set of LS chains in the W-orbit of λ is an admissible system with generating series $\chi(\lambda)$.*

Proof. Let x be the map corresponding to some LS chain $\mu_0 <_{b_1} \mu_1 <_{b_2} \cdots <_{b_l} \mu_l$.

To verify (A0), note that if $\nu <_b \mu$, then every Bruhat chain from ν to μ must have length $\leqslant \mathrm{ht}(b(\mu - \nu))$ (reduce to the case of a covering pair). Bearing in mind $\mu_l \leqslant_1 \lambda$, it follows via (8.3) that if $\mu(x)$ has height $> h$, then every Bruhat chain from μ_0 to λ must have length $< \mathrm{ht}(\lambda) - h$. Thus for a given h, there are at most finitely many choices for l and $\mu_0, \ldots, \mu_l$. Moreover, given $\nu < \mu$, there are only finitely many rationals $0 < b < 1$ such that $\nu <_b \mu$, so (A0) holds.

Axiom (A1) is immediate from the discussion surrounding (8.4) and (8.5).

To prove that F_i is bijective as in (A2), we construct the inverse E_i explicitly. Given a map x such that $\delta(x,i) < 0$, we define

$$E_i(x)(t) = \begin{cases} s_i x(t) & \text{if } t_1 < t \leqslant t_2, \\ x(t) & \text{otherwise,} \end{cases}$$

where t_2 is the smallest value of t such that equality occurs in (8.4) and (8.5), and t_1 is the largest value of $t < t_2$ such that $\int_0^t \langle x(s), \alpha_i^\vee \rangle \, ds = \delta(x,i) + 1$. If we define the dual of x to be the map x^* corresponding to the LS chain

$$-\mu_l <_{a_l} \cdots <_{a_2} -\mu_1 <_{a_1} -\mu_0 \qquad (a_i = 1 - b_i),$$

then $x^*(t) = -x(1-t)$ (aside from a set of measure zero), and $E_i(x^*) = F_i(x)^*$. Hence $E_i(x)$ is the map corresponding to an LS chain (Lemma 8.2), and parts (c) and (d) of the lemma and its dual version show that E_i and F_i are inverses.

Lemma 8.2(b) proves (A3).

To construct a coherent timing pattern, let x be a map such that $\delta(x,i) < 0$ and define $t(x,i)$ to be the least value of t such that equality occurs in (8.4) and (8.5). Assuming $F_i(x)$ is defined (i.e., $\varepsilon(x,i) > 0$), Lemma 8.2(c) shows that $t(x,i) < t(F_i(x), i)$. By iteration, it follows that $x'(t) = x(t)$ for all $x' \preccurlyeq_i x$ and $t \leqslant t(x,i)$. Therefore, given a map $y \succcurlyeq_j x$ for some $j \neq i$ such that $\delta(y,j) = \delta < 0$ and $t(y,j) \leqslant t(x,i)$, we have $y(t) = x(t) = F_i(x)(t)$ for $t \leqslant t(y,j)$, and therefore $\delta(F_i(x), j) \leqslant \delta$. Hence there is a map $y' \succcurlyeq_j F_i(x)$ such that $\delta(y', j) = \delta$, and we claim that $t(y', j) = t(y, j)$. Indeed, we must have $y'(t) = F_i(x)(t) = x(t) = y(t)$ for $t \leqslant \min(t(y',j), t(y,j))$, so a discrepancy between $t(y',j)$ and $t(y,j)$ would contradict their definitions. Similar reasoning proves conversely that if there is a map $y' \succcurlyeq_j F_i(x)$ such that $\delta(y',j) = \delta < 0$ and $t(y',j) \leqslant t(x,i)$, then there is a map $y \succcurlyeq_j x$ such that $\delta(y,j) = \delta$ and $t(y,j) = t(y',j)$, so (A4) holds.

Thus we have an admissible system.

Finally, note that if the initial term μ_0 of an LS chain is not dominant; say, $\langle \mu_0, \alpha_i^\vee \rangle < 0$, then (8.4) implies $\delta(x,i) < 0$. Hence the only maximal object in this system is the singleton chain of weight λ, and the generating series must be $\chi(\lambda)$ (Theorem 2.4). □

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