ON LOCALLY CONVEX HYPERSURFACES WITH BOUNDARY

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ABSTRACT.   In this paper we give some geometric characterizations of locally convex hypersurfaces. In particular, we prove that for a given locally convex hypersurface $\mathcal{M}$ with boundary, there exists $r > 0$ depending only on the diameter of $\mathcal{M}$ and the principal curvatures of $\mathcal{M}$ on its boundary, such that the r-neighbourhood of any given point on $\mathcal{M}$ is convex. As an application we prove an existence theorem for a Plateau problem for locally convex hypersurfaces of constant Gauss curvature.

1. Introduction

Among all hypersurfaces in the $(n + 1)$-dimensional Euclidean space, $\mathbf{R}^{n+1}$ ($n \geq 2$), the locally convex ones form a natural class, and those of constant Gauss curvature are of particular interest. A complete, locally convex hypersurface containing at least one strictly convex point is known to be convex, that is it lies on the boundary of a convex body [5,12]. Therefore it can be represented as a radial graph over the unit sphere $\mathbf{S}^n$. For locally convex hypersurfaces with boundary, the situation can be much more complicated. In this paper we give some geometric characterizations for locally convex hypersurfaces with boundary. Roughly speaking, we will prove that if a locally convex hypersurface $\mathcal{M}$ behaves nicely near its boundary, so does it globally. A typical result is that if $\mathcal{M}$ has positive curvatures on its boundary, then there exists $r > 0$ such that the r-neighbourhood of any point on the hypersurface is convex. As an application we prove the existence of locally convex hypersurfaces of constant Gauss curvature, extending earlier work in [4, 7], as well as giving an affirmative answer to the conjecture in [13].

Definition 1. *A locally convex hypersurface $\mathcal{M}$ in $\mathbf{R}^{n+1}$ is an immersion of an n-dimensional oriented and connected manifold $\mathcal{N}$ (possibly with boundary) in $\mathbf{R}^{n+1}$, i.e., a mapping $T \colon \mathcal{N} \to \mathcal{M} \subset \mathbf{R}^{n+1}$, such that for any $p \in \mathcal{N}$, there exists a neighbourhood*

$\omega_p \subset \mathcal{N}$ such that: (i) T is a homeomorphism from ω_p to $T(\omega_p)$; (ii) $T(\omega_p)$ is a convex graph; (iii) the convexity of $T(\omega_p)$ agrees with the orientation.

The assumption (iii) is to rule out hypersurfaces such as $x_{n+1} = x_1 \max(|x_1| - 1, 0)$. This condition is not necessary if one considers complete, locally convex hypersurfaces containing at least one strictly convex point [5].

The above definition permits $\mathcal{M}$ to be nonsmooth. We say $\mathcal{M}$ is C^k smooth if T is locally a C^k diffeomorphism. We say $\mathcal{M}$ is locally strictly convex if the local graph in (ii) above is strictly convex. Our treatment in this paper is based on the following fundamental result for locally convex hypersurfaces for which we give a complete proof.

Main Lemma. *Let $\mathcal{M}$ be a compact, locally convex hypersurface. Suppose the boundary $\partial \mathcal{M}$ lies in the hyperplane $\{x_{n+1} = 0\}$. Then any connected component of $\mathcal{M} \cap \{x_{n+1} < 0\}$ is convex.*

A direct consequence is the above mentioned classical result that a complete, locally convex hypersurface with a strictly convex point is convex [5]. For applications to locally convex hypersurfaces with boundary we first need to clarify some notions used below.

Since $\mathcal{M}$ is immersed, a point $x \in \mathcal{M}$ may correspond to more than one point in $\mathcal{N}$ with x as their image (under the mapping T). For simplicity we agree with that when referring to a point $x \in \mathcal{M}$ we actually mean a point $p \in \mathcal{N}$ such that $x = T(p)$. Similarly we say $\omega_x \subset \mathcal{M}$ is a neighbourhood of x if it is the image of a neighbourhood in $\mathcal{N}$ of p. We say γ is a curve on $\mathcal{M}$ if it is the image of a curve on $\mathcal{N}$, and a set $E \subset \mathcal{M}$ is connected if it is the image of a connected set in $\mathcal{N}$, and so on. For a given point $x \in \mathcal{M}$, the r-neighbourhood of x, $\omega_r(x)$, is the connected component of $\mathcal{M} \cap B_r(x)$ containing the point x.

In the following we will suppose $\mathcal{N}$ is a connected, compact manifold with boundary, except otherwise specified. It follows that $\mathcal{M}$ is also compact, $\partial \mathcal{M} \neq \emptyset$, and for any $x \in \mathcal{M}$, there are finitely many points in $\mathcal{N}$ with x as their image under the mapping T.

We say $\mathcal{M}$ is *convex of reach r* if for any point $x \in \mathcal{M}$, $\omega_r(x)$ is convex. By definition, a locally convex hypersurface is convex of reach r for $r > 0$ sufficiently small. Let $\mathcal{C}_{x,\xi,r,\alpha}$ denote the cone with vertex x, axis ξ, radius r, and aperture α, that is,

$$\mathcal{C}_{x,\xi,r,\alpha} = \{y \in \mathbf{R}^{n+1} \mid |y - x| < r, \ \langle y - x, \xi \rangle \geq \cos \alpha \, |y - x|\}.$$

By definition, $\mathcal{M}$ satisfies a cone condition for sufficiently small r. That is for any point $x \in \mathcal{M}$ (corresponding to a point $p \in \mathcal{N}$), there exists a cone $\mathcal{C}_{x,\xi,r,\alpha}$ lying on the concave side of $\omega_r(x)$, i.e. the cone and $\omega_r(x)$ lie on the same side of any tangent hyperplane of

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$\mathcal{M}$ at x and $\mathcal{C}_{x,\xi,r,\alpha} \cap \omega_r(x) = \{x\}$. This cone will be called *inner contact cone* of $\mathcal{M}$ at x. We say $\mathcal{M}$ satisfies the *uniform cone condition* with radius r and aperture α if $\mathcal{M}$ has an inner contact cone at all points with the same r and α. We have

Theorem A. *Let $\mathcal{M} \subset B_R(0)$ be a locally convex hypersurface with C^2 boundary $\partial\mathcal{M}$. Then there exist $r, \alpha > 0$ depending only on n,R, $\partial\mathcal{M}$, and the upper and lower bounds of the principal curvatures of $\mathcal{M}$ on $\partial\mathcal{M}$ such that $\mathcal{M}$ is convex of reach r, and satisfies the uniform cone condition with radius r and aperture α.*

Note that we do not require any regularity condition on $\mathcal{M}$ except that the curvatures of $\mathcal{M}$ are well defined on the boundary $\partial\mathcal{M}$. The positive curvature condition on $\partial\mathcal{M}$ can be replaced by a strict convexity condition near $\partial\mathcal{M}$.

Theorem A enables us to treat locally convex hypersurfaces as graphs. Indeed for any $x \in \mathcal{M}$, since $\mathcal{M}$ is convex of reach r, the r-neighbourhood of x can be represented as a radial graph over a domain in $\mathbf{S}^n$, and the uniform cone condition prevents the graph from collapse. As we pointed out that Theorem A holds automatically for $r > 0$ sufficiently small. The main point of Theorem A is that r depends only on the boundary behaviour of $\mathcal{M}$. Therefore Theorem A holds with the same r and α for a family of locally convex hypersurfaces. More precisely, if one deforms $\mathcal{M}$ in the local convexity category, the resulting hypersurface is convex of reach r and satisfies the uniform cone condition with radius r and aperture α (see Theorem 4.1).

Theorem A finds applications when locally convex hypersurfaces are involved, such as problems of prescribing Gauss curvature or harmonic curvature, or the immersion in $\mathbf{R}^3$ of the unit disc of positive curvature. Theorem A is also useful in affine geometry where a prime object is the study of locally convex hypersurfaces of which the affine metric is positive definite, such as the affine Plateau problem, proposed by Chern and Calabi, which we plan to address in a future work. Using the Main Lemma above we proved in [16] that an affine complete locally convex hypersurface is also Euclidean complete, which implies, by virtue of our solution of the affine Bernstein problem in [17], that an affine complete, affine maximal surface in $\mathbf{R}^3$ is an elliptic paraboloid. This latter result also improved Calabi's results on the affine Bernstein problem.

In this paper we investigate the existence of hypersurfaces of prescribed Gauss curvature. This problem leads to a fully nonlinear equation of Monge-Ampere type, which is elliptic when the hypersurface is locally convex. A basic question is the existence of locally convex hypersurfaces of positive constant Gauss curvature K (briefly K-*hypersurfaces*), with prescribed boundary Γ, where Γ is a smooth disjoint finite collection of closed codimension 2 submanifolds in $\mathbf{R}^{n+1}$. This problem was studied in [4,6,13]. In [13] Spruck

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made the conjecture that *if Γ bounds a strictly locally convex hypersurface $\mathcal{M}_0$ with Gauss curvature $K(\mathcal{M}_0) > K_0 > 0$, then Γ bounds a K_0-hypersurface.* If $\mathcal{M}_0$ can be represented as a radial graph over a domain $\Omega \subset \mathbf{S}^n$ such that Ω does not contain any hemisphere, then the existence of a K_0-hypersurface is proven in [4]. In this paper we prove the conjecture holds in its full generality.

Theorem B. *Let Γ be a smooth disjoint finite collection of closed codimension 2 submanifolds in $\mathbf{R}^{n+1}$. Suppose Γ bounds a locally strictly convex hypersurface $\mathcal{M}_0$ with Gauss curvature $K(\mathcal{M}_0) \geq K_0 > 0$. Then Γ bounds a K_0-hypersurface.*

The existence and regularity of convex hypersurfaces with prescribed Gauss curvature have been well investigated for closed hypersurfaces or those which can be represented as a graph over a convex domain [2,3,8,10,14,15,18,19]. If the domain is nonconvex, a necessary and also natural condition is the existence of a subsolution [4,6]. In the locally convex setting this condition is equivalent to the one in Theorem B, that is Γ bounds a locally strictly convex hypersurface. As for when Γ can bound a locally convex hypersurface is a delicate question, there are geometric and topological obstructions. The smallness assumption of the Gauss curvature K_0 is also necessary. Unlike the mean curvature case where one can give an upper bound of the mean curvature in terms of the magnitude of the boundary, an upper bound for the Gauss curvature depends also on the geometric structure of the boundary. We refer the reader to [4,7,11] for discussions on these questions. The solution in Theorem B is usually not unique, as is easily seen by using a plane to cut a sphere.

Theorem B cannot be reduced to the Dirichlet problem since a locally convex hypersurface cannot be represented as a graph in general. To prove Theorem B we will use the well known Perron method, since by Theorem A, we can treat locally convex hypersurfaces as graphs. By the Perron liftings we obtain a sequence of "monotone", locally convex hypersurfaces which converges to a K-hypersurface.

A more general problem is the existence of Weingarten hypersurfaces with prescribed boundary, such as the existence of locally convex hypersurfaces with prescribed harmonic curvature, which was treated in the papers [7, 9].

This paper is organized as follow. In Section 2 we introduce the generalized Gauss mapping for locally convex hypersurfaces and discuss some basic properties of the mapping. In Section 3 we prove the Main Lemma. Using a moving hyperplane to cut off a connected piece from a locally convex hypersurface, we prove this connected piece is convex if it contains no boundary point. In Sections 4 and 5 we then prove Theorems A and B, respectively.

2. The Gauss mapping

For a locally convex hypersurface $\mathcal{M}$, not necessarily smooth, we introduce the generalized *Gauss mapping* $G : \mathcal{M} \to \mathbf{S}^n$. Strictly speaking, the Gauss mapping is defined on $\mathcal{N}$. Indeed, for any interior point $p \in \mathcal{N}$ there is a neighbourhood $\omega_p \subset \mathcal{N}$ such that $T(\omega_p)$ is a convex graph. A hyperplane $\mathcal{L}$ is a *tangent hyperplane* of $\mathcal{M}$ at $x = T(p)$ if $\mathcal{L}$ passes through x and $T(\omega_p)$ lies on one side of $\mathcal{L}$. The Gauss mapping G of $\mathcal{N}$ at p is the set of normals (on the convex side of $\mathcal{M}$) of such tangent hyperplanes. However for convenience we refer to a Gauss mapping defined on $\mathcal{M}$ when no confusion arises. For any interior point $x \in \mathcal{M}$, there is a neighbourhood $\omega_x \subset \mathcal{M}$, which can be represented as a graph of a convex function by Definition 1. The image of the Gauss mapping at x, $G(x)$ (with respect to $p \in \mathcal{N}$, where $T(p) = x$), is the set of normals of tangent hyperplanes of ω_x at x. A vector $\nu \in G(x)$ is called a normal of $\mathcal{M}$ at x. For a C^1 smooth convex hypersurface the generalized Gauss mapping coincides with the Gauss mapping in the classical sense.

First we give some simple properties of the Gauss mapping.

(i) If ν_k is a normal of $\mathcal{M}$ at x_k, $k = 1, 2, \cdots$, such that $\nu_k \to \nu$ and $x_k \to x$, then ν is a normal of $\mathcal{M}$ at x.

(ii) For any point $x \in \mathcal{M}$, $G(x)$ is a closed, convex set strictly contained in a hemisphere.

(iii) For any point $x \in \mathcal{M}$, there is a neighbourhood ω_x such that $G(\omega_x) = \bigcup_{z \in \omega_x} G(z)$ is strictly contained in a hemisphere.

Properties (i)-(iii) follow immediately since near x, $\mathcal{M}$ can be represented as a convex graph. In particular we see that $G(x)$ is strictly contained in the southern hemisphere if and only if locally $\mathcal{M}$ can be represented as

$$x_{n+1} = g(x_1, \cdots, x_n) \tag{2.1}$$

for a convex function g. It is easily seen that (2.1) holds if and only if $\mathcal{M}$ satisfies the cone condition at x with e_{n+1}, the positive x_{n+1}-axis direction, is the axial direction of the cone.

If z is a boundary point of $\mathcal{M}$ and if $\partial \mathcal{M}$ is C^1 smooth, we introduce a unique normal of $\mathcal{M}$ at z as follows. Namely we take z as the origin and choosing the coordinates properly such that locally $\mathcal{M}$ is represented by (2.1) for a convex function g, and the x_i-axes, $i = 1, \cdots, n-1$, are tangent to $\partial \mathcal{M}$ at z. Since g is convex, $\partial_{x_n} g(0, \cdots, 0, x_n)$ is monotone and well defined a.e. as a function of x_n. Hence $\alpha = \lim_{x_n \to 0} \partial_{x_n} g(0, \cdots, 0, x_n)$

exists. By choosing the coordinates properly we may also suppose $\alpha = 0$ such that $g(0, \cdots, 0, x_n) = o(x_n)$ as $x_n \to 0$. Since the x_i-axes, $i = 1, \cdots, n-1$, are tangent to $\partial \mathcal{M}$ at z, we have, for $x = (x_1, \cdots, x_{n+1}) \in \partial \mathcal{M}$,

$$|g(x_1, \cdots, x_n)| = o(r) \quad \text{as} \quad r = (\sum_{i=1}^{n} x_i^2)^{1/2} \to 0. \qquad (2.2)$$

By the convexity of g, we see that (2.2) holds for $x \in \mathcal{M}$ near the origin. It is easily seen that the x_{n+1}-axis is uniquely determined by (2.2). We define the negative x_{n+1}-direction as the normal of $\mathcal{M}$ at z.

Lemma 2.1. *Let $\mathcal{M}$ be a locally convex hypersurface in $\mathbf{R}^{n+1}$. Suppose the origin $\mathcal{O} \in \mathcal{M}$ and the vectors $\pm e_{n+1}$ are not normals of $\mathcal{M}$ at $\mathcal{O}$. Let Γ be the connected component of $\mathcal{M} \cap \{x_{n+1} = 0\}$ containing $\mathcal{O}$. If $\Gamma \cap \partial \mathcal{M} = \emptyset$, then the vectors $\pm e_{n+1}$ are not normals of $\mathcal{M}$ at any points on Γ, and Γ is a locally convex $(n-1)$-dimensional hypersurface in the hyperplane $\{x_{n+1} = 0\}$.*

Proof. First we prove that the vectors $\pm e_{n+1}$ are not normals of $\mathcal{M}$ at any point on Γ. We argue by contradiction. Let E denote the (closed) subset of Γ such that $x \in \Gamma$ if and only if either e_{n+1} or $-e_{n+1}$ is a normal of $\mathcal{M}$ at x. Since $\pm e_{n+1}$ are not normals of $\mathcal{M}$ at the origin, they are not normals of $\mathcal{M}$ at any point nearby. Let z be a point in E and let ℓ be a path in Γ connecting z and the origin. We may suppose $\ell \cap E = \{z\}$, otherwise we may replace z by a point $z' \in \ell \cap E$ such that E has no other point on ℓ between $\mathcal{O}$ and z'. Suppose $-e_{n+1}$ is a normal of $\mathcal{M}$ at z. Then $\{x_{n+1} = 0\}$ is a tangent hyperplane of $\mathcal{M}$ at z and locally $\mathcal{M}$ stays on the side $x_{n+1} \geq 0$. On the other hand, since between $\mathcal{O}$ and z there is no other point of E lying on ℓ, $\mathcal{M}$ intersects with $\{x_{n+1} = 0\}$ at any point in $\ell - \{z\}$. Hence $\mathcal{M}$ can not lie on one side of $\{x_{n+1} = 0\}$ in any neighbourhood of z. We reach a contradiction.

For any point $z \in \Gamma$, by Definition 1 there is a neighbourhood ω_z of z in $\mathcal{M}$ which can be represented as a convex graph. Since $\pm e_{n+1}$ are not normals of $\mathcal{M}$ at z, $\omega_z \cap \{x_{n+1} = 0\}$ is a convex graph as a hypersurface in $\mathbf{R}^n = \{x_{n+1} = 0\}$. Let $\mathcal{N}' = T^{-1}(\Gamma)$. Since T is locally a homeomorphism, $\mathcal{N}'$ is an $(n-1)$-dimensional, connected manifold (not necessarily smooth). It is easy to see that the triple $(\Gamma, \mathcal{N}', T)$ satisfies the conditions in Definition 1. Hence Γ is locally convex. $\square$

For a locally convex hypersurface $\mathcal{M}$, next we introduce a continuous vector field γ_x ($x \in \mathcal{M}$) on $\mathcal{M}$, which will be used in Section 5. If $\mathcal{M}$ is smooth, the normal ν is a continuous vector field on $\mathcal{M}$, and for any $x \in \mathcal{M}$, the line segment γ_x connecting x to

$x - t\nu(x)$, where $t > 0$ is a small constant, gives a mutually disjoint, continuous vector field on $\mathcal{M}$. If $\mathcal{M}$ is nonsmooth, we introduce a continuous vector field γ_x on $\mathcal{M}$ as follows.

Let $r, \alpha > 0$ small such that $\mathcal{M}$ is convex of reach r and satisfies the uniform cone condition with radius r and aperture α. Then for any point $z \in \mathcal{M}$, the ball $B_{r'}(z')$ with radius $r' = \frac{1}{2}r \sin \alpha$ and centre $z' = z + \frac{1}{2}r\xi$ is contained in the inner contact cone $\mathcal{C}_{z,\xi,r,\alpha}$. Therefore one can choose the coordinates properly such that locally near z, $\mathcal{M}$ is represented as a graph by (2.1) for a convex function g such that $|Dg| < \infty$ in B'_σ, the ball in $\mathbf{R}^n$, $(= \{x_{n+1} = 0\})$, of radius σ, where $\sigma \leq \frac{1}{2}r \sin \alpha$.

Let $F_{z,\sigma}$ denote the graph of g. If $F_{z,\sigma}$ contains no boundary point of $\mathcal{M}$, then g is well defined in B'_σ. If $F_{z,\sigma}$ contains boundary points of $\mathcal{M}$, then g is defined in a subdomain B'' of B'_σ, where B'' is the projection of $F_{z,\sigma}$ in $\{x_{n+1} = 0\}$. In both cases let B'' denote the domain of definition of g. Let $D_z = \{(x_1, \cdots, x_{n+1}) \mid x_{n+1} > g(x'), \ x' \in B''\}$, where $x' = (x_1, \cdots, x_n)$. D_z is a convex domain in $\mathbf{R}^{n+1}$. Let $D_{z,t} = \{x \in D_z \mid \text{dist}(x, \partial D_z) > t\}$, where $t < \frac{1}{4}\sigma$. Since $D_{z,t}$ is convex, there is a unique point $y \in \partial D_{z,t}$ such that

$$|y - z| = \inf\{|x - z| \mid x \in \partial D_{z,t}\}. \tag{2.3}$$

We define a mapping ψ from $\mathcal{M}$ to $\mathbf{R}^{n+1}$ such that $\psi(z) = y$. Let γ_z denote the line segment connecting z to $\psi(z)$, closed at z and open at $\psi(z)$ (namely γ_z contains the endpoint z but not the endpoint $\psi(z)$). Then γ_z is a vector field defined on $\mathcal{M}$.

For fixed $\sigma > 0$ and $t < \frac{1}{4}\sigma$, observe that for any $z' \in \omega_{\sigma/2}(z)$, the $\frac{1}{2}\sigma$-neighbourhood of z in $\mathcal{M}$, we have $\psi(z') \in \partial D_{z,t}$ and

$$|\psi(z') - z'| = \inf\{|x - z'| \mid x \in \partial D_{z,t}\}.$$

Therefore by the convexity of $D_{z,t}$, we see that γ_z and $\gamma_{z'}$ are disjoint when $z' \in \omega_{\sigma/2}(z)$, and the direction of γ_z depends continuously on z. Therefore we have.

Lemma 2.2. *Suppose $\mathcal{M}$ is convex of reach r and satisfies the uniform cone condition with radius r and aperture α. Then for any $t \in (0, \frac{1}{8}r \sin \alpha)$, there is a continuous vector field γ_x on $\mathcal{M}$, with $|\gamma_x| = t$ such that γ_x and $\gamma_{x'}$ are disjoint for $x' \in \omega_{2t}(x)$, where $|\gamma_x|$ is the length of the line segment γ_x.*

3. Proof of the Main Lemma

We will use the moving plane method to prove the Main Lemma. We need the concept of *extreme point*. For a convex set $E \subset \mathbf{R}^k$, a boundary point $x \in \partial E$ is an extreme point of E if there exists a tangent hyperplane $\mathcal{L} \subset \mathbf{R}^k$ such that $\mathcal{L} \cap \overline{E} = \{x\}$. Obviously any bounded convex set has extreme points. It is well known that any interior point in E can be represented as a linear combination of extreme points of E. Moreover, if F is a bounded, closed set and E the convex closure of F, that is $E = \cap\{D \mid D \supset F, D \text{ is convex}\}$, then an extreme point of E is a point in F.

Let $\mathcal{M}$ be a locally convex hypersurface whose boundary $\partial \mathcal{M} \subset \{x_{n+1} = \tau^*\}$ for some $\tau^* > 0$. First we suppose $\mathcal{M} \subset \{x_{n+1} \geq 0\}$, the origin $\mathcal{O} \in \mathcal{M}$, $-e_{n+1}$ is a normal of $\mathcal{M}$ at $\mathcal{O}$ with $\mathcal{M}$ strict convex at $\mathcal{O}$. Then $\mathcal{L} = \{x_{n+1} = 0\}$ is a tangent hyperplane of $\mathcal{M}$ at the origin $\mathcal{O}$.

For any $t \in [0, \tau^*)$, let Λ_t denote the connected component of $\{0 \leq x_{n+1} \leq t\} \cap \mathcal{M}$ containing the origin $\mathcal{O}$. Obviously Λ_t is a closed set, and $\Lambda_t \subset \Lambda_{\bar{t}}$ for any $0 < t < \bar{t}$. If $\Lambda_{\bar{t}}$ is convex, we have $\Lambda_t = \Lambda_{\bar{t}} \cap \{x_{n+1} \leq t\}$ for any $0 < t < \bar{t}$. Let D_t be the convex closure of Λ_t. We have $D_t \subset \{0 \leq x_{n+1} \leq t\}$.

The proof of the Main Lemma consists of two steps. First we prove that Λ_t is convex if the north pole e_{n+1} is not in $G(\Lambda_t)$ (Lemma 3.1). We then prove $\mathcal{N}$ is homeomorphic to the unit sphere if e_{n+1} is a normal of $\mathcal{M}$ at some point on $\Lambda_t \cap \{x_{n+1} = t\}$ (Lemmas 3.2 and 3.3), which in turn implies e_{n+1} is not in $G(\Lambda_t)$ for any $t < \tau^*$, by our assumption that $\partial N \neq \emptyset$.

Lemma 3.1. *Suppose $G(\Lambda_t)$ has a positive distance from the north pole of $\mathbf{S}^n$. Then Λ_t is convex, namely $\Lambda_t \subset \partial D_t$.*

Proof. First we show that the south pole of $\mathbf{S}^n$ is not a normal of $\mathcal{M}$ at any point in $\Lambda_t - \{\mathcal{O}\}$. Suppose to the contrary that $-e_{n+1}$ is a normal of $\mathcal{M}$ at some point $x^* \in \Lambda_t - \{\mathcal{O}\}$. Suppose $x^* \in \{x_{n+1} = s\}$ for some $0 < s \leq t$. By our construction of Λ_s there is a curve $\gamma \subset \{0 \leq x_{n+1} \leq s\}$ connecting x^* to the origin $\mathcal{O}$. Let γ' denote the segment of $\gamma \cap \{x_{n+1} = s\}$ containing x^*. By Lemma 2.1, $-e_{n+1}$ is a normal of $\mathcal{M}$ at any point in γ'. Since $\mathcal{M}$ is locally convex, there exists a neighbourhood $\omega \subset \mathcal{M}$ of x^* such that ω lies above the tangent plane $\{x_{n+1} = s\}$. But since $\gamma \subset \{x_{n+1} \leq s\}$, we see that there is a segment of γ near x^* contained in $\{x_{n+1} = s\}$. It follows that γ' is open as a subset of γ. Obviously it is also closed. Hence the whole curve γ lies in the hyperplane $\{x_{n+1} = s\}$. This is a contradiction.

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By assumption, $\mathcal{M}$ is strictly convex at the origin. Hence Λ_s is convex for $s \geq 0$ small. Suppose $\overline{s} \leq t$ is the largest constant such that Λ_s is convex for any $s \in (0, \overline{s})$. Then $\Lambda_{\overline{s}}$ is also convex. We want to prove that $\Lambda_{\overline{s}+\varepsilon}$ is convex for sufficiently small $\varepsilon > 0$ if $\overline{s} < t$.

Let $\Gamma_s = \Lambda_t \cap \mathcal{L}_s$, where $\mathcal{L}_s = \{x_{n+1} = s\}$. By Lemma 2.1, Γ_s is closed, locally convex for any $0 < s \leq t$. Since Γ_s is convex for $s \in (0, \overline{s}]$, it is topologically a sphere for $s \in (0, \overline{s}]$, namely it is homeomorphic to the sphere S^{n-1}. It follows that $\Gamma_{\overline{s}}$, and also $\Gamma_{\overline{s}+\varepsilon}$ for $\varepsilon > 0$ small, can be represented as a radial graph. Hence $\Gamma_{\overline{s}+\varepsilon}$ is topologically a sphere. Let Ω_s be the convex closure of Γ_s. By the convexity of Γ_s for $s \leq \overline{s}$, we have $\partial \Omega_s = \Gamma_s$ for $s \leq \overline{s}$. Hence $\Gamma_{\overline{s}+\varepsilon}$ is in an ε'-neighbourhood of $\partial \Omega_{\overline{s}}$ with $\varepsilon' \to 0$ as $\varepsilon \to 0$. Note that Ω_s has nonempty interior since Γ_s is locally convex.

We first prove $\partial \Omega_{\overline{s}+\varepsilon} = \Gamma_{\overline{s}+\varepsilon}$ for $\varepsilon > 0$ sufficiently small (regarding Ω_s as a set in $\mathbf{R}^n$ ($\{x_{n+1} = s\}$)). Let $E = \partial \Omega_{\overline{s}+\varepsilon} - \Gamma_{\overline{s}+\varepsilon}$. Note that if z is an extreme point of $\Omega_{\overline{s}+\varepsilon}$, then $z \in \Gamma_{\overline{s}+\varepsilon}$. Hence E consists of line segments. Let γ be a line segment in E. Choosing the coordinates properly we suppose $\gamma \subset \{x_n = 0\}$ and $\partial \Omega_{\overline{s}+\varepsilon} \subset \{x_n \geq 0\}$. Then $\gamma \subset F =: \{x_n = 0\} \cap \partial \Omega_{\overline{s}+\varepsilon}$. It is easy to see that F is convex and any extreme points of F belong to $\Gamma_{\overline{s}+\varepsilon}$. Therefore there is a line segment, which we suppose is exactly γ, with both endpoints lying in $\Gamma_{\overline{s}+\varepsilon}$.

For any point $z \in \gamma$, let $z' \in \Gamma_{\overline{s}+\varepsilon}$ satisfy $|z' - z| = \inf\{|y - z| \mid p \in \Gamma_{\overline{s}+\varepsilon}\}$. By the definition of locally convex hypersurface, there is a $\delta > 0$ such that the δ-neighbourhood of any point in $\mathcal{M}$ is convex. Hence z' is unique if ε is sufficiently small. At the endpoints of γ we have $|z - z'| = 0$. By the local convexity of $\mathcal{M}$ it is easy to see that $|z - z'|$ can not attain a strict maximum at interior points of γ. Therefore $\gamma \subset \Gamma_{\overline{s}+\varepsilon}$; a contradiction.

We have proved that if $\Gamma_{\overline{s}}$ is convex, so is $\Gamma_{\overline{s}+\varepsilon}$ for sufficiently small $\varepsilon > 0$. It follows that Γ_s is convex for any $s \in (0, t]$ and Ω_s is the convex set enclosed by Γ_s.

To prove that Λ_t is convex, we first observe that for any $s \in (0, t]$, $\partial D_s \cap \{x_{n+1} = s\} = \Omega_s$. Indeed, since Γ_s is convex and $G(\Lambda_s)$ has a positive distance from the north pole, we see that for any point $z \in \Gamma_s$, there is a hyperplane $\{x_{n+1} = a \cdot x + a_0\}$ for some $a \in \mathbf{R}^n$, $|a| > 0$ small, such that z belongs to the hyperplane and D_s lies on the lower side of the hyperplane.

Let $E_s = \partial D_s - (\Lambda_s \cup \Omega_s)$. Then $E_s \subset \{x_{n+1} < s\}$. Since Λ_s is convex for $s > 0$ small, E_s is empty for $s > 0$ small. Let $\overline{s} \leq t$ be the largest number such that Λ_s is convex for all $s \in (0, \overline{s}]$. If $\overline{s} < t$ we want to prove $\Lambda_{\overline{s}+\varepsilon}$ is convex for sufficiently small $\varepsilon > 0$, namely $E_{\overline{s}+\varepsilon}$ is an empty set for sufficiently small ε. If this is not true, then similar to the above, $E_{\overline{s}+\varepsilon}$ consists of line segments. Let γ be a line segment in $E_{\overline{s}+\varepsilon}$.

As above we may suppose both endpoints of γ belong to $\Lambda_{\bar{s}+\varepsilon}$. For any $z \in \gamma$, let z' be the point in $\Lambda_{\bar{s}+\varepsilon}$ closest to z. Then by the local convexity of $\mathcal{M}$, $|z - z'|$ cannot attain a strict maximum at interior points of γ. But since $|z - z'| = 0$ at the endpoints, we have $\gamma \subset \Lambda_{\bar{s}+\varepsilon}$, a contradiction. Hence $\Lambda_{\bar{s}+\varepsilon}$ is convex. This completes the proof. $\square$

Since $\mathcal{M}$ is strictly convex at the origin, $G(\Lambda_t)$ has a positive distance from the north pole for $t \geq 0$ small. Suppose at height $t > 0$, $G(\Lambda_t)$ has a positive distance from the north pole. Then by (i) in Section 2, $G(\Lambda_{t+\varepsilon})$ has a positive distance from the north pole for $\varepsilon > 0$ small. Hence by Lemma 3.1, $\Lambda_{t+\varepsilon}$ is convex. That is we can move the hyperplane $\mathcal{L}_t = \{x_{n+1} = t\}$ further upward to height $\bar{t} > t$ such that $G(\Lambda_{\bar{t}})$ has a positive distance from the north pole and $\Lambda_{\bar{t}} \cap \partial \mathcal{M}$ is empty. Therefore we can move the hyperplane $\mathcal{L}$ up to level $\tau \leq \tau^*$ such that for any $t < \tau$, Λ_t is convex, $\Lambda_t \subset \partial D_t$, and $G(\Lambda_t)$ has a positive distance from the north pole. We want to prove $\tau = \tau^*$.

If $\tau < \tau^*$, let $\Lambda' = \cup_{t<\tau} \Lambda_t$ and Λ the closure of Λ'. We claim that there is a point $z \in \partial \Lambda$ ($z \in \{x_{n+1} = \tau\}$) such that e_{n+1} is a normal of $\mathcal{M}$ at z. For if not, then $G(\Lambda)$ has a positive distance from the north pole, and so also $G(\Lambda_{\tau+\varepsilon})$ for some $\varepsilon > 0$ small, a contradiction with our definition of τ.

To proceed further we need to examine the set A_t, the connected component of $T^{-1}(\Lambda_t)$ containing p_0, where $p_0 \in \mathcal{N}$ is such that $T(p_0) = \{\mathcal{O}\}$. We want to prove A_t is topologically an n-ball for $t < \tau$, namely A_t is homeomorphic to an n-dimensional ball.

Lemma 3.2. *For any $t < \tau$, T is a homeomorphism from A_t to Λ_t.*

Proof. Since $\mathcal{M}$ is strictly convex at the origin $\mathcal{O}$, by the definition of locally convex hypersurface, Lemma 3.2 is obviously true for $t > 0$ small.

Suppose Lemma 3.2 is true for t, we show that it is true for $t+\varepsilon$ if $t+\varepsilon < \tau$ and $\varepsilon > 0$ is sufficiently small. Indeed, if it is not the case, then there exist sequences $p_k, q_k \in A_{t+\varepsilon_k}$, where $\varepsilon_k \to 0$, such that $T(p_k) = T(q_k) = x_k \in \Lambda_{t+\varepsilon_k}$. Since Λ_t is compact, we may suppose $x_k \to x_0 \in \Lambda_t$. Suppose $p_k \to p$ and $q_k \to q$, then $T(p) = T(q) = x_0$. By the assumption that Lemma 3.2 holds at t, we have $p = q$. However since T is locally a homeomorphism, we must have $p_k = q_k$ when k is large. Hence Lemma 3.2 holds for an open set $t \in (0, t^*)$.

Lemma 3.2 has been proven if $t^* \geq \tau$. If $t^* < \tau$ we show that Lemma 3.2 holds for $t = t^*$. If this is not true, then there exist two points $p, q \in A_{t^*}$ such that $T(p) = T(q) = x \in \Lambda_{t^*}$. By assumption that Lemma 3.2 holds for $t < t^*$, the points p, q cannot be both interior points of A_{t^*}. If p is an interior point and q is a boundary point of A_{t^*}, then

$x = T(p)$ is an interior point of Λ_{t^*} and $x = T(q)$ is simultaneously a boundary point of Λ_{t^*}. This is impossible. If both p and q are boundary points of A_{t^*}, we choose arbitrary two points $p_k, q_k \in A_{t_k}$, where $t_k \nearrow t^*$, such that $p_k \to p$ and $q_k \to q$. Then both $T(p_k)$ and $T(q_k)$ converge to x. Since Λ_{t^*} is convex, there are curves $\ell_k \subset \Lambda_{t^*}$ connecting $T(p_k)$ to $T(q_k)$ such that the arclength of ℓ_k converges to zero. In other words, both $T(p_k)$ and $T(q_k)$ are in the r-neighbourhood of x with $r \to 0$ as $k \to \infty$. By Definition 1, T is locally a homeomorphism. It follows $\mathrm{dist}(p_k, q_k) \to 0$, whence $p = q$. Hence Lemma 3.2 holds. $\square$

As above denote by D_t the convex closure of Λ_t. Let $D = \cup_{t < \tau} D_t$, $\Omega = \partial D - \Lambda' \subset \{x_{n+1} = \tau\}$. Let $\widetilde{\Lambda}$ be the connected component of $\mathcal{M} \cap \partial D$ containing Λ'. Then $\Lambda \subset \widetilde{\Lambda} \subset \Lambda_\tau$, $\widetilde{\Lambda} \subset \partial D$, and Ω is a closed convex set.

Let $A' = \cup_{t < \tau} A_t$, A the closure of A', and $\widetilde{A}$ be the connected component of $T^{-1}(\widetilde{\Lambda})$ containing A'. We want to prove that $\widetilde{\Lambda} = \partial D$, and T is a homeomorphism from $\widetilde{A}$ to ∂D. Since ∂D is a closed, convex hypersurface, it follows $\mathcal{N} = \widetilde{A}$ is a closed manifold. However this is in contradiction with the assumption that $\partial \mathcal{M} \neq \emptyset$. Hence we must have $\tau = \tau^*$.

Lemma 3.3. *We have $\widetilde{\Lambda} = \partial D$ and T is a homeomorphism from $\widetilde{A}$ to ∂D.*

Proof. We have shown that e_{n+1} is a normal of $\mathcal{M}$ at some point in $\partial \Lambda$. Since $\partial \Lambda$ contains no boundary point, by Lemma 2.1, e_{n+1} is a normal of $\mathcal{M}$ at any point $z \in \partial \Lambda$.

If $\Omega = \{z\}$ is a single point, then obviously we have $\widetilde{\Lambda} = \partial D$, and as in the proof of Lemma 3.2 we see that $T^{-1}(z)$ has only one point in $\widetilde{A}$ since T^{-1} is locally a homeomorphism. Hence $\widetilde{A}$ is homeomorphic to ∂D.

If Ω is a convex set, let z be an extreme point of Ω. Let $\mathcal{L} = \{\sum_{i=1}^n a_i x_i = b\}$ be a hyperplane parallel to the x_{n+1} axis, such that $z \in \mathcal{L}$ and $\Omega \subset \{\sum_{i=1}^n a_i x_i < b\}$. Let $C_{z,\xi,r,\alpha}$ be an inner contact cone of $\mathcal{M}$ at z. Let θ be the angle between ξ and e_{n+1}. Since e_{n+1} is a normal of $\mathcal{M}$ at z, we have $\theta < \pi/2$. Hence we can make a linear transformation

$$\widetilde{x}_i = \sum_{j=1}^{n+1} a_{ij} x_i \quad i = 1, \cdots, n,$$
$$\widetilde{x}_{n+1} = x_{n+1},$$

for some constants a_{ij}, such that e_{n+1} is the axial direction of the cone. Therefore we can suppose that near z, $\mathcal{M}$ is represented by $x_{n+1} = g(x_1, \cdots, x_n)$ for a concave function g such that $g \leq \tau$. Since $g = \tau$ on $\partial \Omega$, and Ω is strictly convex at z (i.e., z is an extreme

point of Ω), We have $g = \tau$ for $x \in \Omega$ near z. That is there exists $\delta > 0$ such that $g = \tau$ for $x \in \omega_1 =: \Omega \cap \{b - \delta < \sum_{i=1}^{n} a_i x_i \leq b\}$. Hence $\omega_1 \subset \mathcal{M}$ and $\Lambda_1 = \Lambda' \cup \omega_1$ is connected. Obviously Λ_1 is topologically an n-ball.

Let $\Omega_1 = \Omega - \omega_1$. Then Ω_1 is also a closed convex set. Let A_1 be the connected component of $T^{-1}(\Lambda_1)$ containing A'. We claim that A_1 is homeomorphic to Λ_1. Indeed, we have shown in Lemma 3.2 that A' is homeomorphic to Λ'. From the last paragraph we see that ω_1 is homeomorphic to $A_1 - A'$. Note that $\omega_1 \cap \Lambda' = \emptyset$. Hence T is 1-1 mapping from A_1 to Λ_1. By our construction, T is locally a homeomorphism on $\partial \Lambda' \cap \partial \omega_1$. Hence T is a homeomorphism on A_1.

We continue the above argument. Let z be an extreme point of Ω_1. Suppose $z \in \mathcal{L}$, where $\mathcal{L} = \{\sum_{i=1}^{n} a_i x_i = b\}$ for different a, b, such that $\Omega_1 \subset \{\sum_{i=1}^{n} a_i x_i < b\}$. As above there exists $\delta > 0$ such that $\omega_2 =: \Omega_1 \cap \{b - \delta < \sum_{i=1}^{n} a_i x_i \leq b\}$ is a piece of $\mathcal{M}$. Let $\Lambda_2 = \Lambda_1 \cup \omega_2$, and $\Omega_2 = \Omega_1 - \omega_2$. Then Ω_2 is closed convex set. If $\Omega_2 \neq \emptyset$, Λ_2 is topologically an n-ball. Let A_2 be the connected component of $T^{-1}(\Lambda_2)$ containing A_1. Similar to the above we see that Λ_2 is connected and T is a homeomorphism from A_2 to Λ_2.

We claim that there is an extreme point z of Ω_1 such that we can choose $\delta > \delta_0$ for some δ_0 depending only on $\mathcal{N}$ and the mapping T. Indeed, since Ω_1 is bounded, there is a ball $B'_R(0)$ in $\mathbf{R}^n$ ($= \{x_{n+1} = \tau\}$) such that $\Omega_1 \subset B'_R$. Let R be the smallest constant such that $\Omega_1 \cap \partial B'_R \neq \emptyset$ (note that Ω_1 is a closed convex set). Let $z \in \Omega_1 \cap \partial B_R$. By definition there is an $r > 0$ depending only on $\mathcal{N}$ and T such that the r-neighbourhood of z in $\mathcal{M}$ can be represented by $x_{n+1} = g(x_1, \cdots, x_n)$ for a concave function g such that $g \leq \tau$. Then it is easy to see that one can choose $\delta \geq \delta_0 =: \frac{1}{2}(R - \sqrt{R^2 - r^2})$ such that $g = \tau$ in ω_2.

We proceed further as above. At each step we choose an extreme point $z \in \Omega_k$ such that $\delta > \delta_0 / 2$. Therefore in finitely many steps we exhaust the set Ω. That is $\widetilde{\Lambda} = \partial D$, and ∂D is homeomorphic to $\widetilde{A}$. $\square$

The Main Lemma is thus proved if $\mathcal{M}$ is strictly convex at the origin. If $\mathcal{M}$ is not strictly convex at the origin, we suppose $\mathcal{M}$ lies above the graph $x_{n+1} = \alpha \sum x_i^2 + \beta$, where $\alpha > 0, \beta < 0$ are constants, α sufficiently small. We move the graph upwards until it touches a point $z \in \mathcal{M}$. Then $\mathcal{M}$ is strictly convex at z. Suppose $\mathcal{L} = \{x_{n+1} = \sum_{i=1}^{n} a_i x_i + a_0\}$ is a tangent plane of $\mathcal{M}$ at z. Then the above argument shows that for any $h \geq a_0$, the connected component of $\mathcal{M} \cap \{x_{n+1} < \sum a_i x_i + h\}$ containing z is convex as long as it does not contain boundary points. In particular this means that

the connected component of $\mathcal{M} \cap \{x_{n+1} < \tau^* - \delta\}$ containing z is convex for any $\delta > 0$, provided we choose $\alpha > 0$ sufficiently small such that $\sum a_i^2$ is also sufficiently small. Hence $\mathcal{M}$ is convex. This completes the proof of the Main Lemma.

From the Main Lemma we have

Corollary 3.1. *Let $\mathcal{M}$ be a complete, locally convex hypersurface with a strictly convex point. Then $\mathcal{M}$ is convex.*

Indeed, let the origin $\mathcal{O}$ be a strictly convex point of $\mathcal{M}$ such that $-e_{n+1}$ is a normal of $\mathcal{M}$ at $\mathcal{O}$. As above let Λ_t denote the connected component of $\mathcal{M} \cap \{x_{n+1} < t\}$ containing the origin. Then Λ_t is convex. Since $\Lambda_t \subset \Lambda_{\bar{t}}$ for any $t < \bar{t}$, $\mathcal{M} = \cup_{t<\infty} \Lambda_t$ is convex. Similarly we have

Corollary 3.2. *Let $\mathcal{M}$ be a closed locally convex hypersurface. Then $\mathcal{M}$ is convex.*

Remark 3.1. For smooth hypersurfaces different proofs for Corollaries 3.1 and 3.2 are available, see [12]. For nonsmooth locally convex hypersurfaces, the only proof we know is given in [5], where the details are given only for $n = 2$ and are very difficult to follow.

The above argument also produces a similar result for locally convex hypersurface with arbitrary boundary. More precisely, let $\mathcal{M}$ be a locally convex hypersurface such that the origin $\mathcal{O}$ belongs to $\mathcal{M}$ and $-e_{n+1}$ is a normal of $\mathcal{M}$ at the origin. Let Λ_t denote the connected components of $\mathcal{M} \cap \{x_{n+1} \leq t\}$ containing the origin. Let $\tau \geq 0$ be the largest number such that $\Lambda_t \cap \partial\mathcal{M} = \emptyset$ for all $t < \tau$. Then we have the following extension of the Main Lemma.

Lemma 3.4. *Let τ be as above. Then Λ_t is convex for all $t < \tau$.*

From the proof of Lemma 3.3 we see that the connected component of $\Lambda_\tau \cap \partial D$ containing $\mathcal{O}$ intersects with the boundary $\partial\mathcal{M}$.

4. Proof of Theorem A

Let $\widetilde{\mathcal{N}} = \mathcal{N} \cup (\partial\mathcal{N} \times [0, \theta])$ be an extension of $\mathcal{N}$. Let $\mathcal{M} = T(\mathcal{N})$ be a locally convex hypersurface with C^2 boundary. First we extend the mapping T to $\partial\mathcal{N} \times [0, \theta]$ such that for any $p \in \partial\mathcal{N}$ and $t \in [0, \theta]$,

$$T(p, t) = x + te_n - \frac{1}{2}kt^2\nu, \tag{4.1}$$

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where $x = T(p)$, ν is the normal of $\mathcal{M}$ at x, and e_n is a unit vector in the tangent plane of $\mathcal{M}$ at x, perpendicular to $\partial\mathcal{M}$, towards the outside of $\mathcal{M}$. The constants $\theta > 0$ small and $k > 1$ large will be chosen such that $\widetilde{\mathcal{M}} = T(\widetilde{\mathcal{N}})$ is locally convex.

For any given point $z_0 \in \partial\mathcal{M}$, we choose a new coordinate system such that z_0 is the origin, the south pole of $\mathbf{S}^n$ is a normal of $\mathcal{M}$ at z_0 and locally $\mathcal{M}$ can be represented as

$$x_{n+1} = g(x_1, \cdots, x_n) \tag{4.2}$$

for a convex function g with $\nabla g(0) = 0$. Furthermore we may suppose the x_i-axes, $i = 1, \cdots, n-1$, are tangent to $\partial\mathcal{M}$ at z_0 and the x_n axis is perpendicular to $\partial\mathcal{M}$, directed towards the outside of $\mathcal{M}$. Then the projection of $\partial\mathcal{M}$ on $\{x_{n+1} = 0\}$ can be represented as

$$x_n = \varphi(x_1, \cdots, x_{n-1}) \tag{4.3}$$

such that $\nabla\varphi(0) = 0$. Therefore the boundary $\partial\mathcal{M}$ can locally be represented by

$$
\begin{aligned}
x_n &= \varphi(x_1, \cdots, x_{n-1}) \\
x_{n+1} &= g(x_1, \cdots, x_{n-1}, \varphi).
\end{aligned}
\tag{4.4}
$$

Suppose locally $T(\partial\mathcal{N} \times [0, \theta])$ is represented by

$$x_{n+1} = u(x_1, \cdots, x_n). \tag{4.5}$$

Then $Du(0) = 0$. From (4.1) we have $u_{ii} = g_{ii}$ and $u_{in} = g_{in}$ at the origin, $i = 1, \cdots, n-1$. In order that the Hessian D^2u is positively definite, it suffices to assume $k \geq g_{nn}(0)$. Therefore we can choose k large enough and $\theta > 0$ small enough, depending only on n, the curvatures of $\partial\mathcal{M}$, and the upper and (positive) lower bounds of the principal curvatures of $\mathcal{M}$ on $\partial\mathcal{M}$, such that $\widetilde{\mathcal{M}}$ is locally convex. Furthermore $\widetilde{\mathcal{M}}$ is locally uniformly convex near $\partial\widetilde{\mathcal{M}}$ and $\partial\widetilde{\mathcal{M}}$ is Lipschitz regular.

Now we are in position to prove Theorem A. For any given point $z \in \mathcal{M}$, by choosing proper coordinates we may suppose z is the origin, the south pole of $\mathbf{S}^n$ is a normal of $\mathcal{M}$ at z and the hyperplane $\{x_{n+1} = 0\}$ is a tangent hyperplane of $\mathcal{M}$ at the origin. Let Λ_t, Λ etc be as in Section 3.

If z is an interior point of $\mathcal{M}$, then Λ_t contains no boundary points for $t \geq 0$ sufficiently small, because of the positive curvature condition of $\mathcal{M}$ on $\partial\mathcal{M}$. By Lemma 3.4 there exists $\tau > 0$ such that Λ_t is convex, $\Lambda_t \cap \partial\mathcal{M} = \emptyset$ for $t < \tau$ and Λ contains at least one boundary point of $\mathcal{M}$.

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Let z_0 be a boundary point in Λ. Let ν be the normal of $\mathcal{M}$ at z_0. We claim

$$|\nu - e_{n+1}| \geq \sigma \tag{4.6}$$

for some $\sigma > 0$ depending only on the curvatures of $\mathcal{M}$ at z_0. In other words, $G(\Lambda)$ has a positive distance from the north pole.

To prove (4.6) we choose a new coordinate system such that z_0 is the origin, the south pole of $\mathbf{S}^n$ is a normal of $\mathcal{M}$ at z_0. Then locally near z_0, $\mathcal{M}$ is represented by (4.2) for a convex function g with $\nabla g(0) = 0$, and $\partial\mathcal{M}$ is given by (4.4) with $D\varphi(0) = 0$. Since $\mathcal{M}$ has positive curvatures on $\partial\mathcal{M}$, we have

$$\frac{\partial^2}{\partial x_i^2} g(x_1, \cdots, x_{n-1}, \varphi) \geq c_0 > 0, \quad i = 1, \cdots, n-1. \tag{4.7}$$

It follows

$$x_{n+1} \geq \frac{c_0}{4} \sum_{i=1}^{n-1} x_i^2 \tag{4.8}$$

near the origin. Since $|\varphi| \leq C_0 \sum_{i=1}^{n-1} x_i^2$, we also have

$$x_{n+1} \geq c_0' |x_n|. \tag{4.9}$$

Therefore (4.6) follows from (4.8) (4.9). Note that if z is a boundary point, then by our choice of coordinates we have $\nu = -e_{n+1}$ and (4.6) holds automatically.

We go back to the coordinates where z is the origin and $\{x_{n+1} = 0\}$ is a tangent plane of $\mathcal{M}$ at z. In the following we apply Lemma 3.4 to the hypersurface $\widetilde{\mathcal{M}}$, regarding z as an interior point of $\widetilde{\mathcal{M}}$. If z is a boundary point of $\mathcal{M}$, we have $\tau = 0$ and $\Lambda = \{z\}$. If z is an interior point, then by (4.6) we see that $G(\Lambda)$ has a positive distance from the north pole. Therefore by Lemma 3.4, we can move the hyperplane $\mathcal{L}_\tau = \{x_{n+1} = \tau\}$ further upward to a height $\widetilde{\tau} > \tau$ such that $\widetilde{\Lambda}_t$ is convex, $\widetilde{\Lambda}_t \cap \partial\widetilde{\mathcal{M}} = \emptyset$ for any $t < \widetilde{\tau}$, and $\widetilde{\Lambda}$ contains at least one boundary point of $\widetilde{\mathcal{M}}$, where $\widetilde{\Lambda}_t$ is the connected component of $\widetilde{\mathcal{M}} \cap \{x_{n+1} < t\}$ containing the point z, $\widetilde{\Lambda}$ is the closure of $\cup_{t<\widetilde{\tau}} \widetilde{\Lambda}_t$. By (4.6) we have

$$\widetilde{\tau} > \widetilde{\tau} - \tau \geq r, \tag{4.10}$$

where $r > 0$ depends on θ, k in (4.1), and σ in (4.6). Therefore $\mathcal{M}$ is convex of reach r.

To show that $\mathcal{M}$ satisfies the uniform cone condition, let $\widetilde{D}_t$ be the convex closure of $\widetilde{\Lambda}_t$, $\widetilde{D} = \cup_{t<\widetilde{\tau}} D_t$, and $\widetilde{\Omega} = \partial\widetilde{D} \cap \{x_{n+1} = \widetilde{\tau}\}$. Then $\widetilde{\Omega}$ is convex and closed. By Lemma 3.4, $\widetilde{\Omega}$ contains a boundary point $\widetilde{z} \in \partial\widetilde{\mathcal{M}}$. Since $\partial\mathcal{M}$ is Lipschitz, there exists a ball

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$B'_\varepsilon(y) \subset \widetilde{\Omega}$ for some $\varepsilon > 0$ depending only on the boundary of $\partial \widetilde{\mathcal{M}}$. Hence $\mathcal{M}$ satisfies the cone condition at z for a cone of radius r and axial direction $y - x$. The aperture of the cone depends only on ε, $\widetilde{\tau}$, and R, where $R > 0$ is such that $B_R(0) \supset \widetilde{\mathcal{M}}$. Note that r and α are independent of the point z. Hence Theorem A holds.

In the above proof we reduced Theorem A to the local strict convexity of $\widetilde{\mathcal{M}} - \mathcal{M}$. Therefore the positive curvature condition in Theorem A can be replaced by the assumption that $\mathcal{M}$ can be extended to $\widetilde{\mathcal{M}}$ such that $\widetilde{\mathcal{M}} - \mathcal{M}$ is locally strictly convex. This condition can also be replaced by the local strict convexity condition of $\mathcal{M}$ near $\partial \mathcal{M}$. That is, there exists $\delta > 0$ such that for any $z \in \partial \mathcal{M}$, the δ-neighbourhood of z is strictly convex.

As a consequence we see that Theorem A holds not only for $\mathcal{M}$, but for a family of locally convex hypersurfaces. Indeed, let $\widetilde{\Phi}$ denote the set of locally convex hypersurfaces $\mathcal{M}_1 = T_1(\widetilde{\mathcal{N}})$ such that $T_1 = T$ on $\partial \mathcal{N} \times [0, \theta]$. Let $\Phi = \{T_1(\mathcal{N}) \mid T_1(\widetilde{\mathcal{N}}) \in \widetilde{\Phi}\}$. That is, $\mathcal{M}_1 \in \Phi$ if and only if $\mathcal{M}_1 \cup \{\widetilde{\mathcal{M}} - \mathcal{M}\}$ is a locally convex hypersurface in $\widetilde{\Phi}$. Then for any $R > 0$, there exist $r, \alpha > 0$, depending only on n, R, $\partial \mathcal{M}$, and the curvatures of $\mathcal{M}$ on $\partial \mathcal{M}$, such that any locally convex hypersurface $\mathcal{M}_1 \in \Phi$ is convex of reach r and satisfies the uniform cone condition of radius r and aperture α if $\mathcal{M} \subset B_R(0)$. This result will be used in proving Theorem B, so we state it as a theorem.

Theorem 4.1. *There exist $r, \alpha > 0$, depending only on n, R, $\partial \mathcal{M}$, and the curvatures of $\mathcal{M}$ on $\partial \mathcal{M}$, such that for any locally convex hypersurface $\mathcal{M}_1 \in \Phi$, if $\mathcal{M}_1 \subset B_R(0)$, then $\mathcal{M}_1$ is convex of reach r and satisfies the uniform cone condition of radius r and aperture α.*

If $\mathcal{M}$ is convex of reach r, then for any point $z \in \mathcal{M}$, the r-neighbourhood of z can be represented as a radial graph over a domain in a unit sphere $\mathbf{S}^n$. The uniform cone condition ensures furthermore that the ball of radius $\frac{1}{2} r \sin \alpha$ with centre at $z + \frac{1}{2} r \xi$ is contained in the cone, where ξ is the axial direction of the cone. Therefore one can choose the coordinates properly such that locally near z, $\mathcal{M}$ can be represented as a graph $x_{n+1} = g(x_1, \cdots, x_n)$, such that g is well defined in $B'_{\delta_0} = \{x' = (x_1, \cdots, x_n) \mid |x'| < \delta_0\}$, where $\delta_0 = \frac{1}{2} r \sin \alpha$.

5. Proof of Theorem B

Theorem B cannot be reduced to a Dirichlet problem directly, but we will use the existence of generalized solutions (in Alexandrov's sense) and the regularity of solutions to the Dirichlet problem for the prescribed Gauss curvature equation. For clarity we divide this section into several subsections.

5.1. Surface area

Let u, v be two convex functions defined in a bounded domain $\Omega \subset \mathbf{R}^n$ such that $u \geq v$ in Ω and $u = v$ on $\partial\Omega$. Let A_u and A_v denote the surface area of the graphs of u and v respectively. We claim that

$$A_u \leq A_v \tag{5.1}$$

with equality if and only if $u = v$ in Ω. Indeed, let $v_t = v + t(u - v)$. The surface area of the graph of v_t is given by

$$A_t = \int_\Omega (1 + |Dv_t|^2)^{1/2}.$$

We have

$$\begin{aligned}
\frac{d}{dt}A_t &= \int_\Omega \frac{Dv_t}{(1 + |Dv_t|^2)^{1/2}} D(u - v) \\
&= -\int_\Omega (u - v) H_{v_t},
\end{aligned}$$

where for any convex u, H_u is the mean curvature of the graph of u in the weak sense. Hence

$$A_u - A_v = -\int_\Omega (u - v)\left(\int_0^1 H_{v_t}\right) \leq 0.$$

If $u \neq v$, let h be a linear function such that $u \geq h$ in Ω and $h > v$ in a subdomain $\Omega' \subset \Omega$. Let $w = \max(h, v)$. Then $A_u \leq A_w < A_v$.

¿From the above formula we also see that if $\{u_k\}$ is a sequence of convex functions converging to u uniformly in a convex domain Ω, then $A_{u_k} \to A_u$. Indeed we have

$$\int_\Omega H_u \leq |\partial\Omega|$$

for any convex function u and convex domain Ω.

5.2. Perron method

Next we briefly describe the well known Perron method for the Dirichlet problem for the

prescribed Gauss curvature equation

$$\frac{\det D^2 u}{(1+|Du|^2)^{(n+2)/2}} = K \quad \text{in } \Omega,$$

$$u = g \quad \text{on } \partial\Omega,$$

(5.2)

where K is a positive constant, Ω is a bounded, Lipschitz domain in $\mathbf{R}^n$ (not necessarily convex), and g is a convex function defined on $\overline{\Omega}$. For a convex function u defined in Ω, we define the normal mapping N_u by setting, for $x \in \Omega$,

$$N_u(x) = \{p \in \mathbf{R}^n \mid u(y) \geq x \cdot p + u(x) \ \forall \ y \in \Omega\}$$

and $N_u(E) = \cup_{x \in E} N_u(x)$. For any Borel set $E \subset \Omega$, let $\mu_u(E) = |N_u(E)|$. Then μ_u is a nonnegative measure on Ω [10]. A convex function u, continuous up to the boundary, is a subsolution of (5.2) (Aleksandrov's sense) if $u = g$ on $\partial\Omega$ and for any open set E,

$$\mu_u(E) \geq K \int_E (1+|Du|^2)^{(n+2)/2}.$$

(5.3)

We say u is a generalized solution if equality in (5.3) holds for any Borel set E. Note that Du is a.e. well defined since it is convex. Obviously if u_1, u_2 are subsolutions, so is $\max(u_1, u_2)$.

Suppose u_0 is a subsolution of (5.2). Denote by Ψ the set of all subsolutions of (5.2). Let

$$u(x) = \sup\{w(x) \mid w \in \Psi\}.$$

Then u is a solution of (5.2). Indeed, by convexity there exists a sequence of subsolutions w_k such that $w_k \nearrow u$ uniformly in Ω. By the weak convergence of μ_{w_k}, see [10], u is a subsolution of (5.2). u is indeed a solution, for otherwise we can replace u in any ball $B_r \subset \Omega$ by the solution of (5.2) with $\Omega = B_r$ and boundary value u. The existence of solutions of (5.2) on strictly convex domains is well known [10].

5.3. Monotone sequences

We introduce a *monotone relation* for two locally convex hypersurfaces $\mathcal{M}_0 = T_0(\mathcal{N})$ and $\mathcal{M}_1 = T_1(\mathcal{N})$ in Φ, where Φ is the set in Theorem 4.1. We denote $\mathcal{M}_0 \prec \mathcal{M}_1$ if there is a continuous immersion T_t, $t \in [0,1]$, of the manifold $\mathcal{N}$ in $\mathbf{R}^{n+1}$ such that $T_t(\mathcal{N}) \in \Phi$ and T_t satisfies the monotone condition: for any $t \in [0,1]$ and $p \in \mathcal{N}$, there is a neighbourhood $\omega_p \subset \mathcal{N}$ and $\varepsilon > 0$ such that $T_s(q)$ lies in the concave side of $T_t(\omega_p)$ for $s \in [t, t+\varepsilon]$ and $q \in \omega_p$.

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We say a sequence of locally convex hypersurfaces $\mathcal{M}_k = T_k(\mathcal{N}) \in \Phi$ is *monotone* if $\mathcal{M}_k \prec \mathcal{M}_{k+1}$ for all k. We say the sequence $\mathcal{M}_k$ is *convergent* if there exists a sequence of homeomorphism φ_k from $\mathcal{N}$ to itself such that $T_k \cdot \varphi_k : \mathcal{N} \to \mathbf{R}^{n+1}$ is convergent. Note that if there exists a mapping $\widetilde{T}_k : \mathcal{N} \to \mathbf{R}^{n+1}$ satisfying the conditions in Definition 1 such that $\mathcal{M}_k = \widetilde{T}_k(\mathcal{N})$, then $\widetilde{T}_k = T_k \cdot \varphi_k$ for a homeomorphism φ_k from $\mathcal{N}$ to itself.

Lemma 5.1. *Let $\mathcal{M}_0, \mathcal{M}_1 \in \Phi$ such that $\mathcal{M}_0 \prec \mathcal{M}_1$. If $M_0 \subset \overline{B}_R(0)$, then $M_1 \in \overline{B}_R(0)$.*

Proof. Let $T_t, t \in [0,1]$, be the monotone deformation from $\mathcal{M}_0$ to $\mathcal{M}_1$. Then if $M_t = T_t(\mathcal{N}) \subset \overline{B}_R(0)$ for $t < t_0$, so is $\mathcal{M}_{t_0}$. Therefore it suffices to show that $\mathcal{M}_t \subset \overline{B}_R(0)$ when $t > 0$ small. For any point $p \in \mathcal{N}$, if $T_0(p) \in B_R(0)$, then $T_t(q) \in B_R(0)$ for q sufficiently close to p and $t > 0$ sufficiently close to 0. If $T_0(p) \in \partial B_R(0)$, we may suppose $T_0(p) = (0, \cdots, 0, -R)$. Then $\mathcal{M}_0$ can be represented as a graph $x_{n+1} = g(x_1, \cdots, x_n)$ for some convex function g such that $g \geq \sqrt{R^2 - \sum x_i^2}$. Since T_t is a monotone deformation, by definition we have that for q close to p and $t > 0$ small, $T_t(q)$ lies above the graph of g. Since $\mathcal{N}$ is compact, by the finite covering theorem we conclude that $\mathcal{M}_t \subset \overline{B}_R(0)$ when $t > 0$ is small. $\square$

Therefore if $\mathcal{M}_k$ is a sequence of monotone, locally convex hypersurfaces in Φ, we have $\mathcal{M}_k \subset B_R(0)$ for some $R > 0$ large enough. One can also prove the surface area of $\mathcal{M}_k$ is monotone decreasing.

5.4. Construction of a monotone sequence

Next we use the Perron lifting to construct a sequence of monotone, locally convex hypersurfaces in Φ. Let $\mathcal{M}_0$ be the locally convex hypersurface in Theorem B, given by the immersion $\mathcal{M}_0 = T_0(\mathcal{N})$. Let $\widetilde{\mathcal{N}}, \widetilde{\mathcal{M}_0}$ be the extension of $\mathcal{N}, \mathcal{M}_0$, as in Section 4.

For any point $x_0 \in \mathcal{M}_0$, by Theorem 4.1 we may choose the coordinates properly such that locally near x_0, $\widetilde{\mathcal{M}_0}$ is represented by

$$x_{n+1} = u_0(x_1, \cdots, x_n), \tag{5.4}$$

and u_0 is nonnegative, convex function well defined in $B'_\delta = \{x' = (x_1, \cdots, x_n) \,|\, |x'| < \delta\}$, where

$$\delta = \frac{1}{8}\varepsilon, \qquad \varepsilon \leq \frac{1}{20} r \sin(\alpha/2)$$

are fixed constants. Since in Theorem 4.1 we are concerned with the r-neighbourhood of z, we may also suppose $u_0 \leq r$. By the convexity we have

$$|Du_0| \leq 2r/\delta \quad \text{in} \quad B'_{\delta/2}. \tag{5.5}$$

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Let $D_0 = \{x' \in B'_{\delta/4} \mid (x', u_0(x')) \in \mathcal{M}_0\}$, and $F_0 \subset \mathcal{M}_0$ be the graph of u_0 over D_0. In D_0 we consider the Dirichlet problem for the Gauss curvature equation

$$\frac{\det D^2 u}{(1 + |Du|^2)^{(n+2)/2}} = K_0 \quad \text{in } D_0 \tag{5.6}$$
$$u = u_0 \quad \text{on } \partial D_0$$

where K_0 is the positive constant in Theorem B. By assumption $K(\mathcal{M}_0) \geq K_0$, u_0 is a subsolution of (5.6). By the Perron method in §5.2, there is a solution u_1 of (5.6) such that $u_1 \geq u_0$ in D_0. The projection in the x_{n+1}-axis direction is a natural 1-1 mapping from F_0 to F_1, where F_1 is the graph of u_1. Denote this mapping by ψ_0. Then ψ_0 and its inverse is Lipschitz by (5.5). On $\mathcal{M}_0 - F_0$ let ψ_0 be the identity mapping. Then $T_1 =: \psi_0 \cdot T_0$ defines an immersion of $\mathcal{N}$ such that $\mathcal{M}_1 = T_1(\mathcal{N})$ is a locally convex hypersurface in Φ. We have obviously $K(\mathcal{M}_1) \geq K_0$.

Denote by $A(u_0)$ $(= A(u_0, z))$ and $A(u_1)$ $(= A(u_1, z))$ the surface areas of F_0 and F_1, respectively. By §5.1 we have $A(u_1) \leq A(u_0)$, with equality if and only if $F_0 = F_1$, namely, the Gauss curvature $K(F_0) = K_0$. We choose the point $z_0 \in \mathcal{M}_0$ such that

$$A(u_0, z_0) - A(u_1, z_0) \geq \frac{1}{2} \sup\{A(u_0, z) - A(u_1, z) \mid z \in \mathcal{M}_0\}. \tag{5.7}$$

Note that for a given point $z \in \mathcal{M}_0$, there are different coordinates such that locally $\mathcal{M}_0$ can be represented as a graph by formula (5.4). The supremum (5.7) is also taken among all such possible coordinate systems. By choosing z_0 as in (5.7), we obtain a locally convex hypersurface $\mathcal{M}_1 \in \Phi$. Obviously $\mathcal{M}_0 \prec \mathcal{M}_1$ and $A(\mathcal{M}_0) \geq A(\mathcal{M}_1)$.

Continuing the above procedure, with δ fixed at all steps, and $\mathcal{M}_0$ replaced by $\mathcal{M}_{k-1}$ at the kth step, we obtain a sequence of monotone, locally convex hypersurfaces $\mathcal{M}_k \in \Phi$, $\mathcal{M}_k = T_k(\mathcal{N})$, such that $K(\mathcal{M}_k) \geq K_0$ and the surface area $A(\mathcal{M}_k)$ is uniformly bounded. By Lemma 5.1, we have $\mathcal{M}_k \subset B_R(0)$ for all k.

5.5. Convergence of the monotone sequence $\{\mathcal{M}_k\}$

Let $\cup_{j=1}^m B_\varepsilon(z_j) \supset B_R(0)$, where $\varepsilon \leq \frac{1}{20} r \sin(\alpha/2)$, be a finite covering of $B_R(0)$. For any given j and k, we claim there are finitely many connected components in $B_\varepsilon(z_j) \cap \mathcal{M}_k$. Indeed, let $\omega_i, i = 1, 2, \cdots$, be the connected components of $B_\varepsilon(z_j) \cap \mathcal{M}_k$. Let $\widetilde{\omega}_i$ be the connected components of $B_{r/5}(z_j) \cap \mathcal{M}_k$ containing ω_i. Regarding $\widetilde{\omega}_i$ as a subset of the extended hypersurfaces $\widetilde{M}_k = \mathcal{M}_k \cup (\widetilde{\mathcal{M}}_0 - \mathcal{M}_0)$, we see by Lemma 3.4 that $\widetilde{\omega}_i$ are mutually disjoint. By the uniform cone condition in Theorem 4.1, the surface area of $\widetilde{\omega}_i$ is large than a_0 for some $a_0 > 0$ depending only on n, r, and α. Since the surface area of

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$\mathcal{M}_k$ is uniformly bounded, i.e., $A(\mathcal{M}_k) \le A_0$ for all k, there are at most $\left[\frac{A_0}{a_0}\right]$ connected components of $B_\varepsilon(z_j) \cap \mathcal{M}_k$.

Choose an arbitrary boundary point $y^0 \in \Gamma = \partial\mathcal{M}_0$. We take it as the origin and suppose $y^0 \in B_0$ for some B_0 among the balls $\{B_\varepsilon(z_j)\}$. Let $\omega_k^0(t)$ be the connected component of $B_{0,t} \cap \mathcal{M}_k$ containing y^0, where $B_{0,t}$ is a ball of radius t, concentrated with B_0. By Theorem 4.1, $\omega_k^0 = \omega_k^0(r)$ is convex and there is a cone $\mathcal{C}_k$ of radius r and aperture α, with vertex at y^0 and axial direction ξ_k, such that $\mathcal{C}_k$ lies on the concave side of ω_k^0. By choosing a subsequence we may suppose $\xi_k \to e_{n+1}$, the north pole of $\mathbf{S}^n$. Let $\mathcal{C}$ denote the cone of radius r and aperture $\alpha/2$, with vertex y^0 and axial direction e_{n+1}. Then $\mathcal{C}$ lies in the concave side of ω_k^0 for all large k by the monotone condition. Suppose locally near y^0, ω_k^0 is represented by

$$x_{n+1} = u_k^0(x'), \quad x' = (x_1, \cdots, x_n).$$

If ω_k^0 contains no boundary point of $\mathcal{M}_0$, then u_k^0 is well defined in $B_{\delta'}'$, where $\delta' = \frac{1}{2} r \sin\frac{\alpha}{2}$. Otherwise it is defined in a subdomain of $B_{\delta'}'$ (which is independent of k since $\Gamma = \partial\mathcal{M}_0$ is independent of k). In both cases let D^0 be the domain of definition. Since $\mathcal{M}_k$ is monotone, u_k^0 is a sequence of monotone, convex function. Hence u_k^0 converges to a convex function u^0 in D^0. Let F^0 denote the graph of u^0 (over D^0). We claim

$$K(F^0) = K_0, \tag{5.8}$$

where $K(\mathcal{M})$ denotes the Gauss curvature of $\mathcal{M}$. Indeed, since $K(\mathcal{M}_k) \ge K_0$, we have $K(F^0) \ge K_0$. If F^0 is not a K_0-hypersurface, there is a ball $B_\delta' \subset D^0$ ($\delta = \varepsilon/8$) such that

$$\mu_{u^0}(B_\delta') > K_0 \int_{B_\delta'} (1 + |Du^0|^2)^{(n+2)/2}.$$

Let u_k^* (u^*, resp) be the solution of (5.6) with domain B_δ' and boundary value u_k^0 (u^0, resp.), and let $u_k^* = u_k^0$ ($u^* = u^0$, resp.) in $D^0 - B_\delta'$. Then $u^* \ge u^0$, $u^* \not\equiv u^0$, and $K(F_{u^*}) \ge K_0$, where F_{u^*} is the graph of u^*. It follows the surface area $A(F_{u^*}) \le A(F_{u^0}) + \varepsilon'$ for some $\varepsilon' > 0$. Since $u_k^* \to u^*$ and $u_k^0 \to u^0$ uniformly, we have $A(F_{u_k^*}) \le A(F_{u_k^0}) + \frac{1}{2}\varepsilon'$, which implies by (5.7) that $A(\mathcal{M}_{k+1}) \le A(\mathcal{M}_k) + \frac{1}{2}\varepsilon'$ for all k large. This is impossible. Hence (5.8) holds.

We proceed as above. Suppose at the jth step we have a sequence of connected components $\omega_k^{j-1} \subset \mathcal{M}_k$, which can be represented as graphs $x_{n+1} = u_k^{j-1}(x')$ for $x' \in D^{j-1}$ such that $u_k^{j-1} \to u^{j-1}$. Let F^{j-1} denote the graph of u^{j-1}. Then at the $(j+1)^{th}$ step we choose a point $y^j \in \partial F^0 \cup \cdots \cup \partial F^{j-1}$, and y^j is not an interior point of F^i for

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all $i = 1, \cdots, j - 1$. Suppose $y^j \in \partial F^{j-1}$. As above we take y^j as the origin and let B_j be a ball among $\{B_\varepsilon(z_i), i = 1, \cdots, m\}$ such that $y^j \in B_j$. Let $\widetilde{y}_k = \widetilde{y}_k^j$ be the (unique) point in ω_k^{j-1} such that $\widetilde{y}_k \to y^j$. Then $\widetilde{y}_k \in B_j$ when k is large enough. Let $\omega_k^j(t)$ be the connected component of $B_{j,t} \cap \mathcal{M}_k$ containing $\widetilde{y}_k$, where $B_{j,t}$ is the ball of radius t, concentrated with B_j. As above suppose locally near $\widetilde{y}_k$, ω_k^j is represented by

$$x_{n+1} = u_k^j(x'), \quad x' \in D^j. \tag{5.9}$$

Then u_k^j is monotone, convex, and converges to a convex function u^j. Let F^j denote the graph of u^j. Then we have $K(F^j) = K_0$.

This procedure finishes in finitely many steps (say at the m^*-th step) since we have shown above that for any $k \geq 1$ and $j = 1, \cdots, m$, there are at most $\left[\frac{A_0}{a_0}\right]$ connected components of $B_\varepsilon(z_j) \cap \mathcal{M}_k$. Therefore we obtain a collection of K_0-hypersurface F^i, $i = 0, \cdots, m^* - 1$.

For k sufficiently large, we define a mapping η from $\mathcal{M}_k$ to $\mathcal{M} = \cup F^i$ such that $\eta \cdot T_k$ is locally a homeomorphism, where T_k is the mapping for $\mathcal{M}_k$. Since $\{B_\varepsilon(z_j), j = 1, \cdots, m\}$ is a finite covering of $\mathcal{M}_k$, $\omega_k^j(\varepsilon)$, $j = 0, \cdots, m^* - 1$, is a covering of $\mathcal{M}_k$. Let k large enough (but fixed) such that $|u_k^j - u^j| < \varepsilon''$ for some ε'' small. For any $x \in \mathcal{M}_k$, we have $x \in \omega_k^j(\varepsilon)$ for some j. Hence x is an interior point of $\omega_k^j = \omega_k^j(r)$. Let γ_x be the vector field on $\mathcal{M}_k$ introduced in Lemma 2.2. By Theorem 4.1, $\mathcal{M}_k$ is convex of reach r and satisfies the uniform cone condition with radius r and aperture α. Hence we have $|\gamma_x| \geq \varepsilon''$ when ε'' is chosen sufficiently small. It follows that $\gamma_x \cap F^j$ contains a unique point $\{y\}$. We define a mapping η such that $y = \eta(x)$. Obviously η is well defined on $\mathcal{M}_k$ and locally it is a homeomorphism. Hence $\mathcal{M} = \eta \cdot T_k(\mathcal{N})$ is a locally convex hypersurface. Since F^j are K_0-hypersurfaces, $\mathcal{M}$ is also a K_0-hypersurface.

Similarly we can define a mapping η_i from $\mathcal{M}_k$ to $\mathcal{M}_i$ as above, where k is fixed, $i > k$. Then $\eta_i \to \eta$ and $\eta_i \cdot T_k \to \eta \cdot T_k$. That is, $\mathcal{M}_i$ converges to $\mathcal{M}$.

5.6. Regularity

Let $\mathcal{M}_0$ be the locally convex hypersurface in Theorem B, given by the immersion $\mathcal{M}_0 = T_0(\mathcal{N})$. Let $\widetilde{\mathcal{M}}_0$ be the extension of $\mathcal{M}_0$ as in Section 4, and let $\widetilde{\mathcal{M}} = \mathcal{M} \cup \{\widetilde{\mathcal{M}}_0 - \mathcal{M}_0\}$ be the extension of $\mathcal{M}$, where $\mathcal{M}$ is the K_0-hypersurface obtained in §5.5, such that both $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{M}}_0$ are locally convex. For any boundary point $z \in \Gamma$, one can choose a nearby point $z' \in \widetilde{\mathcal{M}}_0 - \mathcal{M}_0$, and take zz' as the x_{n+1}-axis direction, such that locally near z, $\mathcal{M}_0$ can be represented as

$$x_{n+1} = u^0(x_1, \cdots, x_n) \tag{5.10}$$

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and $\mathcal{M}$ can be represented as

$$x_{n+1} = u(x_1, \cdots, x_n). \tag{5.11}$$

By our construction of $\mathcal{M}_k$, we have $u \geq u^0$ near z.

To prove the regularity of $\mathcal{M}$ we first prove $\mathcal{M}$ is locally strictly convex. Let $z_0 \in \mathcal{M}$ and let $\mathcal{L}$ be a tangent hyperplane of $\mathcal{M}$ at z_0. If $\mathcal{M}$ is not strictly convex at z_0, there is a line segment γ, which passes through the point z_0, such that $\gamma \subset \Lambda_0$, where for $t \geq 0$, Λ_t is the connected component of $\mathcal{M} \cap \{x_{n+1} \leq t\}$ containing z_0.

By the extension in Section 4, $\widetilde{\mathcal{M}}_0 - \mathcal{M}_0$ is locally strictly convex. Hence for $t > 0$ small, Λ_t, as a set in $\widetilde{\mathcal{M}}$, contains no boundary points of $\widetilde{\mathcal{M}}$, and hence is convex by the Main Lemma. Hence Λ_0 is also convex. From [1], an extreme point of Λ_0 is not an interior point of $\mathcal{M}$ since $\mathcal{M}$ is a K_0-hypersurface. Hence Λ_0 contains a boundary point $z_1 \in \partial\mathcal{M} (= \Gamma)$, and there is a line segment $\ell \subset \Lambda_0$ such that z_1 is an endpoint of ℓ.

Suppose near z_1, $\mathcal{M}_0$ and $\mathcal{M}$ are represented by (5.10) and (5.11) respectively. Then ℓ is not tangent to Γ at z_1. Indeed, if ℓ is tangent to Γ, then $u_{\xi\xi}^0 = 0$ since $u \geq u^0$, where ξ is the direction of ℓ. This is a contradiction.

Choosing a new coordinate system we can then suppose z_1 is the origin and ℓ is the x_n axis, such that u is a nonnegative convex function. Since ℓ is not tangent to Γ at z_1, we have by the smoothness of Γ,

$$0 \leq u(x_1, \cdots, x_{n-1}, x_n) \leq C \sum_{i=1}^{n-1} x_i^2 \tag{5.12}$$

for any fixed $x_n > 0$. But since the Gauss curvature of u is a positive constant, one can easily construct a supersolution to show that (5.12) is impossible, using the comparison principle. Therefore $\mathcal{M}$ is locally strictly convex, and so it is smooth [10].

The regularity on the boundary of a K-hypersurface is a local property and has been proven in [4,6]. Hence $\mathcal{M}$ is globally smooth. This completes the proof.

Finally we point out that Theorem B can be extended to the case when the Gauss curvature depends on the position of the hypersurface. That is if Γ can bound a locally convex hypersurface $\mathcal{M}_0$ such that $K(\mathcal{M}_0)(x) \geq f(x)$ for any $x \in \mathcal{M}_0$, then there exists a locally convex hypersurface $\mathcal{M}$ with boundary Γ such that $K(\mathcal{M})(x) = f(x)$, where $f \in C^2(\mathbf{R}^{n+1})$ is a positive function.

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