MONOTONICITY FORMULAE AND CURVATURE EQUATIONS

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ABSTRACT. We derive local integral estimates and monotonicity formulae for certain curvature quantities related to hypersurfaces of prescribed k-th mean curvature. We use these to improve interior curvature bounds established in previous work, and to derive a local Hölder gradient estimate for admissible solutions of the equation of prescribed scalar curvature.

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1. Introduction

In this paper we derive various interior estimates for k-admissible hypersurfaces of $\mathbf{R}^{n+1}$, and more particularly, for graphs of k-admissible solutions of the equation of prescribed k-th mean curvature. These estimates include local integral bounds for certain curvature quantities in terms of boundary integrals of same quantities, and related local monotonicity formulae. We will use these estimates to improve the curvature bounds established in [13], and to derive a local Hölder gradient estimate for 2-admissible solutions of the equation of prescribed scalar curvature.

The k-th mean curvature H_k of a C^2 hypersurface $M \subset \mathbf{R}^{n+1}$ is given by the k-th elementary symmetric function of the principal curvatures $\lambda_1, \ldots, \lambda_n$ of M,

$$H_k = S_k(\lambda_1, \ldots, \lambda_n) = \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k}. \tag{1.1}$$

We say that a C^2 orientable hypersurface M in $\mathbf{R}^{n+1}$ is *k-admissible* if at each point its vector of principal curvatures $\lambda = (\lambda_1, \ldots, \lambda_n)$ belongs to the cone

$$\Gamma_k = \{\lambda \in \mathbf{R}^n : S_j(\lambda) > 0,\ j = 1, \ldots, k\}.$$

We compute the second fundamental form with respect to the upwards pointing normal if M is the graph of some function defined over a subdomain of $\mathbf{R}^n$.

All the estimates of this paper will be proved under the assumption that we have control of the modulus of continuity of the normal vector field ν (actually, a somewhat weaker condition suffices). Thus we are in effect considering only hypersurfaces M that can be represented as graphs with small gradient on small enough neighbourhoods of any point of M. We shall nevertheless state most of our results in a more geometric fashion.

If $M = \text{graph } u$, $H_k = H_k[u]$ is given by an expression depending on Du and D^2u. It is well known that the k-curvature equation

$$H_k[u] = \psi \tag{1.2}$$

is an elliptic equation and $H_k[u]^{1/k}$ is a concave function of D^2u if the graph of u is k-admissible (see [2]). Such solutions are called *k-admissible.*

We shall use the following notation. X denotes the position vector on M, and $X^\top$ and $X^\perp$ denote the tangential and normal components of X. ∇ denotes covariant differentiation on M. Various integrals that appear below will be assumed to be with respect to the natural measures, without these always being indicated. Thus if the integral is over a relatively open subset of M, the measure is n-dimensional Hausdorff measure $\mathcal{H}^n$, while integrals over boundaries of such domains are with respect to $n-1$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$.

Our first result is the following local integral bound.

Theorem 1.1. *Let M be a k-admissible hypersurface in $\mathbf{R}^{n+1}$, $n \ge 2$, such that $M = $ graph u for some function $u \in C^4(\overline{B}_{d_0}(0))$ with $u(0) = 0$, $Du(0) = 0$. Let*

$g := H_k^{1/k}$ for some $k \in \{2, \cdots, n\}$. Then for any $p > 1$ there is a number $\rho_0 \in (0, d_0]$, depending only on p and the modulus of continuity of ν on $M \cap (\overline{B}_{d_0}(0) \times \mathbf{R})$, such that for any $\rho \in (0, \rho_0]$ we have

$$\begin{aligned} \int_{M_\rho} v^{2p} H_1^p H_{k-1} \leq \rho \int_{\partial M_\rho} v^{2p} H_1^p H_{k-1} \\ + C_1 \rho \int_{M_\rho} H_1^p \left(g^k + \rho g^{k-1} |\nabla g|\right) + C_2 \int_{M_\rho} H_1^{p-1} \left(g^k + \rho^2 g^{k-2} |\nabla g|^2\right), \end{aligned} \tag{1.3}$$

where $M_\rho = M \cap B_\rho^{n+1}(0)$, C_1 and C_2 depend only on k, n and p, and $v = \sqrt{1 + |Du|^2}$.

Remarks. (i) The restriction on ρ amounts to requiring $|Du| \leq c(p)$ for a positive constant $c(p) < 1$ depending only on p. Thus $1 \leq v \leq \sqrt{2}$ on M_ρ. The factor v^{2p} could be therefore be removed, at the expense of introducing a constant $C(p)$ in front of the boundary integral.

(ii) If $\Delta g \geq 0$ on M, then (1.3) can be written more explicitly as

$$\int_{M_\rho} v^{2p} H_1^p H_{k-1} \leq \rho \int_{\partial M_\rho} v^{2p} H_1^p H_{k-1} - \frac{k}{n-k+1} \int_{M_\rho} g^k v^{2p} H_1^p \langle X, \nu \rangle \tag{1.4}$$

Above we have incorporated the last term of (1.4) into the second last term of (1.3).

(ii) If M is a convex hypersurface, then the term requiring the introduction of v^{2p} is automatically nonnegative and can be discarded at a suitable point in the proof. In this case (1.3) (and (1.4) if $\Delta g \geq 0$) holds with v^{2p} replaced by 1. Furthermore, we do not need to assume that ρ is small; all we need is $\sup_{B_{d_0}} |Du| < 1$. However, the convexity of M is an unnatural assumption unless $k = n$.

Theorem 1.1 is essentially a differential version of the following local monotonicity formula.

Theorem 1.2. *Let M, g, p and ρ_0 be as in Theorem 1.1. Then for any $0 < r \leq R \leq \rho_0$ we have*

$$\begin{aligned} \frac{1}{r} \int_{M_r} v^{2p} H_1^p H_{k-1} \leq \frac{1}{R} \int_{M_R} v^{2p} H_1^p H_{k-1} \\ + C_1 \int_r^R \left(\frac{1}{\rho} \int_{M_\rho} H_1^p \left(g^k + \rho g^{k-1} |\nabla g|\right) \right) d\rho \\ + C_2 \int_r^R \left(\frac{1}{\rho^2} \int_{M_\rho} H_1^{p-1} \left(g^k + \rho^2 g^{k-2} |\nabla g|^2\right) \right) d\rho, \end{aligned} \tag{1.5}$$

where C_1, C_2 are the constants from (1.3).

Remarks. (i) This generalizes to k-th mean curvature equations the monotonicity formulae established in [12] for $W^{2,p}$ solutions of k-Hessian equations.

(ii) If $\Delta g \geq 0$ on M, we obtain the monotonicity formula

$$\begin{aligned}\frac{1}{r}\int_{M_r} v^{2p}H_1^pH_{k-1} \leq \frac{1}{R}\int_{M_R} v^{2p}H_1^pH_{k-1}\\ -\frac{k}{n-k+1}\int_r^R \rho^{-2}\int_{M_\rho} g^k v^{2p}H_1^p\langle X,\nu\rangle.\end{aligned} \tag{1.6}$$

In the special case $k=2$ we obtain from Theorem 1.2 an a priori local Hölder gradient estimate for 2-admissible solutions of the equation of prescribed scalar curvature. This extends our results [15] for solutions of degenerate two dimensional Monge-Ampère equations and [12] for solutions of the 2-Hessian equation. We state a version involving only L^∞ norms of g and ∇g.

Theorem 1.3. *Let M, g and ρ_0 be as in Theorem 1.1, with $k=2$. Then for any $q>n-1$ and any $r\in(0,\rho_0/2]$ we have an estimate*

$$\begin{aligned}\sup_{\substack{x,y\in B_r^n\\ x\neq y}}\frac{|Du(x)-Du(y)|}{|x-y|^\alpha} \leq C(n,q)r^{-1/q}\bigg\{\int_{M_{4r}} H_1^q + r^{n+q}\sup_{M_{4r}} g^{2q}\\ +r^{3q+n}\sup_{M_{4r}}|\nabla g|^{2q} + r^n\sup_{M_{4r}} g^q + r^{q+n}\sup_{M_{4r}}|\nabla g|^q\bigg\}^{\frac{1}{q}}\end{aligned} \tag{1.7}$$

where $\alpha = 1-(n-1)/q$.

In Section 3 we shall prove the following interior curvature bound.

Theorem 1.4. *Let M, g be as in Theorem 1.1, $n\geq 3$, and suppose that $g := H_k^{1/k}$ satisfies*

$$\mu^{-1}\leq g\leq \mu, \qquad |\nabla g|\leq \mu \quad \text{on} \quad M \tag{1.8}$$

for some positive constant μ. Then for any $s>k(n-1)/2$ we have

$$|A(0)|\leq C \tag{1.9}$$

where A is the second fundamental form of M and C depends only on k, n, s, μ, d_0, the modulus of continuity of ν on $M\cap(\overline{B}_{d_0}(0)\times\mathbf{R})$, and on $\int_M H_1^s$.

Remarks. (i) This improves the curvature bound of [13], which is essentially the same result with $s>kn/2$.

(ii) In the case $k=n$ the lower bound on q is known to be essentially sharp (see [10]). We do not know whether the lower bound $s>k(n-1)/2$ is optimal if $k<n$. Examples in [11] show that $s\geq k(k-1)/2$ is necessary for corresponding interior second derivative bounds for k-Hessian equations. We expect that the optimal bounds for q should be the same for k-Hessian and k-curvature equations.

(iii) A purely local interior curvature bound for the scalar curvature case $k=2$ has been proved by Nelli [7] under the strict ellipticity assumption $[F_{ij}]\geq \delta I$ for a positive constant δ (see the following section for the definition of F_{ij}).

2. Monotonicity formulae

In this section we shall prove Theorem 1.1. Let $\mathbf{e}_1, \ldots, \mathbf{e}_{n+1}$ denote the usual orthonormal basis of $\mathbf{R}^{n+1}$. Near $(0, u(0))$ we may choose a local orthonormal frame field $\hat{\mathbf{e}}_1, \cdots, \hat{\mathbf{e}}_n$ on $M = \text{graph } u$. We denote covariant differentiation on M in the direction $\hat{\mathbf{e}}_i$ by ∇_i. We let $\hat{\mathbf{e}}_{n+1} = \nu$ be the unit normal vector field. Let $A = [h_{ij}]$ denote the second fundamental form of M, relative to the frame $\hat{\mathbf{e}}_1, \cdots, \hat{\mathbf{e}}_n$. The k-curvature operators H_k on M are then defined by

$$H_k[A] = S_k(\lambda_1, \cdots, \lambda_n) \tag{2.1}$$

where $\lambda_1, \cdots, \lambda_n$ are the principal curvatures of M, which are the eigenvalues of A. It is well known that

$$F_{ij} = \frac{\partial H_k}{\partial a_{ij}}[A]$$

is a positive matrix, and $H[A]^{1/k}$ is a concave function of A if M is k-admissible (see [2]).

We first write the equation

$$H_k[A] = g^k \tag{2.2}$$

in the form

$$G_k[A] := H_k[A]^{1/k} = g. \tag{2.3}$$

Differentiating (2.3), and writing $G_{ij} = \frac{\partial G_k}{\partial a_{ij}}[A]$, we obtain

$$G_{ij} \nabla_l h_{ij} = \nabla_l g, \tag{2.4}$$

$$G_{ij} \nabla_l \nabla_l h_{ij} \geq \nabla_l \nabla_l g = \Delta g, \tag{2.5}$$

where in the last inequality we have used the concavity of G to discard a term which is quadratic in ∇A, and where Δ denotes the Laplace-Beltrami operator on M. As usual we assume summation over $l = 1, \ldots, n$ in (2.5). Using the standard formula for commuting covariant derivatives, together with the Gauss equations

$$R_{ijkl} = h_{ik} h_{jl} - h_{il} h_{jk},$$

where R_{ijkl} denotes the Riemann curvature tensor, and the Codazzi equations, which tell us that $\nabla_l h_{ij}$ is symmetric in all indices, we find that

$$\begin{aligned}
\nabla_l \nabla_l h_{ij} &= \nabla_l \nabla_i h_{jl} \\
&= \nabla_i \nabla_l h_{jl} + R_{lijm} h_{ml} + R_{lilm} h_{mj} \\
&= \nabla_i \nabla_j h_{ll} + h_{lj} h_{im} h_{ml} - h_{lm} h_{ij} h_{ml} \\
&\quad + h_{ll} h_{im} h_{mj} - h_{lm} h_{il} h_{mj} \\
&= \nabla_i \nabla_j h_{ll} - h_{lm} h_{ij} h_{ml} + h_{ll} h_{im} h_{mj}.
\end{aligned}$$

Using this in (2.5) we obtain

$$G_{ij}\nabla_i\nabla_j h_{ll} \geq G_{ij}h_{ij}h_{lm}h_{lm} - G_{ij}h_{im}h_{jm}h_{ll} + \Delta g. \tag{2.6}$$

We have

$$G_{ij} = \frac{1}{k}H_k^{(1-k)/k}F_{ij} = \frac{1}{k}g^{1-k}F_{ij} \tag{2.7}$$

where $F_{ij} = \frac{\partial H_k}{\partial a_{ij}}[A]$, so using this in (2.6) we obtain

$$\begin{aligned} F_{ij}\nabla_i\nabla_j H_1 = F_{ij}\nabla_i\nabla_j h_{ll} &\geq F_{ij}h_{ij}h_{lm}h_{lm} - H_1F_{ij}h_{im}h_{jm} + kg^{k-1}\Delta g \\ &=: Q + kg^{k-1}\Delta g. \end{aligned} \tag{2.8}$$

Observe that if $\Delta g \geq 0$, the last term can be dropped; the subsequent computations involving this term are then unnecessary, and we shall see that we obtain (1.4) in place of (1.3). Furthermore, as shown in [3], if M is convex, then Q is nonnegative and can be dropped. However, the convexity of M is not a natural assumption if $k < n$.

From [13], Lemma 2.1, we know that $v = \sqrt{1+|Du|^2} = \nu_{n+1}^{-1}$ satisfies the equation

$$F_{ij}\nabla_i\nabla_j v = vF_{ij}h_{im}h_{jm} + \frac{2}{v}F_{ij}\nabla_i v\nabla_j v + v^2\langle kg^{k-1}\nabla g, \mathbf{e}_{n+1}\rangle, \tag{2.9}$$

where $\mathbf{e}_{n+1}$ is the $n+1$-st standard coordinate vector of $\mathbf{R}^{n+1}$.

We now compute a differential inequality for $W = \phi(v)H_1^p$ where ϕ is a smooth positive function to be chosen and $p > 1$ is a constant. We have

$$\nabla_i W = \phi'(v)H_1^p\nabla_i v + p\phi(v)H_1^{p-1}\nabla_i H_1 \tag{2.10}$$

and

$$\begin{aligned} \nabla_i\nabla_j W = \phi'(v)H_1^p\nabla_i\nabla_j v &+ \phi''(v)H_1^p\nabla_i v\nabla_j v \\ &+ p\phi'(v)H_1^{p-1}(\nabla_i v\nabla_j H_1 + \nabla_j v\nabla_i H_1) + p\phi(v)H_1^{p-1}\nabla_i\nabla_j H_1 \\ &+ p(p-1)\phi(v)H_1^{p-2}\nabla_i H_1\nabla_j H_1. \end{aligned} \tag{2.11}$$

Consequently, using (2.8) and (2.9) we have

$$\begin{aligned} &F_{ij}\nabla_i\nabla_j W \\ \geq&\phi'(v)H_1^p\left\{vF_{ij}h_{im}h_{jm} + \frac{2}{v}F_{ij}\nabla_i v\nabla_j v + v^2\langle kg^{k-1}\nabla g, \mathbf{e}_{n+1}\rangle\right\} \\ &+ \phi''(v)H_1^pF_{ij}\nabla_i v\nabla_j v + 2p\phi'(v)H_1^{p-1}F_{ij}\nabla_i v\nabla_j H_1 \\ &+ p\phi(v)H_1^{p-1}\left\{F_{ij}h_{ij}h_{lm}h_{lm} - H_1F_{ij}h_{im}h_{jm} + kg^{k-1}\Delta g\right\} \\ &+ p(p-1)\phi(v)H_1^{p-2}F_{ij}\nabla_i H_1\nabla_j H_1. \end{aligned} \tag{2.12}$$

For any $\epsilon > 0$ we now estimate

$$\begin{aligned} &2p\phi'(v)H_1^{p-1}F_{ij}\nabla_i v\nabla_j H_1 \\ &\geq -\frac{p(\phi')^2}{\epsilon(p-1)\phi}H_1^p F_{ij}\nabla_i v\nabla_j v - \epsilon p(p-1)\phi H_1^{p-2}F_{ij}\nabla_i H_1\nabla_j H_1. \end{aligned} \tag{2.13}$$

Using this in (2.12) and rearranging terms we obtain

$$\begin{aligned} F_{ij}\nabla_i\nabla_j W \geq\ & H_1^p F_{ij}h_{im}h_{jm}\,[\phi' v - p\phi] \\ &+ H_1^p F_{ij}\nabla_i v\nabla_j v\left[\frac{2\phi'}{v} + \phi'' - \frac{p(\phi')^2}{\epsilon(p-1)\phi}\right] \\ &+ (1-\epsilon)p(p-1)\phi(v)H_1^{p-2}F_{ij}\nabla_i H_1\nabla_j H_1 \\ &+ v^2\phi'(v)H_1^p\langle kg^{k-1}\nabla g, \mathbf{e}_{n+1}\rangle + kp\phi(v)H_1^{p-1}g^{k-1}\Delta g, \end{aligned} \tag{2.14}$$

where we have discarded the nonnegative term $p\phi(v)H_1^{p-1}F_{ij}h_{ij}h_{lm}h_{lm}$.

We now set

$$\phi(v) = v^q$$

for $q > 0$ to be chosen. Then

$$\phi' v - p\phi = (q-p)v^q \geq \frac{1}{2}qv^q \quad \text{if} \quad q \geq 2p. \tag{2.15}$$

Furthermore

$$\frac{2\phi'}{v} + \phi'' = q(q+1)v^{q-2}. \tag{2.16}$$

Next we have

$$\frac{p(\phi')^2}{\epsilon(p-1)\phi} = \frac{pq^2}{\epsilon(p-1)}v^{q-2}. \tag{2.17}$$

Eventually we want to fix $\epsilon \in (0,1)$ so that the coefficient of the third term on the right hand side of (2.14) is positive. It is apparent then that (2.17) cannot be controlled from above by the right hand side of (2.16) for all $p > 1$, so we argue a little differently.

We recall from [13] that

$$\nabla_i v = \nabla_i(\nu_{n+1}^{-1}) = -\nu_{n+1}^{-2}\nabla_i\nu_{n+1} = v^2\sum_{k=1}^n h_{ik}\langle\hat{\mathbf{e}}_k, \mathbf{e}_{n+1}\rangle.$$

$\hat{\mathbf{e}}_k$ is a unit tangent vector field to $M = \operatorname{graph} u$, so for each $k = 1, \ldots, n$ there is a unit vector field ξ_k defined near $0 \in \mathbf{R}^n$ such that

$$\hat{\mathbf{e}}_k = \frac{(\xi_k, D_{\xi_k}u)}{\sqrt{1 + |D_{\xi_k}u|^2}}.$$

Then

$$\langle \hat{\mathrm{e}}_k, \mathrm{e}_{n+1}\rangle = \frac{D_{\xi_k} u}{\sqrt{1+|D_{\xi_k} u|^2}}$$

and consequently

$$\begin{aligned} F_{ij}\nabla_i v \nabla_j v &= v^4 F_{ij} h_{ik} h_{jl} \frac{D_{\xi_k} u D_{\xi_l} u}{\sqrt{1+|D_{\xi_k} u|^2}\sqrt{1+|D_{\xi_l} u|^2}} \\ &\leq v^4 |Du|^2 F_{ij} h_{ik} h_{jk}. \end{aligned}$$

Thus, using (2.17)

$$\begin{aligned} \frac{p(\phi')^2}{\epsilon(p-1)\phi} H_1^p F_{ij}\nabla_i v\nabla_j v &\leq \frac{pq^2}{\epsilon(p-1)} |Du|^2 v^{q+2} H_1^p F_{ij} h_{ik} h_{jk} \\ &\leq \frac{1}{4} q v^q H_1^p F_{ij} h_{ik} h_{jk} \end{aligned}$$

provided

$$\frac{pq}{\epsilon(p-1)} |Du|^2 v^2 \leq \frac{1}{4}. \tag{2.18}$$

We now fix $\epsilon = 1/2$ and set $q = 2p$, so that (2.15) is valid. Then (2.18) holds whenever

$$|Du|^2 v^2 \leq \frac{p-1}{16p^2};$$

in particular, it holds if

$$|Du|^2 \leq \min\left\{1, \left(\frac{p-1}{32p^2}\right)\right\}. \tag{2.19}$$

And this is clearly valid for all $X \in M_{\rho_0} = M \cap B_{\rho_0}^{n+1}$ with sufficiently small $\rho_0 > 0$, depending only p and the modulus of continuity of ν.

Combining the above estimates we see that for any $p > 1$ we have

$$\begin{aligned} F_{ij}\nabla_i\nabla_j W \geq{}& \frac{p}{2} H_1^p F_{ij} h_{im} h_{jm} + (4p^2 + 2p) H_1^p F_{ij}\nabla_i v\nabla_j v \\ &+ \frac{1}{2} p(p-1) H_1^{p-2} F_{ij}\nabla_i H_1 \nabla_j H_1, \\ &+ 2pv^{2p+1} H_1^p \langle kg^{k-1}\nabla g, \mathrm{e}_{n+1}\rangle + kpv^{2p} H_1^{p-1} g^{k-1}\Delta g \end{aligned} \tag{2.20}$$

in M_{ρ_0}, for sufficiently small $\rho_0 > 0$ depending only p and the modulus of continuity of ν.

We now set $\eta(X) = \rho^2 - |X|^2$ where $\rho \in (0, \rho_0]$. Let $M_\rho = M \cap B_\rho^{n+1}$. Then $\partial M_\rho = M \cap \partial B_\rho^{n+1}$ because M and ∂B_ρ^{n+1} intersect transversally, since $B_{\rho_0}^{n+1} \cap M$ is a graph with small gradient; in fact, we may assume that

$$\sup_{B_{d_0}} |Du| \leq \frac{1}{2}. \tag{2.21}$$

We now multiply (2.20) by η and integrate over M_ρ. Integrating by parts twice and using the fact that F_{ij} is divergence free (see [8]),

$$\nabla_i F_{ij} = 0, \tag{2.22}$$

we obtain

$$\begin{aligned}
&\frac{p}{2}\int_{M_\rho} \eta H_1^p F_{ij} h_{im} h_{jm} + (4p^2+2p)\int_{M_\rho} \eta H_1^p F_{ij}\nabla_i v \nabla_j v\\
&\quad + \frac{1}{2}p(p-1)\int_{M_\rho} \eta H_1^{p-2} F_{ij}\nabla_i H_1 \nabla_j H_1\\
&\quad + 2p\int_{M_\rho} \eta v^{2p+1} H_1^p \langle k g^{k-1}\nabla g, \mathbf{e}_{n+1}\rangle + kp\int_{M_\rho} \eta v^{2p} H_1^{p-1} g^{k-1}\Delta g\\
&\le \int_{M_\rho} \eta F_{ij}\nabla_i\nabla_j W\\
&= -\int_{M_\rho} F_{ij}\nabla_i \eta \nabla_j W\\
&= \int_{M_\rho} W F_{ij}\nabla_i\nabla_j \eta - \int_{\partial M_\rho} W F_{ij}\nabla_i \eta \mathcal{N}_j
\end{aligned} \tag{2.23}$$

where $\mathcal{N}$ denotes the outer unit normal to ∂M_ρ in M, i.e., $\mathcal{N}$ is tangent to M and normal to ∂B_ρ^{n+1}.

We now proceed to estimate the last two integrals on the left hand side of (2.23). Using the bounds $1 \le v \le C(p)$ on M_ρ as appropriate without further mention, we estimate

$$\left|\int_{M_\rho} \eta v^{2p+1} H_1^p \langle k g^{k-1}\nabla g, \mathbf{e}_{n+1}\rangle\right| \le C(p) k\rho^2 \int_{M_\rho} H_1^p g^{k-1}|\nabla g|.$$

Next we integrate the last integral on the left hand side of (2.23) by parts. We have

$$\begin{aligned}
&\int_{M_\rho} \eta v^{2p} H_1^{p-1} g^{k-1}\Delta g\\
&= -(k-1)\int_{M_\rho} \eta v^{2p} H_1^{p-1} g^{k-2}|\nabla g|^2\\
&\quad - \int_{M_\rho} v^{2p} H_1^{p-1} g^{k-1}\nabla_i \eta \nabla_i g\\
&\quad - 2p\int_{M_\rho} \eta v^{2p-1} H_1^{p-1} g^{k-1}\nabla_i v \nabla_i g\\
&\quad - (p-1)\int_{M_\rho} \eta v^{2p} H_1^{p-2} g^{k-1}\nabla_i H_1 \nabla_i g.\\
&=: -(k-1)I_1 - I_2 - 2pI_3 - (p-1)I_4.
\end{aligned} \tag{2.24}$$

We estimate these as follows:

$$0 \le I_1 \le C(p)\rho^2 \int_{M_\rho} H_1^{p-1} g^{k-2} |\nabla g|^2;$$

$$\begin{aligned} |I_2| &\le C(p)\rho \int_{M_\rho} H_1^{p-1} g^{k-1} |\nabla g| \\ &\le C(p) \int_{M_\rho} H_1^{p-1} (g^k + \rho^2 g^{k-2} |\nabla g|^2); \end{aligned}$$

$$\begin{aligned} |I_3| &\le C(p)\theta \int_{M_\rho} \eta H_1^p F_{ij} \nabla_i v \nabla_j v + C(p)\theta^{-1} \int_{M_\rho} \eta H_1^{p-2} g^{2k-2} F_{ij}^{-1} \nabla_i g \nabla_j g \\ &\le C(p)\theta \int_{M_\rho} \eta H_1^p F_{ij} \nabla_i v \nabla_j v + C(p)\theta^{-1}\rho^2 \int_{M_\rho} H_1^{p-1} g^{k-2} |\nabla g|^2 \end{aligned}$$

for any $\theta > 0$;

$$\begin{aligned} |I_4| &\le C(p)\theta \int_{M_\rho} \eta H_1^{p-2} F_{ij} \nabla_i H_1 \nabla_j H_1 + \theta^{-1} \int_{M_\rho} \eta H_1^{p-2} g^{2k-2} F_{ij}^{-1} \nabla_i g \nabla_j g \\ &\le C(p)\theta \int_{M_\rho} \eta H_1^{p-2} F_{ij} \nabla_i H_1 \nabla_j H_1 + C(p)\theta^{-1}\rho^2 \int_{M_\rho} H_1^{p-1} g^{k-2} |\nabla g|^2 \end{aligned}$$

for any $\theta > 0$. In these estimates $[F_{ij}^{-1}]$ denotes the inverse matrix of $[F_{ij}]$, and we have used the inequality

$$[F_{ij}] \ge \frac{g^k}{H_1} I, \tag{2.25}$$

or more precisely, its equivalent form

$$[F_{ij}^{-1}] \le \frac{H_1}{g^k} I \tag*{(2.25)$'$}$$

(see [13]).

Using these estimates in (2.23) and fixing $\theta > 0$ sufficiently small, depending on p, we obtain

$$\begin{aligned} &- \int_{M_\rho} W F_{ij} \nabla_i \nabla_j \eta + \int_{\partial M_\rho} W F_{ij} \nabla_i \eta \mathcal{N}_j \\ \le& C(k,p)\rho^2 \int_{M_\rho} H_1^p g^{k-1} |\nabla g| + C(k,p) \int_{M_\rho} H_1^{p-1} (g^k + \rho^2 g^{k-2} |\nabla g|^2). \end{aligned} \tag{2.26}$$

Next we compute the derivatives of η. Clearly $\{\mathbf{e}_i\}$ and $\{\hat{\mathbf{e}}_i\}$ are related by

$$\mathbf{e}_i = \sum_{j=1}^{n+1} c_{ij} \hat{\mathbf{e}}_j, \qquad \hat{\mathbf{e}}_k = \sum_{i=1}^{n+1} c_{ik} \mathbf{e}_i, \tag{2.27}$$

where $c_{ij} = \langle e_i, \hat{e}_j \rangle$ is an orthogonal matrix. We have

$$\nabla_i X_k = c_{ki} \tag{2.28}$$

and by Gauss's formula

$$\nabla_i \nabla_j X_k = h_{ij} \nu_k = h_{ij} c_{k,n+1}. \tag{2.29}$$

Therefore

$$\begin{aligned}\nabla_i \nabla_j \eta &= -2\langle \nabla_i X, \nabla_j X \rangle - 2\langle X, \nabla_i \nabla_j X \rangle \\ &= -2 \sum_{k=1}^{n+1} c_{ki} c_{kj} - 2h_{ij} \langle X, \nu \rangle \\ &= -2\delta_{ij} - 2h_{ij} \langle X, \nu \rangle\end{aligned}$$

by the orthogonality of $[c_{ij}]$. Consequently

$$\begin{aligned}F_{ij} \nabla_i \nabla_j \eta &= -2 \sum_{i=1}^{n} F_{ii} - 2F_{ij} h_{ij} \langle X, \nu \rangle \\ &= -2(n-k+1) H_{k-1} - 2kg^k \langle X, \nu \rangle.\end{aligned} \tag{2.30}$$

In addition, by (2.28) we have, on ∂M_ρ,

$$\begin{aligned}-F_{ij} \nabla_i \eta \mathcal{N}_j &= 2F_{ij} \langle X, \nabla_i X \rangle \mathcal{N}_j \\ &= 2F_{ij} \left(\sum_{k=1}^{n+1} X_k c_{ki} \right) \mathcal{N}_j \\ &= 2F_{ij} X_i^\top \mathcal{N}_j \\ &\leq 2\rho \sum_{i=1}^{n} F_{ii} \\ &= 2\rho(n-k+1) H_{k-1},\end{aligned}$$

where $X^\top$ is the tangential part of X, i.e., the orthogonal projection of X onto the tangent space $T_X M$. Using this in (2.26) and estimating the last term (2.30) we arrive at

$$\begin{aligned}\int_{M_\rho} v^{2p} H_1^p H_{k-1} &\leq \rho \int_{\partial M_\rho} v^{2p} H_1^p H_{k-1} \\ &+ C\rho \int_{M_\rho} H_1^p (g^k + \rho g^{k-1} |\nabla g|) + C \int_{M_\rho} H_1^{p-1} (g^k + \rho^2 g^{k-2} |\nabla g|^2),\end{aligned} \tag{2.31}$$

where C depends only on k, n and p. This is inequality (1.3). This completes the proof of Theorem 1.1.

Theorem 1.2 follows easily from Theorem 1.1. For $\rho \in (0, \rho_0)$ we have

$$\nabla|X| = \frac{X^\top}{|X|} \quad \text{and} \quad \frac{1}{2} < |\nabla|X|| \leq 1 \quad \text{in} \quad M_\rho - \{0\}, \tag{2.32}$$

making ρ_0 smaller if necessary. Therefore

$$\int_{\partial M_\rho} v^{2p} H_1^p H_{k-1} \leq \int_{\partial M_\rho} \frac{v^{2p} H_1^p H_{k-1}}{|\nabla|X||} = \frac{d}{d\rho} \int_{M_\rho} v^{2p} H_1^p H_{k-1}. \tag{2.33}$$

The last line follows by differentiation with respect to ρ of the coarea formula

$$\int_{M_\rho} v^{2p} H_1^p H_{k-1}\, d\mathcal{H}^n = \int_0^\rho \int_{\partial M_t} \frac{v^{2p} H_1^{p+1} H_{k-1}}{|\nabla|X||}\, d\mathcal{H}^{n-1} dt$$

(see [9], Chapter 10, or [4] Theorem 3.2.22). We now get the monotonicity formula (1.5) by using (2.33) in (2.31), dividing by ρ^2 and integrating with respect to ρ from r to R. This completes the proof of Theorem 1.2.

We now show how Theorem 1.3 follows from Theorem 1.2. We assume now that $k = 2$ and $q > n - 1$. Then $2q > 2n - 2 > n$, so $g \in W^{1,2q}(M)$ implies that $g \in L^\infty_{loc}(M)$, by the Sobolev embedding theorem. We now estimate the last two terms in (1.3). By straightforward computation using the Hölder and Young inequalities we find (setting $q = p + 1$) that

$$\begin{aligned} &\int_r^R \left(\frac{1}{\rho} \int_{M_\rho} H_1^p (g^2 + \rho g |\nabla g|) \right) d\rho \\ \leq &\frac{C}{R} \left\{ \int_{M_R} H_1^q + R^{n+q} \sup_{M_R} g^{2q} + R^{3q+n} \sup_{M_R} |\nabla g|^{2q} \right\} \end{aligned} \tag{2.34}$$

and

$$\begin{aligned} &\int_r^R \left(\frac{1}{\rho^2} \int_{M_\rho} H_1^{p-1} (g^2 + \rho^2 |\nabla g|^2) \right) d\rho \\ \leq &\frac{C}{R} \left\{ \int_{M_R} H_1^q + R^n \sup_{M_R} g^q + R^{q+n} \sup_{M_R} |\nabla g|^q \right\}. \end{aligned} \tag{2.35}$$

Since v^{2p} is bounded between two positive constants on M_R, we see that for any $r \in (0, R]$, $r^{-1} \int_{M_r} H_1^q$ is controlled by the terms appearing on the right hand sides of (2.34) and (2.35). Furthermore, H_1 is equivalent to the length of the second fundamental form of M, because M is 2-admissible (see [13]). Since $|Du| \leq 1$, we see that for $r \in (0, R/2]$, $r^{-1} \int_{B_r^n} |D^2 u|^q$ is controlled by the same quantities. A Hölder gradient estimate

$$\begin{aligned} \sup_{\substack{x,y \in B_r^n \\ x \neq y}} \frac{|Du(x) - Du(y)|}{|x-y|^\alpha} \leq C r^{-1/q} \Bigg\{ &\int_{M_{4r}} H_1^q + r^{n+q} \sup_{M_{4r}} g^{2q} \\ &+ r^{3q+n} \sup_{M_{4r}} |\nabla g|^{2q} + r^n \sup_{M_{4r}} g^q + r^{q+n} \sup_{M_{4r}} |\nabla g|^q \Bigg\}^{\frac{1}{q}} \end{aligned}$$

with $\alpha = 1 - (n-1)/q$ and any $r \in (0, R/4]$ then follows from Morrey's estimate [5], Theorem 7.19.

3. Interior curvature bounds

In this section we will use the estimate of Theorem 1.1 to improve our interior curvature estimates in [13]. We will use some results from that paper.

The following result was obtained in [13] in the course of proving the curvature bounds.

Lemma 3.1. *Let M, g be as in Theorem* 1.4. *Then for any $q > 0$ there is a number $\rho_1 \in (0, d_0]$, depending only on q and the modulus of continuity of ν on $M \cap B^{n+1}_{d_0}$, such that for any $\rho \in (0, \rho_1]$ we have*

$$\int_{M_\rho} H_1^{q-1} F_{ij} \nabla_i H_1 \nabla_j H_1 \leq \frac{C(q+1)^3}{q^2 \rho^2} \int_{M_{2\rho}} H_1^{q+1} H_{k-1}, \tag{3.1}$$

where C depends only on k, n and μ, where μ is the constant from (1.8).

The estimate (3.1) is essentially inequality (2.30) from [13], which is

$$\int_{M_{2\rho}} \eta^2 H_1^{q-1} F_{ij} \nabla_i H_1 \nabla_j H_1 \leq \frac{C(q+1)^3}{q^2 \rho^2} \int_{M_{2\rho}} H_1^{q+k} \tag*{(3.1)$'$}$$

for a nonnegative function $\eta \in C_0^\infty(B^{n+1}_{2\rho})$ with $\eta = 1$ on B^{n+1}_ρ and $|D\eta| \leq C\rho^{-1}$. The differences between the two arise because in (3.1) we have not used the estimate

$$H_{k-1} \leq C(k,n) H_1^{k-1} \tag{3.2}$$

on the right hand side, and we have estimated the left hand side of (3.1)$'$ from below by the left hand side of (3.1). These estimates are proved by multiplying (2.8) by $\eta^2 \phi(v) H_1^{q+1}$, integrating by parts, making a suitable choice of ϕ, and estimating various terms, very much as we did in Section 2. Inequality (3.2) is a consequence of the Maclaurin inequalities

$$\left[\binom{n}{m}^{-1} S_m(\lambda)\right]^{1/m} \leq \left[\binom{n}{l}^{-1} S_l(\lambda)\right]^{1/l} \quad \text{for } 1 \leq l \leq m \leq n,\ \lambda \in \Gamma_m. \tag{3.3}$$

In [13] we also assumed that the $C^{0,\alpha}$ norm of ν was under control for some $\alpha > 0$; consequently we obtained a more explicit bound for ρ_1. However, it is clear from the proof that any modulus of continuity for ν suffices.

In [13] we proceeded from (3.1)$'$ by applying the Sobolev inequality of Allard [1] and Michael and Simon [6] to eventually get

$$\left(\frac{1}{\mathcal{H}^n(M_r)} \int_{M_r} H_1^{\beta q}\right)^{1/\beta} \leq C(q+1)^\gamma \left(\frac{1}{\mathcal{H}^n(M_{2r})} \int_{M_{2r}} H_1^{q+k}\right), \tag{3.4}$$

where $\beta = n/(n-2)$ and γ, C are positive constants independent of p and r, without requiring r to be small (for this we require a $C^{0,\alpha}$ modulus of continuity for ν). The inequality (3.4) was then iterated to obtain the curvature bound. The condition $q + k > kn/2$ guarantees that the exponent of integrability of H_1 improves at each step of the iteration. The number β is determined by the Sobolev exponent in n dimensions, while the exponent of H_1 on the right hand side of (3.4) is $k + q$ rather than q because of the ellipticity bounds

$$\frac{c_0}{H_1} I \le [F_{ij}] \le C_1 H_{k-1} I \le C_2 H_1^{k-1} I, \tag{3.5}$$

Here we show that with the aid of an estimate such as (1.3) it is possible to obtain a variant of (3.4) with $\beta = (n-1)/(n-3)$, which comes from the Sobolev exponent in $n-1$ dimensions. An analogous procedure was used in [14] to obtain a corresponding improvement for k-Hessian equations.

We will perform a finite iteration to improve the exponent of integrability of H_1 enough to appeal to [13] to deduce the curvature bound. Since only a finite iteration will be used, we can ignore the precise dependence of various constants on q; however, the dependence on ρ needs to be kept explicit for part of the proof. We assume therefore that q is always bounded from above by some large number $q^* < \infty$; then $\rho < \rho^*$ for some small positive number ρ^* depending only on q^* and the modulus of continuity of ν. A further smallness condition on ρ will arise in the subsequent proof.

We begin by simplifying our key estimates. First, after estimating the integrand on the left hand side of (3.1) from below using (3.5), we obtain

$$\int_{M_\rho} \left|\nabla\left(H_1^{\frac{q}{2}}\right)\right|^2 \le C \int_{M_{2\rho}} H_1^{q+1} H_{k-1}, \tag{3.6}$$

for any $\rho \in (0, \rho^*]$, where C depends on k, n, q, μ, ρ and q^*.

Next, the estimate (1.3) reduces to

$$\int_{M_\rho} H_1^p H_{k-1} \le C_1 \rho \int_{\partial M_\rho} H_1^p H_{k-1} + C_2 \rho \int_{M_\rho} H_1^p + C_3 \int_{M_\rho} H_1^{p-1} \tag{3.7}$$

for any $p > 1$, where C_1, C_2 and C_3 depend only on k, n and p, and $\rho \le \rho_0$ where ρ_0 depends only on p and the modulus of continuity of ν. Using the fact that

$$H_{k-1} \ge c(k,n)(H_k)^{(k-1)/k} = c(k,n) g^{k-1}, \tag{3.8}$$

for ρ sufficiently small, say $0 < \rho \le \rho_2$, we can absorb the second and third terms on the right side of (3.7) into the left side, to obtain

$$\int_{M_\rho} H_1^p H_{k-1} \le C_1 \rho \int_{\partial M_\rho} H_1^p H_{k-1} + C_2, \tag{3.9}$$

for new constants C_1, C_2. We will use this with a suitable choice of p, depending on q.

We will use the Sobolev inequality of Allard [1] and Michael and Simon [6], but it will be applied a little differently than in [13]. We will use it in the following form: for any smooth submanifold $\Sigma \subset \mathbf{R}^{n+1}$ of dimension $n-1$, any function $w \in C_0^1(\Sigma)$ and any $r \in [1, n-1)$ we have

$$\begin{aligned} &\left(\int_\Sigma |w|^{(n-1)r/(n-1-r)}\right)^{(n-1-r)/(n-1)r} \\ \leq &C(n,r)\left(\int_\Sigma |\nabla w|^r + \int_\Sigma |\mathbf{H}_\Sigma w|^r\right)^{1/r}, \end{aligned} \tag{3.10}$$

where $\mathbf{H}_\Sigma$ denotes the mean curvature vector of Σ and ∇^Σ denotes the tangential gradient operator on Σ.

Let us assume for the moment that $n \geq 4$; we will indicate the modifications that need to be made in the case $n = 3$ later. We apply (3.10) with $\Sigma = \Sigma_t = M \cap \partial B_t^{n+1}$, $t \in [\rho/2, \rho]$, $w = H_1^{q/2}$ and $r = 2$ to get

$$\begin{aligned} \left(\int_{\Sigma_t} H_1^{\frac{q(n-1)}{n-3}}\right)^{\frac{n-3}{n-1}} &\leq C(n)\left\{\int_{\Sigma_t}\left|\nabla^{\Sigma_t}\left(H_1^{\frac{q}{2}}\right)\right|^2 + \int_{\Sigma_t}|\mathbf{H}_{\Sigma_t}|^2 H_1^q\right\} \\ &\leq C\left\{\int_{\Sigma_t}\left|\nabla\left(H_1^{\frac{q}{2}}\right)\right|^2 + \rho^{-2}\int_{\Sigma_t} H_1^q + \int_{\Sigma_t} H_1^{q+2}\right\}, \end{aligned} \tag{3.11}$$

with C independent of ρ. To obtain the last two terms we have used the fact that

$$|\mathbf{H}_{\Sigma_t}| \leq C(\rho^{-1} + H_1), \tag{3.12}$$

because $t \in [\rho/2, \rho]$, and because M and ∂B_t^{n+1} can be assumed to have intersection angle bounded away from zero at each point of intersection. We defer the proof of (3.12) to the end of the paper.

Next we need to deal with the term $\int_{\Sigma_t} H_1^{q+2}$ in (3.11). This could be done slightly more simply at a later stage of the proof by using the Sobolev inequality (3.10) in n dimensions, but the argument we use now would still be needed to deal with the case $n = 3$. If the term in question is left as it is, the iteration inequality that we eventually obtain has a term $\int_{M_{4\rho}} H_1^{q+2}$ on the right hand side, forcing us to start the iteration at too large a value of q. For $k = 2$, however, this term causes no difficulties. We will now show that for $k \geq 3$, the term in question can be absorbed into the left hand side of (3.11) for a sufficiently large set of $t \in [\rho/2, \rho]$. This will be sufficient for the proof.

By Hölder's inequality we have

$$\int_{\Sigma_t} H_1^{q+2} \leq \left(\int_{\Sigma_t} H_1^{\frac{q(n-1)}{n-3}}\right)^{\frac{n-3}{n-1}}\left(\int_{\Sigma_t} H_1^{n-1}\right)^{\frac{2}{n-1}}. \tag{3.13}$$

Consequently it is sufficient to show that

$$\left(\int_{\Sigma_t} H_1^{n-1}\right)^{\frac{2}{n-1}} \leq \frac{1}{2C} \tag{3.14}$$

where C is the constant from (3.11).

We will use the following two facts. First, since M_ρ is a graph with small gradient, we have

$$\mathcal{H}^n(M_t) \leq C_1\rho^n \tag{3.15}$$

for each $t \in (0,\rho]$. Second, we are assuming that for some $s > k(n-1)/2 \geq n-1$ we have

$$\int_{M_\rho} H_1^s \leq C_2. \tag{3.16}$$

By the coarea formula and the fact that $|\nabla|X|| \leq 1$ we have

$$\int_{\rho/2}^{\rho} \mathcal{H}^{n-1}(\Sigma_t)dt = \int_{M_\rho - M_{\rho/2}} |\nabla|X|| \leq \mathcal{H}^n(M_\rho) \leq C_1\rho^n.$$

Thus for any $\epsilon \in (0,1)$ we have

$$\mathcal{H}^n(\Sigma_t) \leq \frac{2C_1\rho^{n-1}}{\epsilon} \tag{3.17}$$

for t belonging to a subset $I = I(\epsilon) \subset [\rho/2,\rho]$ of measure at least $(1-\epsilon)\rho/2$. By the coarea inequality again we have

$$\int_{\rho/2}^{\rho}\left(\int_{\Sigma_t} H_1^s\right)dt = \int_{M_\rho - M_{\rho/2}} |\nabla|X||H_1^s \leq \int_{M_\rho} H_1^s \leq C_2.$$

Therefore for any $\epsilon' \in (0,1)$ we have

$$\int_{\Sigma_t} H_1^s \leq \frac{2C_2}{\epsilon'\rho} \tag{3.18}$$

for t belonging to a subset $J = J(\epsilon') \subset [\rho/2,\rho]$ of measure at least $(1-\epsilon')\rho/2$.

Let us now fix $\epsilon = \epsilon' = 1/8$. Then $|I \cap J| \geq 3\rho/8$, and for $t \in I \cap J$ we have, by Hölder's inequality and (3.17), (3.18),

$$\begin{aligned}\int_{\Sigma_t} H_1^{n-1} &\leq \left(\int_{\Sigma_t} H_1^s\right)^{\frac{n-1}{s}} \left(\mathcal{H}^{n-1}(\Sigma_t)\right)^{1-\frac{n-1}{s}} \\ &\leq \left(\frac{16C_2}{\rho}\right)^{\frac{n-1}{s}} \left(16C_1\rho^{n-1}\right)^{1-\frac{n-1}{s}} \\ &= C_3\rho^{(n-1)\left(1-\frac{n}{s}\right)}.\end{aligned}$$

The exponent of ρ in the last line is positive because $s > k(n-1)/2$ and $k \geq 3$. Therefore (3.14) follows for sufficiently small ρ, for all $t \in I \cap J \subset [\rho/2,\rho]$.

Returning to (3.11), we now have, for $k \geq 3$ and for sufficiently small ρ,

$$\left(\int_{\Sigma_t} H_1^{\frac{q(n-1)}{n-3}}\right)^{\frac{n-3}{n-1}} \leq C\left\{\int_{\Sigma_t}\left|\nabla\left(H_1^{\frac{q}{2}}\right)\right|^2 + \int_{\Sigma_t} H_1^q\right\}, \tag{3.19}$$

for all $t \in I \cap J \subset [\rho/2, \rho]$, where now the constant C depends also on ρ. For $k = 2$ we have instead

$$\left(\int_{\Sigma_t} H_1^{\frac{q(n-1)}{n-3}}\right)^{\frac{n-3}{n-1}} \leq C\left\{\int_{\Sigma_t} \left|\nabla\left(H_1^{\frac{q}{2}}\right)\right|^2 + \int_{\Sigma_t} H_1^{q+2}\right\}, \qquad (3.19)'$$

for all $t \in [\rho/2, \rho]$.

Let us now assume that $k \geq 3$. By Hölder's inequality, the estimate (3.4) and Young's inequality, for any $p > 0$ we have

$$\begin{aligned}\int_{\Sigma_t} H_1^p H_{k-1} &\leq \left(\int_{\Sigma_t} H_1^{\frac{p(q+k)}{q+1}}\right)^{\frac{q+1}{q+k}} \left(\int_{\Sigma_t} H_{k-1}^{\frac{q+k}{k-1}}\right)^{\frac{k-1}{q+k}} \\ &\leq C\left\{\int_{\Sigma_t} H_1^{\frac{p(q+k)}{q+1}} + \int_{\Sigma_t} H_1^{q+1}H_{k-1}\right\}.\end{aligned}$$

Applying this with

$$p = \frac{q(q+1)(n-1)}{(q+k)(n-3)} > 1$$

(this condition will be satisfied by our eventual choice of q) and using (3.9) and (3.19) we get, for all $t \in I \cap J$,

$$\begin{aligned}&\int_{M_t} H_1^{\frac{q(q+1)(n-1)}{(q+k)(n-3)}} H_{k-1} \\ \leq &\int_{\Sigma_t} H_1^{\frac{q(q+1)(n-1)}{(q+k)(n-3)}} H_{k-1} + C \\ \leq &C\left\{\int_{\Sigma_t} H_1^{\frac{q(n-1)}{n-3}} + \int_{\Sigma_t} H_1^{q+1}H_{k-1} + 1\right\} \\ \leq &C\left\{\left[\int_{\Sigma_t}\left|\nabla\left(H_1^{\frac{q}{2}}\right)\right|^2 + \int_{\Sigma_t} H_1^q\right]^{\frac{n-1}{n-3}} + \int_{\Sigma_t} H_1^{q+1}H_{k-1} + 1\right\}.\end{aligned}$$

Therefore

$$\begin{aligned}&\left(\int_{M_t} H_1^{\frac{q(q+1)(n-1)}{(q+k)(n-3)}} H_{k-1}\right)^{\frac{n-3}{n-1}} \\ \leq &C\left\{\int_{\Sigma_t}\left|\nabla\left(H_1^{\frac{q}{2}}\right)\right|^2 + \int_{\Sigma_t} H_1^{q+1}H_{k-1} + 1\right\}\end{aligned}$$

for all $t \in I \cap J$.

Next, integrating with respect to t over $I \cap J$ and using the coarea formula we get

$$\begin{aligned}&\int_{I\cap J}\left(\int_{M_t} H_1^{\frac{q(q+1)(n-1)}{(q+k)(n-3)}} H_{k-1}\right)^{\frac{n-3}{n-1}} dt \\ \leq &C\left\{\int_{M_\rho}\left|\nabla\left(H_1^{\frac{q}{2}}\right)\right|^2 + \int_{M_\rho} H_1^{q+1}H_{k-1} + 1\right\} \qquad (3.20) \\ \leq &C\int_{M_{2\rho}} H_1^{q+1}H_{k-1},\end{aligned}$$

where we have used (3.6) to estimate the gradient term, and where we have used that H_1 and H_{k-1} are bounded away from zero.

We now estimate the left hand side from below in an obvious way, using the facts that $I \cap J \subset [\rho/2, \rho]$ and $|I \cap J| \geq 3\rho/8$. After replacing $\rho/2$ by ρ, we finally arrive at the iteration inequality

$$\left(\int_{M_\rho} H_1^{\frac{q(q+1)(n-1)}{(q+k)(n-3)}} H_{k-1} \right)^{\frac{n-3}{n-1}} \leq C \int_{M_{4\rho}} H_1^{q+1} H_{k-1}. \tag{3.21}$$

A straightforward calculation now shows that the exponent of H_1 on the left is greater than the exponent of H_1 on the right, provided $q + k > k(n-1)/2$. Moreover, the improvement in the exponent increases as q increases. Therefore we may iterate (3.21) finitely many times to obtain a bound for the L^p norm of H_1 on M_r for small enough $r > 0$, for some $p > kn/2$. A bound for $H_1(0)$ then follows by appealing to curvature bound proved in [13]. Finally, as shown in [13], for 2-admissible hypersufaces a bound for H_1 is equivalent to a bound for the second fundamental form.

In the case $k = 2$ an almost identical argument leads to the estimate (3.21), because $H_1^{q+2} = H_1^{q+1} H_{k-1}$ if $k = 2$.

We now indicate the minor modifications that need to be made in the case $n = 3$. In this case, by (3.10) and Hölder's inequality we have, for any $r \in [0, 2)$,

$$\begin{aligned}
&\left(\int_{\Sigma_t} H_1^{\frac{qr}{2-r}} \right)^{\frac{2-r}{r}} \\
\leq & C(r) \left\{ \int_{\Sigma_t} \left| \nabla^{\Sigma_t} \left(H_1^{\frac{q}{2}} \right) \right|^r + \int_{\Sigma_t} |\mathbf{H}_{\Sigma_t}|^r H_1^{\frac{qr}{2}} \right\}^{\frac{2}{r}} \\
\leq & C(r) \left(\mathcal{H}^2(\Sigma_t) \right)^{\frac{2}{r}-1} \left\{ \int_{\Sigma_t} \left| \nabla^{\Sigma_t} \left(H_1^{\frac{q}{2}} \right) \right|^2 + \int_{\Sigma_t} |\mathbf{H}_{\Sigma_t}|^2 H_1^q \right\} \\
\leq & C(r) \left(\frac{2C_1\rho}{\epsilon} \right)^{\frac{2}{r}-1} \left\{ \int_{\Sigma_t} \left| \nabla \left(H_1^{\frac{q}{2}} \right) \right|^2 + \rho^{-2} \int_{\Sigma_t} H_1^q + \int_{\Sigma_t} H_1^{q+2} \right\}
\end{aligned} \tag{3.22}$$

for all $t \in I(\epsilon) \subset [\rho/2, \rho]$, where we have used (3.17). The positive power of ρ in the coefficient causes no difficulties in the subsequent argument. We now proceed exactly as before, with $(n-1)/(n-3)$ replaced by $r/(2-r)$ for any $r \in [1, 2)$ such that

$$\frac{q(q+1)r}{(q+k)(2-r)} > 1;$$

this is automatically satisfied for any $q > 0$, provided r is sufficiently close to 2. We note that r needs to be made sufficiently close to 2, depending on s, in the argument used to remove the term $\int_{\Sigma_t} H_1^{q+2}$.

We arrive at the inequality

$$\left(\int_{M_\rho} H_1^{\frac{q(q+1)r}{(q+k)(2-r)}} H_{k-1} \right)^{\frac{2-r}{r}} \leq C \int_{M_{4\rho}} H_1^{q+1} H_{k-1} \tag{3.23}$$

for all $r \in [1, 2)$ sufficiently close to 2, where now C depends on r in addition to the other quantities. There clearly is no need to iterate the inequality in this case. Notice, however, that $C \to \infty$ and $\rho \to 0$ as $r \to 2$.

This completes the proof of Theorem 1.4 except for the proof of the geometric estimate (3.12). This is a consequence of the following simple lemma.

Lemma 3.1. *Let M and N be two smooth n-dimensional submanifolds of $\mathbf{R}^{n+1}$ and let $\Sigma = M \cap N \neq \emptyset$. Suppose that that near a point $X_0 \in \Sigma$, M and N intersect transversally, so that*

$$|\langle \nu_M, \nu_N \rangle| \geq \lambda > 0 \quad on \quad \Sigma \cap B_r^{n+1}(X_0) \tag{3.24}$$

for some positive constants λ and r, where ν_M and ν_N are the normal vector fields to M and N respectively. Then the mean curvature vector $\mathbf{H}_\Sigma$ of Σ satisfies

$$|\mathbf{H}_\Sigma(X_0)| \leq C(n, \lambda)(|A_M(X_0)| + |A_N(X_0)|) \tag{3.25}$$

where A_M and A_N are the second fundamental forms of M and N respectively.

Proof. Let $\mathbf{e}_1, \ldots, \mathbf{e}_{n-1}$ be a local orthonormal frame field on Σ near X_0. On Σ near X_0 we choose an orthonormal basis field $\mathbf{n}_1, \mathbf{n}_2$ for the normal space to Σ by defining $\mathbf{n}_1 = \nu_M$ and $\mathbf{n}_2 = a\nu_M + b\nu_N$, where a and b are chosen so that $\mathbf{n}_1$ and $\mathbf{n}_2$ are orthogonal and $|\mathbf{n}_2| = 1$. (3.24) guarantees that this can be done with a, b bounded by a constant depending only on λ. We may assume that $\mathbf{e}_1, \ldots, \mathbf{e}_{n-1}$, $\mathbf{n}_1$ and $\mathbf{n}_2$ have been extended to a neighbourhood in $\mathbf{R}^{n+1}$ of X_0.

Let D denote the standard connection on $\mathbf{R}^{n+1}$. Then by definition

$$\mathbf{H}_\Sigma = \sum_{j=1}^{2} \left\langle \sum_{\alpha=1}^{n-1} D_{\mathbf{e}_\alpha} \mathbf{e}_\alpha, \mathbf{n}_j \right\rangle \mathbf{n}_j.$$

Substituting the expressions for $\mathbf{n}_1$ and $\mathbf{n}_2$ into this we find that

$$\begin{aligned}
\mathbf{H}_\Sigma &= (1 + a^2) \left\langle \sum_{\alpha=1}^{n-1} D_{\mathbf{e}_\alpha} \mathbf{e}_\alpha, \nu_M \right\rangle \nu_M + ab \left\langle \sum_{\alpha=1}^{n-1} D_{\mathbf{e}_\alpha} \mathbf{e}_\alpha, \nu_M \right\rangle \nu_N \\
&\quad + ab \left\langle \sum_{\alpha=1}^{n-1} D_{\mathbf{e}_\alpha} \mathbf{e}_\alpha, \nu_N \right\rangle \nu_M + b^2 \left\langle \sum_{\alpha=1}^{n-1} D_{\mathbf{e}_\alpha} \mathbf{e}_\alpha, \nu_N \right\rangle \nu_N \\
&= (1 + a^2) \sum_{\alpha=1}^{n-1} h^M_{\alpha\alpha} \nu_M + ab \left(\sum_{\alpha=1}^{n-1} h^M_{\alpha\alpha} \nu_N + \sum_{\alpha=1}^{n-1} h^N_{\alpha\alpha} \nu_M \right) \\
&\quad + b^2 \sum_{\alpha=1}^{n-1} h^N_{\alpha\alpha} \nu_N,
\end{aligned}$$

where ν_M, ν_N and h^M_{ij}, h^N_{ij} are the normal vector fields and components of the second fundamental forms of M, N respectively. The estimate (3.25) now follows.

The estimate (3.12) follows by applying the lemma with $N = \partial B_t^{n+1}$. Since $t \in [\rho/2, \rho]$, $|A_N| \leq C(n)\rho^{-1}$. In addition, $|A_M| \leq CH_1$ because M is 2-admissible.

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