

# Solutions of the periodic Toda lattice via the projection procedure and by the algebra-geometric method

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*Dedicated to 75-th birthday of Vladimir Yakovlevich Fainberg*

## Abstract

In this short review we compare different ways to construct solutions of the periodic Toda lattice. We give two recipes that follow from the projection method and compare them with the algebra-geometric construction of Krichever.

## 1 Introduction

A large class of classical finite integrable systems can be derived from free systems by the symplectic reduction procedure. This class contains the rational and trigonometric Calogero-Moser systems [?, ?, ?], the open Toda Lattice [?, ?], elliptic Calogero-Moser systems [?, ?]. These systems are particular examples of the Hitchin type systems. These systems are constructed by the symplectic reduction from a free two-dimensional field theory defined over basic spectral curves [?]. As by product, this approach allows to construct the Lax representations and to describe solutions of the Cauchy problem by the projection method.

The projection method is working in the following way. Let  $\mathcal{R}$  will be the upstairs phase space, describing a free hierarchy and  $\mathcal{R}^{red}$  be the reduced phase space. It means that there exists a gauge group symmetry  $\mathcal{G}$  of the Hamiltonian hierarchy on  $\mathcal{R}$ . Then  $\mathcal{R}^{red}$  is defined in two steps. First step is the gauge fixing with respect to  $\mathcal{G}$ . Then one should solve the corresponding Gauss law (the moment constraint equation). The set of its solutions describes  $\mathcal{R}^{red}$ . This procedure is denoted as  $\mathcal{R}^{red} \sim \mathcal{R}/\mathcal{G}$ .

To solve the Cauchy problem one lifts the initial data  $x_0 \in \mathcal{R}^{red}$  to a point in the upstairs space  $\mathcal{R}$   $X_0 = \pi^{-1}(x_0)$ . The time evolution on  $\mathcal{R}$  is linear

$$X(t_1, t_2, \dots) = X_0 + \sum_j F_j(X_0)t_j.$$

The solution  $x(t_1, t_2, \dots)$  on  $\mathcal{R}^{red}$  is defined as the gauge transformation  $\pi$  (the projection) of  $X(t_1, t_2, \dots)$  to a point of the reduced phase space  $\mathcal{R}^{red}$

$$x(t_1, t_2, \dots) = \pi[X(t_1, t_2, \dots)].$$

The last step is the most difficult part of the projection method, but for some systems it becomes extremely simple. In particular, the rational and trigonometric Calogero-Moser systems and the open Toda lattice belong to this class. In these cases the solution of the Cauchy problem is reduced to purely algebraic calculations without quadratures [?, ?, ?, ?]. The specific feature of these systems is the simple dependence of the Lax operator on the spectral parameter, and as a result, the spectral curves are the rational curves with singularities.

On the other hand there exist algebro-geometric methods based on the finite gap integration (see, for example, the review [?]). For the just mentioned systems the algebra-geometric integration has been proposed fairly recently [?, ?]. The procedure looks more complicated than the projection method but leads to similar expressions. For more complicated Lax matrices the calculation of gauge coinvariant quantities that are needed in the projection method is a highly nontrivial problem and the algebra-geometric methods become preferable.

Here we compare these two approaches for an intermediate system. It is the periodic Toda lattice where the dependence on the spectral parameter is not very complicated. The projection method can be formulated in terms of the Riemann-Hilbert problem or the Gauss decomposition of infinite matrices. On the other hand the Krichever formula obtained by the algebra-geometric method and reproduced in [?, ?] gives an explicit solution to these problems.

## 2 Periodic Toda lattice as a reduced system

### 1. Definition of the periodic Toda lattice.

The periodic Toda hierarchy describes the one-dimensional  $N$ -body system with the coordinates  $\mathbf{u} = (u_1, \dots, u_N)$  and the momenta  $\mathbf{v} = (v_1, \dots, v_N)$ . It is a completely integrable Hamiltonian system with the canonical symplectic form

$$\omega = (\delta\mathbf{v}, \delta\mathbf{u}) = \sum_{j=1}^N \delta v_j \delta u_j \quad (2.1)$$

on the phase space of the Toda lattice

$$\mathcal{R}^T = (\mathbf{v}, \mathbf{u}), \quad \sum_{j=1}^N v_j = 0, \quad \sum_{j=1}^N u_j = 0.$$

The second order in momenta Hamiltonian is

$$H_2 = \frac{1}{2} \sum_{j=1}^N v_j^2 + \sum_{j=1}^{N-1} \exp(u_j - u_{j-1}) + \exp(u_N - u_1). \quad (2.2)$$

There are in addition  $N - 2$  commuting Hamiltonians providing the complete integrability of the system. They have the form

$$H_3 = \frac{1}{3} \sum_{j=1}^N v_j^3 + v_j (\exp(u_{j+1} - u_j) + \exp(u_j - u_{j-1})), \quad (u_{N+1} = u_N)$$

...

$$H_k = \frac{1}{k} \sum_{j=1}^N v_j^k + \dots$$

If the last term in the potential (??) is absent then the corresponding system is called the non-periodic Toda lattice.

The equations of motion can be represented in the Lax form

$$\partial_k = [L, M_k], \quad \partial_k = \partial_{t_k}, \quad (2.3)$$

where  $t_k$  is a time corresponding to the evolution with respect to the Hamiltonian  $H_k$  and  $L = L(\mathbf{v}, \mathbf{u})$ ,  $M_k(\mathbf{v}, \mathbf{u})$  are matrices of the  $N$ -th order depending on the dynamical variables (see below).

## 2. Reduction procedure.

Here we describe the reduction procedure for the periodic Toda lattice. In this way the periodic Toda lattice is described as a result of the symplectic reduction from a free hierarchy on the cotangent bundle to the loop group  $L(\mathrm{SL}(N, \mathbf{C}))$ . The loop group  $L(\mathrm{SL}(N, \mathbf{C}))$  is the space of maps  $C^* \rightarrow \mathrm{SL}(N, \mathbf{C})$

$$L\mathrm{SL}(N, \mathbf{C}) = \{g(z) |, \quad g(z) = \sum_{a=-\infty}^{\infty} g_a z^a, \quad g_a \in \mathrm{SL}(N, \mathbf{C}), \quad z \in C^*\},$$

where the only finite number of the matrix coefficients  $g_a$  non vanish. Let  $\Phi$  be a one form on  $C^*$  taking values in  $\mathrm{Lie}^*(\mathfrak{sl}(N, \mathbf{C})) \sim \mathfrak{sl}(N, \mathbf{C})$ . Then the pair  $(\Phi, g)$  parametrizes the cotangent bundle  $T^*L(\mathrm{SL}(N, \mathbf{C}))$ . The canonical symplectic form on  $T^*(L\mathrm{SL}(N, \mathbf{C}))$  is

$$\omega = \frac{1}{2\pi i} \oint_{|z|=1} \mathrm{tr}(\delta(\Phi g^{-1} \delta g) \frac{dz}{z}). \quad (2.4)$$

The form is invariant under the left and right multiplications

$$g \rightarrow g f^{(r)}, \quad \Phi \rightarrow (f^{(r)})^{-1} \Phi f^{(r)}, \quad f^{(r)} \in L(\mathrm{SL}(N, \mathbf{C})); \quad (2.5)$$

$$g \rightarrow f^{(l)} g, \quad \Phi \rightarrow \Phi, \quad f^{(l)} \in L(\mathrm{SL}(N, \mathbf{C})). \quad (2.6)$$

There is a set of gauge invariant Hamiltonians

$$H_k = \frac{1}{k} \oint_{|z|=1} \mathrm{tr}(\Phi^k) \frac{dz}{z}, \quad (k = 2, \dots, N). \quad (2.7)$$

Evidently, these Hamiltonians are in involution.

The equations of motion for the hamiltonian hierarchy (??),(??) is

$$\partial_j \Phi = 0, \quad g^{-1} \partial_j g = \Phi^{j-1}, \quad (\partial_j = \partial_{t_j}), \quad (2.8)$$

where  $t_j$  corresponds to the hamiltonian  $H_j$ . The integration of the equations gives

$$\Phi(t_j) = \Phi_0, \quad g(t_2, \dots, t_N) = g_0 \exp \left( \sum_{j=2}^N t_j \Phi_0^{j-1} \right). \quad (2.9)$$

Consider in  $\mathrm{SL}(N, \mathbf{C})$  the nilpotent subgroups  $N^{(+)}$  ( $N^{(-)}$ ) of the upper (lower) triangular matrices with the units on the diagonal. Let  $L^{(-)}$  be the subgroup of  $L(\mathrm{SL}(N, \mathbf{C}))$

$$L^{(-)} = \{g(z) |_{z \rightarrow \infty} \rightarrow n_- \in N^{(-)}\}, \quad (2.10)$$

and

$$L^{(+)} = \{g(z)|_{z \rightarrow 0} \rightarrow n_+ \in N^{(+)}\}. \quad (2.11)$$

Since the right action (??) of  $L^{(-)}$  is the canonical transformations of  $\omega$  (??), one can write down the corresponding hamiltonian generating (??)

$$\text{tr}(\Phi\epsilon), \quad \epsilon \in \text{Lie}(L^{(-)}),$$

and the moment map

$$\begin{aligned} \mu_R : T^*L(\text{SL}(N, \mathbf{C})) &\rightarrow \text{Lie}^*(L^{(-)}) \sim \text{Lie}(L^{(+)}) \\ \mu_R(\Phi, g) &= \text{Pr}|_{\text{Lie}(L^{(+)})}(\Phi). \end{aligned} \quad (2.12)$$

Similarly, the left action (??) of  $L^{(+)}$  leads to the the Hamiltonian

$$\text{tr}(g\Phi g^{-1}\epsilon), \quad \epsilon \in \text{Lie}(L^{(+)}),$$

and the moment map

$$\mu_L(\Phi, g) = \text{Pr}|_{\text{Lie}(L^{(-)})}(g\Phi g^{-1}). \quad (2.13)$$

We choose the following values of the momenta

$$\mu_R = \delta_{j,k-1} + z\delta_{j,N}\delta_{k,1}, \quad \mu_L = \delta_{j,k+1} + z^{-1}\delta_{j,1}\delta_{k,N}, \quad (2.14)$$

and thereby come to the moment constraints

$$\text{Pr}|_{\text{Lie}(L^{(+)})}(\Phi) = \delta_{j,k-1} + z\delta_{j,N}\delta_{k,1}, \quad (2.15)$$

$$\text{Pr}|_{\text{Lie}(L^{(-)})}(g\Phi g^{-1}) = \delta_{j,k+1} + z^{-1}\delta_{j,1}\delta_{k,N}. \quad (2.16)$$

This choice of the momenta is minimal, since the whole subgroup  $L^{(-)}$  preserves  $\mu_R$ , while  $L^{(+)}$  preserves  $\mu_L$ .

For an open dense subset of the loop group  $L(\text{SL}(N, \mathbf{C}))$  one has the Birkhoff decomposition [?].

$$g = r_+ h r_-, \quad r_+ \in L^{(+)}, \quad r_- \in L^{(-)}, \quad h \in \text{diagonal subgroup } D \subset \text{SL}(N, \mathbf{C}). \quad (2.17)$$

Let  $U_+, U_-$  be neighborhoods of the points  $z = 0$  and  $z = \infty$  such that  $\mathbf{C}P^1 = U_+ \cup U_-$ . The matrix function  $g(z)$  is the transition function of a trivial holomorphic vector  $\text{SL}(N, \mathbf{C})$  bundle over  $\mathbf{C}P^1$ . The left multiplications of  $g(z)$  by the holomorphic matrix functions on  $U_+$  are restricted to those that belong to  $L^{(+)}$ . Similarly, the admissible right multiplications are restricted to the functions from  $L^{(-)}$ . The conditions (??),(??) mean that the bundle has fixed quasi-parabolic structure in the points  $z = 0$  and  $z = \infty$ . The moduli space of these bundles are described by the elements of the Cartan subgroup  $D$ .

As we have mentioned the matrices from  $L^{(-)}$  and  $L^{(+)}$  diagonalizing  $g$  preserve the moments (??). Let us fix the gauge as

$$h = \text{diag}(e^{u_1}, \dots, e^{u_N}), \quad \sum_j u_j = 0. \quad (2.18)$$

Simultaneously, the matrix  $\Phi$  is transformed to the form

$$\Phi = r_-^{-1} L r_-. \quad (2.19)$$

Since the moments (??) are invariant under the gauge transform, one can substitute  $h$  and  $L$  in the constraints equations (??), (??)

$$\text{Pr}|_{\text{Lie}(L(+))}(L) = \delta_{j,k-1} + z\delta_{j,N}\delta_{k,1},$$

$$\text{Pr}|_{\text{Lie}(L(-))}(hLh^{-1}) = \delta_{j,k+1} + z^{-1}\delta_{j,1}\delta_{k,N}.$$

Then  $L$  is defined up to the main diagonal. Its matrix elements can be considered as the moduli of solutions of the constraint equations

$$\text{diag}(L) = \text{diag}(v_1, \dots, v_N), \quad \sum_j v_j = 0.$$

Finally, we come to the following solutions

$$\begin{aligned} L_{j,k} = & v_j\delta_{j,k} + \delta_{j,k-1} + \exp(u_j - u_{j-1})\delta_{j,k+1} + \\ & + z\delta_{j,N}\delta_{k,1} + z^{-1}\exp(u_N - u_1)\delta_{j,1}\delta_{k,N}. \end{aligned} \quad (2.20)$$

Therefore, the reduced phase space is described by a pair of matrices

$$\mathcal{R}^{red} = T^*(\text{LSL}(N, \mathbf{C})) // L^{(+)} \oplus L^{(-)} \sim (L, h).$$

It follows from (??) and (??) that the form (??) on  $\mathcal{R}^{red}$  is just the canonical form (??) of the Toda lattice. The Hamiltonians (??) being restricted on  $\mathcal{R}^{red}$  produce the Hamiltonians of the Toda hierarchy. They are in involution, because the original Hamiltonians commute. Thus the reduced system is the periodic Toda hierarchy.

Substituting (??) in the first equation (??) we conclude that  $L$  satisfies the Lax equation

$$\partial_j L = [L, M_j], \quad M_j = \partial_j r_- r_-^{-1}. \quad (2.21)$$

The matrices  $M_j$  can be found from the second equation (??). It leads to the relation between  $L$  and  $M_j$

$$h^{-1}r_+^{-1}\partial_j r_+ h + h^{-1}\partial_j h + M_j = L^{j-1}.$$

Since  $r_- \in L^{(-)}$ , it follows from (??) that  $M_j \in \text{Lie}(L^{(-)})$  and thereby

$$M_j = \text{Pr}|_{\text{Lie}(L^{(-)})} L^{j-1}. \quad (2.22)$$

In particular,

$$M_2 = \exp(u_j - u_{j-1})\delta_{j,k+1} + z^{-1}\exp(u_1 - u_N)\delta_{j,1}\delta_{k,N}.$$

The matrices  $L$  (??) and  $M_j$  (??) are the Lax pairs for the periodic Toda hierarchy.

The open Toda lattice can be derived in the similar way if one starts with the finite-dimensional group  $\text{SL}(N, \mathbf{C})$  instead of the loop group and use the Gauss decomposition instead of the Birkhoff decomposition. The corresponding Lax operator is given by (??), where the last two terms depending on the spectral parameter are absent. The  $M_j$  operators are expressed through the  $L$  in the same way (??).

This derivation can be repeat in the same way for any (twisted) affine algebra  $L(G)$ .

### 3 Solutions via the projection method

#### 1. Linearization.

Let  $\mathbf{u}^0 = (u_1^0, \dots, u_N^0)$  and  $\mathbf{v}^0 = (v_1^0, \dots, v_N^0)$  be the Cauchy data for the equations of motions for the Toda hierarchy. We assume that at  $t_j = 0$ ,  $j = 2, \dots, N$

$$g(t_2, \dots, t_N)|_{t_j=0} = h(t_2, \dots, t_N)|_{t_j=0} = h_0,$$

$$\Phi(t_2, \dots, t_N)|_{t_j=0} = L(t_2, \dots, t_N)|_{t_j=0} = L_0.$$

Then, according to (??), we have the linear evolution in the upstairs space

$$g(\mathbf{t}) = X(\mathbf{v}^0, \mathbf{u}^0, \mathbf{t}, z),$$

where  $\mathbf{t} = (t_2, \dots, t_n)$  and

$$X(\mathbf{v}^0, \mathbf{u}^0, \mathbf{t}, z) = h_0 \exp \sum_{j=2}^N t_j L_0^{j-1}(z). \quad (3.1)$$

The Toda coordinates are obtained from the diagonal part of the Birkhoff decomposition (??).

$$h(\mathbf{t}) = r_+ X(\mathbf{v}^0, \mathbf{u}^0, \mathbf{t}, z) r_- \rightarrow \mathbf{u}(\mathbf{t}) = \log(r_+ X r_-). \quad (3.2)$$

The first step in the projection procedure is the calculation  $X(\mathbf{v}^0, \mathbf{u}^0, \mathbf{t}, z)$  from the Cauchy data (??). Then one should diagonalize  $X(\mathbf{v}^0, \mathbf{u}^0, \mathbf{t}, z)$  by means of the Birkhoff formula. This can be achieved by the two ways - through the solution of the Riemann problem and by the diagonalization in the Gauss decomposition of  $GL(\infty)$ . Before describing these methods we solve the Cauchy problem for the simpler case - the open Toda lattice.

#### 2. Solutions of the open Toda Lattice.

In this case as we explained the matrix  $X(\mathbf{v}^0, \mathbf{u}^0, \mathbf{t})$  does not depend on the spectral parameter and in (??)  $r_+, r_-$  are the constant triangular matrices  $r_+ \in N^{(+)}$ ,  $r_- \in N^{(-)}$ . The diagonal elements  $h(\mathbf{t}) = \text{diag}(h_1(\mathbf{t}), \dots, h_N(\mathbf{t})) \in D$  are extracted from the Gauss decomposition

$$SL(N, \mathbf{C}) = N^{(+)} D N^{(-)}.$$

In fact, the Gauss decomposition is valid for an open dense subset of  $SL(N, \mathbf{C})$ . Let  $\Delta_j(g)$  be the principal lower minors of order  $j$ , ( $j = 1, \dots, N$ ,  $\Delta_N = 1$ ) of  $g \in SL(N, \mathbf{C})$ . Then this subset is described by the condition  $\Delta_j(g) \neq 0$ ,  $j = 1, \dots, N-1$ . Moreover,  $\Delta_j(g)$  are invariant with respect to the multiplication of  $g$  by a matrix from  $N^{(+)}$  from the right and by a matrix from  $N^{(-)}$  from the left. The relations of the minors define the diagonal elements of the Gauss decomposition

$$h_j = \frac{\Delta_{N-j+1}(X(\mathbf{v}^0, \mathbf{u}^0, \mathbf{t}, z))}{\Delta_{N-j}(X(\mathbf{v}^0, \mathbf{u}^0, \mathbf{t}))}.$$

Then we obtain

$$u_j = \log \frac{\Delta_{N-j+1}(X(\mathbf{v}^0, \mathbf{u}^0, \mathbf{t}, z))}{\Delta_{N-j}(X(\mathbf{v}^0, \mathbf{u}^0, \mathbf{t}))}. \quad (3.3)$$

In particular, it follows from (??) that the minors  $\Delta_j(\exp X(\mathbf{v}^0, \mathbf{u}^0, \mathbf{t}, z))$  are exponential polynomials on times.

### 3. Solutions via the Riemann problem.

Let us rewrite the Birkhoff decomposition (??) as

$$g = b\tilde{r}_-^{-1}, \quad (3.4)$$

where  $\tilde{r}_-^{-1}$  is a holomorphic matrix outside the contour  $|z| = 1$  that satisfies the normalization conditions

$$\lim_{z \rightarrow \infty} \tilde{r}_- \rightarrow \text{Id}. \quad (3.5)$$

In other words  $\tilde{r}_- = \text{Id} + \sum_{j>0} r_j z^{-j}$ . The matrix  $b$  is holomorphic inside the contour  $|z| = 1$ . Our ultimate goal is to find the diagonal,  $z$ -independent component of  $b$ . We do it through the solution of the Riemann problem. Namely, we find first  $\tilde{r}_-(z)$  starting from  $g(z)$ . Let us rewrite (??) as

$$b - \text{Id} = (g - \text{Id})\tilde{r}_- + \tilde{r}_- - \text{Id}.$$

Since  $b - \text{Id}$  is holomorphic in  $U_+$ ,  $\tilde{r}_-$  satisfies the matrix singular integral equation for  $\tilde{r}_-$

$$\frac{1}{2\pi i} \oint_{|w|=1} \frac{(g(w) - \text{Id})\tilde{r}_-(w)}{z - w} + \tilde{r}_-(z) - \text{Id} = 0. \quad (3.6)$$

with the normalization condition (??). Then the solution  $\tilde{r}_-(z)$  allows to find  $b$ .

Let  $b = b_0 + \sum_{j>0} b_j z^j$ . Then  $b_0 = \lim_{z \rightarrow 0} (g\tilde{r}_-)$ . One can use the Gauss decomposition of  $b_0$  to obtain  $h = \text{diag}(e^{u_1}, \dots, e^{u_N})$  in (??), as it was already done for the open Toda lattice (??).

Substituting in (??)  $g(w) = X$  (??) we find  $\tilde{r}_-(z)$ , and then

$$b(\mathbf{v}^0, \mathbf{u}^0, \mathbf{t}, z) = X(\mathbf{v}^0, \mathbf{u}^0, \mathbf{t}, z)\tilde{r}_-(z) = h(0, \dots, 0) \exp\left(\sum_{j=2}^N t_j L_0^{j-1}(z)\right) \tilde{r}_-(z).$$

Finally the solution takes the form

$$u_j(\mathbf{t}) = \lim_{z \rightarrow 0} \log \frac{\Delta_{N-j+1}(b(\mathbf{v}^0, \mathbf{u}^0, \mathbf{t}, z))}{\Delta_{N-j}(b(\mathbf{v}^0, \mathbf{u}^0, \mathbf{t}, z))}. \quad (3.7)$$

### 4. Solutions via embedding in $\text{GL}(\infty)$ .

We define  $\text{GL}(\infty)$  in algebraic terms. Details can be found in [?, ?]. Let  $\mathcal{H}$  be the Hilbert space of function on  $S^1$ . The group  $\text{GL}(\infty)$  is subgroup of all invertible bounded operators in  $\mathcal{H}$  such that the matrix elements  $M_{j,k}$  of  $\text{GL}(\infty)$  in the Fourier basis  $e^{im\varphi}$  vanish for large  $|j - k|$ .

The group  $L(\text{SL}(N, \mathbf{C}))$  can be mapped in  $\text{GL}(\infty)$ . Consider in  $\text{GL}(\infty)$  the subgroup of periodic matrices

$$\text{GL}_{\text{per}}(\infty) = \{M_{j+mN, k+mN} = M_{j,k}\}.$$

Let

$$g(z) \in L(\text{SL}(N, \mathbf{C})), \quad g(z) = \sum_n (g_{jk})_n z^n, \quad (j, k = 1 \dots N).$$

Then the image  $M \in \text{GL}_{\text{per}}(\infty)$  of  $g(z)$  is defined as

$$M_{s,t} = (g_{jk})_n, \quad \text{for } s = j + mN, \quad t = k + (m + n)N, \quad s, t, m \in \mathbf{Z}.$$

Under this map the image of  $L^{(-)}$  (??) belongs to the lower triangular matrices in  $\text{GL}_{\text{per}}(\infty)$ , and the image of  $L^{(+)}$  (??) belongs to the upper triangular matrices. Thus, the Birkhoff decomposition (??) corresponds to the Gauss decomposition in  $\text{GL}_{\text{per}}(\infty)$ . The constant diagonal loops give rise to the periodic diagonal matrices.

Let  $\tilde{X}(\mathbf{v}^0, \mathbf{u}^0, \mathbf{t})$  be the image of  $X(\mathbf{v}^0, \mathbf{u}^0, \mathbf{t}, z)$  (??) in  $GL_{per}(\infty)$ , and  $\det_j(\tilde{X})$  be the determinant of the semi-infinite matrix  $\tilde{X}_{m,n}$ ,  $m, n \in \mathbf{Z}$ ,  $m, n \leq j$ . This determinant is bad defined. But relations of them are well defined. As it follows from (??), the solutions can be expressed through the logarithm of the fraction

$$u_j(\mathbf{t}) = \log \frac{\Delta_{j+1}(\tilde{X})}{\Delta_j(\tilde{X})}. \quad (3.8)$$

This expression is related to the  $\tau$ -functions of the two-dimensional Toda hierarchy introduced in [?] (see below).

Thus we have two implicit formulae for the solutions of the Cauchy problem for (??) and (??) obtained by the projection method. We represent in next section the explicit expression in terms of the Riemann theta function.

## 4 Algebra-geometric calculations

Holomorphic vector bundles arise in a natural way in descriptions of integrable systems. One of this bundles related to the periodic Toda lattice was described above. We have demonstrated that the coordinate space is related to the moduli space of the bundle. The vector bundle can be reconstructed from the spectral characteristics of the Lax operator. Its open a way to to construct solutions starting from the spectral data (see, for example, the review [?]). Here we reproduce the solutions of the periodic Toda lattice constructed by Krichever [?, ?].

Let  $C_N$  be the spectral curve defined by the characteristic polynomial  $C_N(\lambda)$  of the Lax matrix  $L$

$$C_N(\lambda, z) = 0, \quad C_N(\lambda, z) = \det(L(z) - \lambda \text{Id}). \quad (4.1)$$

It follows from (??) that (??) is the gauge invariant equation. Due to the special form of the Lax operator (??) the curve is hyperelliptic

$$C_N(z, \lambda) = z + z^{-1} + R_N(\lambda), \quad R_N(\lambda) = \lambda^N + \sum_{k=2}^N I_k \lambda^{N-k},$$

where  $I_k$  are another types of the integrals of motion. The curve has genus  $N - 1$ . It is two-sheets cover of the  $\lambda$ -plane with the branching points  $\lambda_1, \dots, \lambda_{2N}$  as the roots of the equation  $R_N(\lambda) = 4$ .

The space of holomorphic differentials on  $C_N$  is generated by  $\lambda^j d\lambda/dy$ ,  $j = 0, \dots, N - 2$ , and  $y = z + \frac{1}{2}R_N(\lambda)$ . Take the linear combinations of them

$$\omega_i = \sum_{j=0}^{N-2} a_{ij} \lambda^j d\lambda/dy, \quad i = 1, \dots, N - 1$$

in a such a way that

$$2 \int_{\lambda_{2k-1}}^{\lambda_{2k}} \omega_i = \delta_{ik}, \quad k = 1, \dots, N - 1,$$

where the integration is taken over the upper sheet. The differentials  $\omega_i$  are the normalized differentials of the first kind. The matrix  $B$  of the  $b$ -periods

$$B_{jk} = 2 \int_{\lambda_{2j}}^{\lambda_{2N}} \omega_k$$



gives rise to the Jacobian  $\mathcal{J}(\mathcal{C}_N)$  of the curve  $\mathcal{C}_N$

$$\mathcal{J}(\mathcal{C}_N) = \mathbf{C}^{N-1} / (\oplus_j \mathbf{Z}e_j) \oplus (\oplus_k \mathbf{Z}B_k),$$

$$e_j = (0, \dots, \overset{j}{1}, 0, \dots, 0), \quad B_k = (B_{1k}, \dots, B_{N-1,k}).$$

The vector  $(\omega_1, \dots, \omega_{N-1})$  with the components

$$\omega_k(\gamma) = \int_{\lambda_{2N}}^{\gamma} \omega_k$$

defines the Abel map  $\mathcal{C}_N \rightarrow \mathcal{J}(\mathcal{C}_N)$ .

There is an isomorphism between the submanifold  $M_f$  in the phase space  $\mathcal{R}^T$  of levels of integrals and the Jacobian  $\mathcal{J}(\mathcal{C}_N)$ . In this way the Jacobian plays the role of the Liouville torus. For a point  $(\mathbf{v}, \mathbf{u}) \in \mathcal{R}^T$  define  $N-1$  points  $\gamma_j = (\mu_j, z_j)$  on the spectral curve  $\mathcal{C}_N$ . Here  $\mu_j$  is a root of the characteristic polynomial of the left principal minor of order  $N-1$  of the matrix  $L$  (??), while  $z_j$  is a root of (??) at  $\lambda = \mu_j$ , for which the right principle minor of  $L$  is non-degenerate. The isomorphism  $M_f \rightarrow \mathcal{J}(\mathcal{C}_N)$  is defined by the vector  $\mathbf{w} = (w_1, \dots, w_{N_1})$

$$w_k = - \sum_{n=1}^{N-1} \omega_k(\gamma_n) + \frac{1}{2} - \frac{1}{2} \sum_{j \neq k} \int_{\lambda_{2j-1}}^{\lambda_{2j}} \omega \left( \int_{\lambda_{2N}}^t \omega_k \right) dt. \quad (4.2)$$

Introduce two differentials

$$s^1 = \frac{\lambda^{N-1} d\lambda}{y(z, \lambda)} + \sum_{i=0}^{N-2} \alpha_i \frac{\lambda^i d\lambda}{y(z, \lambda)}, \quad s^2 = \frac{\lambda^N d\lambda}{y(z, \lambda)} + \sum_{i=0}^{N-2} \beta_i \frac{\lambda^i d\lambda}{y(z, \lambda)}.$$

They are normalized as

$$\int_{\lambda_{2j-1}}^{\lambda_{2j}} s^{1,2} = 0.$$

Let  $S^1, S^2, Z^\pm$  be the vectors with coordinates

$$S_j^{1,2} = \frac{1}{i\pi} \int_{\lambda_{2j}}^{\lambda_{2N}} s^{1,2}, \quad Z_j^\pm = \int_{\lambda_{2N}}^\pm \omega_j.$$

In the first two integrals the integration is performed on the upper sheet of the spectral curve  $\mathcal{C}_N$ . In the last integrals the integration is performed on the upper and lower sheet.

Now we are ready to define the solution of the Cauchy problem for the evolution with respect to the quadratic Hamiltonian  $H_2$ . The initial data  $(\mathbf{v}_0, \mathbf{u}_0)$  define the spectral curve, the matrix of periods  $B$ , the vectors  $S^{1,2}, Z^\pm$  and  $\mathbf{w}^0 = \mathbf{w}(\mathbf{v}_0, \mathbf{u}_0)$  (??). Let  $\Theta(\mathbf{x})$  be the Riemann theta function

$$\Theta(\mathbf{x}) = \sum_{\mathbf{m} \in \mathbf{Z}^{N-1}} \exp\{i\pi(B\mathbf{m}, \mathbf{m}) + 2i\pi(\mathbf{m}\mathbf{x})\}, \quad B = B_{jk},$$

Then the solution is defined as

$$u_j(t) = \log \frac{\Theta(Z^+ + \frac{1}{2}S^2t + S^1(j-1) + \mathbf{w}^0)\Theta(Z^- + \mathbf{w}^0)}{\Theta(Z^- + \frac{1}{2}S^2t + S^1(j-1) + \mathbf{w}^0)\Theta(Z^+ + \mathbf{w}^0)} + (j-1)C + u_j(0), \quad (4.3)$$

where

$$Z^\pm = (Z_1^\pm, \dots, Z_N^\pm), \quad S^{1,2} = (S_1^{1,2}, \dots, S_N^{1,2}),$$

and the constant  $C$  is chosen in a such a way that  $u_{N+1} = u_1$ .

This formula can be compared with two expressions obtained above by the projection method (??) and (??). It can be supposed that the relations of the theta functions are just the relation of the normalized determinant of the semi infinite matrix  $\tilde{X}$  in (??).

## 5 Other constructions

There exist two other approaches describing solutions of the periodic Toda lattice. So formally their starting points differ from the projection method they eventually are formulated in terms of the Riemann problem.

### 1. The tau-functions of the periodic Toda system.

In two papers [?, ?] the hierarchy of Toda field theory with infinite numbers of fields was considered. It was formulated in terms of  $GL(\infty)$ . This hierarchy depends on two infinite sets of times  $\mathbf{x} = (x_1, x_2, \dots)$ ,  $\mathbf{y} = (y_1, y_2, \dots)$ . The one-dimensional version depends only on the combinations  $\tilde{t}_2 = \frac{1}{2}(x_1 - y_1)$ ,  $\tilde{t}_3 = \frac{1}{2}(x_2 - y_2), \dots$ . In this case the infinite Toda hierarchy can be reduced to the finite set of the Lax equations. In fact, the times  $\tilde{t}_j$  and related to them operators  $\tilde{M}_j$  differ from (??). In terms of [?]

$$L = B_1 + C_1, \quad \tilde{M}_{j+1} = B_j - C_j,$$

where  $C_1 = M_2 = L_-$ ,  $B_1 = L_+ + \text{diag}L$ . Consider the following linear problem

$$LV(\mathbf{t}) = (\mu_L + \mu_R)V(\mathbf{t}), \quad (5.1)$$

$$\partial_j V(\mathbf{t}) = \tilde{M}_j V(\mathbf{t}), \quad j = 2, \dots, N, \quad \partial_j = \partial_{\tilde{t}_j},$$

where the matrices  $\mu_L$  and  $\mu_R$  are the moments (??). The Lax equations for  $L$  and  $\tilde{M}_j$  are the compatibility conditions for this linear system. The matrix function  $V(\mathbf{t})$  can be considered as a matrix wave-function dressing the naked operators  $\mu_L + \mu_R$  and  $\partial_j$ . Their matrix elements can be expressed as relations of the tau-functions  $\tau(\mathbf{x}, \mathbf{y})$ , defined in [?] for the whole two-dimensional hierarchy. The tau-functions for the one-dimensional periodic Toda lattice is a special subclass of the two-dimensional tau-functions, that independent on times  $\mathbf{x} + \mathbf{y}$  and satisfy certain periodicity conditions.

The solutions of the Cauchy problem in these terms, proposed in [?] can be constructed by the following steps. First, the initial data gives rise in (??) to the the wave function  $V^0 = V(\mathbf{t} = 0)$ . There exists a representation for the tau-functions for arbitrary values of times depending on the matrix elements of  $V^0$ . The wave-functions  $V(\mathbf{t})$  are expressed as relations of  $\tau_j(\mathbf{t})$ . The last step is the solution of the inverse problem in (??). Thus, the solving of the Cauchy problem can be illustrated by the following scheme

$$(\mathbf{v}^0, \mathbf{u}^0) \rightarrow (L^0, \tilde{M}_j^0) \rightarrow V^0 \rightarrow \tau(\mathbf{t}) \rightarrow V(\mathbf{t}) \rightarrow L(\mathbf{t}).$$

In these scheme the Riemann problem arises when one constructs the tau functions.

### 2. The R-matrix method.

As it follows from (??),(??) there exists the decomposition of the Lie algebra

$$\text{Lie}(L(\text{SL}(N, \mathbf{C}))) = \text{Lie}(L^{(+)}) \oplus \mathcal{H} \oplus \text{Lie}(L^{(-)}) = \text{Lie}(B^{(+)}) \oplus \text{Lie}(L^{(-)}), \quad (5.2)$$

where  $\mathcal{H}$  is the diagonal subalgebra of  $\text{SL}(N, \mathbf{C})$  and  $\text{Lie}(B^{(+)})$  is the Borel subalgebra. Define the projection

$$M_j^{(+)} = -\text{Pr}_{\text{Lie}(B^{(+)})} L^{j-1}.$$

Then

$$L^{j-1} = -M_j^{(+)} + M_j^{(-)},$$

where  $M_j^{(-)} = M_j$  (??). In other words

$$M_j^{(\pm)} = -\frac{1}{2}(R \pm 1)L^{j-1},$$

where

$$R = \text{Pr}|_{\text{Lie}(L^{(-)})} - \text{Pr}|_{\text{Lie}(B^{(+)})}$$

is the classical  $R$ -matrix corresponding to the decomposition (??). Therefore there exists a pair of the Lax equations for every time  $t_k$

$$\partial_k L = [L, M_j^{(\pm)}]. \quad (5.3)$$

In fact, (??) corresponds to the Birkhoff decomposition (??). The solutions of the Lax hierarchy (??) can be expressed in terms of the Birkhoff decomposition and thereby we come again to the Riemann problem discussed in Section 2.

For general Lax equations defined on  $\mathbf{C}P^1$  the interrelations between the classical  $R$  matrices, the Riemann problem and the algebra-geometric methods were discussed in [?].

#### Acknowledgments.

*I am grateful to the Isaac Newton Institute for Mathematical Studies (Cambridge) for hospitality where this work was completed. I would like to thank V.Novokshenov for useful discussions. The work is supported in part by grants RFFI-00-02-16530, INTAS 99-01782 and 96-15-96455 for support of scientific schools.*

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