ELLIPTIC SELBERG INTEGRALS

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ABSTRACT. We introduce new Selberg-type multidimensional integrals built of Ruijsenaars' elliptic gamma functions. We show that the vanishing of our integrals for a specific parameter hypersurface implies closed evaluation formulas valid for the full parameter space. The resulting integration formulas contain the Macdonald-Morris constant term identities for nonreduced root systems as special limiting cases.

1. INTRODUCTION

In 1944 Selberg introduced the following remarkable and highly nontrivial multidimensional generalization of the celebrated beta integral $\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$ [Se]:

$$\int_0^1 \cdots \int_0^1 \prod_{1\leq j\leq n} x_j^{\alpha-1}(1-x_j)^{\beta-1} \prod_{1\leq j<k\leq n}(x_j-x_k)^{2\gamma}\, dx_1\cdots dx_n$$
$$= \prod_{1\leq j\leq n}\frac{\Gamma(\alpha+(j-1)\gamma)\Gamma(\beta+(j-1)\gamma)\Gamma(1+j\gamma)}{\Gamma(\alpha+\beta+(n+j-2)\gamma)\Gamma(1+\gamma)}, \qquad (1.1)$$

where $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$ and $\mathrm{Re}(\gamma) > -\min(1/n, \mathrm{Re}(\alpha)/(n-1), \mathrm{Re}(\beta)/(n-1))$ (with $\Gamma(\cdot)$ representing the gamma function). Since that time various elegant new proofs for this integration formula were presented [A, O, An, AAR]. In a nutshell, the common idea underlying these proofs is to derive first a functional equation for the dependence of the integral on the parameters and then solve this equation to produce the evaluation constant on the r.h.s.

A very influential subsequent development was set in motion by Macdonald, who presented conjectures for families of Selberg-type integration formulas associated to the integral root systems [M1, M3]. (From this perspective, Selberg's original integral in (1.1) corresponds to the nonreduced root system of type BC_n.) These conjectures were subsequently proven by Opdam by means of a technique involving shift operators that has its origin in the Heckman-Opdam theory of generalized hypergeometric functions associated to root systems [O, HS].

As was observed by Macdonald, the Selberg integrals for root systems may be alternatively formulated in terms of constant term identities of a type studied by Andrews, Macdonald, and Morris; this connection led to the formulation of further generalizations of the Selberg integrals associated to root systems involving a modular deformation parameter denoted by the basic nome q, see [M1, M3] and

references therein. The Macdonald-Morris conjectural q-deformed Selberg-type integrals (or, equivalently, constant term identities) arising this way had been checked on a case by case basis by several authors for all but the exceptional E series [BZ, H, G1, GG, K1, K2], when Cherednik came up with a uniform method of proof valid for all reduced root systems (including the E series) via a generalization of the shift-operator approach of Opdam [C, M2, M3].

At the one-dimensional level, the presently most general q-deformed beta-type integration formula is given by the Nassrallah-Rahman integral [NR, R]

$$\frac{1}{2\pi i}\int_T \frac{(z^2, z^{-2}, z\prod_{r=0}^4 t_r, z^{-1}\prod_{r=0}^4 t_r; q)_\infty}{\prod_{r=0}^4 (t_r z, t_r z^{-1}; q)_\infty}\frac{dz}{z}$$
$$= \frac{2\prod_{r=0}^4 (t_r^{-1}\prod_{s=0}^4 t_s; q)_\infty}{(q; q)_\infty \prod_{0\leq r<s\leq 4}(t_r t_s; q)_\infty}, \tag{1.2}$$

where $|q|, |t_r| < 1$ $(r = 0,\ldots,4)$, T denotes the unit circle with positive orientation, and $(a_1,\ldots,a_l; q)_\infty \equiv \prod_{r=1}^l (a_r; q)_\infty$ with $(a; q)_\infty \equiv \prod_{k=0}^\infty (1 - aq^k)$. This integral generalizes the well-known Askey-Wilson integral, which corresponds to the degeneration $t_0 \to 0$ [AW, GR]. Selberg-type multivariate generalizations of the Nassrallah-Rahman integral (1.2) were introduced by Gustafson [G2, G3]. Gustafson's integration formulas reduce for special parameter values to the Macdonald-Morris q-deformed Selberg integrals associated to the nonreduced root systems.

In a more recent development, one of us found a generalization of the Nassrallah-Rahman integral in which the role of the q-shifted factorials is taken over by Ruijsenaars' elliptic gamma function [S1, S2]. (See [Ru, FV] for information on the elliptic gamma function and [B1, B2] for Barnes' general theory of related multiple gamma functions.) This introduces a next level of complexity in the beta integration formulas in which q is complemented (symmetrically) by a second modular deformation parameter p. The purpose of the present paper is to put forward a similar elliptic generalization of Gustafson's Selberg-type multivariate Nassrallah-Rahman integrals. In previous work, we conjectured such elliptic Selberg integration formulas and showed that they imply—via multidimensional residue calculus—certain identities between Jacobi modular functions (in the sense of Eichler-Zagier [EZ]) that were first formulated by Warnaar as summation conjectures for Frenkel-Turaev type multiple modular hypergeometric series [DS1, DS2, W]. A complete proof of these summation identities was found recently by Rosengren [Ro]. At the one-dimensional level, the modular hypergeometric series have their origin in the theory of exactly solvable statistical models, where they pop up in the construction of elliptic solutions of the Yang-Baxter equation [D-O1, D-O2, FT]; these series have also appeared in the theory of special functions, where they are used to represent new types of biorthogonal rational functions on elliptic grids [SZ] and related biorthogonal functions with continuous orthogonality structures [S3]. In the one-dimensional context, Warnaar's sum reduces to a previously known summation formula for a very-well-poised modular hypergeometric series due to Frenkel and Turaev [FT].

The plan of the paper reads as follows. First we formulate two kinds of elliptic Selberg-type integration formulas (Type I and Type II) that generalize Gustafson's multiple Nassrallah-Rahman integrals. Next we show that the vanishing of the Type I integral on a specific parameter hypersurface (the *Vanishing Hypothesis*) implies a closed evaluation formula valid for the full parameter space for both the Type I integral as well as the Type II integral. Our method of proof is modelled on

the ideas of Gustafson and (to lesser extent) Anderson [G2, G3, An, AAR]. The reasoning is the following. First it is shown that the Type I integral implies the Type II integral. Next we derive a system of difference equations for the Type I integral in the parameters. These difference equations hinge on some theta function identities that have been collected in an appendix at the end of the paper. With the aid of the difference equations and a residue formula we then show that our Vanishing Hypothesis implies the evaluation formula for the Type I integral in the full parameter space.

2. NOTATIONAL PRELIMINARIES: THE ELLIPTIC GAMMA FUNCTION

In this section we collect some elementary properties of Ruijsenaars' elliptic gamma function. A more elaborate treatment can be found in [Ru] and [FV].

Let p and q be complex parameters inside the open unit disc $|p|, |q| < 1$. We consider the converging double product

$$(a; p, q)_\infty \equiv \prod_{j,k=0}^{\infty} (1 - ap^j q^k), \tag{2.1}$$

which, for $p = 0$, collapses to the standard q-shifted factorial $(a; q)_\infty$ [GR]. The elliptic gamma function is defined as the quotient

$$\Gamma(z; p, q) = \frac{(pqz^{-1}; p, q)_\infty}{(z; p, q)_\infty}. \tag{2.2}$$

It is symmetric in p and q and satisfies the first-order difference equations

$$\Gamma(qz; p, q) = \theta(z; p)\Gamma(z; p, q), \quad \Gamma(pz; p, q) = \theta(z; q)\Gamma(z; p, q) \tag{2.3a}$$

and the reflection equation

$$\Gamma(z; p, q)\Gamma(z^{-1}; p, q) = \frac{1}{\theta(z; p)\theta(z^{-1}; q)}, \tag{2.3b}$$

where the theta function is defined as

$$\theta(z; p) = (z, pz^{-1}; p)_\infty. \tag{2.4}$$

This theta function satisfies the functional equations

$$\theta(pz; p) = \theta(z^{-1}; p) = -z^{-1}\theta(z; p), \tag{2.5}$$

and is related to the Jacobi θ_1-function [WW] via the Jacobi triple product identity

$$\theta_1(x|\tau) = 2 \sum_{m=0}^{\infty} (-1)^m p^{(2m+1)^2/8} \sin \pi(2m+1)x \tag{2.6a}$$

$$= p^{1/8} i e^{-\pi i x} (p; p)_\infty \theta(e^{2\pi i x}; p), \tag{2.6b}$$

where $p = e^{2i\pi\tau}$. Quotients of elliptic gamma functions give rise to elliptic Pochhammer symbols defined by

$$\theta(z; p; q)_m = \frac{\Gamma(zq^m; p, q)}{\Gamma(z; p, q)} = \prod_{j=0}^{m-1} \theta(zq^j; p), \quad m \in \mathbb{N}. \tag{2.7}$$

For $p = 0$ the elliptic gamma function and elliptic Pochhammer symbol reduce to the q-shifted factorials $\Gamma(z; 0, q) = 1/(z; q)_\infty$ and $\theta(z; 0; q)_m = (z; q)_\infty/(zq^m; q)_\infty =$

$(z;q)_m = \prod_{j=0}^{m-1}(1 - zq^j)$, respectively. Following the standard short-hand conventions of basic hypergeometric analysis for the products of q-shifted factorials $(a_1,\ldots,a_l;q)_m = \prod_{r=1}^l (a_r;q)_m$, we will employ the compact notation:

$$\Gamma(a_1,\ldots,a_l;p,q) = \prod_{r=1}^l \Gamma(a_r;p,q), \tag{2.8a}$$

$$\theta(a_1,\ldots,a_l;p;q)_m = \prod_{r=1}^l \theta(a_r;p;q)_m, \tag{2.8b}$$

$$\theta(a_1,\ldots,a_l;p) = \prod_{r=1}^l \theta(a_r;p). \tag{2.8c}$$

3. THE VANISHING HYPOTHESIS

In this paper we will assume the following nontrivial hypothesis for the vanishing of an elliptic Selberg integral.

Hypothesis. Let $0 < p,q < 1$ and let $t_0,\ldots,t_{2n+1}$ be complex parameters such that $0 < |t_r| < 1$ for $r = 0,\ldots,2n$, $t_{2n+1} = \prod_{r=0}^{2n} t_r^{-1}$, and with generic argument values in the sense that $\#\{\arg(t_r),\arg(t_r^{-1}) \mid r = 0,\ldots,2n+1\} = 4n+4$. Then

$$\int_{C^n} \prod_{1 \le j < k \le n} \Gamma^{-1}(z_j z_k, z_j z_k^{-1}, z_j^{-1} z_k, z_j^{-1} z_k^{-1}; p, q)$$

$$\times \prod_{j=1}^n \frac{\prod_{r=0}^{2n+1} \Gamma(t_r z_j, t_r z_j^{-1}; p, q)}{\Gamma(z_j^2, z_j^{-2}; p, q)} \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n} = 0, \tag{3.1}$$

where the contour $C \subset \mathbb{C}$ is a positively oriented Jordan curve around zero such that (i) the interior is star shaped around the origin: every half-line parting from zero intersects C just *once*, (ii) $C^{-1} := \{z \in \mathbb{C} \mid z^{-1} \in C\} = C$, and (iii) the points t_r $(r = 0,\ldots,2n+1)$ all lie in the interior of C.

For $n = 1$ the Vanishing Hypothesis (3.1) reads

$$\int_C \frac{\prod_{r=0}^3 \Gamma(z t_r, z^{-1} t_r; p, q)}{\Gamma(z^2, z^{-2}; p, q)} \frac{dz}{z} = 0 \tag{3.2}$$

(with $t_0 t_1 t_2 t_3 = 1$). In this special case the hypothesis is a consequence of the elliptic beta integral in Refs. [S1, S2] (cf. Eq. (4.3) and Remark 4.4 below).

For $p = 0$ the hypothesis degenerates to

$$\int_{C^n} \prod_{1 \le j < k \le n} (z_j z_k, z_j z_k^{-1}, z_j^{-1} z_k, z_j^{-1} z_k^{-1}; q)_\infty$$

$$\times \prod_{1 \le j \le n} \frac{(z_j^2, z_j^{-2}; q)_\infty}{\prod_{r=0}^{2n+1} (t_r z_j, t_r z_j^{-1}; q)_\infty} \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n} = 0 \tag{3.3}$$

(with $\prod_{r=0}^{2n+1} t_r = 1$). In this degenerate situation the hypothesis is a consequence of the $Sp(n)$ Selberg-type Nassrallah-Rahman integral of Gustafson [G3] (cf. Eq. (4.4a) and Remark 4.4 below).

4. Main Consequences: Integration Formulas

Let

$$\Delta_n^I(\mathbf{z}; p, q) = \frac{1}{(2\pi i)^n} \prod_{1 \le j < k \le n} \Gamma^{-1}(z_j z_k, z_j z_k^{-1}, z_j^{-1} z_k, z_j^{-1} z_k^{-1}; p, q)$$

$$\times \prod_{j=1}^n \frac{\prod_{r=0}^{2n+2} \Gamma(t_r z_j, t_r z_j^{-1}; p, q)}{\Gamma(z_j^2, z_j^{-2}, A z_j, A z_j^{-1}; p, q)}, \tag{4.1a}$$

with $A \equiv \prod_{r=0}^{2n+2} t_r$, and let

$$\Delta_n^{II}(\mathbf{z}; p, q) = \frac{1}{(2\pi i)^n} \prod_{1 \le j < k \le n} \frac{\Gamma(t z_j z_k, t z_j z_k^{-1}, t z_j^{-1} z_k, t z_j^{-1} z_k^{-1}; p, q)}{\Gamma(z_j z_k, z_j z_k^{-1}, z_j^{-1} z_k, z_j^{-1} z_k^{-1}; p, q)}$$

$$\times \prod_{j=1}^n \frac{\prod_{r=0}^4 \Gamma(t_r z_j, t_r z_j^{-1}; p, q)}{\Gamma(z_j^2, z_j^{-2}, B z_j, B z_j^{-1}; p, q)}, \tag{4.1b}$$

with $B \equiv t^{2n-2} \prod_{s=0}^4 t_s$. Furthermore, let T denote the unit circle with positive orientation. As the main results of this paper we will show that the Vanishing Hypothesis of the previous section implies the following two elliptic Selberg-type integration formulas.

Theorem 4.1 (Type I Elliptic Selberg Integral). *Let $|p|, |q|$ and $|t_r|$ (with $r = 0, \ldots, 2n + 2$) be smaller than 1 such that $|pq| < |\prod_{s=0}^{2n+2} t_s|$. Then the Vanishing Hypothesis implies that*

$$\int_{T^n} \Delta_n^I(\mathbf{z}; p, q) \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n} =$$

$$\frac{2^n n!}{(p; p)_\infty^n (q; q)_\infty^n} \frac{\prod_{0 \le r < s \le 2n+2} \Gamma(t_r t_s; p, q)}{\prod_{r=0}^{2n+2} \Gamma(t_r^{-1} A; p, q)}. \tag{4.2a}$$

Theorem 4.2 (Type II Elliptic Selberg Integral). *Let $|p|, |q|, |t|$ and $|t_r|$ (with $r = 0, \ldots, 4$) be smaller than 1 such that $|pq| < |t^{2n-2} \prod_{s=0}^4 t_s|$. Then the Vanishing Hypothesis implies that*

$$\int_{T^n} \Delta_n^{II}(\mathbf{z}; p, q) \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n} =$$

$$\frac{2^n n!}{(p; p)_\infty^n (q; q)_\infty^n} \prod_{j=1}^n \frac{\Gamma(t^j; p, q)}{\Gamma(t; p, q)} \frac{\prod_{0 \le r < s \le 4} \Gamma(t^{j-1} t_r t_s; p, q)}{\prod_{r=0}^4 \Gamma(t^{1-j} t_r^{-1} B; p, q)}. \tag{4.2b}$$

For $n = 1$ the integration formulas of Theorems 4.1 and 4.2 specialize both to the elliptic beta-type integral

$$\frac{1}{2\pi i} \int_T \frac{\prod_{r=0}^4 \Gamma(z t_r, z^{-1} t_r; p, q)}{\Gamma(z^2, z^{-2}, z \prod_{r=0}^4 t_r, z^{-1} \prod_{r=0}^4 t_r; p, q)} \frac{dz}{z}$$

$$= \frac{2 \prod_{0 \le r < s \le 4} \Gamma(t_r t_s; p, q)}{(p; p)_\infty (q; q)_\infty \prod_{r=0}^4 \Gamma(t_r^{-1} \prod_{s=0}^4 t_s; p, q)}. \tag{4.3}$$

This beta integration formula was proved by one of us, first for discrete parameter values in [S1], and then for general parameter values in [S2].

For $p = 0$ the integration formula (4.3) degenerates to the Nassrallah-Rahman integral (1.2). More generally, for arbitrary n the integration formulas of Theorems 4.1 and 4.2 amount for $p = 0$ to Gustafson's $Sp(n)$ Selberg-type multivariate Nassrallah-Rahman integrals [G2]

$$\frac{1}{(2\pi i)^n} \int_{T^n} \prod_{1 \leq j < k \leq n} (z_j z_k, z_j z_k^{-1}, z_j^{-1} z_k, z_j^{-1} z_k^{-1}; q)_\infty$$

$$\times \prod_{1 \leq j \leq n} \frac{(z_j^2, z_j^{-2}, A z_j, A z_j^{-1}; q)_\infty}{\prod_{r=0}^{2n+2} (t_r z_j, t_r z_j^{-1}; q)_\infty} \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n}$$

$$= \frac{2^n n!}{(q; q)_\infty^n} \frac{\prod_{r=0}^{2n+2}(t_r^{-1} A; q)_\infty}{\prod_{0 \leq r < s \leq 2n+2}(t_r t_s; q)_\infty} \tag{4.4a}$$

(with $|q|$ and $|t_r| < 1$ for $r = 0, \ldots, 2n + 2$) and [G3]

$$\frac{1}{(2\pi i)^n} \int_{T^n} \prod_{1 \leq j < k \leq n} \frac{(z_j z_k, z_j z_k^{-1}, z_j^{-1} z_k, z_j^{-1} z_k^{-1}; q)_\infty}{(t z_j z_k, t z_j z_k^{-1}, t z_j^{-1} z_k, t z_j^{-1} z_k^{-1}; q)_\infty}$$

$$\times \prod_{1 \leq j \leq n} \frac{(z_j^2, z_j^{-2}, B z_j, B z_j^{-1}; q)_\infty}{\prod_{r=0}^{4}(t_r z_j, t_r z_j^{-1}; q)_\infty} \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n}$$

$$= \frac{2^n n!}{(q; q)_\infty^n} \prod_{j=1}^{n} \frac{(t; q)_\infty \prod_{r=0}^{4}(t^{1-j} t_r^{-1} B; q)_\infty}{(t^j; q)_\infty \prod_{0 \leq r < s \leq 4}(t^{j-1} t_r t_s; q)_\infty} \tag{4.4b}$$

(with $|q|, |t|$ and $|t_r| < 1$ for $r = 0, \ldots, 4$), respectively. For special choices of the parameters $t_r, r = 0, \ldots, 4$, the Type II Gustafson integral (4.4b) specializes to Macdonald's q-deformed Selberg integrals associated to the nonreduced (i.e. BC-type) root systems, or, alternatively, to the Macdonald-Morris constant term identities for the nonreduced root systems (cf. Remark 4.1 below) [M1, M3, K1].

Remark 4.1. The integration formulas of Theorems 4.1 and 4.2 state that the constant term of the Laurent series of the integrand in the variables $z_1, \ldots, z_n$ has a value given by the r.h.s. This gives rise to an alternative formulation of the integration formulas as constant term identities. Since the Type II Gustafson integral (4.4b) amounts from this perspective to a generalization of the Macdonald-Morris constant term identity associated to the root system BC_n [M1, M3, K1], we may view the constant term identity stemming from Theorem 4.2 in turn as a further generalization of this Macdonald-Morris identity from the q-deformed level to the elliptic (or (p, q)-deformed) level.

Remark 4.2. The reflection equation (2.3b) for the theta function permits rewriting of the Type I integrand $\Delta_n^I(\mathbf{z}; p, q)$ (4.1a) in the form

$$\Delta_n^I(\mathbf{z}; p, q) = \frac{1}{(2\pi i)^n} \prod_{1 \leq j < k \leq n} \theta(z_j z_k, z_j z_k^{-1}; p) \theta(z_j^{-1} z_k, z_j^{-1} z_k^{-1}; q)$$

$$\times \prod_{j=1}^{n} \frac{\theta(z_j^2; p) \theta(z_j^{-2}; q) \prod_{r=0}^{2n+2} \Gamma(t_r z_j, t_r z_j^{-1}; p, q)}{\Gamma(A z_j, A z_j^{-1}; p, q)}. \tag{4.5}$$

The Type II integrand $\Delta_n^{II}(\mathbf{z}; p, q)$ (4.1b) can also be rewritten analogously.

Remark 4.3. The integrand $\Delta_n^I(\mathbf{z}; p, q)$ (4.1a) has poles in z_j inside the unit circle at $\{t_r p^l q^m\}_{l,m \in \mathbb{N}}$ $(r = 0, \ldots, 2n + 2)$ and $\{A^{-1} p^{l+1} q^{m+1}\}_{l,m \in \mathbb{N}}$. Furthermore, due to the $z_j \to z_j^{-1}$ reflection-invariance of the integrand, the poles located outside the unit circle are related to these by inversion. Let us assume that $0 < p, q < 1$ and that $t_0, \ldots, t_{2n+2}$ are generic such that $\#\{\arg(t_r), \arg(t_r^{-1}) \mid r = 0, \ldots, 2n + 2\} = 4n + 6$ and $t_r^{-1} A \notin [1, +\infty[$ for $r = 0, \ldots, 2n + 2$. We can then deform the integration contour (without altering the value of the integral) from the unit circle T to any (smooth) positively oriented Jordan curve $C \subset \mathbb{C}$ around zero such that (i) the interior is star shaped around the origin: every half-line parting from zero intersects C just *once*, (ii) $C^{-1} := \{z \in \mathbb{C} \mid z^{-1} \in C\} = C$, and (iii) C separates the poles in z_j at $\{t_r p^l q^m\}_{l,m \in \mathbb{N}}$ $(r = 0, \ldots, 2n+2)$ and $\{A^{-1} p^{l+1} q^{m+1}\}_{l,m \in \mathbb{N}}$ (all in the interior of C) from those related to it by inversion (all in the exterior of C). Indeed, the conditions on C guarantee that one does not cross over poles when deforming from T to C, so the value of the integral remains unchanged. This observation permits an extension of the parameter domain of Theorem 4.1 (assuming the above reality and genericity conditions) through analytic continuation. Indeed, we can perform a radial dilation of one or more parameters t_r from the interior of the unit circle to the exterior while simultaneously deforming the integration contour C so as to maintain the above conditions (i)–(iii) satisfied.

A similar extension of the parameter domain for the Type II integral of Theorem 4.2 was described in [DS1, Section 4] (see also [DS2]).

Remark 4.4. Let us assume $0 < p, q < 1$ and t_r $(r = 0, \ldots, 2n + 2)$ nonzero, inside the open unit disc, and with generic argument values as described in the previous Remark 4.3. Then we see that, by letting t_{2n+1} tend to $\prod_{r=0}^{2n} t_r^{-1}$ (so $t_{2n+2} \to A$) while simultaneously deforming the integration contour from the unit circle T to a Jordan curve C respecting the conditions (i)–(iii) of Remark 4.3, the integration formula of Theorem 4.1 reduces to the Vanishing Hypothesis of Section 3. Indeed, the r.h.s. of the integration formula (4.2a) tends to zero in this limit due to the pole of the denominator factor $\Gamma(t_{2n+2}^{-1} A; p, q)$ at $t_{2n+2} = A$. This checks that the evaluation formula of Theorem 4.1 is indeed compatible with the Vanishing Hypothesis (3.1). In other words, one of the main results of this paper is—in a nutshell—that we show that the vanishing of the Type I integral for parameters on the hypersurface $t_{2n+2} = A$ (or, equivalently, $t_0 \cdots t_{2n+1} = 1$) necesarily extends to the evaluation formula (4.2a) on the full parameter space. Phrased still differently: the Vanishing Hypothesis (3.1) and the integration formula (4.2a) follow from each other.

Remark 4.5. The integration formulas of Theorems 4.1 and 4.2 are not the only possible elliptic generalizations of the Selberg integral (1.1) considered in the literature. In [F] Forrester found different elliptic generalizations connected to the theory of random matrices. Forrester's elliptic Selberg integrals are characterized by an integrand composed of products of theta functions rather than elliptic gamma functions.

5. TYPE I $\Rightarrow$ TYPE II

To prove the statements of the previous section we will first show that the Type II integral (4.2b) follows from the Type I integral (4.2a) by means of a technique

due to Gustafson for $p = 0$ [G1, G3]. A similar technique was also employed independently by Anderson in his proof of the classical Selberg integral (1.1) [An].

Theorem 5.1. *The Type I elliptic Selberg integration formula of Theorem 4.1 implies the Type II elliptic Selberg integration formula of Theorem 4.2.*

Proof. The main idea is to consider the composite integral

$$
\frac{1}{(2\pi i)^{2n-1}} \int_{T^n} \int_{T^{n-1}} \prod_{1 \le j < k \le n} \Gamma^{-1}(z_j z_k, z_j z_k^{-1}, z_j^{-1} z_k, z_j^{-1} z_k^{-1}; p, q)
$$

$$
\times \prod_{j=1}^{n} \frac{\prod_{r=0}^{4} \Gamma(t_r z_j, t_r z_j^{-1}; p, q)}{\Gamma(z_j^2, z_j^{-2}, z_j t^{n-1} \prod_{0 \le s \le 4} t_s, z_j^{-1} t^{n-1} \prod_{0 \le s \le 4} t_s; p, q)}
$$

$$
\times \prod_{\substack{1 \le j \le n \\ 1 \le k \le n-1}} \Gamma(t^{1/2} z_j w_k, t^{1/2} z_j w_k^{-1}, t^{1/2} z_j^{-1} w_k, t^{1/2} z_j^{-1} w_k^{-1}; p, q)
$$

$$
\times \prod_{1 \le j < k \le n-1} \Gamma^{-1}(w_j w_k, w_j w_k^{-1}, w_j^{-1} w_k, w_j^{-1} w_k^{-1}; p, q)
$$

$$
\times \prod_{j=1}^{n-1} \frac{\Gamma(w_j t^{n-3/2} \prod_{0 \le s \le 4} t_s, w_j^{-1} t^{n-3/2} \prod_{0 \le s \le 4} t_s; p, q)}{\Gamma(w_j^2, w_j^{-2}, w_j t^{2n-3/2} \prod_{0 \le s \le 4} t_s, w_j^{-1} t^{2n-3/2} \prod_{0 \le s \le 4} t_s; p, q)}
$$

$$
\frac{dw_1}{w_1} \cdots \frac{dw_{n-1}}{w_{n-1}} \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n}, \tag{5.1}
$$

with p, q, t, and t_r $(r = 0, \ldots, 4)$ inside the open unit disc such that $|pq| < |t^{2n-2} \prod_{r=0}^{4} t_r|$. Let us abbreviate the integral on the l.h.s. of (4.2b) by $I_n^{II}(t, t_r; p, q)$. Integration over the w-cycles by means of formula (4.2a) produces an evaluation for the composite integral (5.1) as

$$
\frac{2^{n-1}(n-1)!}{(p;p)_\infty^{n-1}(q;q)_\infty^{n-1}} \frac{\Gamma^n(t;p,q)}{\Gamma(t^n;p,q)} I_n^{II}(t, t_r; p, q). \tag{5.2a}
$$

Similarly, integration over the z-cycles by means of formula (4.2a) evaluates the integral (5.1) as

$$
\frac{2^n n!}{(p;p)_\infty^n (q;q)_\infty^n} \frac{\Gamma^{n-1}(t;p,q) \prod_{0 \le r < s \le 4} \Gamma(t_r t_s; p, q)}{\prod_{r=0}^{4} \Gamma(t^{n-1} t_r^{-1} \prod_{s=0}^{4} t_s; p, q)} I_{n-1}^{II}(t, t^{1/2} t_r; p, q). \tag{5.2b}
$$

Comparing the expressions (5.2a) and (5.2b) for the composite integral yields the following recurrence for I_n^{II} in the dimension n:

$$
I_n^{II}(t, t_r; p, q) = \tag{5.3}
$$

$$
\frac{2n}{(p;p)_\infty (q;q)_\infty} \frac{\Gamma(t^n; p, q)}{\Gamma(t; p, q)} \frac{\prod_{0 \le r < s \le 4} \Gamma(t_r t_s; p, q)}{\prod_{r=0}^{4} \Gamma(t^{n-1} t_r^{-1} \prod_{s=0}^{4} t_s; p, q)} I_{n-1}^{II}(t, t^{1/2} t_r; p, q).
$$

Iteration of the recurrence—starting from the known value (4.3) for $n = 1$ taken from [S1, S2]—entails

$$
I_n^{II}(t, t_r; p, q) = \frac{2^n n!}{(p;p)_\infty^n (q;q)_\infty^n} \prod_{j=1}^{n} \frac{\Gamma(t^j; p, q)}{\Gamma(t; p, q)} \frac{\prod_{0 \le r < s \le 4} \Gamma(t^{j-1} t_r t_s; p, q)}{\prod_{r=0}^{4} \Gamma(t^{1-j} t_r^{-1} B; p, q)}, \tag{5.4}
$$

which is precisely the formula of Theorem 4.2. $\qquad\square$

6. Difference Equations

In the remainer of the paper we will derive Theorem 4.1 via a generalization of Gustafson's method in [G2] from the trigonometric case $p = 0$ to the generic elliptic case $0 < |p| < 1$. The first step is to exhibit a system of difference equations for the Type I integral in the t_r parameters. The proof of these difference equations hinges on identities for the theta function (2.4) that are collected and proved in Appendix A below.

Let us write $I_n^{(r)}(t_r; p, q)$, $I_n^{(l)}(t_r; p, q)$, and $\Delta_n(z; t_r; p, q)$ for the r.h.s., the l.h.s., and the integrand of the integration formula stated in Theorem 4.1.

Theorem 6.1. *Both the l.h.s. $I_n^{(l)}(t_r; p, q)$ and the r.h.s. $I_n^{(r)}(t_r; p, q)$ of the integration formula stated by Theorem 4.1 satisfy the q-difference equation*

$$\sum_{r=0}^n \prod_{\substack{0 \le s \le n \\ s \ne r}} \frac{\theta(At_s, At_s^{-1}; p)}{\theta(t_r t_s, t_r t_s^{-1}; p)} I_n(t_0, \ldots, qt_r, \ldots, t_{2n+2}; p, q)$$

$$= I_n(t_0, \ldots, t_{2n+2}; p, q) \qquad (6.1)$$

and a dual p-difference equation with the role of p and q interchanged.

Proof. From the q-shift property (2.3a) of the elliptic gamma function it is immediate that the integrand $\Delta_n(z; t_0, \ldots, t_{2n+2}; p, q)$ and the r.h.s. $I_n^{(r)}(t_0, \ldots, t_{2n+2}; p, q)$ of the Type I integration formula satisfy the first-order difference equations

$$\frac{\Delta_n(z; t_0, \ldots, qt_r, \ldots, t_{2n+2}; p, q)}{\Delta_n(z; t_0, \ldots, t_{2n+2}; p, q)} = \prod_{1 \le j \le n} \frac{\theta(t_r z_j, t_r z_j^{-1}; p)}{\theta(Az_j, Az_j^{-1}; p)} \qquad (6.2a)$$

and

$$\frac{I_n^{(r)}(t_0, \ldots, qt_r, \ldots, t_{2n+2}; p, q)}{I_n^{(r)}(t_0, \ldots, t_{2n+2}; p, q)} = \prod_{\substack{0 \le s \le 2n+2 \\ s \ne r}} \frac{\theta(t_r t_s; p)}{\theta(At_s^{-1}; p)}, \qquad (6.2b)$$

respectively. It now follows from (6.2a) and the theta-function identity of Proposition A.3 (Appendix A) that the integrand (and thus the integral) satisfies the q-difference equation (6.1). The fact that the r.h.s. also solves this q-difference equation is inferred similarly with the aid of (6.2b) and the theta-function identity of Proposition A.4. The dual p-difference equation now follows by the symmetry in p and q. $\qquad \square$

Remark 6.1. In view of the permutation symmetry in the parameters $t_0, \ldots, t_{2n+2}$, it is clear that both sides of the integration formula of Theorem 4.1 in fact satisfy $\binom{2n+3}{n+1}$ (the number of ways to select $n+1$ out of these $2n+3$ parameters) q-difference equations of the type in Theorem 6.1 and an equal number of dual p-difference equations.

Remark 6.2. In the proof of Theorem 6.1 it was used that the r.h.s. of the integration formula of Theorem 4.1 satisfies the difference equation (6.2b). It is instructive to note that for proving the integration formulas (4.2a) and (4.2b)—without an appeal to the Vanishing Hypothesis of Section 3—it would at this point already be sufficient to demonstrate that the l.h.s. of (4.2a) also satisfies this difference equation. Indeed, it follows from the difference equation, the $p \leftrightarrow q$ symmetry (and an irrationality argument), together with the analyticity in the parameters, that

the l.h.s. and the r.h.s. of the Type I integration formula must be equal up to a constant factor $c_n(p,q)$ not depending on $t_0,\ldots,t_{2n+2}$. Following the reasoning of Section 5, one is then led to formula (5.4) for the evaluation of the type B integral up to multiplication by the same constant factor $c_n(p,q)$ (which does not depend on t and t_r $(r = 0,\ldots,t_4)$). Here one uses along the way that for $n = 1$ both types of integrals coincide. From the fact that for $t \to 1$ the integral reduces to the n-fold product of the $n = 1$ elliptic beta integral (4.3), one furthermore deduces that the proportionality factor $c_n(p,q)$ must in fact be identical to 1. (To infer the $t \to 1$ limiting behavior of the r.h.s. one uses that $\lim_{t\to 1} \Gamma(t^j;p,q)/\Gamma(t;p,q) = 1/j$.)

7. RESIDUE CALCULUS

We will now infer the Type I integral by induction on the dimension n. First we consider the case $t_{2n+1} = q^{l+1}A^{-1}$, $t_{2n+2} = q^{-l}A$ with $l \in \mathbb{N} \setminus \{0\}$ (where $A \equiv \prod_{r=0}^{2n+2} t_r = q\prod_{r=0}^{2n} t_r$). The integrand $\Delta(z;t_r;p,q) \equiv \Delta_n^I(z;p,q)$ (4.1a) becomes for this special choice of the parameters t_{2n+1}, t_{2n+2} of the form

$$\Delta_n^{(l)}(z) = \Delta_{n,n}(z)\,\delta_n^{(l)}(z), \tag{7.1}$$

where

$$\Delta_{n,m}(z) = \frac{1}{(2\pi i)^n} \prod_{1\le j<k\le n} \Gamma^{-1}(z_jz_k, z_jz_k^{-1}, z_j^{-1}z_k, z_j^{-1}z_k^{-1};p,q) \tag{7.2a}$$

$$\times \prod_{j=1}^n \frac{\prod_{r=0}^{2m}\Gamma(t_rz_j, t_rz_j^{-1};p,q)}{\Gamma(z_j^2, z_j^{-2}, Az_j, Az_j^{-1};p,q)} \qquad (0 \le m \le n+1),$$

$$\delta_n^{(l)}(z) = \prod_{j=1}^n \Gamma(q^{l+1}A^{-1}z_j, q^{l+1}A^{-1}z_j^{-1}, q^{-l}Az_j, q^{-l}Az_j^{-1};p,q). \tag{7.2b}$$

Apart from the difference equations of Theorem 6.1, the main tool for computing the integral $\int_{T^n} \Delta_n^{(l)}(z)\frac{dz_1}{z_1}\cdots\frac{dz_n}{z_n}$ consists of the following residue formula.

Proposition 7.1 (A Residue Formula). *Let the parameters $t_0,\ldots,t_{2n}$ be inside the punctured open unit disc $\{w \in \mathbb{C} \mid 0 < |w| < 1\}$ with generic argument values in the sense that $\#\{\arg(t_r),\arg(t_r^{-1}) \mid r = 0,\ldots,2n\} = 4n+2$, $\arg(t_r),\arg(t_r^{-1}) \ne \arg(t_0\cdots t_{2n})$ for $r = 0,\ldots,2n$, and $t_0\cdots t_{2n} \notin \mathbb{R}$. Then we have for $0 < p < q^l < |\prod_{r=0}^{2n} t_r| < q^{l-1} \le 1$, with $l \in \mathbb{N} \setminus \{0\}$, that*

$$\int_{T^n} \Delta_n^{(l)}(z)\frac{dz_1}{z_1}\cdots\frac{dz_n}{z_n} = (q^lA^{-1})^{2n}\int_{T^n} \Delta_n^{(l-1)}(z)\frac{dz_1}{z_1}\cdots\frac{dz_n}{z_n}$$

$$= (q^lA^{-1})^{2n}\int_{C_{(l-1)}^n} \Delta_n^{(l-1)}(z)\frac{dz_1}{z_1}\cdots\frac{dz_n}{z_n}$$

$$-2n\kappa^{(l)}\int_{T^{n-1}} \Delta_{n-1,n}(z)\nu_{n-1}^{(l)}(z)\frac{dz_1}{z_1}\cdots\frac{dz_{n-1}}{z_{n-1}},$$

where $\Delta_n^{(l)}(z)$ is given by (7.1)–(7.2b), $A = q\prod_{r=0}^{2n} t_r$, and

$$\nu_{n-1}^{(l)}(z) = \prod_{j=1}^{n-1} \theta(q^{-l}Az_j, q^{-l}Az_j^{-1};p),$$

$$\kappa^{(l)} = \frac{(q^lA^{-1})^{2n}}{(p;p)_\infty(q;q)_\infty} \frac{\theta(q^{-2l}A^2;p)}{\theta(q;p,q)_{l-1}} \frac{\prod_{r=0}^{2n}\Gamma(t_rq^lA^{-1}, t_rq^{-l}A;p,q)}{\Gamma(q^{-l}A^2;p,q)}.$$

Here T denotes the unit circle with positive orientation and $C_{(l')} \subset \mathbb{C}$, $l' \in \mathbb{N}$, is a positively oriented Jordan curve around zero such that (i) every half-line parting from zero intersects $C_{(l')}$ just once, (ii) $C_{(l')}^{-1} := \{z \in \mathbb{C} \mid z^{-1} \in C_{(l')}\} = C_{(l')}$, and (iii) $C_{(l')}$ separates the points $t_0, \dots, t_{2n}$, $q^{l'+1}A^{-1}$ and $q^{-l'}A$ (all in the interior) from the points related to these by inversion (all in the exterior).

Proof. The equality on the first line is immediate from the shift property $\delta_n^{(l)}(\mathbf{z}) = (q^l A^{-1})^{2n} \delta_n^{(l-1)}(\mathbf{z})$. To pass to the expression on the second line, we first observe that by deforming the integration contour for the z_n variable from T to $C_{(l-1)}$ one crosses over a pair of poles at $z_n = q^l A^{-1}$ (entering the interior of the contour) and $z_n = q^{-l}A$ (leaving the interior of the contour), respectively. The residues of the integrand $\Delta_n^{(l-1)}(\mathbf{z})$ at these poles are equal to $\pm(2\pi i)^{-1}\Delta_{n-1,n}(\mathbf{z})\nu_{n-1}^{(l)}(\mathbf{z})$, where the plus sign corresponds to $z_n = q^l A^{-1}$ and the minus sign corresponds to $z_n = q^{-l}A$. Since the above residue is holomorphic as a function of z_j (with $1 < j < n-1$) on the symmetric difference of the interiors of $C_{(l-1)}$ and T, and the integrand is permutation-invariant, it is clear that the proposition now follows by successive deformation of the integration contours for the variables $z_1, z_2, \dots, z_n$ from T to $C_{(l-1)}$ and application of the Cauchy residue theorem. $\square$

The residue formula of Proposition 7.1 enables the evaluation of the integral $\int_{T^n} \Delta_n^{(1)}(\mathbf{z}) \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n}$ upon invoking of the Vanishing Hypothesis from Section 3. It is precisely at this point (and only at this point) that we actually employ the Vanishing Hypothesis.

Proposition 7.2 (The case $l = 1$). *Let $t_0, \dots, t_{2n}$ be complex parameters inside the punctured open unit disc $\{w \in \mathbb{C} \mid 0 < |w| < 1\}$ subject to the genericity conditions in Proposition 7.1, and let $0 < p < q < |\prod_{r=0}^{2n} t_r| < 1$. Then the Vanishing Hypothesis implies that*

$$
\int_{T^n} \Delta_n^{(1)}(\mathbf{z}) \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n} =
$$
$$
\frac{2^n n!}{(p;p)_\infty^n (q;q)_\infty^n} \frac{\prod_{0 \le r < s \le 2n} \Gamma(t_r t_s; p, q) \prod_{r=0}^{2n} \Gamma(t_r q^2 A^{-1}, t_r q^{-1} A; p, q)}{\prod_{r=0}^{2n} \Gamma(t_r^{-1} A; p, q) \, \Gamma(q^{-2} A^2; p, q)}
$$

(with $A = q \prod_{r=0}^{2n} t_r$).

Proof. We consider the residue formula of Proposition 7.1 for $l = 1$. Evaluation of the $(n-1)$-dimensional integral with the aid of the formula of Theorem 4.1 (here we use induction on the number of variables) readily entails that

$$
\int_{T^n} \Delta_n^{(1)}(\mathbf{z}) \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n} = (qA^{-1})^{2n} \int_{C_{(0)}^n} \Delta_n^{(0)}(\mathbf{z}) \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n} +
$$
$$
\frac{2^n n!}{(p;p)_\infty^n (q;q)_\infty^n} \frac{\prod_{0 \le r < s \le 2n} \Gamma(t_r t_s; p, q) \prod_{r=0}^{2n} \Gamma(t_r q^2 A^{-1}, t_r q^{-1} A; p, q)}{\prod_{r=0}^{2n} \Gamma(t_r^{-1} A; p, q) \, \Gamma(q^{-2} A^2; p, q)},
$$

where the contour $C_{(0)}$ is in accordance with the conditions stated in Proposition 7.1. By the Vanishing Hypothesis we have that $\int_{C_{(0)}^n} \Delta_n^{(0)}(\mathbf{z}) \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n} = 0$, whence the integration formula follows. $\square$

8. Iteration

With the help of the difference equation of Theorem 6.1 and a theta function identity from Appendix A (below), we arrive at the extension of Proposition 7.2 to the case of arbitrary positive integral l.

Proposition 8.1 (The case $l \geq 1$). *Let $t_0, \ldots, t_{2n}$ be complex parameters inside the punctured open unit disc $\{w \in \mathbb{C} \mid 0 < |w| < 1\}$ subject to the genericity conditions in Proposition 7.1, and let $0 < p < q^l < |\prod_{r=0}^{2n} t_r| < q^{l-1} \leq 1$ with $l \in \mathbb{N} \setminus \{0\}$. Then the Vanishing Hypothesis implies that*

$$\int_{T^n} \Delta_n^{(l)}(\mathbf{z}) \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n} = \tag{8.1}$$

$$\frac{2^n n!}{(p;p)_\infty^n (q;q)_\infty^n} \frac{\prod_{0 \leq r < s \leq 2n} \Gamma(t_r t_s; p, q) \prod_{r=0}^{2n} \Gamma(t_r q^{l+1} A^{-1}, t_r q^{-l} A; p, q)}{\prod_{r=0}^{2n} \Gamma(t_r^{-1} A; p, q) \, \Gamma(q^{-l-1} A^2; p, q) \, \theta(q; p; q)_{l-1}}$$

(where $A = q \prod_{r=0}^{2n} t_r$).

Proof. For $l = 1$ the statement of the proposition reduces to that of Proposition 7.2. In the rest of the proof we will therefore restrict ourselves to the case $l > 1$. Starting point is the residue formula of Proposition 7.1. Proposition A.3 with $B = q^{-l} A$ enables us to expand the factor $\nu_{n-1}^{(l)}(\mathbf{z})$ as

$$\prod_{j=1}^{n-1} \theta(q^{-l} A z_j, q^{-l} A z_j^{-1}; p) = \tag{8.2}$$

$$\sum_{r=0}^{n-1} \prod_{\substack{0 \leq s \leq n-1 \\ s \neq r}} \frac{\theta(q^{-l} A t_s, q^{-l} A t_s^{-1}; p)}{\theta(t_r t_s, t_r t_s^{-1}; p)} \prod_{j=1}^{n-1} \theta(t_r z_j, t_r z_j^{-1}; p).$$

Substitution of this expansion in the residue formula yields

$$\int_{T^n} \Delta_n^{(l)}(\mathbf{z}) \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n} = (q^l A^{-1})^{2n} \int_{C_{(l-1)}^n} \Delta_n^{(l-1)}(\mathbf{z}) \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n}$$

$$- 2n \kappa^{(l)} \sum_{r=0}^{n-1} \prod_{\substack{0 \leq s \leq n-1 \\ s \neq r}} \left(\frac{\theta(q^{-l} A t_s, q^{-l} A t_s^{-1}; p)}{\theta(t_r t_s, t_r t_s^{-1}; p)} \right. \tag{8.3}$$

$$\left. \times \int_{T^{n-1}} \Delta_{n-1,n}(\mathbf{z}; t_0, \ldots, q t_r, \ldots, t_{2n}; q t_0 \cdots t_{2n}; p, q) \frac{dz_1}{z_1} \cdots \frac{dz_{n-1}}{z_{n-1}} \right),$$

where $\Delta_{n,m}(\mathbf{z}; t_0, \ldots, t_{2m}; A; p, q) \equiv \Delta_{n,m}(\mathbf{z})$ is given by the r.h.s. of Eq. (7.2a). The $(n-1)$-dimensional integral on the last line of Eq. (8.3) is evaluated through the formula of Theorem 4.1 by induction on the number of variables:

$$\int_{T^{n-1}} \Delta_{n-1,n}(\mathbf{z}; t_0, \ldots, q t_r, \ldots, t_{2n}; q t_0 \cdots t_{2n}; p, q) \frac{dz_1}{z_1} \cdots \frac{dz_{n-1}}{z_{n-1}} = \tag{8.4}$$

$$\frac{2^{n-1}(n-1)!}{(p;p)_\infty^{n-1} (q;q)_\infty^{n-1}} \frac{\prod_{0 \leq s < s' \leq 2n} \Gamma(t_s t_{s'}; p, q)}{\prod_{s=0}^{2n} \Gamma(t_s^{-1} A; p, q)} \theta(q^{-1} t_r^{-1} A; p) \prod_{\substack{0 \leq s \leq 2n \\ s \neq r}} \theta(t_r t_s; p).$$

We will now use Eq. (8.3) to compute the desired integral $\int_{T^n} \Delta_n^{(l)}(\mathbf{z}) \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n}$ by induction on l starting from the known value for $l = 1$ from Proposition 7.2.

Evaluation of the integral $\int_{T^n} \Delta_n^{(l-1)}(\mathbf{z}) \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n}$ in the regime $0 < p < q^{l-1} < |\prod_{r=0}^{2n} t_r| < q^{l-2} \leq 1$, with an appeal to the induction hypothesis, and performing the analytic continuation to the regime $0 < p < q^l < |\prod_{r=0}^{2n} t_r| < q^{l-1} < 1$ while deforming the contour from T to $C_{(l-1)}$ so as to avoid crossing over the poles at $z_j = q^l A^{-1}$ and $z_j = q^{-l} A$ (cf. Remark 4.3), produces the evaluation

$$\int_{C_{(l-1)}^n} \Delta_n^{(l-1)}(\mathbf{z}) \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n} = \tag{8.5}$$

$$\frac{2^n n!}{(p;p)_\infty^n (q;q)_\infty^n} \frac{\prod_{0 \leq r < s \leq 2n} \Gamma(t_r t_s; p, q) \prod_{r=0}^{2n} \Gamma(t_r q^l A^{-1}, t_r q^{1-l} A; p, q)}{\prod_{r=0}^{2n} \Gamma(t_r^{-1} A; p, q) \Gamma(q^{-l} A^2; p, q) \theta(q; p; q)_{l-2}}.$$

Substitution of Eqs. (8.4) and (8.5) in the expansion (8.3) gives rise to an explicit evaluation of the integral $\int_{T^n} \Delta_n^{(l)}(\mathbf{z}) \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n}$ in terms of a complicated sum:

$$\int_{T^n} \Delta_n^{(l)}(\mathbf{z}) \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n} =$$

$$\frac{2^n n! \, (q^l A^{-1})^{2n}}{(p;p)_\infty^n (q;q)_\infty^n} \frac{\prod_{0 \leq r < s \leq 2n} \Gamma(t_r t_s; p, q) \prod_{r=0}^{2n} \Gamma(t_r q^l A^{-1}, t_r q^{-l} A; p, q)}{\prod_{r=0}^{2n} \Gamma(t_r^{-1} A; p, q) \Gamma(q^{-l} A^2; p, q) \theta(q; p; q)_{l-1}}$$

$$\times \left(\theta(q^{l-1}; p) \prod_{r=0}^{2n} \theta(t_r q^{-l} A; p) - \right.$$

$$\left. \theta(q^{-2l} A^2; p) \sum_{r=0}^{n-1} \theta(q^{-1} A t_r; p) \prod_{s=n}^{2n} \theta(t_r t_s; p) \prod_{\substack{0 \leq s \leq n-1 \\ s \neq r}} \frac{\theta(q^{-l} A t_s, q^{-l} A t_s^{-1}; p)}{\theta(t_r t_s^{-1}; p)} \right).$$

Simplification of the part within brackets by means of the summation formula (cf. Proposition A.5 with $n \to n-1$ and $B = q^{1-l}$)

$$\sum_{r=0}^{n-1} \theta(q^{-1} A t_r^{-1}; p) \prod_{s=n}^{2n} \theta(t_r t_s; p) \prod_{\substack{0 \leq s \leq n-1 \\ s \neq r}} \frac{\theta(q^{-l} A t_s, q^{-l} A t_s^{-1}; p)}{\theta(t_r t_s^{-1}; p)} = \tag{8.6}$$

$$\frac{\theta(q^{l-1}; p) \prod_{s=0}^{2n} \theta(q^{-l} A t_s; p)}{\theta(q^{-2l} A^2; p)} - \frac{(q^{-l} A)^{2n} \theta(q^{-l-1} A^2; p) \prod_{s=0}^{2n} \theta(q^l A^{-1} t_s; p)}{\theta(q^{-2l} A^2; p)},$$

finally entails the desired integration formula stated by the proposition. $\qquad\square$

9. INTERPOLATION

Proposition 8.1 states (roughly) that the Vanishing Hypothesis implies that the Type I integral (4.2a) holds for a discrete parameter sequence of the form $t_{2n+1} = q^{l+1} A$, $t_{2n+2} = q^{-l} A$ with $l \in \mathbb{N} \setminus \{0\}$. We will now remove these restrictions on the parameters. For this purpose we first need some detailed information on the structure of the Taylor expansion of the Type I elliptic Selberg integral (4.2a) in the deformation parameter p.

Lemma 9.1. *Let p, q and t_r $(r = 0, \ldots, 2n+2)$ be parameters inside the punctured open unit disc $\{w \in \mathbb{C} \mid 0 < |w| < 1\}$ such that $|pq| < |A|$ (where $A = \prod_{r=0}^{2n+2} t_r$) and, furthermore, let $\mathbf{z} \in T^n$. Then the Taylor expansions around $p = 0$ of the*

integrand $\Delta_n(\mathbf{z}; t_r; p, q) = \Delta_n^I(\mathbf{z}; p, q)$ (4.1a) *and the r.h.s.* $I_n^{(r)}(t_r; p, q)$ *of the Type I integration formula* (4.2a), *given by*

$$\Delta_n(\mathbf{z}; t_r; p, q) = \sum_{m=0}^{\infty} \delta_n(\mathbf{z}; t_r; q; m) \, p^m,$$

$$I_n^{(r)}(t_r; p, q) = \sum_{m=0}^{\infty} \iota_n^{(r)}(t_r; q; m) \, p^m,$$

converge in absolute value. Furthermore, the structure of the expansion coefficients is of the form

$$\delta_n(\mathbf{z}; t_r; q; m) = \Delta_n(\mathbf{z}; t_r; 0, q) L_\Delta(\mathbf{z}, t_r; q; m),$$
$$\iota_n^{(r)}(t_r; q; m) = I_n^{(r)}(t_r; 0, q) L_I(t_r; q; m),$$

where $L_\Delta(\mathbf{z}, t_r; q; m)$ *denotes a permutation-invariant Laurent polynomial in* $z_1, \ldots, z_n$ *and in* $t_0, \ldots, t_{2n+2}$ *with a pole of order* $2m$ *at* $z_j = 0$ *and a pole of order* m *at* $t_r = 0$, *and* $L_I(t_r; q; m)$ *denotes a permutation-invariant Laurent polynomial in* $t_0, \ldots, t_{2n+2}$ *with a pole of order* m *at* $t_r = 0$.

Proof. The fractions $\Delta_n(\mathbf{z}; t_r; p, q)/\Delta_n(\mathbf{z}; t_r; 0, q)$ and $\Delta_n(\mathbf{z}; t_r; p, q)/\Delta_n(\mathbf{z}; t_r; 0, q)$ are built of doubly-infinite products of the type $(ap; p, q)_\infty^{\pm 1}$ (for certain arguments a). From the estimates

$$|(ap; p, q)_\infty| \leq |(-|ap|; |p|, |q|)_\infty|$$

and

$$\frac{1}{|(ap; p, q)_\infty|} \leq \frac{1}{|(|ap|; |p|, |q|)_\infty|},$$

it is clear that the Taylor expansions of these factors around $p = 0$ converge in absolute value provided the relevant quantities ap in the denominators satisfy the restriction $|ap| < 1$. This is guaranteed by the conditions on t_r and $\mathbf{z}$, and hence the Taylor expansions of $\Delta_n(\mathbf{z}; t_r; p, q)$ and $I^{(r)}(t_r; p, q)$ around $p = 0$ converge absolutely as so do the expansions of individual factors of the form $(ap; p, q)_\infty^{\pm 1}$.

The stated structure for the expansion coefficients $\delta_n(\mathbf{z}; t_r; q; m)$ and $\iota_n^{(r)}(t_r; q; m)$ is immediate from the expansion formulas

$$\Delta_n^I(\mathbf{z}; p, q) = \Delta_n^I(\mathbf{z}; 0, q) \exp\left(-\sum_{m=1}^{\infty} \frac{p^m g_m}{m(1-p^m)}\right),$$

where

$$g_m = \sum_{1 \leq i < j \leq n} (z_i^m + z_i^{-m})(z_j^m + z_j^{-m}) + \sum_{1 \leq j \leq n} (z_j^{2m} + z_j^{-2m})$$
$$+ (1-q^m)^{-1}\left(A^m - q^m A^{-m} + \sum_{0 \leq r \leq 2n+2} (q^m t_r^{-m} - t_r^m)\right) \sum_{1 \leq j \leq n} (z_j^m + z_j^{-m}),$$

and

$$I_n^{(r)}(t_r; p, q) = I_n^{(r)}(t_r; 0, q) \exp\left(-\sum_{m=1}^{\infty} \frac{p^m h_m}{m(1-p^m)}\right),$$

where

$$h_m = -n + \sum_{0 \leq r < s \leq 2n+2} \frac{q^m t_r^{-m} t_s^{-m} - t_r^m t_s^m}{1-q^m} + \sum_{0 \leq r \leq 2n+2} \frac{A^m t_r^{-m} - q^m A^{-m} t_r^m}{1-q^m},$$

respectively. These expansion formulas are obtained with the aid of the representation $(a; p)_\infty = \exp\left(\sum_{j=0}^\infty \log(1 - ap^j)\right)$ and the expansion $\log(1 - x) = -\sum_{m=1}^\infty x^m/m$ for the logarithm. □

With the aid of Lemma 9.1, we now lift the discreteness restriction on the parameters t_{2n+1}, t_{2n+2}.

Proposition 9.2. *The Vanishing Hypothesis implies that the Type I elliptic Selberg integration formula (4.2a) holds for parameters in the hypersurface $t_{2n+1}t_{2n+2} = q$ of the parameter domain in Theorem 4.1.*

Proof. Analytic continuation extends the parameter domain of Proposition 8.1 to $0 < |t_r| < 1$ $(r = 0, \ldots, 2n)$, $0 < |p| < |q^l| < |\prod_{r=0}^{2n} t_r| < q^{l-1} \leq 1$ with $l \in \mathbb{N} \setminus \{0\}$. Via the substitution $t_{2n} \to q^l t_{2n}$, we arrive at the integration formula of Theorem 4.1 for parameters of the form

$$t_{2n+2} = qt_{2n+1}^{-1}, \qquad t_{2n} = q^l \prod_{\substack{0 \leq r \leq 2n+1 \\ r \neq 2n}} t_r^{-1} \qquad (9.1)$$

(so $A = q^{l+1}t_{2n+1}^{-1}$), with $0 < |t_r| < 1$ $(r = 0, \ldots, 2n - 1)$, $0 < |q| < |t_{2n+1}| < 1$, $|q^{l-1}| < |\prod_{r=0}^{2n-1} t_r|$, and $|p| < |q^l|$. Hence, for parameters of the form (9.1) (subject to the domain restrictions), the expansion coefficients in Lemma 9.1 are equal:

$$\int_{T^n} \delta_n(\mathbf{z}; t_r; q; m)\frac{d\mathbf{z}}{\mathbf{z}} = \iota_n^{(r)}(t_r; q; m), \qquad (9.2)$$

for an infinite integer sequence of $l \in \mathbb{N} \setminus \{0\}$. Since both sides of (9.2) are meromorphic in t_{2n} in a neighborhood of $t_{2n} = 0$ (this is a consequence of Lemma 9.1) and moreover equal for a discrete sequence of values of t_{2n} converging to zero, it follows that the equality (9.2) of the expansion coefficients in fact holds for arbitrary t_{2n} with $0 < |t_{2n}| < 1$ (while still assuming that $t_{2n+1}t_{2n+2} = q$). The same equality is then true for the l.h.s. and r.h.s. of the Type I integration formula due to the absolute convergence of the Taylor expansions. □

Finally, we remove the hypersurface condition $t_{2n+1}t_{2n+2} = q$ with the aid of the difference equation in Theorem 6.1.

Theorem 9.3. *The Vanishing Hypothesis implies that the Type I elliptic Selberg integration formula (4.2a) holds for the full parameter domain in Theorem 4.1.*

Proof. Swapping the parameters t_n and t_{2n+2} in the basic difference equation of Theorem 6.1 for the l.h.s. $I_n^{(l)}(t_r; p, q)$ and r.h.s. $I_n^{(r)}(t_r; p, q)$ of the Type I integration formula produces the relation:

$$a_{2n+2}(p)I_n(t_0, \ldots, qt_{2n+2}; p, q) = I_n(t_0, \ldots, t_{2n+2}; p, q)$$
$$- \sum_{r=0}^{n-1} a_r(p)I_n(t_0, \ldots, qt_r, \ldots, t_{2n+2}; p, q),$$

with

$$a_{2n+2}(p) = \prod_{s=0}^{n-1} \frac{\theta(At_s, At_s^{-1}; p)}{\theta(t_{2n+2}t_s, t_{2n+2}t_s^{-1}; p)},$$

$$a_r(p) = \frac{\theta(At_{2n+2}, At_{2n+2}^{-1}; p)}{\theta(t_{2n+2}t_r, t_{2n+2}t_r^{-1}; p)} \prod_{\substack{s=0 \\ s \neq r}}^{n-1} \frac{\theta(At_s, At_s^{-1}; p)}{\theta(t_r t_s, t_r t_s^{-1}; p)}, \quad r = 0, \ldots, n-1.$$

Since the coefficient $a_{2n+2}(p)$ does not vanish, it is clear from the difference equation that if $I_n^{(l)}(t_r; p, q) = I_n^{(r)}(t_r; p, q)$ for $t_{2n+1}t_{2n+2} = q^l$, with l some positive integer, then the same equality also holds for $t_{2n+1}t_{2n+2} = q^{l+1}$ (provided $|p| < |t_0 \ldots t_{2n} q^l|$). By an induction argument in l, starting from Proposition 9.2, we arrive at the desired equality for a *finite* discrete sequence of parameter surfaces of the form $t_{2n+1}t_{2n+2} = q^l$ with $l \in \mathbb{N} \setminus \{0\}$. The same type of interpolation argument as in the proof of Proposition 9.2 removes the discreteness condition: first we pass to a corresponding equality for the coefficients of the Taylor expansions at $p = 0$ of both sides of the Type I integration formula, valid for an *infinite* discrete sequence of parameter surfaces of the form $t_{2n+1}t_{2n+2} = q^l$; then the discreteness condition on l is removed by means of an analyticity argument; finally, the equality is lifted to the level of the integration formula by the absolute convergence of the Taylor expansions. $\square$

This completes the induction in the number of variables n. We thus conclude that the Vanishing Hypothesis of Section 3 implies the elliptic Selberg type integration formulas of Theorem 4.1 and Theorem 4.2.

APPENDIX A. SOME THETA FUNCTION IDENTITIES

In this appendix we have collected a number of identities for the theta function (2.4). These identities hold as equalities between analytic functions of the indeterminates. They are used in the proof of the difference equations for the Type I elliptic Selberg integral stated in Theorem 6.1 and in the induction prodecure of Section 8 (viz. the proof of Proposition 8.1). The identities in question may be seen as generalizations of a classical identity due to Weierstrass (cf. Remark A.2 below).

We start off with two preparative identities. Their proof is by induction on the size.

Lemma A.1. *Let $n \in \mathbb{N} \setminus \{0\}$ and let $z_1, \ldots, z_{n-1}$ and $t_0, \ldots, t_n$ be (nonzero) complex indeterminates. Then one has that*

$$\sum_{r=0}^n t_r \frac{\prod_{1 \leq j \leq n-1} \theta(t_r z_j, t_r z_j^{-1}; p)}{\prod_{\substack{0 \leq s \leq n \\ s \neq r}} \theta(t_r t_s, t_r t_s^{-1}; p)} = 0. \tag{A.1}$$

Proof. For $n = 1$ we get

$$\frac{t_0}{\theta(t_0 t_1, t_0 t_1^{-1}; p)} + \frac{t_1}{\theta(t_1 t_0, t_1 t_0^{-1}; p)} = \frac{t_0}{\theta(t_0 t_1, t_0 t_1^{-1}; p)} \left(1 + \frac{t_1}{t_0} \frac{\theta(t_0 t_1^{-1}; p)}{\theta(t_1 t_0^{-1}; p)} \right),$$

which is identically zero in view of the $z \to z^{-1}$ reciprocity relation (2.5) for the theta function. For arbitrary n it is clear from the quasi-periodicity property (2.5) of the theta function that the p-shift $z_j \to pz_j$ amounts to multiplication of the

l.h.s. by an overall factor of the form $1/(pz_j^2)$. Hence, upon setting $z_j = p^{x_j}$ $(j = 1, \ldots, n-1)$, it is clear that the expression on the l.h.s. can be written as $c(t_0, \ldots, t_n; p)p^{-(x_1^2 + \cdots + x_n^2)}$ where $c(t_0, \ldots, t_n; p)$ is a constant not depending on $x_1, \ldots, x_{n-1}$. (Here we have used the quasi-periodicity, entireness, and permutation symmetry in $x_1, \ldots, x_{n-1}$.) By writing the l.h.s. as

$$\sum_{r=0}^{n-1} t_r \frac{\prod_{1 \leq j \leq n-2} \theta(t_r z_j, t_r z_j^{-1}; p)}{\prod_{\substack{0 \leq s \leq n-1 \\ s \neq r}} \theta(t_r t_s, t_r t_s^{-1}; p)} \frac{\theta(t_r z_{n-1}, t_r z_{n-1}^{-1}; p)}{\theta(t_r t_n, t_r t_n^{-1}; p)}$$
$$+ t_n \frac{\prod_{1 \leq j \leq n-1} \theta(t_n z_j, t_n z_j^{-1}; p)}{\prod_{\substack{0 \leq s \leq n \\ s \neq r}} \theta(t_n t_s, t_n t_s^{-1}; p)},$$

we see—upon evaluating at $z_{n-1} = t_n$ and application of the induction hypothesis—that the proportionality constant $c(t_0, \ldots, t_n; p)$ is in fact equal to zero. $\square$

Lemma A.2. *Let $n \in \mathbb{N} \setminus \{0\}$ and let $t_0, \ldots, t_{2n+1}$ be complex indeterminates subject to the constraint $\prod_{r=0}^{2n+1} t_r = 1$. Then one has that*

$$\sum_{r=0}^{n} \frac{\prod_{n+1 \leq s \leq 2n+1} \theta(t_r t_s; p)}{\prod_{\substack{0 \leq s \leq n \\ s \neq r}} \theta(t_r t_s^{-1}; p)} = 0. \tag{A.2}$$

Proof. The proof is similar to that of the previous lemma. For $n = 1$ we have for the l.h.s.

$$\frac{\theta(t_0 t_2, t_0 t_3; p)}{\theta(t_0 t_1^{-1}; p)} + \frac{\theta(t_1 t_2, t_1 t_3; p)}{\theta(t_1 t_0^{-1}; p)} =$$
$$\frac{\theta(t_0 t_2, t_0 t_3; p)}{\theta(t_0 t_1^{-1}; p)} \left(1 + \frac{\theta(t_0 t_1^{-1}, t_1 t_2, t_1 t_3; p)}{\theta(t_1 t_0^{-1}, t_0 t_2, t_0 t_3; p)} \right),$$

which is seen to vanish after elimination of the arguments $t_1 t_2$ and $t_1 t_3$ by means of the relation $t_0 t_1 t_2 t_3 = 1$ and application of the $z \to z^{-1}$ reciprocity relation (2.5). For arbitrary n one infers with the aid of the quasi-periodicity property (2.5) that, after elimination of t_{2n+1} (or t_n) by means of the relation $\prod_{r=0}^{2n+1} t_r = 1$, the p-shift $t_n \to p t_n$ (or $t_{2n+1} \to p^{-1} t_{2n+1}$) amounts to the multiplication of the l.h.s. by the overall factor $t_n t_{2n+1}$. Furthermore, it is clear that the residues of the (generically simple) poles congruent to $t_r = t_s$ $(1 \leq s \neq r \leq n)$ cancel due to the permutation symmetry. Hence, upon setting $t_r = p^{g_r}$ $(r = 0, \ldots, 2n+1)$ with $\sum_{r=0}^{2n+1} g_r = 0$, we see that the l.h.s. is of the form $c(p)p^{(g_0^2 + \cdots + g_n^2 - g_{n+1}^2 - \cdots - g_{2n+1}^2)/2}$ where $c(p)$ is a constant not depending on $g_0, \ldots, g_{2n+1}$. (Here we used the quasi-periodicity, the entireness, and the permutation symmetry in $g_0, \ldots, g_n$ and in $g_{n+1}, \ldots, g_{2n+1}$.) Rewriting the l.h.s. as

$$\sum_{r=0}^{n-1} \frac{\prod_{n+1 \leq s \leq 2n} \theta(t_r t_s; p)}{\prod_{\substack{0 \leq s \leq n-1 \\ s \neq r}} \theta(t_r t_s^{-1}; p)} \frac{\theta(t_r t_{2n+1}; p)}{\theta(t_r t_n^{-1}; p)} + \frac{\prod_{n+1 \leq s \leq 2n+1} \theta(t_n t_s; p)}{\prod_{0 \leq s \leq n-1} \theta(t_n t_s^{-1}; p)},$$

and substitution of $t_n = 1/t_{2n+1}$, reveals that $c(p) = 0$ by the induction hypothesis. $\square$

Armed with these two preparative lemmas, we are in the position to prove the theta-function identities behind the difference equation of Theorem 6.1.

Proposition A.3. *Let $n \in \mathbb{N}$ and let $z_1, \ldots, z_n$, $t_0, \ldots, t_{2n+2}$ and B be (nonzero) complex indeterminates. Then one has that*

$$\sum_{r=0}^{n} \prod_{\substack{0 \le s \le n \\ s \ne r}} \frac{\theta(Bt_s, Bt_s^{-1}; p)}{\theta(t_r t_s, t_r t_s^{-1}; p)} \prod_{1 \le j \le n} \frac{\theta(t_r z_j, t_r z_j^{-1}; p)}{\theta(B z_j, B z_j^{-1}; p)} = 1. \qquad (A.3)$$

Proof. For arbitrary n it follows from the quasi-periodicity property (2.5) that the l.h.s. is invariant with respect to the p-shift $B \to pB$. Furthermore, the residues of the (generically simple) poles in B congruent to $B = z_j$ and $B = z_j^{-1}$ $(j = 1, \ldots, n)$ vanish. This is clear from the permutation symmetry and $z_j \to z_j^{-1}$ reflection-invariance in $z_1, \ldots, z_n$, combined with the observation that the residue at $B = z_n$, which is given explicitly by

$$\frac{z_n^{-2}}{(p;p)_\infty^2 \theta(z_n^2; p)} \frac{\prod_{0 \le s \le n} \theta(z_n t_s, z_n t_s^{-1}; p)}{\prod_{1 \le j \le n-1} \theta(z_n z_j, z_n z_j^{-1}; p)} \sum_{r=0}^{n} t_r \frac{\prod_{1 \le j \le n-1} \theta(t_r z_j, t_r z_j^{-1}; p)}{\prod_{\substack{0 \le s \le n \\ s \ne r}} \theta(t_r t_s, t_r t_s^{-1}; p)},$$

vanishes in view of Lemma A.1. Hence, by the the periodicity and analyticity it follows that the l.h.s. is equal to a constant $c(t_0, \ldots, t_n; z_1, \ldots, z_n; p)$ not depending on B. Substitution of $B = t_0$ reveals that the constant in question must be zero, as at this value the term for $r = 0$ is equal to 1 and the terms for $r > 0$ vanish. $\qquad\square$

Proposition A.4. *Let $n \in \mathbb{N}$ and let $t_0, \ldots, t_{2n+2}$ be (nonzero) complex indeterminates and $A = \prod_{r=0}^{2n+2} t_r$. Then one has that*

$$\sum_{r=0}^{n} \prod_{\substack{0 \le s \le n \\ s \ne r}} \frac{\theta(At_s; p)}{\theta(t_r t_s^{-1}; p)} \prod_{n+1 \le s \le 2n+2} \frac{\theta(t_r t_s; p)}{\theta(At_s^{-1}; p)} = 1. \qquad (A.4)$$

Proof. For $n = 0$ the stated identity holds manifestly. For arbitrary n, it follows from the shift property (2.5) that the l.h.s. is invariant with respect to the p-shift $t_{2n+2} \to pt_{2n+2}$. Furthermore, the residue of the l.h.s. at the (simple) pole congruent to $t_{2n+2} = A$ reads

$$-\frac{A^{-1}}{(p;p)_\infty^2} \frac{\prod_{0 \le s \le n} \theta(At_s; p)}{\prod_{n+1 \le s \le 2n+1} \theta(At_s^{-1}; p)} \sum_{r=0}^{n} \frac{\prod_{n+1 \le s \le 2n+1} \theta(t_r t_s; p)}{\prod_{\substack{0 \le s \le n \\ s \ne r}} \theta(t_r t_s^{-1}; p)},$$

with $\prod_{r=0}^{2n+1} t_r = 1$. Hence, this residue vanishes by Lemma A.2. The upshot is (using the periodicity, analyticity, and permutation symmetry in $t_{n+1}, \ldots, t_{2n+2}$) that the l.h.s. is equal to a constant $c(t_0, \ldots, t_n; p)$ not depending on $t_{n+1}, \ldots, t_{2n+2}$. Writing the l.h.s. as

$$\frac{\theta(At_n; p)}{\theta(At_{2n+2}^{-1}; p)} \sum_{r=0}^{n-1} \frac{\theta(t_r t_{2n+2}; p)}{\theta(t_r t_n^{-1}; p)} \prod_{\substack{0 \le s \le n-1 \\ s \ne r}} \frac{\theta(At_s; p)}{\theta(t_r t_s^{-1}; p)} \prod_{n+1 \le s \le 2n+1} \frac{\theta(t_r t_s; p)}{\theta(At_s^{-1}; p)}$$

$$+ \prod_{0 \le s \le n-1} \frac{\theta(At_s; p)}{\theta(t_n t_s^{-1}; p)} \prod_{n+1 \le s \le 2n+2} \frac{\theta(t_n t_s; p)}{\theta(At_s^{-1}; p)},$$

and evaluation in $t_{2n+2} = 1/t_n$ entails that $c(t_0, \ldots, t_n; p) = 0$ by the induction hypothesis. $\qquad\square$

By dividing out overall factors, the identities of Propositions A.3 and A.4 can be rewritten in the (interpolation) form

$$\frac{\prod_{1\leq j\leq n}\theta(Bz_j,Bz_j^{-1};p)}{\prod_{0\leq r\leq n}\theta(Bt_r,Bt_r^{-1};p)} = \tag{A.5}$$

$$\sum_{r=0}^{n}\frac{1}{\theta(Bt_r,Bt_r^{-1};p)}\prod_{\substack{0\leq s\leq n\\ s\neq r}}\frac{1}{\theta(t_rt_s,t_rt_s^{-1};p)}\prod_{1\leq j\leq n}\theta(t_rz_j,t_rz_j^{-1};p)$$

and

$$\frac{\prod_{n+1\leq s\leq 2n+2}\theta(At_s^{-1};p)}{\prod_{0\leq s\leq n}\theta(At_s;p)}=\sum_{r=0}^{n}\frac{1}{\theta(At_r;p)}\frac{\prod_{n+1\leq s\leq 2n+2}\theta(t_rt_s;p)}{\prod_{\substack{0\leq s\leq n\\ s\neq r}}\theta(t_rt_s^{-1};p)}, \tag{A.6}$$

respectively. The next proposition provides an elliptic summation formula generalizing Eq. (A.6) that was used in the proof of Proposition 8.1.

Proposition A.5. *Let $n\in\mathbb{N}$ and let $t_0,\ldots,t_{2n+2}$, B be (nonzero) complex indeterminates and $A=\prod_{r=0}^{2n+2}t_r$. Then one has that*

$$\frac{\theta(A^2B^2;p)}{\theta(B^{-1};p)}\sum_{r=0}^{n}\frac{\theta(At_r^{-1};p)}{\theta(ABt_r,ABt_r^{-1};p)}\frac{\prod_{n+1\leq s\leq 2n+2}\theta(t_rt_s;p)}{\prod_{\substack{0\leq s\leq n\\ s\neq r}}\theta(t_rt_s^{-1};p)}$$

$$=\frac{\theta(B;p)\prod_{s=0}^{2n+2}\theta(ABt_s;p)-\theta(A^2B;p)\prod_{s=0}^{2n+2}\theta(ABt_s^{-1};p)}{\theta(B;p)\prod_{s=0}^{n}\theta(ABt_s,ABt_s^{-1};p)}. \tag{A.7}$$

Proof. Let us divide both sides of the identity by $\theta(A^2B;p)$ and view the resulting expressions as functions of B. It is seen from the quasi-periodicity property (2.5) that after this division both sides become invariant with respect to the p-shift $B\to pB$. Furthermore, the residues of the (generically simple) poles in B congruent to $B=1$ and to $B=1/A^2$ (caused by the common factor $1/\theta(A^2B;p)$) are equal on both sides by Proposition A.4 (cf. also Remark A.1 below). Since the residues of the (generically simple) poles congruent to $A^{-1}t_r^{\pm1}$ $(r=0,\ldots,n)$ are also manifestly equal on both sides, it follows that the division by $\theta(A^2B;p)$ produced an equation that has both sides differing by at most a constant $c(t_0,\ldots,t_{2n+2};p)$ not depending on B. Evaluation in $B=1/A$ shows that this constant is zero. $\square$

Remark A.1. For $B\to1$ the identity of Proposition A.5 reduces to that of Proposition A.4. Indeed, multiplication of Eq. (A.7) by $\theta(B^{-1};p)/\theta(A^2B;p)$ and letting B tend to 1 reproduces Eq. (A.6).

Remark A.2. For $n=1$ the identities of Proposition A.3 and Proposition A.4 reduce to

$$\frac{\theta(Bt_1,Bt_1^{-1},t_0z,t_0z^{-1};p)}{\theta(t_0t_1,t_0t_1^{-1},Bz,Bz^{-1};p)}+\frac{\theta(Bt_0,Bt_0^{-1},t_1z,t_1z^{-1};p)}{\theta(t_1t_0,t_1t_0^{-1},Bz,Bz^{-1};p)}=1 \tag{A.8}$$

and

$$\frac{\theta(At_1,t_0t_2,t_0t_3,t_0t_4;p)}{\theta(t_0t_1^{-1},At_2^{-1},At_3^{-1},At_4^{-1};p)}+\frac{\theta(At_0,t_1t_2,t_1t_3,t_1t_4;p)}{\theta(t_1t_0^{-1},At_2^{-1},At_3^{-1},At_4^{-1};p)}=1, \tag{A.9}$$

where $A=t_0t_1t_2t_3t_4$. In this special case both identities amount to a well-known three-term equation for the theta function due to Weierstrass (cf. [WW, Sec. 20.53,

Ex. 5])

$$\theta(vw, vw^{-1}, xy, xy^{-1}; p) =$$
$$\theta(vy, vy^{-1}, xw, xw^{-1}; p) - xy^{-1}\theta(vx, vx^{-1}, yw, yw^{-1}; p). \qquad (A.10)$$

Indeed, Eq. (A.8) is recovered via the substitution $x = t_0$, $y = t_1$, $v = B$, $w = z$ and Eq. (A.9) is recovered via the substitution $x = t_0(t_3 t_4)^{1/2}$, $y = t_1(t_3 t_4)^{1/2}$, $v = t_0 t_1 t_2(t_3 t_4)^{1/2}$, $w = (t_3/t_4)^{1/2}$, respectively. Similarly, the identity of Proposition A.5 becomes for $n = 0$:

$$\theta(A^2 B^2, t_0 t_1, t_0 t_2, t_1 t_2; p) = \qquad (A.11)$$
$$\theta(B^{-1}, ABt_0, ABt_1, ABt_2; p) + B^{-1}\theta(A^2 B, Bt_0 t_1, Bt_0 t_2, Bt_1 t_2; p),$$

with $A = t_0 t_1 t_2$. This also amounts to Weierstrass' three-term equation (A.10), but now with $x = (t_0 t_1)^{1/2}t_2$, $y = B(t_0 t_1)^{1/2}t_2$, $v = B(t_0 t_1)^{3/2}t_2$ and $w = (t_0/t_1)^{1/2}$.

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REFERENCES

An.   G. Anderson, *A short proof of Selberg's generalized beta formula*, Forum Math. **3** (1991), 415–417.

AAR.   G.E. Andrews, R. Askey, and R. Roy, *Special Functions*, Encyclopedia of Mathematics and its Applications **71**, Cambridge Univ. Press, Cambridge, 1999.

A.   K. Aomoto, *Jacobi polynomials associated with Selberg's integral*, SIAM J. Math. Anal. **18** (1987), 545–549.

AW.   R. Askey and J. Wilson, *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*, Mem. Amer. Math. Soc. **54** (1985), No. 319.

B1.   E.W. Barnes, *The genesis of the double gamma functions*, Proc. London Math. Soc. **31** (1900), 358–381.

B2.   ———, *On the theory of the multiple gamma function*, Trans. Cambridge Phil. Soc. **19** (1904), 374–425.

BZ.   D. Bressoud and D. Zeilberger, *A proof of Andrews' q-Dyson conjecture*, Discrete Math. **54** (1985), 201–224.

C.   I. Cherednik, *Double affine Hecke algebras and Macdonald's conjectures*, Ann. Math. **141** (1995), 191–216.

D-O1.   E. Date, M. Jimbo, A. Kuniba, T. Miwa, and M. Okado, *Exactly solvable SOS models: local height probabilities and theta function identities*. Nuclear Phys. B **290** (1987), 231–273.

D-O2.   ———, *Exactly solvable SOS models, II: Proof of the star-triangle relation and combinatorial identities*, Conformal Field Theory and Lattice Models, (Eds. M. Jimbo, T. Miwa, and A. Tsuchiya), Advanced Studies in Pure Math. **16** (1988), 17–122.

DS1.   J.F. van Diejen and V.P. Spiridonov, *An elliptic Macdonald-Morris conjecture and modular hypergeometric sums*, Math. Res. Lett. **7** (2000), 729–746.

DS2.   ———, *Elliptic beta integrals and modular hypergeometric sums: an overview*, Proceedings of the NSF Conference Special Functions 2000: Current Perspective and Future Directions, (Tempe, Arizona, May 29–June 9, 2000), submitted.

EZ.   M. Eichler and D. Zagier, *The Theory of Jacobi Forms*, Progress in Math. **55**, Birkhäuser, Boston, 1985.

FV.   G. Felder and A. Varchenko, *The elliptic gamma function and $SL(3,\mathbb{Z}) \ltimes \mathbb{Z}^3$*, Adv. Math. **156** (2000), 44–76.

F.   P.J. Forrester, *Theta function generalizations of some constant term identities in the theory of random matrices*, SIAM J. Math. Anal. **21** (1990), 270–280.

FT.   I.B. Frenkel and V.G. Turaev, *Elliptic solutions of the Yang-Baxter equation and modular hypergeometric functions*, The Arnold-Gelfand Mathematical Seminars (Eds. V.I. Arnold, I.M. Gelfand, V.S. Retakh, and M. Smirnov), Birkhäuser, Boston, MA, 1997, pp. 171–204.

GG.   F. Garvan and G. Gonnet, *Macdonald's constant term conjectures for exceptional root systems*, Bull. Amer. Math. Soc. **24** (1991), 343–347.

G1.   R.A. Gustafson, *A generalization of Selberg's beta integral*, Bull. Amer. Math. Soc. (N.S.) **22** (1990), 97–105.

G2.   _____, *Some q-beta and Mellin-Barnes integrals with many parameters associated to classical groups*, SIAM J. Math. Anal. **23** (1992), 525–551.

G3.   _____, *Some q-beta integrals on $SU(n)$ and $Sp(n)$ that generalize the Askey-Wilson and Nassrallah-Rahman integrals*, SIAM J. Math. Anal. **25** (1994), 441–449.

GR.   G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Encyclopedia of Mathematics and its Applications **35**, Cambridge Univ. Press, Cambridge, 1990.

H.   L. Habsieger, *La q-conjecture de Macdonald-Morris pour G_2*, C. R. Acad. Sci. Paris, Sér. I, Math. **303** (1986), 211–213.

HS.   G.J. Heckman and H. Schlichtkrull, *Harmonic Analysis and Special Functions on Symmetric Spaces*, Perspectives in Math. **16**, Academic Press, San Diego, 1994.

K1.   K.W.J. Kadell, *A proof of the q-Macdonald-Morris conjecture for BC_n*, Mem. Amer. Math. Soc. **108** (1994), no. 516.

K2.   _____, *A simple proof of an Aomoto-type extension of Gustafson's Askey-Wilson Selberg integral*, Methods Appl. Anal. **5** (1998), 125–142.

M1.   I.G. Macdonald, *Some conjectures for root systems*, SIAM J. Math. Anal. **13** (1982), 988–1007.

M2.   _____, *Affine Hecke algebras and orthogonal polynomials*, Séminaire Bourbaki Exp. 797, Astérisque **237** (1996), 189–207.

M3.   _____, *Constant term identities, orthogonal polynomials, and affine Hecke algebras*, Doc. Math. (1998), DMV Extra Volume ICM I, 303–317.

NR.   B. Nassrallah and M. Rahman, *Projection formulas, a reproducing kernel and a generating function for q-Wilson polynomials*, SIAM J. Math. Anal. **16** (1985), 186–197.

O.   E.M. Opdam, *Some applications of hypergeometric shift operators*, Invent. Math. **98** (1989), 1–18.

R.   M. Rahman, *An integral representation of a $_{10}\phi_9$ and continuous bi-orthogonal $_{10}\phi_9$ rational functions*, Can. J. Math. **38** (1986), 605–618.

Ro.   H. Rosengren, *A proof of a multivariable elliptic summation formula conjectured by Warnaar*, Preprint, 2001.

Ru.   S.N.M. Ruijsenaars, *First order analytic difference equations and integrable quantum systems*, J. Math. Phys. **38** (1997), 1069–1146.

Se.   A. Selberg, *Bemerkninger om et multipelt integral*, Norsk Mat. Tiddskr. **26** (1948), 71–78; Collected Papers, vol. **1**, Springer Verlag, Berlin, 1989, pp. 204–213.

S1.   V.P. Spiridonov, *An elliptic beta integral*, Proceedings of the Fifth International Conference on Difference Equations and Applications (Temuco, Chile, January 3–7, 2000), (Eds. S. Elaydi, J. Fenner-Lopez, and G. Ladas), Gordon and Breach, to appear.

S2.   _____, *On an elliptic beta function*, Russ. Math. Surveys **56**(1), (2001), 181–182 (in Russian).

S3.   _____, *Elliptic beta integrals and special functions of hypergeometric type*, Proceedings of the NATO Advanced Research Workshop *Integrable Structures of Exactly Solvable Two-Dimensional Models of Quantum Field Theory*, (Kiev, Ukraine, September 25-30, 2000), (Eds. G. von Gehlen and S. Pakuliak), Kluwer Academic Publishers, to appear.

SZ.   V.P. Spiridonov and A. Zhedanov, *Spectral transformation chains and some new biorthogonal rational functions*, Commun. Math. Phys. **210** (2000), 49–83.

W.   S.O. Warnaar, *Summation and transformation formulas for elliptic hypergeometric series*, Preprint, 2000.

WW.   E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis*, Cambridge University Press, Cambridge, 1986.

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