CONNECTIVITY VIA NONGENERIC PENCILS

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Abstract. We use nongeneric pencils of hypersurfaces in order to prove a new Lefschetz type theorem for singular non compact spaces, at the homotopy level. As applications, we derive results on the topology of the fibres of polynomial functions or the topology of complements of hypersurfaces in $\mathbb{C}^n$.

1. Introduction

The Lefschetz Hyperplane Theorem asserts that, if $X \subset \mathbb{P}^N$ is a projective variety and $H \subset \mathbb{P}^N$ a hyperplane such that $X \setminus H$ is non singular of dimension $\geq n$ (more generally: if $X \setminus H$ has *rectified homotopy depth* greater or equal to n), then $\pi_k(X, X \cap H) = 0$ for all $k < n$.

Lefschetz's original proof ([Lef], see also [La]) uses a generic pencil of hyperplanes to scan the space. Several generalizations to (non compact) spaces with singularities, such as by Goresky and MacPherson [GM], Hamm and Lê [HL1-3], use Morse theory (method first employed by Bott [Bo], Andreotti and Frankel [AF]). For the bibliography up to '88, one can look up [GM].

It appears that in some of these generalizations, under the respective hypotheses, generic pencils do exist and their use yield alternative proofs (e.g. in [GM, Thm. 1.2, pag.199]).

In this paper we extend the method of slicing by pencils to a larger class of "admissible" pencils. This aim is motivated by the fact that, in certain situations, the pencils one is able to use are not generic. As an important example, we mention the study of the topology of a polynomial function $f\colon \mathbb{C}^n \to \mathbb{C}$, which is itself a nongeneric pencil on $\mathbb{C}^n$ (see §5).

So let $X = Y \setminus V$, where Y is a compact complex analytic space and V is a closed complex subspace. For instance, quasi-projective varieties are of this kind.

We call "pencil" the ratio of two sections f and g of a holomorphic line bundle $L \to Y$. It defines a holomorphic function $h := f/g$ over the complement $Y \setminus A$ of the "axis" of the pencil (i.e. the indeterminacy locus) $A := \{f = g = 0\}$. A pencil is called *generic* with respect to X when its axis A is general (i.e. stratified transversal to some Whitney stratification of the pair (Y, V)) and when the holomorphic map $h = f/g : Y \setminus A \to \mathbb{P}^1$ has only stratified double points as singularities. These singularities are finitely many, by the compactness of Y, but part of those might be outside X. Instead of only double

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points, one may consider pencils with any kind of stratified isolated singularities (see [HL3], [GM]).

We mean by *nongeneric pencil* the situation when, in addition, *singularities may occur inside the axis* (e.g. when the axis A is not transversal). In this paper we consider non-generic pencils with isolated singularities in the axis, in the precise meaning of Definition 2.3. The main result we prove is the following:

Theorem 1.1. *Let $X = Y \setminus V$, where Y and $V \subset Y$ are compact complex analytic spaces. Let f, g be sections of a holomorphic line bundle over Y, defining a pencil with at most isolated singularities in the axis (Definition 2.3) and let X_α denote a generic member of the pencil. Let $\operatorname{rhd} X \geq n$, where $n \geq 2$.*

If one of the following two conditions is fulfilled:

(a) *$A \not\subset V$ and the pair $(X_\alpha, A \cap X_\alpha)$ is $(n-2)$-connected,*

(b) *$A \subset V$ and $V \subset \{g = 0\}$,*

then the pair (X, X_α) is $n-1$ connected.

Example 1.2. Let $\hat{f} = x^2y + xz^2$, $\hat{g} = z^3$ define a pencil on $\mathbb{P}^3$ with homogeneous coordinates x, y, z, w. We restrict the pencil to the nonsingular surface $Y \subset \mathbb{P}^3$ given by $yw + x^2 - z^2 = 0$. The axis $\hat{A} \subset \mathbb{P}^3$ consists of two lines, one of which, namely $\{x = z = 0\}$ being the singular locus of each member of the pencil. The axis $A = \hat{A} \cap Y$ of the pencil on Y is two points and one of them is singular. This is clearly a nongeneric pencil, since all of its members have singularities in the axis.

Nevertheless, Theorem 1.1 can be applied to this situation, since the pencil has isolated singularities in the axis (by Proposition 2.4), $\operatorname{rhd} Y \geq 2$ and $(Y_\alpha, A \cap Y_\alpha)$ is 0-connected.

Theorem 1.1 represents a far reaching extension of the Lefschetz theorem on hyperplane sections stated before: although not generic itself, the hyperplane H can be viewed as a member of a generic pencil (by choosing a generic axis inside H, which is possible to do in the projective space). Then one may conclude by using our Remark 3.6, which deals with this particular situation.

The main condition we impose in our Theorem is on the rectified homotopical depth of the space X (abbreviated $\operatorname{rhd} X$). This was introduced by Hamm and Lê [HL3], who proved several Lefschetz type theorems and Grothendieck's conjectures [HL-1,2,3,4]. We only say here that this amounts to a local condition $\operatorname{rhd}_x X \geq n$ which is satisfied, for instance, at points x where (X, x) is a germ of a complete intersection of dimension n. The condition on rhd is recursive, namely $\operatorname{rhd} X_\alpha \geq n - 1$, by [HL3, Theorem 3.2.1].

The condition on the connectivity of $(X_\alpha, A \cap X_\alpha)$ comes in naturally (see [La]) and can also be recursive. Indeed, $A \cap X_\alpha$ may be a (singular) fibre of a second pencil $h' = f/g'$ on the space X_α (which replaces X as total space), with new axis $A_1 := \{f = g' = 0\}$. Let's denote by $(X_\alpha)_\beta$ a generic member of it. We suppose by induction that the pair $((X_\alpha)_\beta, A_1 \cap (X_\alpha)_\beta)$ is $(n-3)$ connected and suppose that Theorem 1.1 can be applied. It follows that $(X_\alpha, (X_\alpha)_\beta)$ is $(n-2)$ connected. When X is compact or when $A \cap X_\alpha$ is generic, this implies in turn, see Remark 3.6, that $(X_\alpha, A \cap X_\alpha)$ is $(n-2)$ connected.

By applying Switzer's result [Sw, Proposition 6.13] to the above theorem, one derives the usual attaching result:

Corollary 1.3. *Under the hypotheses of Theorem 1.1, up to homotopy type, the space X is built from X_α by attaching cells of dimension $\geq n$.*

If X is in addition a Stein space of dimension n, then the attaching cells are of dimension precisely n. If moreover X is $n-1$ connected, then the general hyperplane section X_α has the homotopy type of a bouquet of spheres $\vee S^{n-1}$. □

The proof of our theorem is based on the Nash blowing-up along the axis of the pencil, on homotopy excision and on local Lefschetz type results. We also remark (Proposition 3.5) that, if X_α is not generic, then, in the conclusion of Theorem 1.1 one may replace X_α by X_D, a small "tube" neighbourhood of X_α.

As natural applications, we prove new connectivity estimations on fibres of polynomial functions and complements of affine hypersurfaces.

2. Singularities in the axis

Fix a Whitney stratification $\mathcal{W}$ of Y such that V is a union of strata. Let $\tau = [s : t]$ denote a point on the complex projective line $\mathbb{P}^1$. Let $\mathbb{Y}$ denote the hypersurface $\{(x,\tau) \in Y \times \mathbb{P}^1 \mid sf(x) - tg(x) = 0\}$ in $Y \times \mathbb{P}^1$ obtained by blowing up the axis A and let $\mathbb{X} := \mathbb{Y} \cap (X \times \mathbb{P}^1)$. Consider the projection $p : \mathbb{Y} \to \mathbb{P}^1$ to $\mathbb{P}^1$ and its restriction $p_{|\mathbb{X}} : \mathbb{X} \to \mathbb{P}^1$. We also consider the projection to the first factor $\pi : \mathbb{Y} \to Y$. We use the following notations throughout the paper: for any $M \subset \mathbb{P}^1$, $\mathbb{Y}_M := p^{-1}(M)$ and $\mathbb{X}_M := \mathbb{X} \cap \mathbb{Y}_M$.

Observe that $A \times \mathbb{P}^1 \subset \mathbb{Y}$, that $\mathbb{Y} \setminus (A \times \mathbb{P}^1)$ can be identified (as the graph of h) with $Y \setminus A$, and that the restriction $p_{|\mathbb{Y}\setminus(A\times\mathbb{P}^1)}$ can be identified with h. The stratification $\mathcal{W}$ restricted to the open set $Y \setminus A$ induces a Whitney stratification on $\mathbb{Y} \setminus (A \times \mathbb{P}^1)$, via the above identification.

Definition 2.1. (**Stratification of $\mathbb{Y}$**)
Let $\mathcal{S}$ be the coarsest Whitney stratification on $\mathbb{Y}$ which coincides over $\mathbb{Y} \setminus (A \times \mathbb{P}^1)$ with the one induced by $\mathcal{W}$ on $Y \setminus A$. (This exists, by classical stratification arguments, see e.g. [GLPW].) We refer to it as the canonical stratification of $\mathbb{Y}$ generated by the stratification $\mathcal{W}$ of Y. We also consider as canonical stratification of $\mathbb{X}$ the stratification induced by $\mathcal{S}$ on $\mathbb{X}$.

After endowing $\mathbb{Y}$ and $\mathbb{X}$ with canonical stratifications, the next observation is that both $p : \mathbb{Y} \to \mathbb{P}^1$ and $p_{|\mathbb{X}} : \mathbb{X} \to \mathbb{P}^1$ are *stratified locally trivial fibrations above the complement in $\mathbb{P}^1$ of some finite set* ("bad values") and that this is true in general, regardless of the singularities or the position of the axis A. This type of result is classical (Isotopy Theorem), it goes back to Thom's paper [Th] and it is based on the fact that p is proper analytic and $\mathcal{S}$ has finitely many strata. The problem one has to deal with is what happens in the pencil when one encounters such a critical value. This comes from the singular locus of p. Let us first define it.

Definition 2.2. The *singular locus* of p with respect to $\mathcal{S}$ is the following closed analytic subset of $\mathbb{Y}$:

$$\mathrm{Sing}_{\mathcal{S}}p := \bigcup_{\mathcal{S}_\beta \in \mathcal{S}} \mathrm{Sing}\, p_{|\mathcal{S}_\beta}.$$

The *critical values* of p with respect to $\mathcal{S}$ are the points in the image $p(\mathrm{Sing}_{\mathcal{S}} p)$.

Definition 2.3. We say that the pencil defined by h is a *(nongeneric) pencil with isolated singularities in the axis* if the singularites of the function p at the blown-up axis $A \times \mathbb{P}^1$ are (at most) isolated.

Note that our definition of pencil with isolated singularities in the axis is equivalent to the condition $\dim \mathrm{Sing}_{\mathcal{S}} p \leq 0$ and therefore it implies that the singularities of p outside the axis are also isolated. We shall assume for the remainder of this paper that the singular locus of p is of dimension 0, hence it consists of a finite number of points. This assumption is satisfied for instance in the following particular but very significant case.

Proposition 2.4. *Let $Y \subset \mathbb{P}^N$ be a projective variety endowed with some Whitney stratification and let $\hat{h} = \hat{f}/\hat{g}$ define a pencil of hypersurfaces in $\mathbb{P}^N$. Let B denote the set of points on $\hat{A} \cap Y$ where some member of the pencil is singular or where $\hat{A}$ is not transversal to $\mathcal{W}$. If $\dim B \leq 0$ and the singular points of $h : Y \setminus A \to \mathbb{P}^1$ with respect to $\mathcal{W}$ are isolated then p has isolated singularities.*

Proof. On $\mathbb{Y} \setminus (A \times \mathbb{P}^1)$, p is just h and its singularities are isolated. The notation A stays for $\hat{A} \cap Y$, as usual.

Next, let us remark that $\mathbb{Y} = \mathbb{H} \cap (Y \times \mathbb{P}^1)$, where $\mathbb{H} = \{x \in \mathbb{P}^N, [s : t] \in \mathbb{P}^1 \mid s\hat{f}(x) - t\hat{g}(x) = 0\}$. The singularities of $\mathbb{H}$ are contained into $\Sigma_{\hat{A}} = (\hat{A} \times \mathbb{P}^1) \cap \{s\partial\hat{f} - t\partial\hat{g} = 0\}$, which is at most a collection of lines, by hypothesis. We endow $\mathbb{H}$ with the coarsest Whitney stratification. It follows that $\mathbb{H}$ is transversal to the strata $\mathcal{W} \times \mathbb{P}^1$ of $Y \times \mathbb{P}^1$, except eventually along $B \times \mathbb{P}^1$. By using that a transversal intersection of Whitney stratified sets is Whitney [GLPW, p. 19]), it follows that the canonical stratification $\mathcal{S}$ of $\mathbb{Y}$ restricted to $(A \setminus B) \times \mathbb{P}^1$ contains only stratawhich are products by $\mathbb{P}^1$. Hence the projection p is transversal to these strata. Finally, the stratification $\mathcal{S}$ may distinguish at most a finite number of points, as point strata, out of each line from the collection of projective lines $B \times \mathbb{P}^1$. Then p, being a projection, is still transversal to the complement of these points in $B \times \mathbb{P}^1$. □

3. Proof of the main result and some consequences

Denote in the following $A' := A \cap X$. For any $M \subset \mathbb{P}^1$, let us denote $Y_M := \pi(p^{-1}(M))$ and $X_M := X \cap Y_M$.

We compute here the homotopy groups $\pi_j(X, X_\alpha)$ of the pair space-section, where in the sequel α is supposed to be a general value for p. Let $\mathrm{Sing}_{\mathcal{S}} p = \{b_1, \ldots, b_k\} \subset \mathbb{P}^1$ be the singular values of p.

Lemma 3.1. *For any $W \subset \mathbb{P}^1$, the space Y_W (resp. X_W) is homotopy equivalent to the space $\mathbb{Y}_W$ (resp. $\mathbb{X}_W$), to which one attaches along the product of A (resp. A') by W the product of A (resp. A') by* Cone(W)*. In particular, if W is contractible, then $Y_W \overset{\mathrm{ht}}{\simeq} \mathbb{Y}_W$ and $X_W \overset{\mathrm{ht}}{\simeq} \mathbb{X}_W$.*

Proof. The first statement immediately follows from the definition of the spaces $\mathbb{Y}$ and $\mathbb{X}$. As for the second statement, we have the homotopy equivalences:

$$Y_W \overset{\mathrm{ht}}{\simeq} \mathbb{Y}_W \cup_{A \times W} A \times \mathrm{Cone}(W) \overset{\mathrm{ht}}{\simeq} \mathbb{Y}_W,$$

since $A \times \mathrm{Cone}(W) \overset{\mathrm{ht}}{\simeq} A \times W$. The same argument applies to X_W and we get $X_W \overset{\mathrm{ht}}{\simeq} \mathbb{X}_W$. Notice that in case $A' = \emptyset$, we have $\mathbb{X}_W = X_W$, for any W. □

Take small disjoint closed discs $D_i \subset \mathbb{P}^1$ centered at b_i, a point $\alpha \in \mathbb{P}^1$ exterior to all discs, and simple paths (non self intersecting, mutually non intersecting except at α) $\gamma_i \subset \mathbb{P}^1 \setminus \cup_{i=1}^k \overset{\circ}{D}_i$ from α to some fixed point $c_i \in \partial D_i$, for all $i \in \{1, \dots, k\}$. Denote by $\mathcal{D}$ the subset $\bigcup_{i=1}^k D_i \cup \gamma_i$. Choose a closed disc $K \subset \mathbb{P}^1 \setminus \overset{\circ}{\mathcal{D}}$ such that $K \cap \mathcal{D} = \{\alpha\}$ and consider the decomposition $\mathbb{P}^1 = K \cup \overline{\mathbb{P}^1 \setminus K}$, where $K \cap \overline{\mathbb{P}^1 \setminus K}$ is a circle which we denote by S (therefore $\alpha \in S$). Since $\mathcal{D}$ is a deformation retract of $\overline{\mathbb{P}^1 \setminus K}$, in the following we shall tacitly identify $\mathcal{D}$ to the latter.

By the local triviality of the map $p_{|\mathbb{X}}$, the space $\mathbb{X}_{\overline{\mathbb{P}^1 \setminus K}}$ retracts to $\mathbb{X}_{\mathcal{D}}$ and by Lemma 3.1, the corresponding retraction follows in the space X, namely $X_{\overline{\mathbb{P}^1 \setminus K}}$ retracts to $X_{\mathcal{D}}$. Indeed:

$$X_{\overline{\mathbb{P}^1 \setminus K}} \overset{\mathrm{ht}}{\simeq} \mathbb{X}_{\overline{\mathbb{P}^1 \setminus K}} \cup_{A \times \overline{\mathbb{P}^1 \setminus K}} A \times \mathrm{Cone}(\overline{\mathbb{P}^1 \setminus K}) \overset{\mathrm{ht}}{\simeq} \mathbb{X}_{\mathcal{D}} \cup_{A \times \mathcal{D}} A \times \mathrm{Cone}(\mathcal{D}) \overset{\mathrm{ht}}{\simeq} X_{\mathcal{D}}.$$

The same lemma also shows that X_α is homotopy equivalent to X_K. Hence we have the homotopy equivalence of pairs $(X_K \cup X_{\mathcal{D}}, X_K) \overset{\mathrm{ht}}{\simeq} (X_K \cup X_{\overline{\mathbb{P}^1 \setminus K}}, X_\alpha) = (X, X_\alpha)$.

We now want to apply homotopy excision (Blakers-Massey theorem [BM], see also [Gr, Corollary 16.27]) to the pair $(X_K \cup X_{\mathcal{D}}, X_K)$, where we tacitly use $\mathcal{D}$ instead of $\overline{\mathbb{P}^1 \setminus K}$ when writing $X_S = X_K \cap X_{\mathcal{D}}$. We have clearly that the pair $(X_{\mathcal{D}}, X_S)$ is 0-connected and we need the connectivity level of (X_K, X_S). By considering the triple (X_K, X_S, X_α) and remembering that $X_K \overset{\mathrm{ht}}{\simeq} X_\alpha$, we get, for any i, the isomorphism:

$$\pi_{i+1}(X_K, X_S) \simeq \pi_i(X_S, X_\alpha). \tag{1}$$

Proposition 3.2. *Assume that $A' \neq \emptyset$.*

(a) *If (X_α, A') is m-connected, $m \geq 0$, then:*
  (i) *(X_S, X_α) is at least $m+1$ connected.*
  (ii) *The excision morphism $\pi_j(X_{\mathcal{D}}, X_S) \to \pi_j(X, X_K)$ is an isomorphism for $j \leq m+1$ and an epimorphism for $j = m+2$.*

(b) *If, for any $i \in \{1, \dots, k\}$, the pair $(\mathbb{X}_{D_i}, \mathbb{X}_{c_i})$ is s-connected, then $(X_{\mathcal{D}}, X_\alpha)$ is s-connected too.*

Proof. (a)(i). Note first that X_S is homotopy equivalent to the subset $\mathbb{X}_S \cup A' \times K$ of $\mathbb{X}_K$. Let I and J be two arcs (of angle less than 2π) which cover S. We have the homotopy equivalence $(X_S, X_\alpha) \overset{\mathrm{ht}}{\simeq} (\mathbb{X}_I \cup (A' \times K) \cup \mathbb{X}_J \cup (A' \times K)), \mathbb{X}_J \cup (A' \times K))$.

Then, by homotopy excision (Blakers-Massey theorem), if the (identical!) pairs $(\mathbb{X}_I \cup (A' \times K), \mathbb{X}_{\partial I} \cup (A' \times K))$ and $(\mathbb{X}_J \cup (A' \times K), \mathbb{X}_{\partial J} \cup (A' \times K))$ are $m+1$ connected, then the following morphism:

$$\pi_j(\mathbb{X}_I \cup (A' \times K), \mathbb{X}_{\partial I} \cup (A' \times K)) \to \pi_j(\mathbb{X}_I \cup (A' \times K) \cup \mathbb{X}_J \cup (A' \times K), \mathbb{X}_J \cup (A' \times K))$$

is an isomorphism for $j \leq 2m+1$. This would imply that (X_S, X_α) is $m+1$ connected.

It remains to prove our hypothesis. This follows since the pair $(\mathbb{X}_I \cup (A' \times K), \mathbb{X}_{\partial I} \cup (A' \times K))$ is homotopy equivalent to $(\mathbb{X}_\alpha \times I, \mathbb{X}_\alpha \times \partial I \cup A' \times I)$ and this, in turn, is just

the product of pairs $(\mathbb{X}_\alpha, A') \times (I, \partial I)$. Here we use the assumption in the statement of our Proposition.
(ii). From (1) and from the point (i) it follows that the pair (X_K, X_S) is $m+2$ connected. Hence we've got the needed connectivity level of (X_K, X_S) which makes homotopy excision work. The proof is now complete.

(b). From Lemma 3.1 it follows that $(X_{\mathcal{D}}, X_\alpha) \stackrel{\mathrm{ht}}{\simeq} (\mathbb{X}_{\mathcal{D}}, \mathbb{X}_\alpha)$, since $\mathcal{D}$ is contractible. By Switzer's result [Sw, 6.13], the s-connectivity of the CW-relative complex $(\mathbb{X}_{D_i}, \mathbb{X}_{c_i})$ implies that, up to homotopy equivalence, $\mathbb{X}_{D_i}$ is obtained from $\mathbb{X}_{c_i}$ by attaching cells of dimension $\geq s+1$. Since $\mathbb{X}_{\mathcal{D}} = \cup_i \mathbb{X}_{D_i \cup \gamma_i}$ and $\mathbb{X}_{c_i} \stackrel{\mathrm{ht}}{\simeq} \mathbb{X}_\alpha$, it follows that $\mathbb{X}_{\mathcal{D}}$ is obtained from $\mathbb{X}_\alpha$ by attaching cells of dimension $\geq s+1$. □

Proposition 3.3. (case $V \supset \{g = 0\}$)
Assume that $V \supset \{g = 0\}$. If the pair $(\mathbb{X}_{D_i}, \mathbb{X}_{c_i})$ is s-connected, for any $i \in \{1, \dots, k\}$, then (X, X_α) is s-connected.

Proof. In this case $h_{|X} = f/g : X \to \mathbb{C}$ is a well defined holomorphic function. Since $\mathbb{C}$ retracts to $\mathcal{D}$, the spaces K and S do not occur and we simply have $(X, X_\alpha) \stackrel{\mathrm{ht}}{\simeq} (X_{\mathcal{D}}, X_\alpha)$. We have $A' = \emptyset$, therefore $\mathbb{X}_{D_i} = X_{D_i}$ and $\mathbb{X}_{c_i} \stackrel{\mathrm{diff}}{\simeq} X_\alpha$. The result follows by using Switzer's argument [Sw, 6.13], like in the proof of Proposition 3.2(b). □

Since $A \subset V$, the axis A may be highly non transversal to the stratification $\mathcal{W}$ of (Y, V). However, the singularities of p might be still isolated; remember that Definition 2.2 says that $x \in \mathrm{Sing}_{\mathcal{S}} p$ if and only if $x \in (\mathrm{Sing} Y_t) \cap A$, where $t = p(x)$. This situation occurs when studying polynomial functions on $\mathbb{C}^n$. For instance, the polynomial function $f\colon \mathbb{C}^2 \to \mathbb{C}$, $f(x,y) = x + x^2 y$, as pencil of hypersurfaces, has isolated singularities at infinity. See §4 for applications.

By Propositions 3.2 and 3.3, the connectivity of (X, X_α) depends on the one of $(\mathbb{X}_{D_j}, \mathbb{X}_{c_j})$, for each j, we further study the latter, for some fixed j. Let's then drop the index j and write simply $(\mathbb{X}_D, \mathbb{X}_c)$. We have assumed that p has no singularities over D^* and has only isolated singularities over the center b of D, among the following three possible kinds: singularities on $\mathbb{X}_b \setminus A \times \mathbb{P}^1$, singularities on $\mathbb{Y}_b \setminus (\mathbb{X} \cup A \times \mathbb{P}^1)$ and singularities of p in the axis $A \times \{b\}$.

Say $\mathbb{Y}_b \cap \mathrm{Sing}_{\mathcal{S}} p = \{a_1, \dots, a_r\}$. Let us consider (small) local Milnor-Lê balls at each isolated singularity of $\mathbb{Y}_b$. The existence of such balls was shown by Milnor [Mi] in case of smooth ambient space and by Lê D.T. for singular stratified spaces [Lê1], [Lê2]. These closed balls have the property that, modulo the reducing of the radius of D as much as necessary, their boundaries ∂B_i are transversal to the strata of our stratification $\mathcal{S}$ of the space $\mathbb{Y}$. It is natural now to excise the complement C of the disjoint union $\sqcup_{i \in \{1,\dots,r\}} B_i \cap \mathbb{Y}_{\mathcal{D}}$ from the pair $(\mathbb{Y}_D, \mathbb{Y}_c)$. This has to be related to the fact that $p_| : C \to D$ is a trivial fibration, and moreover, since it is a stratified fibration, the restriction $p_{|\mathbb{X}} : C \cap \mathbb{X} \to D$ is also trivial.

If we perform excision, then we reduce the problem to a local one, around the isolated singularities. In homology, we get the direct sum decomposition $H_*(\mathbb{X}_D, \mathbb{X}_c) = \oplus_i H_*(B_i \cap \mathbb{X}_D, B_i \cap \mathbb{X}_c)$. In homotopy, the excision (Blakers-Massey theorem) implies that the level of connectivity of $(\mathbb{X}_D, \mathbb{X}_c)$ is at least equal to the minimum of the levels of connectivity of $(B_i \cap \mathbb{X}_D, B_i \cap \mathbb{X}_c)$.

We need a condition which implies a certain level of connectivity of each pair $(B_i \cap \mathbb{X}_D, B_i \cap \mathbb{X}_c)$. A condition that fits well is the rectified homotopical depth of the total space $\mathbb{X}$. This condition does not depend on the stratification of the space.

Proposition 3.4. *If* $\mathrm{rhd}\,(X) \geq s+1$*, then, for any* $i \in \{1, \dots, r\}$*, the pair* $(B_i \cap \mathbb{X}_D, B_i \cap \mathbb{X}_c)$ *is at least* s*-connected.*

Proof. Since we work on the space $\mathbb{X}$, we would need a condition on $\mathrm{rhd}\,(\mathbb{X})$. So, we first prove that $\mathrm{rhd}\,(\mathbb{X}) \geq s+1$. Since on the space $X \times \mathbb{P}^1$ we have the product stratification $\mathcal{W} \times \mathbb{P}^1$, the condition $\mathrm{rhd}_{\mathcal{W}}(X) \geq s+1$ implies $\mathrm{rhd}_{\mathcal{W}\times\mathbb{P}^1}(X \times \mathbb{P}^1) \geq s+2$. Our space $\mathbb{X}$ is a hypersurface in $X \times \mathbb{P}^1$ and therefore its rectified homotopical depth is one less, i.e. $\mathrm{rhd}\,\mathbb{X} \geq s+1$, by [HL3, Theorem 3.2.1].

The rectified homotopical depth of $\mathbb{X}$ gives a certain level of connectivity of the complex links of the strata of the stratification $\mathcal{S}$, according to [GM] and [HL3]. One may relate the connectivity of these complex links to the connectivity of the Milnor-Lê data $(B_i \cap \mathbb{X}_D, B_i \cap \mathbb{X}_c)$. This is more special data, especially when the singularity is not in $\mathbb{X}_D$ but on its "boundary" $\mathbb{Y} \cap (V \times \mathbb{P}^1)$. Such relation among connectivities is the local Lefschetz theorem of Hamm and Lê [HL3, Theorem 4.2.1 and Cor. 4.2.2]. This result can be applied for the function $p_| : \mathbb{Y}_D \to D$ with isolated singularities and for the space $\mathbb{X}_D = \mathbb{Y}_D \backslash (V \times \mathbb{P}^1)$ and it tells precisely that, since $\mathrm{rhd}\,\mathbb{X} \geq s+1$, the pair $(B_i \cap \mathbb{X}_D, B_i \cap \mathbb{X}_c)$ is at least s-connected. This proves our statement. □

Proof. of Theorem 1.1. From the long exact sequence of the triple $(X_{\mathcal{D}}, X_S, X_\alpha)$ and since (X_S, X_α) is $n-1$ connected (by Proposition 3.2(a)(i)), it follows that the morphism (induced by inclusion):

$$\pi_j(X_{\mathcal{D}}, X_\alpha) \to \pi_j(X_{\mathcal{D}}, X_S) \tag{2}$$

is an isomorphism for $j \leq n-1$ and an epimorphism for $j = n$.

Next, by Proposition 3.4 and the observations before it, the pairs $(\mathbb{X}_{D_i}, \mathbb{X}_{c_i})$ are $n-1$ connected. Then, applying Proposition 3.2(b) and (a)(i), via the morphism (2), we get the connectivity $n-1$ of (X, X_α). So far for the proof of Theorem 1.1(a).

As for (b), it is now an imediate consequence of Proposition 3.3 together with Proposition 3.4. Note however something important, which we shall use in §4: the hypothesis $\dim \mathrm{Sing}_{\mathcal{S}} p \leq 0$ is too strong. Actually, in Theorem 1.1(b) it is sufficient to assume that $\dim \mathrm{Sing}_{\mathcal{S}} p' \leq 0$, where p' is the restriction of p on $\mathbb{Y}_{\mathbb{C}}$. Our proof only uses this weaker hypothesis. □

We end this section by proving the usual observation that we can replace X_α in the conclusion of the theorem by a "very bad" member of the pencil and then take a good neighbourhood of this. Let $D_\delta \subset \mathbb{P}^1$ denote a small enough disc centered at δ such that $D_\delta \cap p(\mathrm{Sing}_{\mathcal{S}} p) = \{\delta\}$.

Proposition 3.5. *In Theorem 1.1, replace the hypothesis about the singularities of the pencil with the following: "Let* f, g *define a pencil with at most isolated singularities except at one fiber* X_δ*, that is* $\dim(\mathrm{Sing}_{\mathcal{S}} p \cap X_\beta) \leq 0$*, for all* $\beta \neq \delta$*".*

Then (X, X_{D_δ}) *is* $(n-1)$ *connected.*

Proof. We just consider D_δ as one of the small discs D_i within $\mathcal{D}$. We of course still need a generic member X_α and the hypothesis on the connectivity of (X_α, A') in case A' is not empty.

It follows from Proposition 3.2(a)(i) and from (1) that (X_K, X_S) is n-connected. By homotopy excision of $X_{\mathcal{D}}$ from $(X, X_{\mathcal{D}})$, this implies that $(X, X_{\mathcal{D}})$ is n-connected. In turn, via the homotopy exact sequence of the triple $(X, X_{\mathcal{D}}, X_{D_\delta})$, this implies that the morphism induced by inclusion $\pi_i(X_{\mathcal{D}}, X_{D_\delta}) \to \pi_i(X, X_{D_\delta})$ is an isomorphism for $i \leq n-1$. It remains to study $(X_{\mathcal{D}}, X_{D_\delta})$. Since $\mathcal{D}$ and D_δ are contractible, Lemma 3.1 says that $(X_{\mathcal{D}}, X_{D_\delta}) \stackrel{\mathrm{ht}}{\simeq} (\mathbb{X}_{\mathcal{D}}, \mathbb{X}_{D_\delta})$. By the argument in the proof of Proposition 3.2(b) and also by Proposition 3.4, $\mathbb{X}_{\mathcal{D}}$ is obtained from $\mathbb{X}_{D_\delta}$ by attaching cells of dimension $\geq n$. Consequently, $(\mathbb{X}_{\mathcal{D}}, \mathbb{X}_{D_\delta})$ is also $(n-1)$-connected, hence (X, X_{D_δ}) too. □

Suppose now that X is compact. Then, since in generic pencils on such X the tube neighbourhood X_{D_δ} contracts to the fibre X_δ, we have:

Remark 3.6. In the conclusion of the above Proposition 3.5, the tube X_{D_δ} can be replaced by X_δ when the inclusion $X_\delta \subset X_{D_\delta}$ is a homotopy equivalence. This happens for instance if X is compact or if the pencil has no singularities in the axis (i.e. $(A \times \mathbb{P}^1) \cap \mathrm{Sing}_{\mathcal{S}} p = \emptyset$).

As mentioned in the Introduction, this remark is useful in order to make Theorem 1.1 work inductively on dimension, namely to show that the condition on the connectivity of $(X_\alpha, A \cap X_\alpha)$ is recursive. It also clarifies why the "classical" Lefschetz hyperplane theorem (stated in the Introduction) can be proved by using a pencil having a generic axis into the "bad" hyperplane H.

4. Aplications

We give applications of Theorem 1.1(b) concerning the topology of fibres of polynomial functions $f : \mathbb{C}^n \to \mathbb{C}$ and complements of hypersurfaces in $\mathbb{C}^n$. The asymptotic behaviour of f was studied by several authors in the last years (see e.g. [ST], [Pa], [Ti2]).

A polynomial function may be extended to a meromorphic function on a compact space; this embedding is however not unique. We consider here the embedding of $\mathbb{C}^n$ into a weighted projective space $\mathbb{P}_w := \mathbb{P}(w_1, \ldots, w_n, 1)$, as follows. Associate to each coordinate x_i a positive weight w_i, for $i \in \{1, \ldots, n\}$ and write $f = f_d + f_{d-k} + \cdots$ where f_i is the degree i weighted-homogeneous part of f and $f_{d-k} \neq 0$. Then take a new variable z of weight 1. We get a meromorphic function $\tilde{f}/z^d$ on $\mathbb{P}_w$ and the Nash blown up space $\mathbb{Y} := \{s\tilde{f} - tz^d = 0\} \subset \mathbb{P}_w \times \mathbb{P}^1$, where $\tilde{f}$ is the degree d homogenized of f. We consider the coarsest Whitney stratification $\mathcal{S}$ on $\mathbb{Y}$. Here V is the hyperplane at infinity $\{z = 0\}$ and $g := z^d$.

We say that the pencil defined by f *has at most isolated singularities in* $\mathbb{Y}' := \mathbb{Y}_{\mathbb{C}}$ if the singularities of the restriction $p' := p_{|\mathbb{Y}'} : \mathbb{Y}' \to \mathbb{C}$ with respect to the stratification $\mathcal{S}$ are isolated. Let us denote by Σ the weighted projective variety $\{\mathrm{grad}\, f_d = 0, f_{d-k} = 0\} \subset \mathbb{P}_w \cap \{z = 0\}$.

Proposition 4.1. *If* $\dim \mathrm{Sing} f \leq 0$ *and* $\dim \Sigma \leq 0$ *then the pencil defined by* f *has at most isolated singularities in* $\mathbb{Y}'$.

Proof. We prove that $\dim \mathrm{Sing}_{\mathcal{S}} p' \le 0$. Since the singularities of p' on $\mathbb{C}^n = \mathbb{Y}' \setminus \{z = 0\}$ are isolated by hypothesis, we only have to look on $\mathbb{Y}' \cap \{z = 0\}$, which is the product $\{f_d = 0\} \times \mathbb{C} \subset (\mathbb{P}_w \cap \{z = 0\}) \times \mathbb{C}$.

We need to know the stratified structure of $\mathbb{Y}'$ in the neighbourhood of $\mathbb{Y}' \cap \{z = 0\}$. Consider first $\mathbb{Y}' \cap \{z = 0\} \cap (\{\mathrm{grad}\, f_d \neq 0\} \times \mathbb{C})$. This is a subspace of the quotient by the $\mathbb{C}^*$ action of the nonsingular part of the hypersurface $\{\tilde{f} - tz^d = 0\}$ considered as subset of $(\mathbb{C}^{n+1} \setminus \{0\}) \times \mathbb{C}$. We claim that the stratification by the orbit type of this nonsingular part is Whitney regular. Indeed, the $\mathbb{C}^*$ action reduces, within Zariski-open subsets $\{x_i \neq 0\}$, to the action of a finite group. For quotients by finite groups, the natural orbit type stratification is Whitney regular (see e.g. [GLPW, p. 21], [Fe]).

Since the action on the factor $\mathbb{C}$ is trivial, the strata within $\mathbb{Y}' \cap \{z = 0\} \cap (\{\mathrm{grad}\, f_d \neq 0\} \times \mathbb{C})$ are products by $\mathbb{C}$. It follows that p' (which is the projection to $\mathbb{C}$) is transversal to these strata.

Next we look to points $q \in \{\mathrm{grad}\, f_d = 0\} \setminus \{f_{d-k} = 0\} \subset \mathbb{P}_w \cap \{z = 0\}$. We have $\tilde{f}(x, z) - tz^d = f_d(x) + z^k g$, where $g(q, z, t) = f_{d-k}(q) + zh(q, z, t)$ and $f_{d-k}(q) \neq 0$.

Locally at q, our hypersurface $\mathbb{Y}' = \{\tilde{f}(x, z) - tz^d = 0\}$ is equivalent, via an analytic (and $\mathbb{C}^*$-equivariant) change of coordinates, modulo a choice of the k-root, to the product of $\{f_d(x) + (z')^k = 0\}$ by the coordinate t, where z' is the new coordinate z and $\{z' = 0\}$ is equal to $\{z = 0\}$ at q. Since we have again a product by $\mathbb{C}$, we may deduce that, locally, the line $\{q\} \times \mathbb{C}$ is included into a Whitney stratum of $\mathcal{S}$. Hence the map p' has no singularities on $\{q\} \times \mathbb{C}$.

This proves that $\mathrm{Sing}_{\mathcal{S}} p' \subset \Sigma \times \mathbb{C}$. Since $\dim \Sigma \le 0$, the set $\Sigma \times \mathbb{C}$ is a finite union of complex lines. The map p' is transversal to such a line, so singularities of p' on $\Sigma \times \mathbb{C}$ can occur only if $\Sigma \times \mathbb{C}$ contains point-strata of $\mathcal{S}$. But there can only be finitely many such point-strata. This ends our proof. □

Note 4.2. The above proof shows that the condition $\dim \Sigma \le 0$ implies that $\dim \mathrm{Sing} f \le 1$.

Corollary 4.3. *If the polynomial $f : \mathbb{C}^n \to \mathbb{C}$ has isolated singularities and $\dim \Sigma \le 0$ then its general fibre X_α is homotopy equivalent to a bouquet of spheres of dimension $n-1$. Moreover, any atypical fibre of f is at least $n-3$ connected.*

Proof. By applying Theorem 1.1(b) via Proposition 4.1, we get that $(X, X_\alpha) = (\mathbb{C}^n, X_\alpha)$ is $n-1$ connected. Since X_α is Stein of dimension $n-1$, our first statement follows by Corollary 1.3.

For the second statement, we use an argument similar to the one used to prove [Ti1, Theorem 5.5]. We give here an outline of it. First, we take a sufficiently large ball B centered at the origin such that $X_b \cap B_M$ is homotopy equivalent to X_b, for any ball B_M larger or equal to B.

We next consider the polar locus of the map (p', z) at some isolated singularity in $\mathbb{Y}_b \cap \{z = 0\}$. This locus is a curve or it is void, see *loc. cit.* The polar curves corresponding to the singularities in $\mathbb{Y}_b \cap \{z = 0\}$ intersect a nearby general fibre X_α at a finite number of points. It follows that X_α is homotopy equivalent to $X_\alpha \cap B$ to which one attaches a finite number of $n-1$ cells, corresponding to the intersection multiplicities of the polar curves with X_α. On the other hand, $X_b \cap B$ is homotopy equivalent to $X_\alpha \cap B$ to which one attaches n cells, corresponding to the isolated singularities on X_b. It follows that $(X_\alpha, X_\alpha \cap B)$ is $n-2$ connected and that $(X_b, X_\alpha \cap B)$ is $n-1$ connected. Therefore, since X_α is $n-2$ connected, X_b is $n-3$ connected. □

This represents an extension of results on connectivity of fibres, in the vein of [ST] and [Ti1]. Dimca and Păunescu [DP] proved recently a related result, by different methods. We shall explain in a forthcomming paper [LT] that, with a recursive procedure, the above result recovers completely [DP] and in the same time improves the connectivity estimation for generic fibres.

Example 4.4. $f = x^2y^3 + v^2 + y$ has $\mathrm{Sing} f = \emptyset$ and $\Sigma = \{[1:0:0],[0:1:0]\} \subset \mathbb{P}^2$. Here all the weights are 1 and $\mathbb{P}_w$ is the usual projective space $\mathbb{P}^3$. One can verify that the general fibre is homotopy equivalent to $S^2 \vee S^2 \vee S^2$. It turns out that the only atypical fibre is $f^{-1}(0)$.

We further give an application of our results to the topology of complements of hypersurfaces in $\mathbb{C}^n$. This is a topic which goes back to Zariski and van Kampen [vK], who described a general procedure to compute the fundamental group of the complement to an algebraic curve in $\mathbb{P}^2$ by slicing with linear pencils. Zariski showed that π_1 depends on the type and position of singularities of the curve. More recently, Libgober [Li] proved similar results on the higher homotopy group $\pi_{n-k-1}(\mathbb{C}^n \setminus V)$, where k is the dimension of the singular locus of the hypersurface V, $k < n-2$. This is the first possible nontrivial homotopy group of rank higher than 1, since, by the classical Lefschetz theorem, $\pi_i(\mathbb{C}^n \setminus V) = 0$, for $2 \le i < n-k-1$. Since $\pi_{n-k-1}(\mathbb{C}^n \setminus V) = \pi_{n-k-1}(H_k \cap \mathbb{C}^n \setminus V)$, for a general linear subspace H_k of codimension k, the problem of finding π_{n-k-1} reduces to the case of the complement of a hypersurface V with isolated singularities (see [Li, §1]).

In [Li, Theorem 2.4], Libgober considers the case when V has at most isolated singularities and $\bar{V}$ is transversal to the hyperplane at infinity in $\mathbb{P}^n$. We show here that, under certain circumstances (weaker transversality condition but imposing that V is a generic fibre of f), we may conclude to the triviality of $\pi_{n-k-1}(\mathbb{C}^n \setminus V)$.

Proposition 4.5. *If V is a general fibre of a polynomial $f\colon \mathbb{C}^n \to \mathbb{C}$ and the pencil defined by f has at most isolated singularities in $\mathbb{Y}'$ then $\mathbb{C}^n \setminus V \overset{\mathrm{ht}}{\simeq} S^1 \vee \bigvee S^n$. When $n > 2$, it follows that $\pi_{n-1}(\mathbb{C}^n \setminus V) = 0$.*

Proof. We consider small enough discs $D_i \subset \mathbb{C}$ centered at the "bad values" of the pencil, like in §3, and also a small enough disc D_0 centered at the general value β. Let $\alpha \in \partial \bar{D}_0$. Using the notations in §3, the configuration $D_0^* \cup_i (D_i \cup \gamma_i)$ is a strong deformation retract of $\mathbb{C} \setminus \{\beta\}$. It follows that $\mathbb{C}^n \setminus V \overset{\mathrm{ht}}{\simeq} f^{-1}(D_0^* \cup_i (D_i \cup \gamma_i))$.

Now, denoting by S^1 the boundary of $\bar{D}_0$, the space $f^{-1}(D_0^* \cup_i (D_i \cup \gamma_i))$ is homotopy equivalent to the product $S^1 \times f^{-1}(\alpha)$ to which one attaches $f^{-1}(\cup_i(D_i \cup \gamma_i))$ over $\{\alpha\} \times f^{-1}(\alpha)$. But, the attaching of the space $f^{-1}(\cup_i(D_i \cup \gamma_i))$ to $\{\alpha\} \times f^{-1}(\alpha)$ gives $\mathbb{C}^n$, hence a contractible space. Therefore, we get $\mathbb{C}^n \setminus V \overset{\mathrm{ht}}{\simeq} S^1 \vee S(f^{-1}(\alpha))$, where $S(\cdot)$ denotes the suspension.

Finally, by Corollary 1.3 and the remark following it, $f^{-1}(\alpha)$ is homotopy equivalent to a bouquet of spheres S^{n-1}. This concludes the proof. □

In the Example 4.4, the "singularities at infinity" of a general fibre V, in the sense of [Li] are non isolated (since $\bar{V} \not\pitchfork \{z = 0\}$ along $\{x = 0\} \cup \{y = 0\} \subset \mathbb{P}^2$). The results of [Li] do not give any information in this case; hoewever, our Proposition 4.5 can be applied, since the pencil defined by f has at most isolated singularities in $\mathbb{Y}'$. We get $\mathbb{C}^3 \setminus V \overset{\text{ht}}{\simeq} S^1 \vee S^3 \vee S^3 \vee S^3$.

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