Combinatorial computation of combinatorial formulas for knot invariants

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Abstract

A purely combinatorial algorithm for computing combinatorial formulas of finite type knot invariants is described and illustrated.

This work is a direct continuation of [16], where a general geometrical approach to the construction of combinatorial formulas of cohomology classes of spaces of knots in $\mathbb{R}^n$, $n \geq 3$, was proposed. Here we show in detail how it works in the most classical and well-studied case of zero-dimensional classes, i.e., of invariants of knots in $\mathbb{R}^3$.

We present an algorithm producing combinatorial formulas for finite-type knot invariants. Starting from a proper *weight system* of rank k, i.e. the principal part of a knot invariant of order k, it produces a finite collection of open subvarieties in the space of knots, any of which is distinguished by at most k standard conditions on the geometrical disposition of knots, so that the value of our invariant on a knot in $\mathbb{R}^3$ is equal to the algebraic number of these varieties containing the knot. The work of the algorithm is demonstrated in the case of simplest knot invariants of orders 2 and 3, reduced mod 2.

All known to me other algorithms of explicit calculation of *all* finite type invariants include drawing the planar pictures (diagrams of knots and singular knots) and deforming these pictures. The algorithm proposed below is *purely combinatorial*, i.e. it deals not with planar or spatial pictures but with easily encodable combinatorial objects similar to the chord diagrams. The execution of the algorithm is a chain of linear algebraic operations over these objects, similar to (and starting with) checking the homological 4T- and 1T-conditions for a sum of chord diagrams. In particular, the complexity of the algorithm and its answers depends on the order k only, and not on the complexity of arising knot diagrams. Therefore it is ready for effective computer realization.

Probably the first combinatorial formulas for some finite type knot invariants were proposed by J. Lannes [6]. The most convenient known combinatorial formulas of this

kind are the Polyak-Viro *arrow diagrams* introduced in [9]. By a theorem of M. Goussarov [4] any finite type invariant can be represented by such a formula; the proof of this theorem probably implies also a method of computing such formulas.

The algorithm described below is by now apparently worse than the one following from this approach. Indeed,

a) it is realized over $\mathbb{Z}_2$ only, i.e. without accounting orientations of arising varieties;

b) the answers are more awkward, containing some terms other than just the Polyak-Viro diagrams;

c) the main advantage, the possibility of an effective computerization, is not realized yet.

This list is in fact a program of the further work. Indeed, the accounting of orientations is just a technical problem, cf. §V.3.3 in [12].

The inconvenience b) can be explained by the fact that our approach provides too many choices for spanning the cycles homologous to zero by semialgebraic varieties in the space of curves, see e.g. §3.2 below. Now I do not select the most economical of them, and use the way which has the simplest formulation but causes not the simplest calculations. By this reason the existing algorithm is called *stupid*. In §4 this algorithm calculates a combinatorial formula for the simplest invariant (of order two). This formula consists of three terms. Almost the same algorithm with one nontrivial switch (which is easy for the human eye but is not formalized yet) gives the Polyak-Viro formula consisting of one term, see §2 in [16]. One of nearest problems is to make the existing algorithm not so stupid, i.e. to teach it to select the most economical choices. On the other hand, the plenty of choices provides many comparison results on the combinatorial formulas like in [10]. In §6 I describe one other algorithm referring to both the our homological techniques and the Goussarov's theorem. It has a slightly simpler formulation and simpler results (exactly the Polyak–Viro formulas), however the systems of linear equations to be solved in its execution are exponentially greater (over the order of the invariant). Also, I do not see how it could be extended to the calculation of other cohomology classes of spaces of knots.

I invite the volunteers acquainted with the mathematical programming to solve the problem c). I hope very much that a computer himself will then find some ways to make best choices in problem b), so that it will remain to systematize its experience.

This work is only a step in the general program of realizing arbitrary dimensional cohomology classes of spaces of knots, cf. [16] (which, in its turn, is just a sample of a wide class of similar problems concerning effective methods in the topology of spaces of nonsingular objects).

Like in [16], our algorithm is based on the study of the *discriminant subvariety* Σ in the space $\mathcal{K}$ of smooth parametrized curves $f : \mathbb{R}^1 \to \mathbb{R}^3$ (i.e. of the set of maps $f \in \mathcal{K}$ that are not smooth embeddings). The main tool is the *simplicial resolution* of the discriminant, i.e. a certain topological space σ together with a continuous surjective

map $\pi : \sigma \to \Sigma$. Homology groups of Σ and σ are closely related to one another and to the cohomology groups of the space of knots $\mathcal{K} \setminus \Sigma$. The resolved discriminant σ admits a natural filtration $F_1 \subset F_2 \subset \ldots$ which generates a spectral sequence calculating its homology groups.

As in [16], the algorithm is essentially a conscientious realization of the work of this spectral sequence in the terms of relative chains. We start from a weight system of rank k, i.e. from a relative cycle γ in the term F_k reduced modulo F_{k-1} represented as a sum of open cells of maximal dimension in $F_k \setminus F_{k-1}$. (Such cells are in one-to-one correspondence with *equivalence classes of the chord diagrams*, see [12], [3] and §1 below.) Then we calculate its first boundary $d^1(\gamma) \subset F_{k-1} \setminus F_{k-2}$ and *span* it there, i.e. we construct a chain $\gamma_1 \subset F_{k-1} \setminus F_{k-2}$ such that $\partial \gamma_1 = -d^1(\gamma)$ in $F_{k-1} \setminus F_{k-2}$. Then we define $d^2(\gamma)$ as the boundary of the chain $\gamma + \gamma_1$ in $F_{k-2} \setminus F_{k-3}$, span it by a chain $\gamma_2 \subset F_{k-2} \setminus F_{k-3}$, etc. By the Kontsevich's theorem, the entire this sequence of operations can be accomplished, and we get a cycle $\gamma + \gamma_1 + \ldots + \gamma_{k-1}$ defining an absolute cycle in the one-point compactification of the resolved discriminant σ. Pushing it down, we get a semialgebraic cycle in the non-resolved discriminant Σ. Finally we span it by a relative cycle (mod Σ) in the whole space of curves: tautologically, this relative cycle is the desired combinatorial formula.

In [16] this method was used to realize certain positive dimensional finite type cohomology classes of spaces of knots. This theory has a deep analogy with the homological study of subspace arrangements, especially with the explicit realizations of their homology classes proposed in [18]. More on this analogy see in [17].

In §1, I describe some standard semialgebraic subvarieties in the space of curves and in terms $F_i \setminus F_{i-1}$ of the resolved discriminant: all spanning chains γ_{k-i} and their boundaries will be built of these varieties. In §2 I study the boundaries of these varieties, which is necessary for checking the homological conditions. In §3 the main algorithm is described. In §4 I show how this algorithm calculates a combinatorial formula (mod 2) for the unique invariant v_2 of filtration 2, in §5 the same is done for the next complicated invariant v_3 of filtration 3.

I thank A. B. Merkov very much, whose help and critical attention were very essential: I believe that the idea of this work arose implicitly from our previous conversations. I acknowledge the hospitality of the Isaac Newton Institute, Cambridge, where a main part of the work was accomplished.
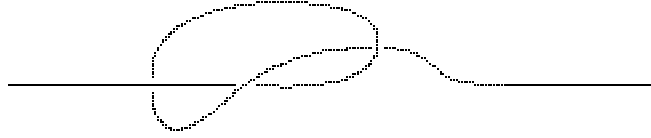
Figure 1: A long knot

1 Zoo of varieties in the space of curves and in the resolved discriminant

1.1 Preliminary remarks

We consider *long knots*, see [16], i.e., smooth embeddings $\mathbb{R}^1 \to \mathbb{R}^3$ coinciding with a standard embedding outside some compact subset in $\mathbb{R}^1$. We denote by $\mathcal{K}$ the space of all maps $\mathbb{R}^1 \to \mathbb{R}^3$ with these boundary conditions, and define the discriminant $\Sigma \subset \mathcal{K}$ as the set of all maps having either self-intersections or singular points. The long knots are exactly the points of the difference $\mathcal{K} \setminus \Sigma$, see Fig. 1. The points of Σ are called *singular knots*.

We work with the space $\mathcal{K}$ as with an affine space of a very large but finite dimension. A justification for this, based on the techniques of finite-dimensional approximations, is described in [11]–[13]. The quotes ' , ' below indicate formally nonstrict statements and terms which need such a justification. In particular, we use the 'Alexander duality' in $\mathcal{K}$,

$$\tilde{H}^i(\mathcal{K} \setminus \Sigma) \simeq \bar{H}_{\omega-i-1}(\Sigma), \tag{1}$$

where $\tilde{H}^i$ is the cohomology group reduced modulo a point, $\bar{H}_j$ is the *Borel–Moore* homology group, i.e. the reduced homology group of the one-point compactification, and ω is the notation for the 'dimension' of $\mathcal{K}$. We realize the homology classes in the right-hand part of (1) by semialgebraic chains of infinite dimensions but finite codimensions.

As in [11]–[13], [16], we use a *simplicial resolution* of Σ, i.e. another space σ together with a 'proper' map $\pi : \sigma \to \Sigma$ that induces a homomorphism of 'Borel–Moore homology groups'. The finite type cohomology classes in $\mathcal{K} \setminus \Sigma$ are defined as those Alexander dual (1) to the 'direct images' under this map of elements of $\bar{H}_*(\sigma)$ in $\bar{H}_*(\Sigma)$. They form an important subgroup $H_f^* \subset H^*(\mathcal{K} \setminus \Sigma)$. The space σ admits a natural filtration $F_1 \subset F_2 \subset \ldots$, generating a spectral sequence $E_{p,q}^r$ calculating the Borel–Moore homology classes of σ, i.e. $E_{p,q}^r \to \bar{H}_{p+q}(\sigma)$, $E_{p,q}^1 \simeq \bar{H}_{p+q}(F_p \setminus F_{p-1})$. The resulting filtration in the Borel–Moore homology group of σ induces a filtration in the 'Alexander dual' group H_f^*. E.g. the 0-dimensional cohomology classes (i.e. knot invariants) of finite filtration are known as finite-type knot invariants, and the filtration of an invariant often is called its *order* or *degree*.
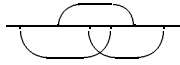
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Figure 2: A chord diagram

The naturality of this construction is clear from the fact that in the case of knots in $\mathbb{R}^n$, $n > 3$, the *entire* (highly non-trivial) cohomology group of the space of such knots comes from the similar construction: $H_f^* = H^*(\mathcal{K} \setminus \Sigma)$. For $n = 3$ this construction gives us *a priori* only a subgroup of $H^*(\mathcal{K} \setminus \Sigma)$ (however no example is known of a cohomology class which cannot be approximated by ones coming from our construction). In the sequel we consider the case of knot invariants in $\mathbb{R}^3$ only.

In this work we use the construction of the resolution σ described in [15]. It is slightly more economical than the one from [11]–[13].

Any term $F_i \setminus F_{i-1}$, $i \geq 2$, of its filtration consists of finitely many open cells (so that its one-point compactification $\bar{F}_i / \bar{F}_{i-1}$ is a cell complex), and its cells of maximal dimension $\omega - 1$ (responsible for the knot invariants) are in one-to-one correspondence with equivalence classes of *chord diagrams* with i chords.

Such a diagram consists of a horizontal line ("Wilson line") and $2i$ distinct points in it matched into pairs. Since [11] such pairs of points are depicted by arcs ("chords") with ends at these points, see Fig. 2. The Wilson line symbolizes the source line $\mathbb{R}^1$ of our long knots and singular knots $f : \mathbb{R}^1 \to \mathbb{R}^3$. We shall consider also some other lines $\mathbb{R}^1$, therefore we denote the Wilson line by $\mathbb{R}^1_w$. (In the parallel theory of *compact knots* $S^1 \hookrightarrow \mathbb{R}^3$ we have not the Wilson line but the Wilson loop S^1, the matched pairs of points in which are connected by segments; this explains the term "chord", see [1], [3].) Two chord diagrams are *equivalent* if they can be transformed one into the other by an orientation-preserving homeomorphism of the Wilson line.

The cell of $F_i \setminus F_{i-1}$ corresponding to an equivalence class of chord diagrams with i chords consists of all triples of type
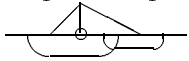
$$(C, f, t), \tag{2}$$

where C is any chord diagram of this equivalence class, f is a smooth map $\mathbb{R}^1 \to \mathbb{R}^3$ gluing together the endpoints of any chord of C, and t is an interior point of a certain $(i - 1)$-dimensional simplex arising in the construction of the resolution: the vertices of this simplex correspond formally to our chords. If $i = 0$, then the unique such cell (corresponding to the empty chord diagram) coincides with the entire space $\mathcal{K}$ of long curves $\mathbb{R}^1 \to \mathbb{R}^3$.

So, we have $(2i)!/(i!2^i)$ maximal cells in $F_i \setminus F_{i-1}$, in correspondence with all possible matchings of given $2i$ points. The dimension of any such cell is equal to $2i + (\omega - 3i) + (i - 1) = \omega - 1$, i.e. the dimension of hypersurfaces in $\mathcal{K}$.

Any cycle of maximal dimension in $F_i \setminus F_{i-1}$ is a linear combination of such cells. The homological condition, which such a linear combination should satisfy to be a cycle, are formulated in terms of cells of vice-maximal dimension. Such cells arise from maps $f : \mathbb{R}^1 \to \mathbb{R}^3$ having either $i-2$ double points and one triple point or $i-1$ double points and one stationary point $*$ at which $f' = 0$. The corresponding homological conditions are called 4T-relations and 1T-relations, respectively.

The vice-maximal cells of the first (4T) type are related with equivalence classes of configurations of $2i-1$ points in $\mathbb{R}^1_w$ separated into $i-2$ unordered pairs ("chords") and one triple. Namely, with any such class three vice-maximal cells are associated: they correspond to the additional choices of one pair of points inside the triple. They are depicted by drawings with tripods like , where the point of the triple *not in the chosen pair* is marked by a small circle. A point of such a cell also has the form (2), where C is any configuration of our equivalence class, f is a map $\mathbb{R}^1 \to \mathbb{R}^3$ gluing together all points inside any pair and the triple of C, and t is a point of a certain $(i-1)$-dimensional simplex arising from the construction of the resolution: some its $i-2$ vertices correspond formally to the chords of C, one vertex more to the triple in C, and the last vertex to the distinguished pair inside this triple.

Similarly, the cells of the second (1T) type are in the one-to-one correspondence with equivalence classes of configurations of $2i-1$ points in $\mathbb{R}^1_w$ separated in $i-1$ pairs and one singular point $*$.

The boundary of a maximal cell in $F_i \setminus F_{i-1}$ consists of $2i-1$ summands corresponding to all segments in $\mathbb{R}^1_w$ bounded by neighboring points of any configuration C of the corresponding equivalence class. Namely, let us contract any such segment. If its endpoints belong to different pairs, then after contraction this couple of pairs degenerates to a triple; the corresponding summand in the boundary of our cell consists of some two vice-maximal cells of 4T type related with the arising configuration. If these bounding points belong to one pair, then we obtain one vice-maximal cell of the 1T type.

Example 1 The term F_1 of the filtration consists of exactly two cells, one of maximal dimension, and the second equal to its boundary:

$$\partial \quad \underset{}{\overline{}} \quad = \quad \underline{*} \quad , \tag{3}$$

thus there are no cohomology classes of filtration 1 of the space of long knots, in particular no knot invariants of order 1.

Example 2 The term $F_2 \setminus F_1$ contains three cells of maximal dimension,

$$\text{} \quad , \quad \text{} \quad , \text{ and } \quad \text{} \quad , \tag{4}$$

three vice-maximal cells od $4T$ type

$$\text{} \quad , \quad \text{} \quad , \text{ and } \quad \text{} \quad , \tag{5}$$

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and three vice-maximal cells of $1T$ type

$$\text{(diagrams)} \quad , \quad \text{(diagrams)} \quad , \text{ and } \quad \text{(diagram)} \tag{6}$$

The boundary operator acting from the maximal cells to the vice-maximal ones is described in the next three equations (7)—(9):

$$\partial \, \text{(diagram)} = \left(\text{(diagram)} + \text{(diagram)} \right) + \left(\text{(diagram)} + \text{(diagram)} \right) + \left(\text{(diagram)} + \text{(diagram)} \right) = 0 \tag{7}$$

(where the sum in the first pair of brackets (respectively, in the second, respectively, in the third) arises from the contraction of the segment between the first and the second (respectively, second and third, respectively, third and fourth) points of the chord diagram;

$$\partial \, \text{(diagram)} = \left(\text{(diagram)} + \text{(diagram)} \right) + \text{(diagram)} + \left(\text{(diagram)} + \text{(diagram)} \right), \tag{8}$$

$$\partial \, \text{(diagram)} = \text{(diagram)} + \left(\text{(diagram)} + \text{(diagram)} \right) + \text{(diagram)}. \tag{9}$$

Therefore the kernel of this operator is generated by the first chord diagram in (4).

Our maximal cell has a boundary not only in the same term $F_i \setminus F_{i-1}$ of the filtration, but also in the lower term $F_{i-1} \setminus F_{i-2}$: it consists of i summands corresponding to all chords of the chord diagram indexing our cell. Namely, let us choose any such chord. Erasing it we obtain an $(i-1)$-chord diagram, i.e. the picture of a maximal cell in $F_{i-1} \setminus F_{i-2}$. The corresponding summand of the boundary of the initial cell is a singular hypersurface in the latter cell: it consists of all triples (C', f, t') in it such that the map f additionally glues together some two points of $\mathbb{R}^1_w$ placed among the $2i - 2$ points of the $(i-1)$-chord diagram C' in the same way as the endpoints of the erased chord. In the notation of §1.4 it will be depicted by a drawing obtained from the initial i-chord diagram by replacing the chosen chord by a broken line (zigzag) with the same endpoints.

In our algorithm, only the subvarieties of maximal and vice-maximal cells are considered.

Any picture in this text, describing such a variety, consists of a chord diagram (maybe with one tripod) indicating the cell in which the variety lies, and some additional furniture (zigzags and arrows with endpoints at the Wilson line of this diagram, and some subscripts) indicating further conditions distinguishing our subvariety. For examples, see Fig. 3 and the rest of the paper.

These pictures are very similar to the *arrow-segment diagrams* introduced by A. Merkov [8] (with some additional features arising from the three-dimensionality of our problem, and also from the fact that we consider subvarieties in the resolved discriminant, and not in the functional space $\mathcal{K}$ only).
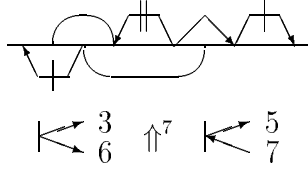
Figure 3: A subvariety in the resolved discriminant

Additional conditions describing our varieties will be described in §§1.2, 1.4, now we give some preliminary explanations.

All endpoints of chords and zigzags in the picture will be called its *active points*, cf. [7], [8]. They will be numbered from the left to the right in $\mathbb{R}^1_w$. The numbers in subscripts mean the numbers of points participating in the corresponding conditions.

Let us fix a complete flag of directions in $\mathbb{R}^3$. The first direction will be called "up". The "table" plane $\mathbb{R}^2$ in which the knot diagrams are drawn will be considered as the quotient of $\mathbb{R}^3$ by this direction. Saying that some point is *above* or *below* another one, we refer to exactly this direction. Further, we choose the direction "to the east" in this quotient space $\mathbb{R}^2$ and say that some point in $\mathbb{R}^3$ is to the east of some other if the projection to $\mathbb{R}^2$ of the vector connecting the latter point to the former one has this direction.

The presence of the single vertical bar | in a subscript or zigzag means that the corresponding condition deals with the projections of some objects (points or tangent vectors) to $\mathbb{R}^2$.

Similarly, we consider the quotient space $\mathbb{R}^1$ of $\mathbb{R}^2$ by the direction "to the east", fix the direction "to the north" in it and mark with the double bar ‖ all conditions referring to the projections to this space along the sum of two directions considered above.

The projection $\mathbb{R}^3 \to \mathbb{R}^2$ along the direction "up" is denoted by p_1, the projection $\mathbb{R}^3 \to \mathbb{R}^1$ along both directions "up" and "to the east" by p_2. For any map $f : \mathbb{R}^1_w \to \mathbb{R}^3$ we denote by f_1 (respectively, by f_2) the composition $p_1 \circ f : \mathbb{R}^1_w \to \mathbb{R}^2$ (respectively, $p_2 \circ f : \mathbb{R}^1_w \to \mathbb{R}^1$).

Any picture with k active points ($2i$ of which are endpoints of chords) defines a subvariety in the corresponding maximal cell of $F_i \setminus F_{i-1}$, in particular if $i = 0$ then in the space $\mathcal{K}$ itself. The desired chains $d^{k-i}(\gamma)$ and γ_{k-i} (see page 3) will be constructed as sums of these varieties.
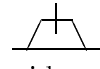
1.2   Subvarieties of full dimension

Here we describe all additional conditions of the "inequality" type, distinguishing subvarieties of full dimension in our maximal cells of $F_i \setminus F_{i-1}$. Such subvarieties are the blocks for constructing the spanning chains γ_j. (In particular, such subvarieties with no arcs are the blocks in $\mathcal{K}$ of which the desired combinatorial formulas are built.) See Fig. 3 for a picture describing such a subvariety in a maximal cell of $F_2 \setminus F_1$: it can occur as

an intermediate step in the calculation of a combinatorial formula for a knot invariant of order 9 or more.

There are five types of such conditions. They are described in paragraphs **1**, ..., **5** of this subsection. The paragraphs **1!** and **2!** describe important linear combinations of varieties distinguished by the conditions from paragraphs **1** and **2** respectively.
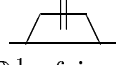
1. The oriented zigzag ⟋△⟍ or ⟋△⟍ with endpoints at the Wilson line denotes the condition "$f(a)$ is below $f(b)$" in the case of the first picture, and "$f(a)$ is above $f(b)$" in the case of the second, where f is our map $\mathbb{R}^1 \to \mathbb{R}^3$ and $a < b$ are the points of $\mathbb{R}^1$ corresponding to the endpoints of the oriented zigzag. Endpoints of different zigzags of this form cannot coincide with one another and with endpoints of chords.

1!. The non-oriented zigzag crossed by one bar, ⟋△̶⟍ , means that the projections of corresponding two points $f(a)$, $f(b)$ to $\mathbb{R}^2$ coincide. The variety defined by this condition is the union of two similar varieties with this fragment replaced by ⟋△⟍ and ⟋△⟍ .

Definition 1 The ×-*points* of the picture are its active points which are endpoints of chords, oriented (non-crossed) zigzags, and once crossed non-oriented zigzags (in contrast with φ-points which will be defined in the next paragraph). ×-points are obviously matched in pairs. Such a pair of points together with the chord or zigzag joining them is called a ×-*pair*.

The points of any ×-pair have one and the same projection to $\mathbb{R}^2$. E.g. in Fig. 3 we have three ×-pairs.

2. The oriented zigzag with one crossing bar, ⟋△̶⟍ or ⟋△̶⟍ , connecting some two points of the Wilson line, says us that the image in $\mathbb{R}^3$ of the point corresponding to its tail lies to the left of the endpoint with arrow. Exactly one endpoint of such a zigzag should be an ×-point of some ×-pair, and the other endpoint will be called a φ-point. A point of the Wilson line can be a φ-point of at most one such zigzag.

2!. The non-oriented twice crossed zigzag ⟋△̶̶⟍ expresses the condition of type $f_2(a) = f_2(b)$, i.e. that the projections to $\mathbb{R}^1$ of images of endpoints a, b of this double crossed zigzag along the plane spanned by both directions "up" and "to the east" coincide. The variety distinguished by this condition is the union of two varieties defined by conditions expressed by once crossed zigzags oriented to different sides and having the same endpoints.

This picture appears in the same circumstances as these defined in **2**, i.e. exactly one endpoint of the double crossed non-oriented zigzag should be a ×-point; the other its point also is called a φ-point and cannot be an φ-point of some other zigzag.

Definition 2 The tree formed by a ×-pair together with all single crossed oriented zigzags and double crossed non-oriented zigzags connecting its ×-points with some other

points is called a *crab*, the zigzags of types described in paragraphs **2** and **2!** its *legs*, and the φ-points of such zigzags its *feet*. Two crabs are *congruent* if they define one and the same condition on maps $\mathbb{R}^1 \to \mathbb{R}^3$, i.e. they have equal $\times$-pairs and equal sets of φ-points, and the orientations of legs with one and the same feet coincide (so that all the difference is that these legs can grow from different points of the $\times$-pair). The *normal* crab has all legs growing from the left $\times$-point of its $\times$-pair.

Obviously, there is exactly one normal crab in any congruence class. However, in §5 we shall draw also non-normal crabs for simplicity of pictures. In Fig. 3 we have two normal crabs (one of which has no legs at all) and one non-normal one.

By definition, if a map $f : \mathbb{R}^1 \to \mathbb{R}^3$ satisfies the conditions expressed by a picture with a crab, then all the images of its feet points are "equally northern" as the image of its basic $\times$-pair, i.e. they have the same projection to $\mathbb{R}^1$.
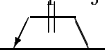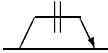
3. The subscript of type

$$ \overset{l}{\underset{j}{\mathrel{\mathop{\mathrm{K}}}}} \quad \text{or} \quad \overset{l}{\underset{j}{\mathrel{\mathop{\mathrm{K}}}}} \quad \text{or} \quad \overset{l}{\underset{j}{\mathrel{\mathop{\mathrm{K}}}}} \quad \text{or} \quad \overset{l}{\underset{j}{\mathrel{\mathop{\mathrm{K}}}}} \tag{10} $$

(where j and l are the numbers of endpoints a_j, a_l of some $\times$-pair in the list of all active points of the picture) means that the direction "to the east" in $\mathbb{R}^2$ is a linear combination of projections to $\mathbb{R}^2$ of derivatives of f at these endpoints, and both coefficients in this combination are positive (respectively, the coefficient at the projection of $f'(a_l)$ is positive and that of $f'(a_j)$ is negative, respectively, the coefficient at the projection of $f'(a_j)$ is positive and that of $f'(a_l)$ is negative, respectively, both are negative).

These pictures say nothing on the orientations of frames formed by these projections in $\mathbb{R}^2$. In particular any of pictures $\overset{l}{\underset{j}{\mathrel{\mathop{\mathrm{K}}}}}$ and $\overset{l}{\underset{j}{\mathrel{\mathop{\mathrm{K}}}}}$ denotes the same condition as the similar picture with numbers l, j permuted, while $\overset{l}{\underset{j}{\mathrel{\mathop{\mathrm{K}}}}}$ denotes the same condition as $\overset{j}{\underset{l}{\mathrel{\mathop{\mathrm{K}}}}}$.

4. The subscript of type $\Uparrow^j$ or $\Downarrow^j$ means that the projection $f'_2(a_j)$ of the derivative $f'(a_j)$ of f at the jth active point to $\mathbb{R}^1$ is directed "to the north" (respectively, "to the south").

5. For some two $\times$-pairs (and their crabs) we can fix the information that the image of one of these pairs is "more northern" than the other (i.e. its projection to the line $\mathbb{R}^1$ is more northern than the projection of the other). This condition is expressed by the double crossed arrow $\diagup\!\!\!\!\!\!\overline{}\diagdown$ or $\diagup\!\!\!\!\!\!\overline{}\diagdown$ directed from the left point of the more southern $\times$-pair to that of the more northern one.

Definition 3 Any chord diagram together with finitely many conditions of types **1–5** drawn at or under it is called a $\odot$-*picture*. The *filtration* of such a picture is the number of chords in it.

Any such picture Θ of filtration i distinguishes some semialgebraic chain $V(\Theta)$ in the cell of $F_i \setminus F_{i-1}$ corresponding to the "chord diagram" part of the picture.

Namely, suppose that we have an identification (i.e. an orientation-preserving homeomorphism) of the Wilson line $\mathbb{R}^1_w$ of the picture and the source line $\mathbb{R}^1$ of our maps $f : \mathbb{R}^1 \to \mathbb{R}^3$. Such a map $f \in \mathcal{K}$ *respects* our $\odot$-picture if it glues together the endpoints of any chord and satisfies all other conditions encoded in the picture.

Similarly to [9], a *representation* of a $\odot$-picture in the singular knot $f : \mathbb{R}^1 \to \mathbb{R}^3$ is any orientation-preserving diffeomorphism $h : \mathbb{R}^1_w \to \mathbb{R}^1$ such that the map $f \circ h$ respects this $\odot$-picture.

Now, consider any point of our maximal cell, i.e. a triple (C, f, t) as in (2). This point participates in our chain $V(\Theta)$ with multiplicity equal to the algebraic number of representations of our picture in the singular knot f. In the present paper we consider only $\mathbb{Z}_2$-homology, therefore the "algebraic number" means just the parity of the number of representations. In a future work the orientations of these varieties will be specified, and the "algebraic numbers" will take any integer values.

In interesting cases (if we have no contradictory conditions, like e.g. the pictures

$\Kappa{}^l_j$, $\Uparrow^l$ and $\Uparrow^j$ simultaneously) the chain $V(\Theta)$ has full dimension in our maximal cell.

Example 3 The arrow diagrams of [9] can be considered as $\odot$-pictures with empty chord diagrams, having only the conditions of type **1**. In [9] a natural accounting of signs of corresponding representations was specified. In our terms, it allows one to define the corresponding integral chains in the space $\mathcal{K}$; the Goussarov's theorem [4] claims that any finite-type knot invariant can be realized as a linear combination of such chains.

Definition 4 Any chain $V(\Theta)$ *of full dimension* in a maximal cell of $F_i \setminus F_{i-1}$, defined by a $\odot$-picture Θ, is called a $\odot$-*chain*. By the *sum* of several $\odot$-pictures with one and the same chord diagram we mean the homological sum of corresponding $\odot$-chains.

Remark. We could consider only the normalized pictures in which signs $\Uparrow^i$ or $\Downarrow^j$ are put at *any* active point. Indeed, any $\odot$-picture, some l points of which are free of such signs, can be decomposed into the sum of 2^l similar pictures with all possible combinations of arrows. Also, we could consider only the pictures with *linear* orderings of crabs from the south to the north (indicated by double crossed oriented zigzags as in paragraph **5**), decomposing any picture with only a partial order into the sum of pictures corresponding to all its extensions to linear orders.

However all this would increase the number of summands exponentially, which we do not want to have. Any planar picture like a knot diagram carries all this garbage information, which makes the algorithms referring to the graphical calculus quite inefficient for real computerization.
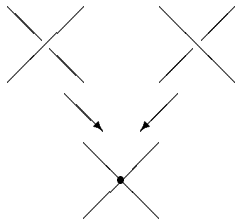
Figure 4: **X**-degenerations

Remark. The *virtual knots* of [5], [4] (and equivalence classes of virtual knots) can be also realized as domains in the space of maps $\mathbb{R}^1 \to \mathbb{R}^3$. Similarly, the singular virtual knots of [4] should be thought of as certain domains in appropriate terms of the simplicial resolution of the discriminant.

1.3 Standard degenerations

We are going to study some subvarieties of codimension one in our maximal cells, namely, the subvarieties forming boundaries of all ⊙-chains considered in the previous subsection. They can be achieved by the following standard degenerations of singular knots from these full-dimension varieties.

X. One additional self-intersection can occur at some pair of points distant from the endpoints of the chords and also from the φ-points, see Fig. 4.

R1, R2, R3: Degenerations occurring during the standard Reidemeister moves, see Fig. 5. In the left bottom picture it is assumed that the derivative of f at the "cusp" point is not equal to zero and is directed "up" (respectively, "down") if the parameter in $\mathbb{R}^1_w$ grows "from the right to the left" (respectively, "from the left to the right") in this picture.

R̃2, R̃3: Reidemeister moves of singular knots, cf. [1]. In the left bottom picture of Fig. 6 it is assumed that the projections of two branches to $\mathbb{R}^2$ are tangent, but these branches form a nonzero angle in the vertical plane at their intersection point.

Since we have fixed the flag of directions in $\mathbb{R}^2$, there are many non-isotopic (via isotopies preserving the foliations into fibers of the projections p_1 and p_2 along these directions, but not the particular fibers) degenerations of these types: in total 4 of type **R1**, 2 of type **R2**, 16 of type **R3**, 4 of type **R̃2** and 12 of type **R̃3**. If we distinguish the orientations of branches of our curve participating in the degeneration, then the number of possibilities will be even more: it should be multiplied additionally by the corresponding power of 2.

M1, M2, M3 (cf. [8]): essential changes of the Morse structure of the function $f_2 \equiv p_2 \circ f : \mathbb{R}^1_w \to \mathbb{R}^1$, see Fig. 7. The crossing points of solid lines in these pictures denote any possible ×-pairs, i.e. over/under crossings corresponding to oriented zigzags or intersection points corresponding to chords.
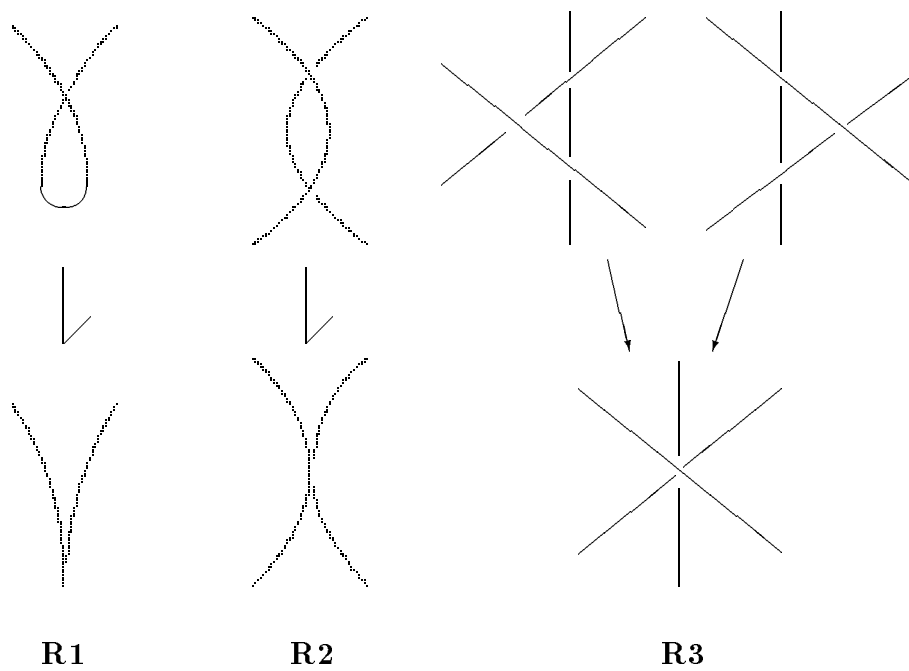
Figure 5: Standard Reidemeister degenerations
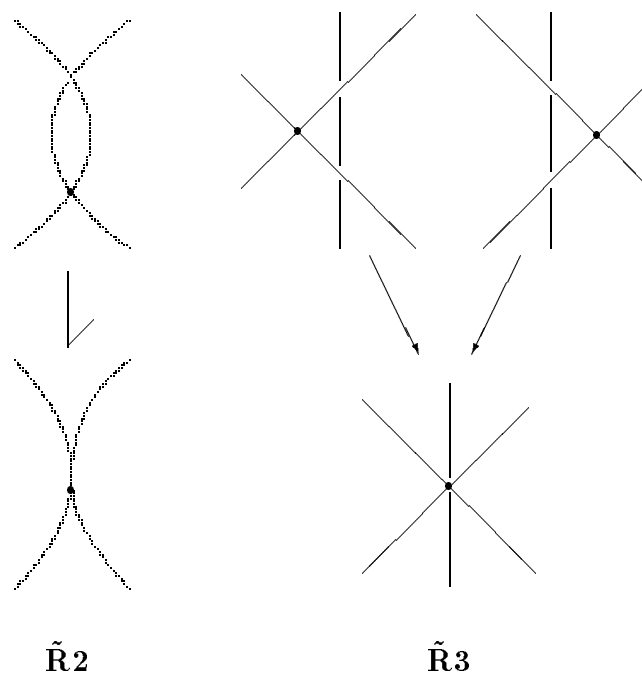
Figure 6: Reidemeister degenerations of singular knots
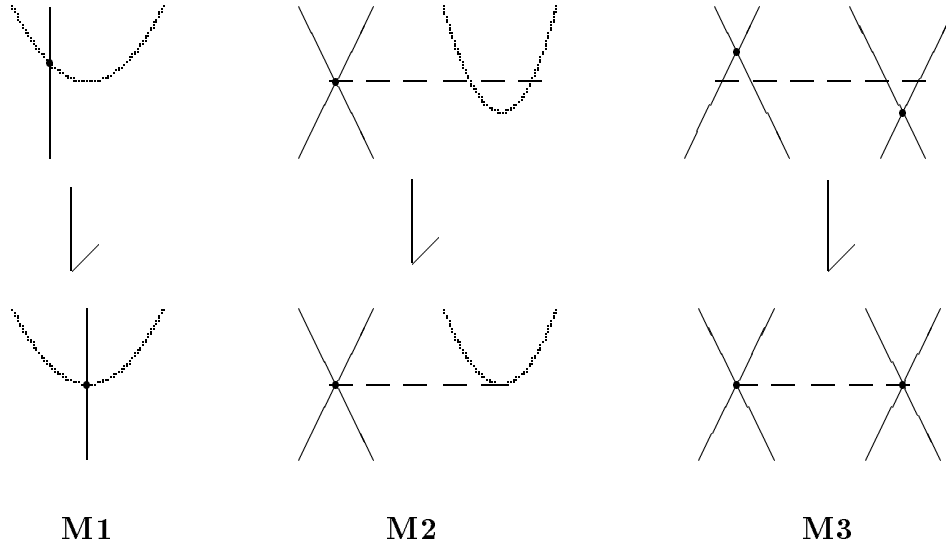
Figure 7: Essential Morse moves

The explicit formula for the homological boundary of any ⊙-chain is a formalization of these degenerations applied to points (2) of these chains, see §2 below. From this point of view, each of degenerations **R3** and **R̃3** should be separated into two essentially different kinds. Namely, in any of these degenerations three double points of the knot diagram meet. It can happen that all these three points correspond to ×-pairs of the initial ⊙-picture, or to only two of them. (If to one or zero then the corresponding degeneration causes no contribution to the boundary.)

1.4 Chains of codimension one
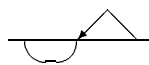
In this subsection we specify a class of semialgebraic chains of codimension one in the maximal cells of $F_i \setminus F_{i-1}$. The boundaries of all ⊙-chains described in §1.2 can be represented as sums of chains of this class (plus something in other cells). Any of the latter chains can be distinguished by all the same conditions as in §1.2 plus exactly one condition of "equality" type or a nongeneric coincidence of some active points participating in the description of ⊙-chains. These standard chains of codimension one will be called ⊖-chains, and the pictures distinguishing them the ⊖-pictures.

These pictures and corresponding conditions are described in the following paragraphs **1̄**–**6̄**. In square brackets we indicate the type(s) of degenerations at which a singular knot satisfying the corresponding condition can occur.

1̄. [**X**]. One non-oriented zigzag ⟋△⟍ or maybe ⟋▱⟍ denotes the condition $f(a) = f(b)$ for its endpoints a, b. These points cannot coincide with endpoints of chords of our picture. A subvariety with such a condition appears as a piece of the

14

boundary of the $\odot$-chain $V(\Theta)$ of full dimension, described by the same picture, but with the non-oriented zigzag replaced by oriented (to either side) one, see Fig. 4 and formula (17) below. Moreover, such non-oriented zigzags appear from the chords when we take boundary operators $\bar{H}_*(F_{i+1} \setminus F_i) \to \bar{H}_*(F_i \setminus F_{i-1})$ of our spectral sequence, see the paragraph after Proposition 14 and formulas (27), (33), (39), and (80)—(88) below.
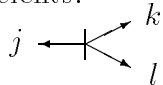
The pair of points connected by such a zigzag should be also considered as a $\times$-pair. It carries all the possibilities specified in the previous §1.2 for such pairs. In particular, it can be the body of a crab, and the conditions (10) can be imposed on the endpoints of such a non-oriented zigzag as well. Also, the conditions of type $\Uparrow^j$ or $\Downarrow^j$ can be attached at them.

$\bar{\mathbf{2}}$. [$\tilde{\mathbf{R}}\mathbf{3}$]. An endpoint of (exactly one) non-crossed oriented zigzag can coincide with an endpoint of a chord. The arising picture like is *congruent* to (i.e. defines the same condition as) another one having the same chord and the same "free" endpoint of the zigzag, but with their common endpoint coinciding with the other endpoint of the chord:

$$\text{} \cong \text{} \quad , \qquad \text{} \cong \text{} \quad . \tag{11}$$
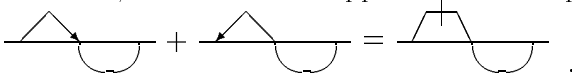
Again, we choose one of these two possibilities as the *normal* one: the corresponding endpoint of our zigzag should coincide with the left endpoint of the chord. In particular, in both equalities (11) the right-hand pictures are normal and the left-hand ones are not. Any condition of type (10) can be posed for any two of three active points participating in the figure formed by our zigzag and the chord.

$\bar{\mathbf{2}}\mathbf{a}$. The additional condition of type (where j, k, l are the numbers of active points a_j, a_k, a_l participating in a configuration of type $\bar{\mathbf{2}}$) means that the vector $f_1'(a_l)$ (i.e. the projection of the derivative $f'(a_l)$ to $\mathbb{R}^2$) is a linear combination of vectors $f_1'(a_j)$ and $f_1'(a_k)$ with positive coefficients.
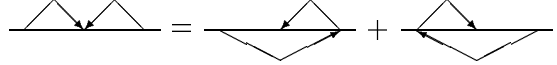
Conversely, the condition says us that no one of these three vectors in $\mathbb{R}^2$ lies in the angle between the other two.
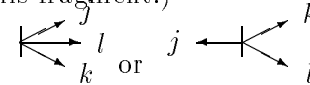
The last two conditions can arise in degenerations of type $\tilde{\mathbf{R}}\mathbf{3}$ if all three double points of the singular knot diagram meeting at this degeneration correspond to $\times$-pairs of our $\odot$-picture. These conditions can be expressed also as Boolean functions of more standard conditions like the ones described in paragraphs 3 and 4 of §1.2.

$\bar{\mathbf{2}}!$. The sum of two -chains whose sets of conditions coincide up to the orientation of non-crossed arrows participating in the degeneration $\bar{\mathbf{2}}$ will be expressed by a single picture with all the same furniture, but with these opposite arrows replaced by the once crossed non-oriented

zigzag: $\quad \text{} + \text{} = \text{} \quad .$

3̄. [R3]. Exactly one endpoint of some non-crossed oriented zigzag can coincide with that of exactly one other. The corresponding varieties satisfy obvious relations like

$$\text{(zigzag picture)} = \text{(zigzag picture)} + \text{(zigzag picture)}$$

. (As usual, this expression assumes that these three pictures coincide outside this fragment.)

Again, the conditions of type (picture with j, l, k) or (picture with j, k, l) can be imposed on the derivatives of maps f_1 at these three points. It happens if all three double points of the planar knot diagram meeting at this degeneration define ×-pairs of our -picture.

3̄!. The sum of all 4 varieties coinciding with our one up to the reverse of one or both zigzags participating in this degeneration will be shown by a similar picture with these oriented zigzags replaced by non-oriented crossed zigzags:

$$\text{(pic)} + \text{(pic)} + \text{(pic)} + \text{(pic)} = \text{(pic)} . \qquad (12)$$

This picture expresses the condition that the projections to $\mathbb{R}^2$ of all three involved points coincide. It is congruent to two other pictures obtained from it by cyclic permutations of vertices:

$$\text{(pic)} \equiv \text{(pic)} \equiv \text{(pic)} . \qquad (13)$$

One of these three pictures is *normal*, namely the one indicated by the very right-hand picture in (13).

4̄. [R̃2]. The conditions of type

$$j \searrow l \quad\text{or}\quad j \searrow l \quad\text{or}\quad l \searrow j \qquad (14)$$

(where j and l are numbers of endpoints a_j, a_l of one and the same chord) means that the direction "up" in $\mathbb{R}^3$ is a linear combination of derivatives $f'(a_l)$ and $f'(a_j)$, both with positive coefficients (respectively, $f'(a_l)$ with a positive coefficient and $f'(a_j)$ with a negative one, respectively, both with negative coefficients).

4̄!. [R2]. a) The condition (picture) $\begin{smallmatrix}l\\j\end{smallmatrix}$ (where l and j are numbers of points a_l, a_j of one and the same ×-pair) means that the projections of $f'(a_l)$ and $f'(a_j)$ to $\mathbb{R}^2$ are co-directed. If our ×-pair is a chord, then the corresponding variety can occur also as the union of two varieties defined in the previous paragraph:

$$\text{(pic)} \begin{smallmatrix}l\\j\end{smallmatrix} = \; l\searrow j \; + \; j\searrow l \; . \qquad (15)$$

b) The condition (picture) $\begin{smallmatrix}l\\j\end{smallmatrix}$ means that the projections of $f'(a_l)$ and $f'(a_j)$ to $\mathbb{R}^2$ are of opposite directions. If a_l, a_j are endpoints of a chord, then the corresponding variety can occur also as the union

$$\text{(pic)} \begin{smallmatrix}l\\j\end{smallmatrix} = \; l\searrow j \; + \; l\searrow j \; . \qquad (16)$$

Any of these two conditions is symmetric over the letters l and j.
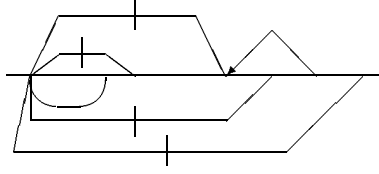
Figure 8: Normal couple of crabs of type $\bar{\mathbf{6}}$!

$\bar{\mathbf{5}}$. [M1, M2]. The condition $j \mapsto$ or $j \leftmapsfrom$, where j is the number (among all active points of the picture) of an $\times$- or φ-point a_j, means that the projection to $\mathbb{R}^2$ of $f'(a_j)$ is directed to the east (respectively, to the west). This $\times$-point cannot be an endpoint of a non-oriented zigzag, otherwise we have two conditions of equality type.

$\bar{\mathbf{5}}$!. The sum of these two conditions is denoted by $\overset{j}{\leftrightarrow}$.

In combination with conditions of type $\bar{\mathbf{5}}$ or $\bar{\mathbf{5}}$! the condition $\Uparrow^j$ or $\Downarrow^j$ can occur. (If the projection of the first derivative to the meridian $\mathbb{R}^1$ is equal to zero, as it follows from the conditions of type $\bar{\mathbf{5}}$ or $\bar{\mathbf{5}}$!, then the projection of the second derivative can be positive or negative.)

An important linear combination of the above-described conditions is expressed by the subscript of type $\overset{j}{\underset{l}{\Rrightarrow}}$. It says us that the vectors $f_2''(a_j), f_2'(a_l)$, i.e. projections to $\mathbb{R}^1$ of the second derivative $f''(a_j)$ and the first derivative $f'(a_l)$, are directed into one and the same side. The similar picture with opposite directions $\overset{j}{\underset{l}{\Lleftarrow}}$ can also appear.

$\bar{\mathbf{6}}$. [M3]. (Clinching crabs). For some crossed zigzag $\triangle$ or $\triangle$ both its endpoints can be endpoints of (different) chords or oriented zigzags. Alternatively, some feet of exactly two different crabs can coincide, or both these conditions can be satisfied simultaneously.

$\bar{\mathbf{6}}$!. The pictures like in Fig. 8, i.e. *trees* consisting of exactly two $\times$-pairs (none of which is a non-oriented non-crossed zigzag) and some more points joined with them by non-oriented single crossed zigzags, indicate that the "longitudes" of all these points (i.e. projections $f_2 \equiv p_2 \circ f(\cdot)$ of their images to the "meridian" line $\mathbb{R}^1$) coincide, and there are no restrictions on the "latitudes" of all these images $f_1(\cdot)$ (i.e. on their orders from the west to the east in $\mathbb{R}^2$). There can be many pictures describing one and the same condition of this type. Only one of them is *normal* (see Fig. 8): all its non-oriented crossed zigzags should grow from the left point of the lexicographically more left $\times$-pair, and the only one of these zigzags that joins it with the other $\times$-pair ends at the left point of the latter pair.

Important remark. Nowhere in our conditions $\mathbf{1}$–$\mathbf{5}$, $\bar{\mathbf{1}}$–$\bar{\mathbf{6}}$ the orientation of $\mathbb{R}^2$ or $\mathbb{R}^3$ can be referred to (say, in the form that the projections to $\mathbb{R}^2$ of derivatives of f at the points of a $\times$-pair form a positive frame). This is related to the fact that the entire theory (if it is good) should be transferable without problems to the theory of knots in $\mathbb{R}^n$ with any $n \geq 3$, where such references will fail.

Definition 5 Any chord diagram together with finitely many requirements of types $\mathbf{1}$–$\mathbf{5}$ of §1.2 and exactly one condition of types $\bar{\mathbf{1}}$–$\bar{\mathbf{6}}$ of §1.4 drawn at or under it is called a *-picture*.

Any such picture defines a 'semialgebraic chain' in this cell: it consists of all points (C, f, t) as in (2) such that the picture has representations in f: these points should be taken with multiplicities equal to the parities of numbers of these representations.

In all interesting cases such a subvariety has codimension one in the cell of $F_i \setminus F_{i-1}$ corresponding to its chord diagram.

Definition 6 Any semialgebraic chain of codimension one defined by a -picture in the cell corresponding to its chord diagram is called a -*chain.*

2   On differentials

The boundary of any -variety consists of three essential parts that lie respectively a) in the same maximal cell of $F_i \setminus F_{i-1}$, b) in vice-maximal cells of $F_i \setminus F_{i-1}$, and c) in the lower term $F_{i-1} \setminus F_{i-2}$ of the main filtration of σ. In §§2.1–2.7 of this section we describe the part a), in §2.8 the part b), and in §2.9 the part c).

Let us recall that for any $i \geq 1$ the maximal cells of the canonical cell decomposition of the term $F_i \setminus F_{i-1}$ of the resolved discriminant are in one-to-one correspondence with i-chord diagrams.

Proposition 1 *For any maximal cell of the canonical cell decomposition of $F_i \setminus F_{i-1}$, $i \geq 0$, and any -chain $V(\Theta)$ in it (see Definition 4), the boundary of this chain in the cell is equal to a finite sum of -chains, defined by -pictures, any of which has no more active points than the initial -chain $V(\Theta)$.*

The proof follows from the list of degenerations given in §1.3. Below we outline these differentials. Although we deal here with $\mathbb{Z}_2$-chains only, the similar statement is true in the case of any coefficient group and follows from the same considerations.

2.1   Degenerations preserving the number of active points

The most important for us are the degenerations that do not decrease the number of active points. They can be formulated as degenerations of the conditions themselves, and not of the configurations of active points. They are listed in the following eight equations:

$$\partial \ \text{⟋⟍} \ = \partial \ \text{⟋⟍} \ = \ \text{⟋⟍} \tag{17}$$

$$\partial \ \text{⟋⊤⟍} \ = \partial \ \text{⟋⊤⟍} \ = \ \text{⟋⊤⟍} \tag{18}$$

$$\partial \ \text{⟋⊤⊤⟍} \ = \partial \ \text{⟋⊤⊤⟍} \ = \ \text{⟋⊤⊤⟍} \tag{19}$$

$$\partial \ \Uparrow^j \quad = \quad \partial \ \Downarrow^j \quad = \quad \overset{j}{\longleftrightarrow\!\!\!+\!\!\!\longleftrightarrow} \tag{20}$$

$$\partial \ \text{⟨}\!\overset{l}{\underset{j}{\cdot}} \ = \ \overset{\longrightarrow}{\longleftarrow}\!\!\!+\!\!\overset{l}{\underset{j}{\cdot}} \ + \ l \mapsto \ + \ j \mapsto \tag{21}$$

$$\partial \ \text{⟨}\!\overset{l}{\underset{j}{\cdot}} \ = \ \overset{\longrightarrow}{\longleftarrow}\!\!\!+\!\!\overset{l}{\underset{j}{\cdot}} \ + \ l \leftarrow \ + \ j \leftarrow \tag{22}$$

$$\partial \; \overset{l}{\underset{j}{\longleftarrow}} \;\; = \;\; \overset{l}{\underset{j}{\Longrightarrow}} \;\; + \;\; l \longmapsto \;\; + \;\; j \longleftarrow \tag{23}$$

$$\partial \; \overset{l}{\underset{j}{\longleftarrow}} \;\; = \;\; \overset{l}{\underset{j}{\Longrightarrow}} \;\; + \;\; l \longleftarrow \;\; + \;\; j \longmapsto \tag{24}$$

Namely, if the picture of our -chain contains some fragment indicated in the left part of some of these equations under the ∂ sign, then its boundary contains the -chain in whose picture this fragment is replaced by any summand of the right-hand part of the same equation (i.e. in the part not containing the sign ∂).

2.2 Other part of the boundary

The other part of the boundary operator, formulated in terms of degenerations of active point configurations, is more difficult to describe because of the plenty of possibilities. In the rest of this section we give a conceptual description of this part. Roughly speaking, we have to contract into points an arbitrary set of intervals separating active points of the -picture, consider the limit set to which our -chain tends when the active points of its elements undergo this collision, check whether the codimension of this limit set is equal to 1, and describe it in terms of our standard conditions **1–5** and **$\bar{1}$–$\bar{6}$**.

In §2.3 we describe the way in which different ×-pairs can coalesce (this can happen in degenerations of types **R2, R3, $\tilde{\text{R}}$2, $\tilde{\text{R}}$3**). In §2.4 we study what can then happen with their legs. In §2.5 we shall see how the segments connecting active points of one and the same crab can be contracted (it happens in degenerations of types **M1, M2**). In §2.6 we study collisions of points of different crabs not reflecting the collisions of their ×-pairs: this happens in degenerations of type **M3**.

Before starting, we introduce several general notions.

Definition 7 Given a -picture Θ of filtration i and a collection of segments in its Wilson line bounded by some its neighboring active points, the *easy contraction* of this collection is the new picture, in which all these segments are replaced by points, all chords and zigzags of Θ joining endpoints of these segments with some other points are replaced by analogous chords and zigzags joining the resulting points, all chord and zigzags joining points which are contracted to one point are erased, and any point obtained as a contraction of segment(s) inherits all conditions of types **3–5** imposed previously on all endpoints of this segment (or these segments).

The easy contraction is *contradictory* if the set of points (2) such that the resulting picture has representations in f is of codimension greater than one in the corresponding maximal cell of $F_i \setminus F_{i-1}$.

The simplest example of a contradictory contraction is a single segment with conditions $\Uparrow^j$ and $\Downarrow^{j+1}$ imposed at its two endpoints. Another one appears if both endpoints of our segment are endpoints of chords: in this case the resulting limit variety lies in another cell. The third important example is as follows. Consider the graph whose vertices are the active points of the initial -picture Θ, and the edges are its chords, oriented non-crossed zigzags, and also our chosen segments of the Wilson line. An *admissible path* along this graph can go along the chords
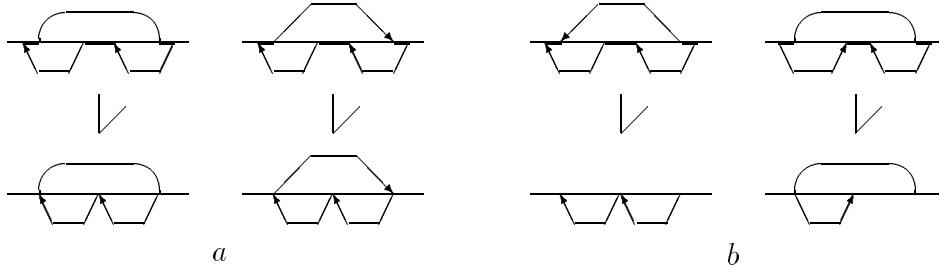
Figure 9: Contradictory and reduced contractions of pictures

and segments in any direction, but going along the zigzags it should follow their orientation. If this graph has a nontrivial admissible *cycle*, then our contraction surely is contradictory.

For instance, the contraction of three thickened segments in any of two -pictures in the upper row of Fig. 9a) is contradictory.

On the other hand, some of conditions expressed by the zigzags of a contracted picture can be corollaries of some other zigzags and chords. Say, if we contract the thickened segments in two pictures of the upper row in Fig. 9b) then we obtain three zigzags (respectively, two zigzags and one chord) where one of zigzags is a corollary of two other (respectively, of the other zigzag and the chord) and can be removed. The *reduced contraction* of a -picture is the composition of its easy contraction and the subsequent elimination of such superfluous zigzags, see the bottom row of Fig. 9*b*).

2.3   Collision of different ×-pairs

In this subsection we consider the boundary components of -chains, occurring in contractions of segments, at least one of which is bounded by ×-points.

We describe here only a very restricted part of such possible collisions. Namely, we suppose that no conditions of type **4** are imposed at the points of the ×-pairs participating in our collisions, and these pairs are not related by conditions of type **5**.

There are the following principal cases.

2.3.1   Collision of three ×-pairs

This collision is a formalization (in terms of - and -pictures) of degenerations **R3** and **R̃3** in the case when all three double points of the knot diagram participating in this degeneration (see upper right pictures of Figs. 5 and 6) are ×-pairs of the initial -picture.

It is illustrated by last summands of formulas (90)—(93), (96), (97), (127)—(129).

Suppose that our -picture contains three ×-pairs (a_{i_1}, a_{i_2}), (a_{i_3}, a_{i_4}), and (a_{i_5}, a_{i_6}), at most one of which is a chord and at least two are non-crossed oriented zigzags, such that $|i_1 - i_3| = 1$, $|i_2 - i_5| = 1$, $|i_4 - i_6| = 1$. (We do not assume that $i_1 < i_2$ or $i_3 < i_4$ or $i_5 < i_6$.)

The easy contraction of all three segments $[a_{i_1}, a_{i_3}]$, $[a_{i_2}, a_{i_5}]$, $[a_{i_4}, a_{i_6}]$ turns these three ×-pairs to a triangle whose vertices correspond to these segments, and the sides are formed by three zigzags (or two zigzags and one chord) inherited from these ×-pairs. Suppose that this

triangle is not contradictory. Then the boundary of our -chain contains a -chain, whose picture is the *reduced contraction* of our three segments, with additional condition of type $\bar{\mathbf{2}}\mathbf{a}$ imposed at the newborn three points, see e.g. the last summands in formulas (90) and (127). Namely, if all three vectors $[a_{i_1}, a_{i_3}]$, $[a_{i_4}, a_{i_6}]$ and $[a_{i_5}, a_{i_2}]$ are equally oriented in $\mathbb{R}^1$, then we have the condition $j \longleftarrow\!\!\!\!\!\mid\!\!\!\!<^{\,k}_{\,l}$, otherwise we have the condition $\mid\!\!\!\!<^{\,j}_{\,l}{}_{\,k}$ where the middle arrow corresponds to the point obtained by the contraction of the segment for which this sign is different from these for other two segments.

2.3.2 Complete collision of two $\times$-pairs

This is a formalization of degenerations **R2** and $\tilde{\mathbf{R}}\mathbf{2}$. It is illustrated by the last summands in (28) and (29).

Suppose that our -picture contains two $\times$-pairs $(a_{i_1} < a_{i_2})$ and $(a_{i_3} < a_{i_4})$, at most one of which is a chord and at least one is a non-crossed oriented zigzag, and $|i_1 - i_3| = 1$, $|i_2 - i_4| = 1$.
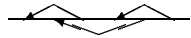
Again, we contract the segments $[a_{i_1}, a_{i_3}]$ and $[a_{i_2}, a_{i_4}]$ into points. This contraction can be contradictory only if both $\times$-pairs are zigzags, and the resulting two zigzags have opposite orientations. If it is not contradictory then the corresponding reduced contraction consists of one chord or one zigzag. In this case, the boundary of our -chain contains the -chain, whose picture is this reduced contraction with one additional condition of type (14) or (15) or (16) (where l and j are the numbers of points of the newborn $\times$-pair in the complete list of active points of the new -picture). Namely, if we had two (non-contradictory) zigzags, so that the reduced contraction is one zigzag, then we have additional condition (15) or (16) depending on the sign of the product $(a_{i_4} - a_{i_2})(a_{i_3} - a_{i_1})$. If we had a chord and zigzag, and the reduced contraction is one chord, then we get an additional condition of type (14).

2.3.3 Partial collision of two $\times$-pairs

This is a formalization of degenerations **R3** and $\tilde{\mathbf{R}}\mathbf{3}$ in the case when only two of three double points of the (singular) knot diagram correspond to $\times$-pairs of the initial -picture.

Suppose that our -picture contains two $\times$-pairs (a_{i_1}, a_{i_2}), (a_{i_3}, a_{i_4}), and $|a_{i_1} - a_{i_3}| = 1$ (again, we do not assume that $a_{i_1} < a_{i_2}$ or $a_{i_3} < a_{i_4}$).

Then the boundary of the initial -chain contains the -chain obtained from it by the easy contraction of the segment $[a_{i_1}, a_{i_3}]$.

Remark. This is true even if the more strong condition considered in §2.3.1 or §2.3.2 is satisfied. For instance, the formula (127) on page 54 shows that the boundary of the -chain contains not only the variety shown in the end of (127) (described in §2.3.1) but also some five varieties obtained by partial collisions of different $\times$-pairs of our picture (see summands 4 through 8 in the right-hand part of (127)). Similarly, the formula (34) on page 34 contains not only its last term (predicted in §2.3.2) but also three terms obtained by partial contractions.

2.4   What can happen with their legs

If points of several $\times$-pairs of a -picture tend to one another, then the feet of their crabs also can coalesce.

The components of the boundary indicated in the subsection 2.3 correspond to the case when it is not so, i.e. all the feet of different crabs tend to different points of the source line $\mathbb{R}^1$ and, moreover, they do not tend to the $\times$-points of these crabs. In the present subsection we count all the other possibilities corresponding to the same collisions of $\times$-pairs.

First we consider a complete collision of two $\times$-pairs as in §2.3.2. Suppose that we can distinguish several feet $a_{j_1}, \ldots, a_{j_r}$ of one of these pairs, and equally many feet $a_{l_1}, \ldots, a_{l_r}$ of the other in such a way that for any $\alpha = 1, \ldots, r$

a) $|j_\alpha - l_\alpha| = 1$,

b) either both corresponding legs are oriented towards these feet or both are oriented from them.

(Again, for simplicity we assume that conditions of type 4 are not imposed at all these points.)

Then the boundary of our -chain contains a -chain in whose picture our two $\times$-pairs are changed in correspondence with §2.3.2, and additionally any couple of legs with endpoints at a_{j_α} and a_{l_α} is replaced by only one such leg.

In the case of the partial collision as in §2.3.3, the situation is almost the same as in the case of a complete collision, with only one possibility more (see e.g. the last summands in formulas (96)–(105), (130)–(134)).

Namely, suppose that the $\times$-pair (a_{i_1}, a_{i_3}) has a foot a_{i_5} such that $|i_2 - i_5| = 1$, $i_5 \notin \{j_1, \ldots, j_r, l_1, \ldots, l_r\}$. Then the boundary of $V(\Theta)$ contains also the -chain in which the $\times$-pairs are changed in correspondence with §2.3.3 (i.e. by the easy contraction of the segment $[a_{i_1}, a_{i_3}]$), any pair of legs with feet $a_{j_\alpha}, a_{l_\alpha}, \alpha = 1, \ldots, r$, is replaced by one leg, and the leg with the foot a_{i_5} disappears, but additionally a condition of type (10) on the tangent vectors of f at the resulting degenerate curves appears. Moreover, in the same way one more leg of the $\times$-pair (a_{i_3}, a_{i_4}) with a foot a_{i_6}, $|i_6 - i_4| = 1$, can perish.

Finally, suppose that we have the collision of type §2.3.1 of three $\times$-pairs, and several disjoint subsets of the union of their feet are distinguished, such that

a) any subset contains two or three feet all whose crabs are different;

b) these subsets consist of neighboring active points of the picture, i.e. they are not separated by other such points;

c) either all feet in any such set are the arrows of corresponding legs, or all these feet are their tails;

d) for all such sets of cardinality three, their middle points belong to one and the same crab.

Then the boundary of our -chain contains a -chain in whose picture our three $\times$-pairs are changed in correspondence with §2.3.1, and additionally any couple or triple of legs with feet at the points of one subset fixed above is replaced by only one leg.

Additionally, some one, two or three legs (no more than one for each of these $\times$-pairs) can disappear imposing instead some conditions on the derivatives of f_1 at the three $\times$-points of the arising configuration. If there are three such perishing legs, then all their feet cannot be

tails or arrows simultaneously; one of them whose type (in this sense) is different from that of other two should belong to the crab mentioned in condition d) above. This leg (if it exists) has one characterization more: its foot lies in one of three contracting segments. There can be only one leg with this property; if it exists then the segment containing this foot should be bounded by the points of ×-pairs different from the ×-pair from which this leg grows.

If there are only two such perishing legs and they have one and the same type (i.e. are oriented to or from their feet simultaneously), then the corresponding two crabs have only marginal feet in any above-considered set of cardinality three, see especially item d) above.

2.5 Collisions inside crabs

If two feet of one and the same crab of our -picture are neighbors (i.e. are not separated by other active points), and both corresponding legs are either directed to them or from them, then the boundary of the corresponding -chain contains a similar picture with these two legs replaced by one leg and the condition of type $\bar{5}!$ imposed at its foot. This is a formalization of the degeneration **M2**.

If a foot of some crab is a neighbor of some ×-point of the same crab, then we can eliminate this foot (and the corresponding leg) imposing instead one of two conditions $\bar{5}$ at this ×-point: the choice of this condition depends on the orientation of this leg. This is a formalization of the degeneration **M1**. See e.g. two last summands in (31)

Remark. *A priori* there is one possibility more, when both these points belong to the ×-pair of the crab, i.e. are connected by an oriented non-crossed zigzag, and the corresponding degeneration is of type **R1**. However, this situation cannot occur in the real calculations related with the knot invariants, because the initial data of our algorithm should satisfy the 1-term relation. On the other hand, this situation can occur in an essential way in the calculation of higher-dimensional cohomology classes, see e.g. [16].

2.6 Clinching crabs

Consider all active points of some two crabs and suppose that there is a non-empty set of non-intersecting pairs $(a_{j_\alpha}, a_{l_\alpha}) \subset \mathbb{R}^1_w$, $\alpha = 1, \ldots, r$, such that for any α

a) all points a_{j_α} are active points of the first crab, and all points a_{l_α} are active points of the second;

b) $|j_\alpha - l_\alpha| = 1$;

c) a_{j_α} and a_{l_α} cannot be ×-points simultaneously;

d) the simultaneous contraction of all segments $[a_{j_\alpha}, a_{l_\alpha}]$ is not contradictory in the sense of §2.2.

Again, we assume for simplicity that the conditions of type **4** are not imposed at all points of these crabs. Then the boundary of the initial -chain contains the variety whose picture is the reduced contraction of all segments $[a_{j_\alpha}, a_{l_\alpha}]$.

$\check{R}1$ $\check{R}3$
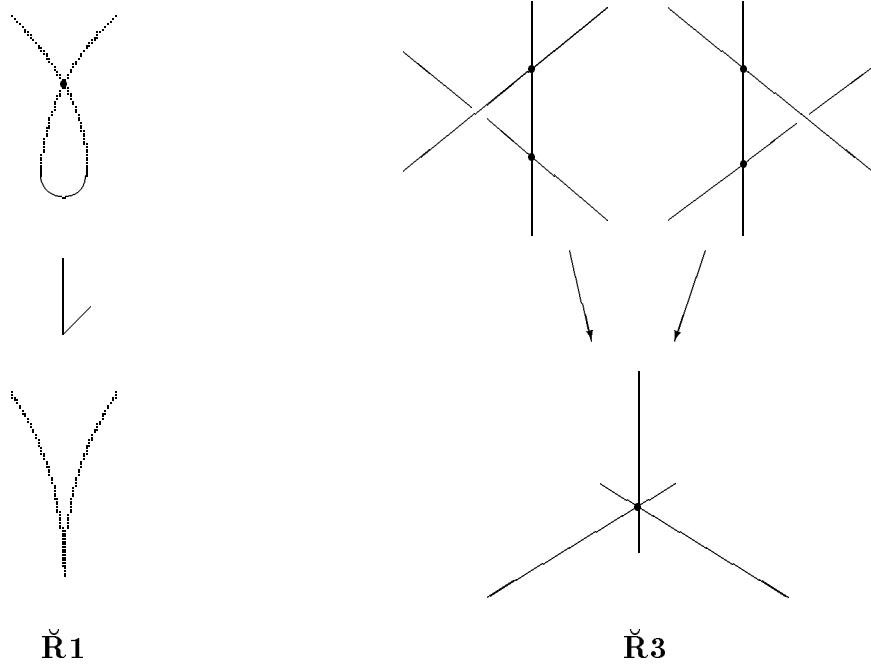
Figure 10: Degenerations into smaller cells

2.7 Impact of other conditions

Finally, suppose that some conditions of type **4** or **5** were imposed at the active points of the initial -chain $V(\Theta)$ participating in the degeneration. Then, generally speaking, the part of its boundary $\partial V(\Theta)$ defined by all possible degenerations of configurations of active points of Θ will be only a part of the -chain described in the previous subsections $2.3 - 2.6$. It is easy to check that anyway the homological boundary of our -chain will be a linear combination of -chains, in correspondence with Proposition 1.

2.8 The boundary in vice-maximal cells

In the previous subsections of §2 we have considered the pieces of the boundary of a -chain that lie in the same maximal cell of $F_i \setminus F_{i-1}$ as the initial chain. Now we consider some other part of this boundary: the one inside the vice-maximal cells of the same term $F_i \setminus F_{i-1}$ of the filtration of the resolved discriminant. It is related with the degenerations of chords of our -picture, see Fig. 10.

In the left bottom picture of Fig. 10 it is assumed (unlike the similar picture of Fig. 5) that the derivative of the parametrizing map $f : \mathbb{R}^1_w \to \mathbb{R}^3$ itself is equal to zero at the "cusp" point, and not of its projection f_1 only. In the right-hand bottom picture it is assumed that the vector "up" in $\mathbb{R}^3$ belongs to the octant formed by three segments oriented from the intersection point towards their "longer" parts.

Proposition 2 *The degeneration* $\check{\mathbf{R}}\mathbf{1}$ *actually cannot occur in our calculation of knot invariants.*

Proof is by induction over our algorithm; here is its idea. The initial data γ of our algorithm (i.e. the "weight system") satisfies the 1T-relation, i.e. the endpoints of any chord of any chord diagram constituting γ are separated in $\mathbb{R}^1_w$ by endpoints of some other chords. On the next steps of the algorithm some of these separating chords can be destroyed, but not traceless. It follows from the construction of the algorithm, that for any chord of any -picture actually occurring in its execution one of two holds:

a) there is an active point of this picture between the endpoints a, b of this chord,

b) the vectors $f'_1(a)$ and $f'_1(b)$, i.e. the projections to $\mathbb{R}^2$ of tangent vectors at these endpoints, are co-directed.

Any of these two conditions *prevents* the degeneration of type $\check{\mathbf{R}}\mathbf{1}$, i.e. the corresponding piece of the boundary will have too large codimension to participate in homological calculations.

A similar consideration allows us not to consider the degeneration $\mathbf{R}\mathbf{1}$, see Remark in §2.5.

The study of the degeneration $\check{\mathbf{R}}\mathbf{3}$ is very similar to that of degenerations $\mathbf{R}\mathbf{3}$ and $\tilde{\mathbf{R}}\mathbf{3}$ above. Again, from the point of view of formal calculations it should be subdivided into two subcases depending on whether the unique overcrossing point of the singular diagram in its upper picture is counted in the -picture by an oriented zigzag or not. These two possibilities are illustrated by two last (respectively, all but two last) summands in the right-hand part of any of equations (59)—(67). As in §2.4, some legs can vanish in this collision, see e.g. two last summands in any of equations (68)—(71) or four last summands in (72).

The notation of pieces of the boundary of a -chain in the cell obtained by this degeneration is very similar to what we had previously: it consists of the notation of the cell with conditions of type $\mathbf{1}, \ldots, \mathbf{5}$ drawn at or under it.

Additionally, we should consider one geometrical condition more. Namely, suppose that we have a vice-maximal cell of 4T type, i.e. its notation consists of $i-2$ chords and one tripod with a marked foot, see (5). Suppose also that the condition $j \longleftarrow\!\!\!\overset{k}{\underset{l}{\diagdown}}$ described in paragraph $\overline{\mathbf{2}}\mathbf{b}$ of section 1.4 is imposed on the three endpoints of this tripod. Then the additional subscript (respectively,) means that the direction "up" (respectively, "down") in $\mathbb{R}^3$ lies in the octant formed by three tangent vectors of f at the endpoints of this triple.

Similarly, if we have the condition $\overset{j}{\underset{k}{\Longleftarrow}}{}^{l}$ on the projections of tangent vectors at these three points, then the condition (respectively,) means that the direction "up" and the vector $f'(a_l)$ penetrate the plane in $\mathbb{R}^3$ spanned by $f'(a_k)$ and $f'(a_j)$ in one and the same direction (respectively, in different directions). For illustrations, see last two terms in any of equations (59)—(67).

2.9   The boundary in the lower term of the filtration

Given a -chain in some maximal cell of $F_i \setminus F_{i-1}$, its boundary in $F_{i-1} \setminus F_{i-2}$ consists of i -chains of filtration $i-1$, corresponding to all chords of the initial -picture and obtained

from them by replacing the chord by an nonoriented noncrossed zigzag.

3   Description of the algorithm

3.1   Hierarchy of subvarieties of codimension 1

Our algorithm will use the induction over the complexity of -chains, i.e. over some partial order on the set of such chains in an arbitrary maximal cell of $F_i \setminus F_{i-1}$. Let us describe this partial order.

First of all, we order these -chains by the number of active points in the corresponding -picture. If these numbers of two pictures are equal, then we subordinate them by the type of the (unique) condition of equality type described in items $\overline{1}$, $\overline{2}$, $\overline{3}$, $\overline{4}$, $\overline{5}$, or $\overline{6}$ of §1.4: the -chain with condition $\overline{1}$ is older than that with condition $\overline{2}$, etc. If these numbers also are equal for some two -chains with equally many active points, then in cases $\overline{1}$—$\overline{5}$ we subordinate these chains by the lexicographic order (in the Wilson line, counting from the left to the right) of the collection of active point participating in such a condition. In the case of conditions $\overline{1}$, $\overline{3}$, $\overline{4}$, $\overline{5}$ this is all, but the case $\overline{2}$ has three subcases. Namely, the "free" (i.e. not belonging to the chord) endpoint of the zigzag can lie to the left (in the Wilson line) of both endpoints of the chord, or between them, or to the right. We subordinate these subcases in this order.

The -chains with degenerations of type $\overline{6}$ are partially ordered in the following complex way. First, we compare the "longitude" (i.e. the projection $f_2 \equiv p_2 \circ f$ to $\mathbb{R}^1$) of the clinching couple of crabs, participating in this degeneration, with that of all other crabs, using only the information contained in the double crossed arrows of our pictures as in part **5** of § 1.2. Namely, the highest order is due to such pictures that no other crab is definitely more southern than this couple. Such pictures that exactly one crab definitely is more southern have the next order, etc. If these data for some two -chains are equal, then we compare the lexicographic orders of the four points participating in two ×-pairs of this degeneration among all active points of the Wilson line.

This is still not a linear order of the set of all possible -pictures. Say, two pictures that differ only by the directions of several non-crossed or once crossed arrows or by the signs in certain conditions of inequality type (described in §1.2) will be incomparable.

3.2   How the algorithm works

It starts from a weight system of rank k, i.e. from a cycle γ in $F_k \setminus F_{k-1}$ given by a linear combination of several maximal cells corresponding to k-chord diagrams. Then we calculate its first boundary $d^1(\gamma)$, which is the sum of several -chains in maximal cells of $F_{k-1} \setminus F_{k-2}$, see the paragraph after Proposition 14 on the page 31 and also formulas (27) and (39). Similarly, the initial data of the rth step of the algorithm, $r = k - i$, is some cycle $d^r(\gamma) \subset F_i \setminus F_{i-1}$ constructed on the previous steps.

On this step we try to span this cycle by a relative chain γ_r such that $\partial \gamma_r = -d^r(\gamma)$ in $F_i \setminus F_{i-1}$. We do it in any maximal cell of $F_i \setminus F_{i-1}$ separately. Namely, we fix such a cell M,

consider the chain $d^r(\gamma) \cap M$ of codimension 1, and try to span it by a chain $\gamma_r^M \subset M$ of full dimension such that $\partial \gamma_r^M + (d^r(\gamma) \cap M) \subset \partial M$.

Denote by Δ this chain $d^r(\gamma) \cap M$ which we should kill by the chain γ_r^M.

Inductive conjecture. Δ is a finite linear combination of -chains in our cell, any of which is distinguished by at most $r - 1$ conditions of inequality type **1–5** and exactly one condition as in item $\bar{\mathbf{1}}$ of §1.4 (i.e. a non-crossed non-oriented zigzag).

Consider the group of all linear combinations of -chains in our cell, distinguished by at most $r - 1$ conditions of inequality type as in §1.2 and one *arbitrary* condition of equality type (i.e. as in items $\bar{\mathbf{1}}$–$\bar{\mathbf{6}}$ of §1.4).

The above-described ordering of -chains defines an increasing filtration in this group. Given any such -chain, the term of the filtration lead by it consists of sums of similar varieties lower than or incomparable with it.

We kill the cycle Δ by an inductive process, any step of which will decrease the filtration of our sum Δ of -chains.

Namely, we select those of -chains in Δ which have the greatest possible number of active points. Denote this number by N. Among -chains with N active points we select those which contain a non-oriented non-crossed zigzag as in condition $\bar{\mathbf{1}}$ of §1.4. Among all such -chains we choose the ones for which this zigzag is (lexicographically) as left as possible; let A be the sum of all such -chains in Δ. Denote by $\tilde{A}$ the sum of -chains whose -pictures are obtained from -pictures of varieties from A by replacing the non-oriented non-crossed zigzag by an zigzag oriented to the left (*or maybe to the right: it is important only that all such replacing zigzags should be directed to one and the same side for all summands in $\tilde{A}$*).

Example 4 If Δ is given by the sum (27) then the leading chain A is the first summand in (27), and $\tilde{A}$ is indicated in the left-hand part of (28). The filtration of the chain $A + \partial \tilde{A}$ is then strictly lower than that of A. The leading term of this sum is equal to the second summand in (27). We kill it by the left-hand part of (29). The remaining sum consists of -chains with no more than 3 active points, in particular its filtration is even smaller.

A similar situation holds also in the general case. Namely, we have the following general statement.

Proposition 3 *If $A \neq 0$ then the filtration of the cycle $\Delta + \partial \tilde{A}$ is strictly smaller than that of Δ.*

Proof. Let W be one of -chains constituting $\tilde{A}$. By Proposition 1 its boundary ∂W is the sum of several -chains whose -pictures have no more than N active points. For any such bounding -chain obtained from W by any degeneration other than of type **X** (see §1.3), its filtration is strictly smaller than that of all components of A. Thus the unique reason by which the chain $\Delta + \partial \tilde{A}$ could have filtration greater than or equal to that of Δ is as follows: the non-oriented zigzags in pictures of components of $\partial \tilde{A}$ arising as degenerations of some oriented non-crossed zigzags of $\tilde{A}$ can be lexicographically "more or equally left" than these from A.

For instance, consider the equation (93). Suppose that A contains the -chain given by the *second* summand of the right-hand part of (93) and we try to kill it by the -chain W shown

in the left-hand part. Then, aside of our killed -chain, ∂W contains the first summand in (93), whose filtration is strictly greater. Fortunately, the sum of such harmful -chains in the boundary of $\tilde{A}$ always vanishes.
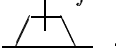
Indeed, denote by Λ the sum of -chains in $\partial \tilde{A} + A$ having N active points and such "more or equally left" zigzags.

Lemma 1 *If the chain Λ is not equal to zero, then $\partial \Delta \neq 0$ in our cell, in contradiction to the definition of Δ.*
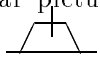
Proof. By construction, the pictures of all summands in Λ have a common oriented zigzag, arising from the common non-oriented zigzag of -chains of A. Replacing the latter oriented zigzag by a non-oriented one in any of these -chains, we obtain a subvariety of codimension 2 in our cell. If $\Lambda \neq 0$, then the chain formed by all such subvarieties also is not equal to zero. But this chain is a part of the *boundary* in the cell M of the *cycle* $\Delta \equiv d^r(\gamma) \cap M$; this part cannot be killed by any other components of boundaries of summands of this cycle. A contradiction.

Applying this trick several times we kill all the summands in Δ whose -pictures contain N active points and a non-oriented zigzag as in paragraph $\bar{\mathbf{1}}$ of §1.2. Denote by Δ_1 the resulting cycle in M, homologous to Δ but having no such summands.

Consider all the -chains in Δ_1 with the same number N of active points and the equality-type condition of type $\bar{\mathbf{2}}$. Denote the sum of them by B.

Proposition 4 *The chain B can be described by -pictures in which all the equality type conditions are of type $\bar{\mathbf{2}}!$ only. I.e., any its summand in which a chord is connected with a third point by a non-crossed oriented zigzag, can be matched by another summand with almost the same -picture (up to an orientation-preserving homeomorphism of the Wilson line), only with the reversed orientation of this zigzag, so that the sum of these two pictures can be replaced by one picture with one crossed non-oriented zigzag*   .

Proof follows immediately from the condition $\partial \Delta_1 \cap M = 0$.

Let us write the chain B in this way. Let B' be the sum of all -chains in B having the maximal possible order in the sense of §3.1. Let $\tilde{B}$ be the sum of similar pictures, obtained from all the pictures of B' by replacing any crossed non-oriented zigzag , connecting a chord with a third point, by the similar zigzag *oriented to the right* in the Wilson line (*or maybe to the left, but uniformly over all pictures of B'*).

Proposition 5 *If $B \neq 0$ then the filtration of the cycle $\Delta_1 + \partial \tilde{B}$ is strictly smaller than that of Δ_1.*

Proof is very similar to that of Proposition 3.

So, we can kill all the summands in Δ_1 with N active points and condition of type $\bar{\mathbf{2}}$. Among the -chains forming the remaining cycle (which we shall call Δ_2), consider the chains with N active points and degeneration of type $\bar{\mathbf{3}}$ only. The sum of all these -chains in Δ_2 will be denoted by C.

Proposition 6 *The chain C can be represented as a sum of -varieties, whose -pictures contain equality type conditions of type $\bar{\mathbf{3}}!$ only.*

Proof is the same as for Proposition 4.

Let us write C in this way. We assume that the condition of type $\bar{\mathbf{3}}!$ of any its -picture is written in the *normal* way, i.e. its two crossed non-oriented zigzags have common left points, see (13). Let C' be the sum of the -chains in C having the maximal possible order in the sense of §3.1. Denote by $\tilde{C}$ the sum of -chains, whose pictures are obtained from pictures of C' by the rule
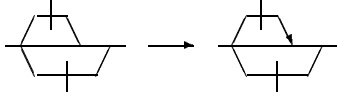
, i.e. by replacing the more left crossed non-oriented zigzag in any such condition by the crossed zigzag oriented to the right (*or maybe to the left, but then uniformly for all -chains in C*).

Proposition 7 *If $C \neq 0$ then the filtration of the cycle $\Delta_2 + \partial\tilde{C}$ is strictly smaller than that of Δ_2.*

Proof coincides with that of Proposition 5.

Repeating this, we obtain a chain Δ_3 homologous to Δ in M, all whose -chains have no more than N active points, and those with exactly N points have no degenerations of types $\bar{\mathbf{1}}$, $\bar{\mathbf{2}}$ or $\bar{\mathbf{3}}$. Denote by D the sum of such -chains in Δ_3 with degeneration of type $\bar{\mathbf{4}}$.

Proposition 8 *The chain D can be represented as a sum of -chains whose -pictures contain equality-type conditions of types $\bar{\mathbf{4}}!a)$ and $\bar{\mathbf{4}}!b)$ only.*

Proof is the same as that of Proposition 6.

Let us write D in this way. Let D' be the sum of the -chains in D having the maximal possible order in the sense of §3.1, i.e. with the most left (lexicographically) possible position of the pair of points participating in these conditions. Denote by $\tilde{D}$ the sum of similar -chains obtained from pictures of D' by replacing all pictures $\begin{smallmatrix}l\\j\end{smallmatrix}$ by $\begin{smallmatrix}j\\l\end{smallmatrix}$ (*or maybe by* $\begin{smallmatrix}l\\j\end{smallmatrix}$, *but then uniformly*), and replacing all pictures $\begin{smallmatrix}l\\j\end{smallmatrix}$ with $l < j$ by $\begin{smallmatrix}j\\l\end{smallmatrix}$ (*or maybe by* $\begin{smallmatrix}l\\j\end{smallmatrix}$ *but then uniformly*).

Proposition 9 *If $D \neq 0$ then the filtration of the cycle $\Delta_3 + \partial\tilde{D}$ is strictly smaller than that of Δ_3.*

Proof is the same as that of Proposition 7.

Repeating this reduction, we obtain a chain Δ_4 homologous to Δ in M, all whose -pictures have no more than N active points, and those with precisely N do not have degenerations of types $\bar{\mathbf{1}}$, $\bar{\mathbf{2}}$, $\bar{\mathbf{3}}$, or $\bar{\mathbf{4}}$. Denote by E the sum of all such pictures in Δ_4 with exactly N active points and degeneration of type $\bar{\mathbf{5}}$.
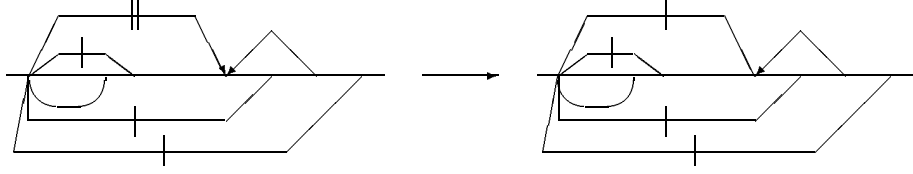
Figure 11: Killing a variety of type $\bar{\mathbf{6}}!$

Proposition 10 *The chain E can be described by the -pictures whose equality-type conditions are of type $\bar{\mathbf{5}}!$ only.*

Proof is the same as that of Proposition 8.

Let us write E in this way. Order the -pictures of E in respect with the number of the point at which the condition of type $\bar{\mathbf{5}}!$ occurs (among all active points) and denote by E' the sum of such varieties with the minimal value j of this number. Let $\tilde{E}$ be the sum of similar varieties obtained from varieties of E' by replacing these conditions $\overset{j}{\leftrightarrow}$ of type $\bar{\mathbf{5}}!$ by $\Uparrow^j$ (*or maybe by $\Downarrow^j$*).

Proposition 11 *If $E \neq 0$ then the filtration of the cycle $\Delta_4 + \partial\tilde{E}$ is strictly smaller than that of Δ_4.*

Proof is the same as that of Proposition 9.

Repeating this reduction, we obtain a chain Δ_5, all whose pictures have no more than N active points, and those with exactly N points have degenerations of type $\bar{\mathbf{6}}$ only. Denote the sum of these varieties by F.

Proposition 12 *The chain F can be combined into a sum of -chains, all whose equality-type conditions are the clinching crabs having no oriented non-crossed zigzags joining their $\times$-pairs to one another or to some φ-points: all links between these $\times$-pairs or between $\times$-pairs and their feet can be described by non-oriented single crossed zigzags as in paragraph $\bar{\mathbf{6}}!$ of §1.4 only.*

Proof coincides with that of Proposition 10.

Let us write the chain F in this form and select those of its summands which have the greatest order in the hierarchy of §3.1 (i.e., with the smallest number of other crabs definitely more southern than these participating in the condition of type $\bar{\mathbf{6}}!$, and among varieties minimizing this number only those with the lexicographically most left position of the four points participating in this degeneration). Denote by F' the sum of these selected summands. Let $\tilde{F}$ be the sum of similar -chains, in which the clinching crabs of -pictures of summands of F' are divorced in the following way: the $\times$-pair having smaller lexicographic order than the other one inherits all the legs, and the other $\times$-pair losses all legs but becomes *more northern* than it.

I.e., in the normal picture of the -chain (see Fig. 8) we replace the unique non-oriented once crossed zigzag, joining the ×-pairs, by a double crossed zigzag oriented from the lexicographically more left ×-pair to the more right one, see Fig. 11. (*Again, we can replace the orientation in this rule by the opposite one, but then to do it uniformly for all -chains in F'.*)

Proposition 13 *If $F \neq 0$ then the filtration of the cycle $\Delta_5 + \partial \tilde{F}$ is strictly smaller than that of Δ_5.*

Proof coincides with that of Proposition 11.

Repeating, we finally kill all the -chains with N active points, i.e. reduce the cycle Δ to a sum of -chains with $\leq N - 1$ such points. Continuing by induction over N, we eliminate all the -chains in our cell $M \subset F_i \setminus F_{i-1}$. The sum of all varieties $\tilde{A}, \tilde{B}, \ldots, \tilde{F}$ participating in this inductive process on all its steps is the desired chain γ_r^M in the cell M.

Remark. On any step of the algorithm we have two choices of the variety of full dimension killing the leading (of highest filtration) summands of a cycle of codimension 1 in M. In our stupid algorithm, the first of these choices is always selected: the other possibilities are indicated in italic letters before the statements of Propositions 3, 5, 7, 9, 11, and 13. A more smart algorithm (working faster and providing more compact answers) should choose them using some criteria which are not clear to me yet. Also, the demands of uniformity of switchings in these possibilities probably can sometimes be weakened, which will give us additional degrees of freedom.

Suppose now that such chains γ_r^M are constructed in all the cells M of maximal dimension in $F_i \setminus F_{i-1}$ (i.e. in the cells corresponding to the i-chord diagrams). The sum of these chains γ_r^M over all cells M provides a homology in $F_i \setminus F_{i-1}$ between our cycle $d^r(\gamma)$ of codimension 1 and a semialgebraic cycle in the union of all other cells. By dimensional reasons, the latter cycle is a linear combination of some cells of vice-maximal dimension. It should be homological to zero in $F_i \setminus F_{i-1}$: otherwise our cycle γ cannot be extended ("integrated") to a knot invariant, and the algorithm stops. By the Kontsevich's stabilization theorem, this cannot happen in similar calculations of rational knot invariants; the homotopy splitting conjecture for knot discriminants (see e.g. §5.1 in [14]) would imply the same for arbitrary coefficients.

Therefore this linear combination of vice-maximal cells is equal to the boundary of a linear combination of maximal cells (corresponding to chord diagrams). Thus we need only to add the latter linear combination to the chain $\sum_M \gamma_r^M$ constructed previously. The obtained sum is the desired chain $\gamma_r \subset F_i \setminus F_{i-1}$ such that $\partial \gamma_r = -d^r(\gamma)$ in $F_i \setminus F_{i-1}$.

Proposition 14 (step of induction) *If all the chains $\gamma \subset F_k \setminus F_{k-1}$, $\gamma_1 \subset F_{k-1} \setminus F_{k-2}$, ..., $\gamma_r \subset F_i \setminus F_{i-1}$ (where $i = k - r$), were constructed as above, then the next boundary $d^{r+1}(\gamma)$ of their sum $\gamma + \gamma_1 + \cdots + \gamma_r$ in $F_{i-1} \setminus F_{i-2}$ is a semialgebraic cycle, whose intersection with any maximal cell of $F_{i-1} \setminus F_{i-2}$ consists of finitely many -chains, any of which is distinguished by no more than r conditions of inequality type as in items **1–5** of §1.2, and exactly one condition of equality type as in item $\bar{1}$ of §1.4.*

More precisely, the sum $\gamma + \ldots + \gamma_{r-1}$ contributes nothing to this homological boundary (as its geometrical boundary has too small dimension there), and any -chain participating in γ_r

contributes precisely i summands (some of which can then annihilate with some others) whose
-pictures are obtained from the -picture of this -chain V by replacing one of chords by a
non-oriented non-crossed zigzag as in item $\bar{\mathbf{1}}$ of §1.4.

The step of induction is complete.

Remark. Exactly the same algorithm will give a combinatorial formula in the case of integer-valued invariants: it remains only to specify the (co)orientations of all subvarieties and conditions from §1.

Problem. Is it possible to realize the entire sequence of spanning chains $\gamma_1, \ldots, \gamma_k$ by the sums of -varieties with conditions of type $\mathbf{1}$ only?

4   Calculation of a combinatorial formula for the knot invariant of second order

In this section we give the first illustration of our algorithm, showing how it calculates a combinatorial formula for the unique order 2 knot invariant v_2 reduced mod 2.

Theorem 1 *The value of v_2 on a generic long knot $f : \mathbb{R}^1 \to \mathbb{R}^3$ is equal (mod 2) to the sum of three numbers (see three summands in (25)):*

a) the number of configurations $\{a < b < c < b\} \subset \mathbb{R}^1$ such that $f(c)$ is above $f(a)$ and $f(d)$ is above $f(b)$;

b) the number of configurations $\{a < b < c\}$ such that $f(c)$ is above $f(a)$ and the projection of $f(b)$ to $\mathbb{R}^2$ lies to the east of the (common) projection of $f(a)$ and $f(c)$;

c) the number of configurations $\{a < b\}$ such that $f(b)$ is above $f(a)$ and the direction "to the east" in $\mathbb{R}^2$ is a linear combination of projections of derivatives $f'(a)$ and $f'(b)$, such that the first of these projections participates in this linear combination with a positive coefficient, and the second with a negative one.

$$(25)$$

The proof takes the rest of this section.

4.1   Principal part

The principal part of v_2 in $F_2 \setminus F_1$ is expressed by the chord diagram

$$(26)$$

4.2 First differential and its homology to zero

The boundary of the principal part (26) in F_1 is equal to

$$\text{(diagram)} \qquad (27)$$

Let us span this boundary by a chain in the maximal cell of F_1. The stupid algorithm associates to the elements of (27) the -chains described in left parts of equations (28), (29).

$$\partial \; \text{(diagram)} = \text{(diagram)} + \text{(diagram)} + \text{(diagram)} \qquad (28)$$

$$\partial \; \text{(diagram)} = \text{(diagram)} + \text{(diagram)} + \text{(diagram)} \qquad (29)$$

(Formally, there should be two summands more in any of these two formulas, in particular in (29) they coincide with two pictures of the left part of the equation (11). But these summands are equal to one another and annihilate; the same will happen in the case of integer coefficients as well.)

According to these equations, the sum of these two -chains provides a homology between the cycle (27) and the chain (30).

$$\text{(diagram)} \qquad (30)$$

The stupid algorithm transforms the latter chain to the sum of left parts of equations (31), (32).

$$\partial \; \text{(diagram)} = \text{(diagram)} + \text{(diagram)} + \text{(diagram)} \qquad (31)$$

$$\partial \; \text{(diagram)} = \text{(diagram)} + \text{(diagram)} + \text{(diagram)} + \text{(diagram)} \qquad (32)$$

According to these equations, the cycle (30) is equal to the boundary of the sum of varieties indicated in left parts of (31), (32).

33

Finally, we get that the chain γ_1 equal to the sum of four -chains indicated in left parts of equations (29), (28), (31) and (32) provides a homology in F_1 between the cycle (27) and the vice-maximal cell of F_1 taken with some coefficient.

Lemma 2 *The latter coefficient is equal to zero, so that the cycle (27) is equal to $\partial \gamma_1$ in F_1.*

Proof. The boundary of the chain γ_1 in the vice-maximal cell of F_1 can be calculated in the following way. For any of four -chains constituting γ_1 we let the endpoints of the unique chord of the -picture tend to one another in $\mathbb{R}^1_w$ and consider the subvariety in the vice-maximal cell swept out by all the limit positions of singular knots from our chains after this contraction. In all four cases this limit variety consists of pairs (a point $* \in \mathbb{R}^1$, a map $f \in \mathcal{K}$ such that $f'(*) = 0$) satisfying one condition more. Namely, for the picture in the left-hand part of (28) (respectively, of (29)) this condition is as follows: there exists a point $a < *$ (respectively, $a > *$) in $\mathbb{R}^1_w$ such that $f(a)$ lies below (respectively, above) $f(*)$ in $\mathbb{R}^3$. For both varieties indicated in left-hand parts of (31) and (32) the limit condition claims that the second derivative $f''_1(*)$ is directed "to the east" in $\mathbb{R}^2$.

All these additional conditions specify subvarieties of codimension one in the vice-maximal cell of F_1, and lemma is proved.

4.3 Second differential and its homology to zero

The projection of the chain γ_1 to Σ is indicated in (33):

$$\tag{33}$$

The stupid algorithm transforms these -chains to -chains indicated in left parts of equalities (34)—(36).

$$\tag{34}$$

$$\tag{35}$$

$$\partial \quad \text{(36)}$$

(36)

The sum of the third, fourth, and fifth terms in the right-hand part of (34) equals the second summand in (35). Therefore the sum of right-hand parts of (34)—(36) is equal to the cycle (33), and the sum of -chains indicated in left-hand parts of (34)—(36) is the desired combinatorial formula.

Theorem 1 is thus proved.

5 Calculation of a combinatorial formula for v_3

This is the next demonstration of the algorithm: the calculation a combinatorial formula for the unique order 3 invariant v_3 reduced mod 2.

Theorem 2 *A combinatorial formula for the third order invariant (mod 2) is given by the sum of 14 -chains indicated in (37).*

$$ \text{(37)} $$

(37)

5.1 Principal part

The principal part γ of v_3 in $F_3 \setminus F_2$ is equal to the sum of three chord diagrams

$$ \text{(38)} $$

(38)

(It is essential here that we consider the mod 2 homology: the similar formula with integral coefficients is true for homology of knots in $\mathbb{R}^n$ with even $n \geq 4$ only.)

5.2 First differential and its homology to zero

The boundary $d^1(\gamma)$ of this principal part in $F_2 \setminus F_1$ is equal to the sum of nine $-$chains

$$\cdots \; + \; \cdots \; + \; \cdots \; + $$

$$+ \; \cdots \; + \; \cdots \; + \; \cdots \; + $$

$$+ \; \cdots \; + \; \cdots \; + \; \cdots \; . \tag{39}$$

These $-$chains lie in all three maximal cells of $F_2 \setminus F_1$: the second chain in the second cell of (4), 6th and 7th chains in the third, and all six remaining chains in the first cell of (4).

Let us span this sum by a chain in $F_2 \setminus F_1$. The stupid algorithm associates to elements of (39) the $-$chains indicated in left parts of equations (40)–(48).

$$\partial \; \cdots \; = \; \cdots \; + \; \cdots \; + $$

$$+ \; \cdots \; + \; \cdots \tag{40}$$

$$\partial \; \cdots \; = \; \cdots \tag{41}$$

$$\partial \; \cdots \; = \; \cdots \; + \; \cdots \; + $$

$$+ \; \cdots \; + \; \cdots \tag{42}$$

$$\partial \; \cdots \; = \; \cdots \; + \; \cdots \; + $$

$$\cdots \; + \; \cdots \; + \; \cdots \tag{43}$$

$$\partial \; \cdots \; = \; \cdots \; + \; \cdots \; + $$

$$+ \; \cdots \; + \; \cdots \; + \; \cdots \; + \; \cdots \tag{44}$$

$$\partial \; \cdots \; = \; \cdots \; + \; \cdots \tag{45}$$

36

$$\partial \quad \text{} \quad = \quad \text{} \quad + \quad \text{} \tag{46}$$

$$\partial \quad \text{} \quad = \quad \text{} \quad + \quad \text{} \quad +$$

$$+ \quad \text{} \quad + \quad \text{} \quad + \quad \text{} \tag{47}$$

$$\partial \quad \text{} \quad = \quad \text{} \quad + \quad \text{} \quad +$$

$$+ \quad \text{} \quad + \quad \text{} \quad + \quad \text{} \quad + \quad \text{} \tag{48}$$

These equations imply easily the following statement.

Proposition 15 *The cycle* (39) *is homologous to the chain* (49) *in the union of maximal cells of $F_2 \setminus F_1$. This homology is provided by the nine varieties indicated in left parts of equations* (40)—(48).

$$\text{} \quad + \quad \text{} \quad + \quad \text{} \quad + \quad \text{} \quad +$$

$$+ \quad \text{} \quad + \quad \text{} \quad + \quad \text{} \tag{49}$$

Indeed, the summands in the right-hand parts of these equations satisfy the following relations (where (a;b) denotes the bth component of the right-hand part of the equation (a)).
   (40;4)=(43;4), (40;3)=(43;3), (45;2)+(46;2)=(49;5), (43;5)+(44;6)=(49;6),
   (47;5)+(48;6)=(49;7), (42;2)=(47;2), (42;3)=(47;3), (42;4)+(48;2)=(49;2),
   (40;2)+(44;5)=(49;1), (44;3)+(48;5)=(49;4), (44;2)+(48;4)=(49;3),
   (44;3)=(47;4), (43;2)=(48;3).
The stupid algorithm transforms the chain (49) into the sum of left parts of equations (50)—(56).

$$\partial \quad \text{} \quad = \quad \text{} \quad + \quad \text{} \quad + \quad \text{} \tag{50}$$

$$\partial \quad \text{} \quad = \quad \text{} \quad + \quad \text{} \quad + \quad \text{} \tag{51}$$

$$\partial \quad = \quad + \quad + \quad \tag{52}$$

$$3 \leftarrow$$

$$\partial \quad = \quad + \quad + \quad \tag{53}$$

$$2 \mapsto$$

$$\partial \quad = \quad \tag{54}$$

$$\partial \quad \overset{3}{\underset{1}{}} \quad = \quad + \quad \overset{1}{\underset{3}{}} \quad + \quad 1 \mapsto \quad + \quad 3 \mapsto \tag{55}$$

$$\partial \quad \overset{4}{\underset{2}{}} \quad = \quad + \quad \overset{2}{\underset{4}{}} \quad + \quad 2 \mapsto \quad + \quad 4 \mapsto \tag{56}$$

According to these equations, the cycle (49) is homologous to the chain

$$\quad + \quad \tag{57}$$

$$3 \qquad 4$$

Indeed, we have (50;3)=(52;2), (51;2) =(53;3) (52;3) + (55;3) =(57;1) and (51;3) + (56;3) =(57;2).

Varieties indicated in (57) are the boundaries of two varieties shown in (58).

$$\quad + \quad \tag{58}$$

$$\Uparrow^3 \qquad \Uparrow^4$$

Finally, we get the following statement.

Proposition 16 *The cycle (39) is homologous in $F_2 \setminus F_1$ to some linear combination of vice-maximal cells of the canonical cell decomposition. This homology is provided by the sum of eighteen -chains indicated in left parts of equations (40)—(48) and (50)—(56), and in the formula (58).*

Proposition 17 *The linear combination of vice-maximal cells mentioned in the previous lemma is equal to zero. In particular the cycle (39) is equal to the boundary in $F_2 \setminus F_1$ of the sum of these eighteen -chains.*

Proof. Consider the part $\bar{\partial}$ of the boundary operator associating to any -variety the intersection of its full boundary with the union of vice-maximal cells (see the beginning of §2).

Accordingly to §2.8 this operator $\bar{\partial}$ acts on our eighteen $\quad$-varieties as is shown in the next equations (59)—(76).

$$\bar{\partial}\ \ \cdots\ =\ \cdots\ +\ \cdots\ +\ \cdots\ +\ \cdots\ + \tag{59}$$

$$\bar{\partial}\ \ \cdots\ =\ \cdots\ +\ \cdots\ +\ \cdots\ +\ \cdots \tag{60}$$

$$\bar{\partial}\ \ \cdots\ =\ \cdots\ +\ \cdots\ +\ \cdots\ +\ \cdots\ + \tag{61}$$

$$\bar{\partial}\ \ \cdots\ =\ \cdots\ +\ \cdots\ +\ \cdots\ +\ \cdots\ + \tag{62}$$

$$\bar{\partial}\ \ \cdots\ =\ \cdots\ +\ \cdots\ +\ \cdots\ +\ \cdots \tag{63}$$

$$\bar{\partial}\ \ \cdots\ =\ \cdots\ +\ \cdots\ +\ \cdots\ +\ \cdots\ + \tag{64}$$

$$\bar{\partial} \quad = \quad + \quad + \quad + \quad +$$

$$+ \quad + \tag{65}$$

$$\bar{\partial} \quad = \quad + \quad + \quad + \quad +$$

$$+ \quad + \tag{66}$$

$$\bar{\partial} \quad = \quad + \quad + \quad + \tag{67}$$

$$\bar{\partial} \quad = \quad + \quad + \quad + \quad +$$

$$+ \quad + \tag{68}$$

$$\bar{\partial} \quad = \quad + \quad + \quad + \quad +$$

$$+ \quad + \tag{69}$$

$$\bar{\partial} \quad = \quad + \quad + \quad + \quad +$$

$$+ \quad + \tag{70}$$

40

$$\bar{\partial} \quad = \quad + \quad + \quad + \quad + \tag{71}$$

$$+ \quad + \quad$$

$$\bar{\partial} \quad = \quad + \quad + \quad + \quad + \tag{72}$$

$$+ \quad + \quad + \quad + \quad$$

$$\bar{\partial} \quad = \quad + \quad + \quad + \quad + \quad + \tag{73}$$

$$\bar{\partial} \quad = \quad + \quad + \quad + \quad + \quad + \tag{74}$$

$$\bar{\partial} \quad = \quad + \quad + \quad + \quad + \quad + \tag{75}$$

$$\bar{\partial} \quad = \quad + \quad + \quad + \quad + \quad + \tag{76}$$

It remains to prove that the sum of right-hand parts of all these operators is equal to zero. They satisfy the the following relations, reducing them. Denote by $(a;b)$ the bth summand of the right-hand part of the equation (a). Then we have

$(59;1) = (65;2)$, $(59;2) = (59;3)$, $(59;4) = (65;1)$, $(59;6) = (61;5)$,
$(60;1) = (63;2)$, $(60;2) = (67;2)$, $(61;1) = (61;3)$, $(61;2) = (64;4)$,
$(61;4) = (64;3)$, $(62;1) = (62;3)$, $(62;2) = (65;4)$, $(62;4) = (65;3)$,
$(63;1) = (67;3)$, $(63;4) = (64;5)$, $(64;1) = (66;4)$, $(64;2) = (66;2)$,
$(65;6) = (67;3)$, $(66;1) = (66;3)$, $(68;1) = (68;)3$, $(69;2) = (69;4)$,
$(68;2) = (70;2)$, $(68;4) = (70;1)$, $(69;1) = (70;4)$, $(69;3) = (70;3)$,
$(70;6) = (74;6)$, $(71;1) = (72;1)$, $(71;2) = (72;2)$, $(71;3) = (72;3)$,
$(71;4) = (72;4)$, $(73;2) = (73;4)$, $(73;6) = (74;1)$, $(74;3) = (74;5)$,
$(75;2) = (75;4)$, $(75;5) = (76;3)$, $(75;6) = (76;1)$, $(76;2) = (76;4)$.

Any remaining summand containing the condition   is matched by a similar summand with this condition replaced by   , and their sum is equal to a similar term without these conditions

at all. The summands remaining after all these reductions can be combined in the sum shown in the next three lines (77)—(79); here a formula consisting of a picture and a sum in cartouch below it denotes the sum of -varieties with one and the same main picture and subscripts shown in the cartouch.

$$
\begin{array}{c}
\triangle \circ \quad +\\[6pt]
\left(\ \mathord{\mapsto}{}^{3}_{2} + \mathord{\mapsto}{}^{2}_{3} + \Uparrow^{2} + \Uparrow^{3} + \mathord{\mapsto}{}^{1}_{2} + \mathord{\mapsto}{}^{1}_{2} + \mathord{\mapsto}{}^{1}_{2} + \mathord{\mapsto}{}^{1}_{2}\ \right)
\end{array}
\tag{77}
$$

$$
\begin{array}{c}
+ \quad \triangle\!\!\circ \quad +\\[6pt]
\left(
\begin{array}{c}
\mathord{\mapsto}{}^{1}_{3} + \mathord{\mapsto}{}^{2}_{3} + \mathord{\mapsto}{}^{1}_{2}{}_{3} + \mathord{\mapsto}{}^{2}_{1}{}_{3} + \mathord{\mapsto}{}^{1}_{3}{}_{2} +\\[8pt]
+ \mathord{\mapsto}{}^{1}_{2} + \mathord{\mapsto}{}^{2}_{1} + \mathord{\mapsto}{}^{1}_{2}
\end{array}
\right)
\end{array}
\tag{78}
$$

$$
\begin{array}{c}
+ \quad \circ\!\!\triangle \\[6pt]
\left(
\begin{array}{c}
\mathord{\mapsto}{}^{1}_{2} + \mathord{\mapsto}{}^{1}_{3} + \mathord{\mapsto}{}^{2}_{3} + \mathord{\mapsto}{}^{1}_{2}{}_{3} + \mathord{\mapsto}{}^{2}_{1}{}_{3} + \mathord{\mapsto}{}^{1}_{3}{}_{2} +\\[8pt]
+ \mathord{\mapsto}{}^{2}_{3} + \mathord{\mapsto}{}^{3}_{2} + \Uparrow^{2} + \Uparrow^{3}
\end{array}
\right)
\end{array}
\tag{79}
$$

It remains to prove that the sum indicated in any cartouch is identically equal to zero on all generic singular knots respecting the picture over this cartouch.

A. The sum of first four terms in the cartouch of (77) is identically equal to $1 \in \mathbb{Z}_2$. Indeed, the sum of first two terms (respectively, of the 3d and 4th terms) is equal to 1 if and only if the vectors $f_2'(a_2)$ and $f_2'(a_3)$ are directed into equal (respectively, different) sides in $\mathbb{R}^1$. The sum of remaining four terms in (77) is obviously also equal to 1, and entire line (77) vanishes.

B. The sum of upper six terms in the cartouch of (79) also is identically equal to 1. Indeed, if all three vectors $f_1'(a_1)$, $f_1'(a_2)$, $f_1'(a_3)$ are directed into some halfplane in $\mathbb{R}^2$ then exactly one of last three conditions is satisfied and none or two of the first three conditions are satisfied. If these three vectors do not lie in one halfplane then exactly one of first three conditions and none of the last three conditions is satisfied.

The lower four terms in (79) coincide with first terms in (77) and also form a tautology.

C. Adding the term to both levels of the cartouch (78) we obtain two functions identically equal to 1. Proposition 17 is completely proved.

Therefore we can take the sum of eighteen -chains mentioned in it for the chain γ_1 spanning the cycle $d^1(\gamma)$ in entire $F_2 \setminus F_1$.

5.3   Second differential and its homology to zero

The boundary of this chain γ_1 in F_1 (see §2.9) is equal to the sum of thirty-six -chains indicated in the next nine lines (80)—(88):

$$(80)$$

$$(81)$$

$$(82)$$

$$(83)$$

$$(84)$$

$$(85)$$

$$(86)$$

$$(87)$$

43

$$+ \quad \cdots \quad + \quad \cdots \quad + \quad \cdots \quad + \quad \cdots$$

$$\Uparrow^3 \qquad\qquad \Uparrow^3 \qquad\qquad \Uparrow^4 \qquad\qquad \Uparrow^4 \tag{88}$$

To span this cycle, the stupid algorithm supplies us with the   -chains indicated in left parts of the next equalities (89)–(115); note that any of first 9 of them kills two of our 36 summands in (80)—(88).

$$\partial \quad \cdots \quad = \quad \cdots \quad + \quad \cdots \quad + \quad \cdots \quad +$$

$$+ \quad \cdots \quad + \quad \cdots \quad + \quad \cdots \quad + \quad \cdots \tag{89}$$

$$\partial \quad \cdots \quad = \quad \cdots \quad + \quad \cdots \quad +$$

$$+ \quad \cdots \quad + \quad \cdots \quad + \quad \cdots \quad + \quad \cdots \tag{90}$$

$$\partial \quad \cdots \quad = \quad \cdots \quad + \quad \cdots \quad +$$

$$+ \quad \cdots \quad + \quad \cdots \quad + \quad \cdots \quad + \quad \cdots \tag{91}$$

$$\partial \quad \cdots \quad = \quad \cdots \quad + \quad \cdots \quad +$$

$$+ \quad \cdots \quad + \quad \cdots \quad + \quad \cdots \quad + \quad \cdots \quad +$$

$$+ \quad \cdots \quad + \quad \cdots \quad + \quad \cdots \tag{92}$$

$$\partial \quad = \quad + \quad +$$

$$+ \quad + \quad + \quad +$$

$$+ \quad +$$

$$1 \qquad 3$$

$$2$$
$$1$$
$$3$$

$$(93)$$

$$\partial \quad = \quad + \quad +$$

$$+ \quad + \quad +$$

$$+ \quad + \quad +$$

$$1 \qquad 3$$

$$(94)$$

$$\partial \quad = \quad + \quad +$$

$$+ \quad + \quad +$$

$$+ \quad + \quad +$$

$$2 \qquad 4$$

$$(95)$$

$$\partial \quad = \quad + \quad +$$

$$+ \quad + \quad + \quad +$$

$$+ \quad + \quad +$$

$$2$$
$$4$$

$$2$$
$$3$$
$$1$$

$$(96)$$

$$(97)$$

$$(98)$$

$$(99)$$

$$(100)$$

$$(101)$$

∂ $\quad=\quad+\quad+\quad+$

$+\quad+\quad+\quad+\quad+$

$3 \leftarrow\!\dashv$

$1 \quad 3$

2
3

(102)

∂ $\quad=\quad+\quad+\quad+$

$+\quad+\quad+\quad+\quad+$

$3 \leftarrow\!\dashv$

$3 \quad 1$

2
3

(103)

∂ $\quad=\quad+\quad+\quad+$

$+\quad+\quad+\quad+\quad+$

$2 \mapsto$

$1 \quad 3$

2
1

(104)

∂ $\quad=\quad+\quad+\quad+$

$+\quad+\quad+\quad+\quad+$

$2 \mapsto$

$3 \quad 1$

1
2

(105)

∂ $\quad=\quad+\quad+\quad+$

$+\quad+\quad+\quad+$

$3 \quad 1$

2
1

3
2

(106)

47

$$\partial \quad = \quad + \quad + \quad +$$

$$+ \quad + \quad + \quad + \tag{107}$$

$$1 \quad 3 \qquad \begin{smallmatrix}2\\1\end{smallmatrix} \qquad \begin{smallmatrix}3\\2\end{smallmatrix}$$

$$\partial \quad = \quad + \quad + \quad +$$

$$\begin{smallmatrix}1\\3\end{smallmatrix} \qquad \begin{smallmatrix}1\\3\end{smallmatrix} \qquad 1 \mapsto \qquad 3 \mapsto$$

$$+ \quad + \tag{108}$$

$$\begin{smallmatrix}1\\3\end{smallmatrix} \qquad \begin{smallmatrix}1\\3\end{smallmatrix}$$

$$\partial \quad = \quad + \quad + \quad + \quad +$$

$$\begin{smallmatrix}1\\3\end{smallmatrix} \qquad \begin{smallmatrix}1\\3\end{smallmatrix} \qquad 1 \mapsto \qquad 3 \mapsto \qquad \begin{smallmatrix}1\\3\end{smallmatrix}$$

$$+ \quad + \quad + \tag{109}$$

$$\begin{smallmatrix}1\\2\end{smallmatrix} \qquad \begin{smallmatrix}1\\2\end{smallmatrix} \qquad \begin{smallmatrix}1\\3\end{smallmatrix}$$

$$\partial \quad = \quad + \quad + \quad + \quad +$$

$$\begin{smallmatrix}2\\4\end{smallmatrix} \qquad \begin{smallmatrix}2\\4\end{smallmatrix} \qquad 2 \mapsto \qquad 4 \mapsto \qquad \begin{smallmatrix}2\\4\end{smallmatrix}$$

$$+ \quad + \quad +$$

$$\begin{smallmatrix}1\\3\end{smallmatrix} \qquad \begin{smallmatrix}2\\3\end{smallmatrix} \qquad \begin{smallmatrix}2\\3\end{smallmatrix} \tag{110}$$

48

$$\partial \quad \ldots \quad = \quad \ldots \quad + \quad \ldots \quad + \quad \ldots \quad +$$

$$+ \quad \ldots \quad + \quad \ldots \tag{111}$$

$$\partial \quad \ldots \quad = \quad \ldots \quad + \quad \ldots \quad + \quad \ldots \quad + \quad \ldots \tag{112}$$

$$\partial \quad \ldots \quad = \quad \ldots \quad + \quad \ldots \quad + \quad \ldots \quad +$$

$$+ \quad \ldots \quad + \quad \ldots \quad + \quad \ldots \tag{113}$$

$$\partial \quad \ldots \quad = \quad \ldots \quad + \quad \ldots \quad + \quad \ldots \quad + \quad \ldots \tag{114}$$

$$\partial \quad \ldots \quad = \quad \ldots \quad + \quad \ldots \quad + \quad \ldots \quad + \quad \ldots \tag{115}$$

All the terms of the sum (80)—(88) are contained in the right-hand parts of equations (89)—(115). Indeed, let us denote by (a;b) the bth component of the right-hand part of the equation (a). Then we have

$$(80;1) = (89;2), \quad (80;2) = (90;2), \quad (80;3) = (90;1), \quad (80;4) = (91;2),$$
$$(81;1) = (91;1), \quad (81;2) = (89;1), \quad (81;3) = (93;2), \quad (81;4) = (92;2),$$
$$(82;1) = (94;2), \quad (82;2) = (92;1), \quad (82;3) = (94;1), \quad (82;4) = (93;1),$$
$$(83;1) = (95;2), \quad (83;2) = (97;2), \quad (83;3) = (96;2), \quad (83;4) = (97;1),$$
$$(84;1) = (96;1), \quad (84;2) = (95;1), \quad (84;3) = (98;1), \quad (84;4) = (99;1), \qquad (116)$$
$$(85;1) = (100;1), \quad (85;2) = (101;1), \quad (85;3) = (102;1), \quad (85;4) = (103;1),$$
$$(86;1) = (104;1), \quad (86;2) = (105;1), \quad (86;3) = (106;1), \quad (86;4) = (107;1),$$
$$(87;1) = (108;1), \quad (87;2) = (109;1), \quad (87;3) = (110;1), \quad (87;4) = (111;1),$$
$$(88;1) = (112;1), \quad (88;2) = (113;1), \quad (88;3) = (114;1), \quad (88;4) = (115;1).$$

Therefore the equations (89)—(115) provide a mod 2 homology between the cycle (80)—(88) and the sum of all remaining terms of right-hand parts of these equations. These terms satisfy many relations which very much reduce this sum. Namely, we have:

$(89;3)+(94;6) =(98;2), (89;4) =(93;4), (89;5)+(93;5)+(97;3) = (107;2),$
$(89;6) =(97;4), (89;7)+(95;4) =(101;2), (90;3)+(90;4)+(92;6) = (99;2),$
$(90;5) =(92;7), (91;3)=(96;3), (91;4)+(91;5)+(96;4) =(100;2),$
$(92;3)+(95;5)+(95;6) =(103;2), (92;4)+(95;7) =(105;2), (92;5)=(97;5),$
$(92;8)=(109;4), (93;3) =(96;5), (93;6)+(94;8) =(108;4), (94;3)+(96;6) =(102;2),$
$(94;4)+(94;5)+(96;7) =(104;2), (94;7)+(95;3) =(106;2), (95;8)+(97;6) =(111;4),$
$(96;8) =(110;4), (98;3) =(108;2), (98;4) =(102;4), (98;5) =(104;3),$
$(99;3) =(109;2), (99;4) =(103;4), (100;3) =(104;5), (100;4)+(110;3) =(114;2),$
$(101;4) =(105;6), (101;5) =(105;5), (101;6)+(111;3) =(115;2), (102;3)=(107;3),$
$(102;5)+(108;3) =(112;2), (102;7) =(104;7), (102;8) =(110;7), (103;3) =(105;3),$
$(103;5)+(109;3) =(113;2), (103;6) =(107;4), (103;7) =(105;7), (104;4) =(110;2),$
$(104;6) =(106;4), (105;4) =(111;2), (112;3) =(114;3), (112;4) =(114;4), (113;6) =(115;4).$

Therefore we obtain that the cycle (80)—(88) is homological in F_1 to the chain consisting of all elements of left parts of equations (89)—(115) not participating in either (116) or in the previous twelve lines. Namely, this are the following terms:

$(90;6), (91;6), (92;9), (93;7), (96;9), (97;7), (98;6), (98;7), (99;5),$
$(100;5), (101;3), (101;7), (102;6), (103;8), (104;8), (105;8), (106;3), (106;5),$
$(106;6), (106;7), (107;5), (107;6), (107;7), (108;5), (109;5), (109;6), (109;7),$
$(110;5), (110;6), (111;5), (113;3), (113;4), (113;5), (115;3).$

It is convenient to combine these terms in the following five lines (117)—(121). Here the picture (117;5) stays for the sum (106;5)+(107;5), and any of lines (118)—(121) represents the sum of several -chains, whose notation consists of one and the same main picture and different subscripts listed in the cartouch under the line.

$$(117)$$

$$(118)$$

$$(119)$$

$$(120)$$

$$(121)$$

It is convenient to think on the subscripts under all terms of these five formulas as on the characteristic functions (mod 2) of certain subvarieties in the manifolds distinguished by the pictures above these subscripts.

This cycle $(117) + \cdots + (121)$ can be reduced very much.

Proposition 18 *The chain* (117)—(121) *is equal in* F_1 *to the chain*

$$(122)$$

The proof consists of Lemmas 3–6 below.

Lemma 3 *The chain* (118) *is equal to zero.*

First proof. The boundary of this chain contains a similar chain with the same expression in the cartouch but with the main picture replaced by a similar picture without arrows. This part of the boundary cannot be killed by the boundaries of any other sums (117) or (119)—(121). Thus the whole chain (117)—(121) has the chance to be a cycle (which it is by its construction) only if the expression in the cartouch defines an identically zero function mod 2.

Second proof. The sum of conditions listed in the cartouch (118) coincides with that of the formula (78). Therefore it is equal to 0 at any generic curve respecting the main picture of (118): the proof of this coincides with the corresponding part of the proof of Proposition 17.

Lemma 4 *The chain (119) is equal to zero.*

The first proof almost repeats that of Lemma 3, and the second refers to the triviality of the line (79), see Proposition 17.

Lemma 5 *The contents of any of two cartouches under the lines* (120), (121) *defines one and the same function on the union of varieties defined by the main pictures of these lines.*

The first proof is almost the same as for the previous two lemmas: the difference of these two functions would define the intersection of the *boundary* of our *cycle* (117)—(121) with the common boundary of varieties defined by these two main pictures. The direct proof is as follows. The sum of first four subscripts under the line (120) defines the function identically equal to 1. The sum of two last subscripts of this line and two first subscripts of the line (121) also is equal to this function. Finally, the remaining terms No. 5 under (120) and No. 3 under (121) coincide and Lemma 5 is proved.

These three lemmas imply that we can just erase both lines (118) and (119), and replace the contents of the cartouch under (120) by that under (121). Therefore, we have reduced the sum of all lines (118)—(121) to the expression given in the second row of the next formula:

$$\tag{123}$$

The first row in (123) is obviously equal to (117), and we have proved that the sum (117)—(121) is equal to the sum (123).

Lemma 6

$$\tag{124}$$

Indeed, the sum of subscripts in the left part of this formula is equal to 1 if and only if the vectors $f_1'(a_2)$, $f_1'(a_3)$ (i.e. the projections to $\mathbb{R}^2$ of derivatives of the map f at the second and the third active points of the configuration) cross the horizontal line in one and the same transversal direction. For a generic curve, this is equivalent to the condition that the segment of this curve between these two points intersects the same line an odd number of times.

Substituting this identity into (123) we complete the proof of Proposition 18.

The stupid algorithm transforms the chain (122) into the left-hand parts of two equalities

$$\tag{125}$$

$$\tag{126}$$

where the second subscript under the last picture is explained in paragraph $\bar{5}!$ of subsection 1.4. This condition is equivalent to the condition that the segment of the plane curve $f_1(\mathbb{R}^1)$ between our two active points a_1, a_2 intersects the horizontal line passing through the point $f_1(a_1) = f_1(a_2)$ an odd number of times. Another description of this condition is equal to the sum of subscripts under the second from the end picture in (125) and the third from the end picture in (126). Thus the sum of right-hand parts of equalities (125), (126) is equal to (122).

We have proved the following statement.

Proposition 19 *The chain representing the second differential $d^2(v_3) \subset F_1$ (and expressed by the formulas (80)—(88)) is equal to the homological boundary of the sum of twenty-nine -chains in F_1 shown in the left parts of equations (89)—(115), (125) and (126).*

5.4   The third differential and the chain spanning it

The third differential $d^3(v_3) \in \bar{H}_{\omega-1}(\Sigma)$ is represented by the image of the spanning chain mentioned in the previous proposition under the canonical projection $\pi : F_1 \to \Sigma$. It consists of twenty-nine   -chains, whose pictures can be obtained from the pictures in the left parts of equations (89)—(115), (125) and (126) by replacing the unique arc of any picture by a zigzag (without arrows).

By the construction, the sum of these 29 varieties is a cycle in Σ.

It remains to *span* this cycle by a relative chain in $\mathcal{K}$ (i.e. to represent it as a boundary of such a chain). The stupid algorithm provides us immediately with the following equations (127)—(140).

$$\partial \;\;=\;\; + \;\; + \;\; + \tag{127}$$

$$+ \;\; + \;\; + \;\;$$

$$+ \;\; + \;\;$$

$$\partial \;\;=\;\; + \;\; + \;\; + \tag{128}$$

$$+ \;\; + \;\; + \;\;$$

$$+ \;\; + \;\;$$

$$\partial \;\;=\;\; + \;\; + \;\; + \tag{129}$$

$$+ \;\; + \;\; + \;\;$$

$$+ \;\; + \;\;$$

∂ ⋯ $=$ ⋯ $+$ ⋯ $+$ ⋯ $+$

$+$ ⋯ $+$ ⋯ $+$ ⋯ $+$

$1 \mapsto$

$+$ ⋯ $+$ ⋯

$\begin{matrix} 2 \\ 3 \end{matrix}$

(130)

∂ ⋯ $=$ ⋯ $+$ ⋯ $+$ ⋯ $+$

$+$ ⋯ $+$ ⋯ $+$ ⋯ $+$

$+$ ⋯ $+$ ⋯

$4 \leftarrowtail$

$\begin{matrix} 1 \\ 2 \end{matrix}$

(131)

∂ ⋯ $=$ ⋯ $+$ ⋯ $+$ ⋯ $+$

$+$ ⋯ $+$ ⋯ $+$ ⋯ $+$

$3 \leftarrowtail$

$+$ ⋯ $+$ ⋯ $+$ ⋯

$\begin{matrix} 1 \\ 3 \end{matrix}$

$\begin{matrix} 2 \\ 3 \end{matrix}$

(132)

∂ ⋯ $=$ ⋯ $+$ ⋯ $+$ ⋯ $+$

$+$ ⋯ $+$ ⋯ $+$ ⋯ $+$

$2 \mapsto$

$+$ ⋯ $+$ ⋯ $+$ ⋯

$\begin{matrix} 1 \\ 3 \end{matrix}$

$\begin{matrix} 1 \\ 2 \end{matrix}$

(133)

$$(134)$$

$$(135)$$

$$(136)$$

$$(137)$$

$$(138)$$

$$(139)$$

$$(140)$$

Proposition 20 *The sum of right-hand parts of equations (127)—(140) is equal to the chain $d^2(v_3)$, i.e. to the sum of projections to Σ of varieties in F_1 encoded in the left-hand parts of equations (89)—(115), (125) and (126).*

Theorem 2 follows immediately from this proposition, because the formula (37) consists of pictures given in the left parts of these fourteen equations.

Proof of proposition. All these projections participate in the formulas (127)—(140). Namely, for any a equal to one of numbers 89, $\ldots$, 115 or 125 or 126, let us denote by $[a]$ the -chain in Σ obtained by the projection π from the -variety in F_1 encoded in the left part of the equation (a). Then we have

$$
\begin{array}{llll}
(127;1) = [91], & (127;2) = [89], & (127;3) = [90], & (128;1) = [94], \\
(128;2) = [93], & (128;3) = [92], & (129;1) = [96], & (129;2) = [95], \\
(129;3) = [97], & (130;2) = [98], & (130;3) = [99], & (131;1) = [100], \\
(131;2) = [101], & (132;1) = [102], & (132;2) = [103], & (133;1) = [104], \\
(133;3) = [105], & (134;1) = [106], & (134;2) = [107], & (135;1) = [108], \\
(135;2) = [109], & (136;1) = [110], & (136;2) = [111], & (137;1) = [112], \\
(137;2) = [113], & (138;1) = [114], & (138;2) = [115], & (139;1) = [125], \\
(140;1) = [126]. & & &
\end{array}
\tag{141}
$$

So we need only to prove that the sum of remaining terms of right-hand parts of equations (127)—(140) is equal to zero. These terms satisfy many obvious relations, which reduce very much this sum. Namely, we have

(127,4)+(127,5)+(128,7)=(130,1), (127,6)+(128,8)+(129,4)=(134,3),
(127,7)+(127,8)+(129,5)=(131,3), (128,4)+(129,6)+(129,7)=(132,3),
(128,5)+(128,6)+(129,8)=(133,2), (130,6)+(130,7)+(132,4)=(139,3),
(131,7)+(136,4)=(138,3), (132,6)+(135,4)=(137,3), (139,4)+(140,3)=(140,6),
(128,9)=(135,5), (129,9)=(136,5), (130,5)=(132,5), (130,4)=(135,3),
(131,6)=(133,6), (132,8)=(133,8), (132,9)=(136,8), (133,4)=(134,4),
(133,5)=(136,3), (133,7)=(134,5), (134,6)=(140,5), (137,6)=(138,6),
(137,7)=(138,7), (139,5)=(140,4).

The remaining summands are the following ones: (127,9), (128,10), (129,10), (130,8), (131,4), (131,5), (131,8), (132,7), (133,9), (134,7), (134,8), (135,6), (135,7), (135,8), (136,6), (136,7), (137,4), (137,5), (138,4), (138,5), (139,2), (140,2).

They can be combined as shown in following four formulas (142)—(145):

$$+ \qquad \qquad \qquad \qquad \qquad + \qquad \qquad + \qquad \qquad + \qquad \qquad + \qquad \qquad + \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad 1 \atop 3 \tag{142}$$

$$+ \qquad \qquad \qquad + \qquad \qquad \left({1 \atop 2} + {2 \atop 1} + {1 \atop 3} + {1 \atop 2} + {2 \atop 3} + \Uparrow^2 + \Uparrow^3 \right) \tag{143}$$

$$+ \qquad \qquad \qquad + \qquad \qquad \left({2 \atop 3} + {2 \atop 3} + {1 \atop 3} \right) \tag{144}$$

$$+ \qquad \qquad \qquad \left({1 \atop 2} + {1 \atop 2} + {1 \atop 2} + {1 \atop 2} + {1 \atop 3} + \Uparrow^2 + \Uparrow^3 \right) \tag{145}$$

Elements of (142) satisfy the following two relations:

$$+ \qquad \qquad + \qquad \qquad = \qquad \qquad , \tag{146}$$

$$+ \qquad \qquad = \qquad \qquad = \qquad \qquad \left({2 \atop 3} + {2 \atop 3} \right) \tag{147}$$

Therefore the sum (142) is equal to

$$\tag{148}$$

The subscript under (148) coincides with that under the line (144). Similarly to the direct proofs of Lemmas 3, 4, 5, it is easy to show that the expressions in subscripts under other two lines, (143) and (145), also are equal to the sum of these three terms. But the varieties given by main pictures (without subscripts) of formulas (143)—(145), (148) satisfy the condition

$$\tag{149}$$

Proposition 20 is completely proved, and the proof of Theorem 2 is complete.

6   Another algorithm

Suppose that a knot invariant of order k is defined by a linear combination of Polyak–Viro arrow diagrams, with $\leq k$ arrows each. The weight system of this invariant can be then easily reconstructed from these diagrams: we take the part of this linear combination consisting of diagrams with exactly k arrows, forget the orientation of arrows in any of these diagrams (so that they become chord diagrams) and additionally multiply these diagrams by certain signs 1 or -1 (if we calculate the integer-valued invariants), see [9].

This implies an easy algebraic method of computing arrow diagrams representing an invariant with a given k-weight system. Indeed, there are only $\sum_{j=0}^{k}(2j)!/j!$ arrow diagrams with $\leq k$ arrows. Consider the space of all linear combinations of these diagrams. Then we have a system of linear equations on these combinations. The first $2k!/(2^k k!)$ equations are, generally, non-homogeneous, and ensure that the weight system of our invariant is the given one. The remaining equations are homogeneous and reflect the fact that our linear combination actually is an invariant. Namely, we consider these arrow diagrams as -pictures (with all arrows replaced by oriented zigzags) and calculate the boundaries of corresponding -chains as indicated in section 2 (and in the future analog of this section "with integer coefficients"). Then the corresponding linear combination of these boundaries should vanish.

The Goussarov's theorem implies that this system of linear equations always can be resolved.

This algorithm is more elementary than the one described in the previous sections, but the systems of linear equations arising in its execution are greater (approximately by the factor 2^k).

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