ON THE PROFILE OF THE CHANGING SIGN MOUNTAIN PASS SOLUTIONS FOR AN ELLIPTIC PROBLEM

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1. Introduction

Consider

$$\begin{cases} -\varepsilon^2 \Delta u = f(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^N with smooth boundary, $\varepsilon > 0$ is a small number.

In recent years, there are a lot of results on the existence and the profile of solutions for (1.1). See for example [2, 4, 7, 8, 10, 11, 12, 18, 20, 24, 25, 27]. This problem arises from biological sciences [17, 21]. It is observed that solutions of (1.1) may exhibit sharp peaks near a certain number of points. In biology, the locations of the peaks correspond to the higher concentration places of chemicals, certain population, etc. Therefore, it is important to know the locations of the peaks of the solutions for (1.1).

In this paper, we consider a kind of nonlinearity $f(u)$, such that the mountain pass type solution for (1.1) will exhibit a new concentration phenomenon. Assume that $f(t)$ satisfies the following conditions:

(f_1) there exists $a < b$, $a < 0$, such that $f(a) = f(b) = 0$ and $f(t) < 0$ for $t \in (a, b)$;
(f_2) $\int_a^0 f(s)\,ds < 0$ if $b < 0$;
(f_3) $f \in C^1([a, +\infty)) \cap C^2((a, +\infty))$ and $f''(t) > 0$ for all $t > a$;
(f_4) There is $\alpha > 0$, such that

$$(t-a)^{1-\alpha} f''(t) \to c_0 > 0, \quad \text{as } t \to a+,$$

for some $c_0 > 0$.

(f_5) there is a $\mu > 1$, such that

$$f'(t)t \geq \mu f(t),$$

for $t > 0$ large.

(f_6) $|f(t)| \leq C(1 + |t|^{p-1})$ for some $p \in (2, 2^*)$, where $2^* = 2N/(N-2)$ if $N \geq 3$, $2^* = +\infty$ if $N = 1, 2$.

A typical example of function satisfying (f_1)–(f_6) is

$$f(t) = (t-a)^{p-1} - (t-a),$$

where $2 < p < 2^*$, $a \in \left(-\left(\frac{p}{2}\right)^{1/(p-2)}, 0\right)$.

Note that (f_1) and $f''(t) > 0$ for all $t > a$, imply $f'(a) < 0$. Moreover, from $f''(t) > 0$, we know that $f(t)$ has exactly two zero points a and b.

By [4], we know that (f_1), (f_2) and (f_3) guarantee the existence of a solution $\underline{u}_\varepsilon$, which is a local minimum of the corresponding functional of (1.1), for $\varepsilon > 0$ small, with $a \leq \underline{u}_\varepsilon \leq 0$ and $\underline{u}_\varepsilon \to a$ as $\varepsilon \to 0$ on any compact subset K of Ω. Since $\underline{u}_\varepsilon$ is a local minimum, we can expect that (1.1) has a mountain pass solution u_ε.

In this paper, we shall analyse the profile of the mountain pass type solution. Let $v_\varepsilon = u_\varepsilon - \underline{u}_\varepsilon$, then v_ε satisfies

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$$\begin{cases} -\varepsilon^2\Delta v - f'(\underline{u}_\varepsilon)v = g_\varepsilon(x,v), & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where

$$g_\varepsilon(x,t) = f(\underline{u}_\varepsilon + t) - f(\underline{u}_\varepsilon) - f'(\underline{u}_\varepsilon)t.$$

Let

$$I_\varepsilon(v) = \frac{1}{2}\int_\Omega (\varepsilon^2|Dv|^2 - f'(\underline{u}_\varepsilon)v^2) - \int_\Omega G_\varepsilon(x,v), \quad v \in H_0^1(\Omega),$$

$$G_\varepsilon(x,t) = F(\underline{u}_\varepsilon + t) - F(\underline{u}_\varepsilon) - f(\underline{u}_\varepsilon)t - \frac{1}{2}f'(\underline{u}_\varepsilon)t^2, \quad F(t) = \int_0^t f(s)ds.$$

We define

$$c_\varepsilon = \inf_{\gamma\in\Gamma}\max_{0\le t\le 1} I_\varepsilon(\gamma(t)), \tag{1.3}$$

where $\Gamma = \{\gamma(t) \in C([0,1], H_0^1(\Omega)), \gamma(0) = 0, \gamma(1) = e\}$, $e \in H_0^1(\Omega)$ is a point with $I_\varepsilon(e) < 0$.

It follows from (f_4) and (f_5) that there is $\bar{\mu}_1 > 2$, such that $\bar{\mu}_1 G_\varepsilon(x,t) \le t g_\varepsilon(x,t)$ for $t \ge 0$ (see the proof of $(g3)$ in section 3). By the mountain pass lemma of Ambrosetti and Rabinowitz [1], we know that (1.2) has a positive solution v_ε with $I_\varepsilon(v_\varepsilon) = c_\varepsilon$. Thus (1.1) has a mountain pass type solution $u_\varepsilon = \underline{u}_\varepsilon + v_\varepsilon$. For the profile of u_ε, we have

Theorem 1.1. *Let $u_\varepsilon = \underline{u}_\varepsilon + v_\varepsilon$ be a mountain pass solution of* (1.1). *There is an $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0]$, we have*

(i) *there is a $\bar{c} > 0$, which is independent of ε, such that $\max_{x\in\overline{\Omega}} u_\varepsilon(x) \ge \bar{c}$;*

(ii) *for any local maximum point x_ε of u_ε with $u_\varepsilon(x_\varepsilon) \ge c' > 0$, we have $d(x_\varepsilon, \partial\Omega) \le C\varepsilon$, and for any $\theta > 0$, $|u_\varepsilon(x) - \underline{u}_\varepsilon(x)| \le Ce^{-\nu|x-x_\varepsilon|/\varepsilon}$ if $x \in \Omega \setminus B_\theta(x_\varepsilon)$. Here $\nu > 0$ is a constant.*

(iii) *for any sequence of ε, there is a subsequence $\varepsilon_j \to 0$, such that $x_j \to x_0 \in \partial\Omega$ with $H(x_0) = \max_{x\in\partial\Omega} H(x)$, where x_j is any local maximum point of u_{ε_j} with $u_{\varepsilon_j}(x_j) \ge c' > 0$, $H(x)$ is the mean curvature of $\partial\Omega$ at x;*

(vi) *$\overline{\{x : u_{\varepsilon_j}(x) > 0\}} \cap \partial\Omega \ne \emptyset$, and for any $\theta > 0$, $\overline{\{x : u_{\varepsilon_j}(x) > 0\}} \subset B_\theta(x_0)$, where $x_0 \in \partial\Omega$ is the point in (iii).*

By Theorem 1.1, we see that (1.1) has a changing sign mountain pass type solution which has a positive peak near a global maximum point x_0 of the mean curvature of the boundary, and is negative away from a small neighbourhood of x_0.

If $f(t)$ has two zero points $b > 0$ and 0, and $f(t) < 0$ for $t \in (0,b)$, Ni and Wei [25], Del Pino and Felmer [12] proved that a positive mountain pass solution of (1.1) has a unique local maximum point which locates near the center of the domain. The same result is still true if $f(t)$ has two zero points $b > a > 0$, and $f(t) < 0$ for $t \in (a,b)$. See [8]. Our results here look very similar to those for the Neumann problem obtained [22, 23]. The main reason that the local maximum point of u_ε, is close to the boundary is that the corresponding problem in a half space possesses a mountain pass solution. See the result in Section 3. But unlike the Neumann problem, the local maximum point of u_ε is inside the domain.

Our result does not claim that u_ε has exactly one local maximum point, because, except for $N = 1$, it is not clear that the solution of the form $U(x) = \underline{u}(x_N) + u(x)$ for problem (3.11) has exactly one local maximum point. But from the proof of the main result, it is easy to see that if u_ε has two local maximum points $x_\varepsilon^{(1)}$ and $x_\varepsilon^{(2)}$, then $|x_\varepsilon^{(1)} - x_\varepsilon^{(2)}| \le C\varepsilon$.

Moreover, both $x_\varepsilon^{(1)}$ and $x_\varepsilon^{(2)}$ lie near (compare with ε) the normal direction of $\partial\Omega$ at $\bar{x}_\varepsilon$, where $\bar{x}_\varepsilon \in \partial\Omega$ with $|\bar{x}_\varepsilon - x_\varepsilon^{(1)}| = d(x_\varepsilon^{(1)}, \partial\Omega)$.

We call a solution u of (3.1) is nondegenerate if the kernel $KerL$ of the linear operator L defined as

$$Lw =: -\Delta w - f'(\underline{u}(x_N))w - \frac{\partial g_1(x_N, u)}{\partial u} w, \quad w \in H_0^1(R_+^N),$$

satisfies $KerL = span\{\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_{N-1}}\}$. It is an open problem whether the mountain pass solution of (3.1) is nondegenerate. If we can prove that (3.1) has a positive solution which is nondegenerate, then using the reduction method, we can easily construct various kinds of boundary peak solutions for (1.2) as in [3, 9, 16, 20]. On the other hand, we know that for certain nonlinearities $f(t)$, such as $f(t) = (t-a)^{p-1} - (t-a)$, the positive solution of

$$-\Delta u = f(u+a), \quad u \in H_0^1(R^N)$$

is unique and nondegenerate. So we can prove the existence of positive interior peak solutions for (1.2) and find lower estimate for the number of such solutions as in [6, 11, 15, 28]. We shall discuss this problem briefly in this paper. Here we stress that it is the nondegeneracy not the uniqueness which is important to us. Although the results on existence of positive interior peak solutions for (1.2) look similar to those for the Neumann problem [15, 28], the locations of the peaks of the positive interior peak solutions for these two problems are different.

It is worth pointing out that by the moving plane method of Gidas, Ni and Nirenberg [13], if Ω is convex, the distance of any local maximum point of a positive solution for (1.1) to the boundary of Ω has a positive lower bound which is independent of the nonlinearities and the solutions. So it is only possible for a changing sign solution to have a positive local maximum point close to the boundary of a convex domain. Therefore, the assumption $a < 0$ is essential in this paper.

This paper is organized as follows. In section 2, we obtain an asymptotic expansion of the local minimum solution $\underline{u}_\varepsilon$ near the boundary of Ω. The estimates in section 2 is essential to the proof of the main results of this paper. In section 3, we study the existence of a mountain pass solution for an elliptic problem on half space, which corresponds to the limit problem when we blow up (1.2) at a boundary point of Ω. ¿From this mountain pass solution, we can construct an approximate solution for (1.2) and thus obtain an upper bound for c_ε. Section 4 is devoted to the proof of Theorem 1.1. In section 5, we discuss briefly the existence of interior peak solutions for (1.2). Appendix A contains a decay estimate of any positive solution of (3.1).

2. THE EXPANSION OF THE LOCAL MINIMUM NEAR THE BOUNDARY

Let $\underline{u}_\varepsilon$ be the solution of (1.1) with $a \le \underline{u}_\varepsilon \le 0$ and $\underline{u}_\varepsilon \to a$ in any compact subset of Ω as $\varepsilon \to 0$. In this section, we shall obtain an asymptotic expansion for $\underline{u}_\varepsilon$ near the boundary of Ω.

Let $\underline{u}(t)$ be the solution of

$$\begin{cases} -u'' = f(u), & t \ge 0, \\ a \le u(t) \le 0, & t \ge 0, \\ u(0) = 0, & u(+\infty) = a. \end{cases} \tag{2.1}$$

Then $\underline{u}(t)$ is decreasing and $|\underline{u}(t) - a| \le Ce^{-\sqrt{-f'(a)}t}$.

Lemma 2.1. *We have that there is a $c_0 > 0$, such that*

$$\int_0^{+\infty} (|v'|^2 - f'(\underline{u})v^2) \ge c_0 \|v\|^2_{H_0^1(R_+^1)},$$

for any $v \in H_0^1(R_+^1)$.

Proof. First, the proof of Proposition 2 in [5] shows that $-y'' - f'(\underline{u})y$ with zero Dirichlet boundary condition has no non-positive eigenvalue and this operator is seen to be Fredholm at points of the spectrum less than $-f'(a)$. Hence we see that

$$\inf\Big\{\int_0^{+\infty}(|v'|^2 - f'(\underline{u})v^2) : v \in H_0^1(R_+^1), \int_0^{+\infty} v^2 = 1\Big\} = \lambda > 0. \tag{2.2}$$

Thus,

$$\int_0^{+\infty}(|v'|^2 - f'(\underline{u})v^2) \geq \lambda \int_0^{+\infty} v^2.$$

So if $c_0 > 0$ is small enough, we see

$$\begin{aligned}
&\int_0^{+\infty}(|v'|^2 - f'(\underline{u})v^2)\\
=&c_0\int_0^{+\infty}|v'|^2 + (1-c_0)\int_0^{+\infty}|v'|^2 - \int_0^{+\infty} f'(\underline{u})v^2\\
\geq&c_0\int_0^{+\infty}|v'|^2 + (1-c_0)(\lambda\int_0^{+\infty}v^2 + \int_0^{+\infty}f'(\underline{u})v^2) - \int_0^{+\infty}f'(\underline{u})v^2\\
=&c_0\int_0^{+\infty}|v'|^2 + (1-c_0)\lambda\int_0^{+\infty}v^2 - c_0\int_0^{+\infty}f'(\underline{u})v^2\\
\geq&c_0\int_0^{+\infty}|v'|^2 + ((1-c_0)\lambda - Cc_0)\int_0^{+\infty}v^2\\
\geq&c_0'||v||^2_{H_0^1(R_+^1)}.
\end{aligned}$$

□

Lemma 2.2. *Let* v_ε *be a solution of*

$$\begin{cases} -\varepsilon^2\Delta v - f'\Big(\underline{u}\Big(\frac{d(x,\partial\Omega)}{\varepsilon}\Big)\Big)v = h, & \text{in } \Omega,\\ v = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Then $|v_\varepsilon|_\infty \leq C|h|_\infty$ *for some* $C > 0$, *independent of* ε, h *and* v_ε.

Proof. We argue by contradiction. Suppose that there are $\varepsilon_m \to 0$, h_m, such that the solution v_m of (2.3) satisfies $|v_m|_\infty \geq m|h_m|_\infty$. Without loss of generality, we assume $|v_m|_\infty = 1$. Then, $|h_m|_\infty \to 0$ as $m \to +\infty$. Let $x_m \in \Omega$ be such that $|v_m(x_m)| = |v_m|_\infty$. Let $u_m(y) = v_m(\varepsilon y + x_m)$, $\varepsilon y + x_m \in \Omega$.

If $\frac{d(x_m,\partial\Omega)}{\varepsilon_m} \to +\infty$, then we see that

$$-\Delta u - f'(a)u = 0, \quad \text{in } R^N,$$

has a nontrivial bounded solution. By averaging over the unit sphere, we see that $-u'' - \frac{N-1}{r}u' - f'(a)u = 0$ has a bounded nontrivial solution in R_+^1, which is impossible since $f'(a) < 0$.

If $\frac{d(x_m,\partial\Omega)}{\varepsilon_m} \leq C$ for some $C > 0$, then

$$-\Delta u - f'(\underline{u})u = 0, \quad \text{in } R_+^N,$$

has a nontrivial bounded solution u with $u|_{x_N=0} = 0$. This is impossible by Proposition 2 of [5].

□

Let $\psi(t)$ be the solution of

$$\begin{cases} -\psi'' - f'(\underline{u}(t))\psi = -\underline{u}', & t > 0 \\ \psi(0) = \psi(+\infty) = 0. \end{cases} \tag{2.4}$$

The existence of such solution is guaranteed by Lemma 2.1. Then $\psi(t) > 0$ if $t > 0$ by the positivity of the operator and $\underline{u}' < 0$.

Now we are ready to obtain an asymptotic expansion for $\underline{u}_\varepsilon$.

Proposition 2.3. *Let $\underline{u}_\varepsilon$ be the local minimum near a, then*

$$\underline{u}_\varepsilon(x) = \underline{u}\Big(\frac{d(x,\partial\Omega)}{\varepsilon}\Big) + \varepsilon(N-1)\psi\Big(\frac{d(x,\partial\Omega)}{\varepsilon}\Big)H(\bar{x}) + O(\varepsilon^{1+\sigma}),$$

for some $\sigma > 0$, where $\underline{u}$ and ψ are defined by (2.1) and (2.4) respectively, $\bar{x} \in \partial\Omega$ is a point such that $|x - \bar{x}| = d(x, \partial\Omega)$.

Proof. Define

$$\underline{u}_\varepsilon^* = \begin{cases} \underline{u}(t), & t \in [0, \delta/\varepsilon], \\ \text{smooth}, & t \in [\delta/\varepsilon, 2\delta/\varepsilon], \\ \underline{u}(2\delta/\varepsilon), & t \in [2\delta/\varepsilon, +\infty). \end{cases} \tag{2.5}$$

In $[\delta/\varepsilon, 2\delta/\varepsilon]$, we can choose $\underline{u}_\varepsilon^*$ such that $a \le \underline{u}_\varepsilon^*(t) \le 0$, $|\underline{u}_\varepsilon^*(t) - a| \le Ce^{-\sqrt{-f'(a)}\delta/\varepsilon}$, $|D^i\underline{u}_\varepsilon^*(t)| \le C\varepsilon^{-i}e^{-\sqrt{-f'(a)}\delta/\varepsilon}$, $i = 1, 2$.

Let $\underline{u}_\varepsilon^{**}(x) = u_\varepsilon^*\big(\frac{d(x,\partial\Omega)}{\varepsilon}\big)$. Then $\underline{u}_\varepsilon^{**}(x)$ is constant for $d(x, \partial\Omega) \ge 2\delta$. By a simple calculation, using the equation satisfied by $\underline{u}(t)$, we see that

$$\begin{aligned} -\varepsilon^2\Delta\underline{u}_\varepsilon^{**} &= -\underline{u}_\varepsilon^{*''}\Big(\frac{d(x,\partial\Omega)}{\varepsilon}\Big) - \varepsilon\underline{u}_\varepsilon^{*'}\Big(\frac{d(x,\partial\Omega)}{\varepsilon}\Big)\Delta d(x,\partial\Omega) \\ &= \begin{cases} f\Big(\underline{u}_\varepsilon^*\big(\frac{d(x,\partial\Omega)}{\varepsilon}\big)\Big) - \varepsilon\underline{u}_\varepsilon^{*'}\big(\frac{d(x,\partial\Omega)}{\varepsilon}\big)\Delta d(x,\partial\Omega), & \text{if } d(x,\partial\Omega) \le \delta, \\ O(\varepsilon^{-2}e^{-\delta'/\varepsilon}) - \varepsilon\underline{u}_\varepsilon^{*'}\big(\frac{d(x,\partial\Omega)}{\varepsilon}\big)\Delta d(x,\partial\Omega), & \text{if } \delta \le d(x,\partial\Omega) \le 2\delta, \\ -\varepsilon\underline{u}_\varepsilon^{*'}\big(\frac{d(x,\partial\Omega)}{\varepsilon}\big)\Delta d(x,\partial\Omega), & \text{if } d(x,\partial\Omega) \ge 2\delta, \end{cases} \\ &= f(\underline{u}_\varepsilon^{**}) - \varepsilon\underline{u}_\varepsilon^{*'}\Big(\frac{d(x,\partial\Omega)}{\varepsilon}\Big)\Delta d(x,\partial\Omega) + O\Big(\varepsilon^{-2}e^{-\delta''/\varepsilon}\Big), \end{aligned}$$

since for $d(x, \partial\Omega) \ge \delta$,

$$f(\underline{u}_\varepsilon^{**}) = f(a) + O(|\underline{u}_\varepsilon^{**} - a|) = O\Big(\varepsilon^{-2}e^{-\delta''/\varepsilon}\Big).$$

Let $\xi_\varepsilon = \underline{u}_\varepsilon^{**} - \underline{u}_\varepsilon$. Then

$$\begin{aligned} -\varepsilon^2\Delta\xi_\varepsilon &= f(\underline{u}_\varepsilon^{**}) - f(\underline{u}_\varepsilon) - \varepsilon\underline{u}_\varepsilon^{*'}\Big(\frac{d(x,\partial\Omega)}{\varepsilon}\Big)\Delta d(x,\partial\Omega) + O\big(\varepsilon^{-2}e^{-\delta'/\varepsilon}\big) \\ &= -c_\varepsilon(x)\xi_\varepsilon - \varepsilon\underline{u}_\varepsilon^{*'}\Big(\frac{d(x,\partial\Omega)}{\varepsilon}\Big)\Delta d(x,\partial\Omega) + O\big(\varepsilon^{-2}e^{-\delta'/\varepsilon}\big), \end{aligned}$$

where $c_\varepsilon = -\int_0^1 f'(\tau\underline{u}_\varepsilon^{**} + (1-\tau)\underline{u}_\varepsilon)\, d\tau$.

Let $\xi_\varepsilon = \varepsilon\psi_\varepsilon$. Then

$$\begin{cases} -\varepsilon^2\Delta\psi_\varepsilon + c_\varepsilon\psi_\varepsilon = -\underline{u}_\varepsilon^{*'}\big(\frac{d(x,\partial\Omega)}{\varepsilon}\big)\Delta d(x,\partial\Omega) + O\big(\varepsilon^{-3}e^{-\delta'/\varepsilon}\big), & \text{in } \Omega, \\ \psi_\varepsilon = 0, & \text{on } \partial\Omega. \end{cases}$$

Similar to the proof of Lemma 2.2, it is not difficult to prove that $|\psi_\varepsilon|_\infty \le C$ (Here we need to use Proposition 2.4 in [4] and Theorem 2 in [26] to prove that $c_\varepsilon(\varepsilon y + x_\varepsilon) \to -f'(\underline{u}(x_N))$ as $\varepsilon \to 0$ if $d(x_\varepsilon, \partial\Omega) \le C\varepsilon$.).

Let

$$\psi_\varepsilon^*(t) = \begin{cases} \psi(t), & \text{if } t \in [0, \delta/\varepsilon], \\ smooth, & \text{if } t \in [\delta/\varepsilon, 2\delta/\varepsilon], \\ 0, & \text{if } t \in [2\delta/\varepsilon, +\infty). \end{cases}$$

For any x with $d(x, \partial\Omega) \leq \delta$, let $\bar{x}$ be the unique point on $\partial\Omega$ such that $|x - \bar{x}| = d(x, \partial\Omega)$. Let

$$\psi_\varepsilon^{**}(x) = -(N-1)\psi_\varepsilon^*\Big(\frac{d(x,\partial\Omega)}{\varepsilon}\Big)H(\bar{x}),$$

where $H(\bar{x})$ is the mean curvature of $\partial\Omega$ at $\bar{x}$. Noting that

$$\Delta\psi_\varepsilon^* = \varepsilon^{-2}\psi_\varepsilon^{*''} + \varepsilon^{-1}\Delta d(x,\partial\Omega)\psi_\varepsilon^{*'} = \varepsilon^{-2}\psi_\varepsilon^{*''} + O(\varepsilon^{-1}),$$

we see

$$\begin{aligned} -\varepsilon^2\Delta\psi_\varepsilon^{**} =& \varepsilon^2(N-1)\Big(H(\bar{x})\Delta\psi_\varepsilon^* + 2\varepsilon^{-1}\psi_\varepsilon^{*'}\langle Dd(x,\partial\Omega), DH(\bar{x})\rangle + \psi_\varepsilon^*\Delta H(\bar{x})\Big) \\ =&(N-1)H(\bar{x})\psi_\varepsilon^{*''} + O(\varepsilon) \\ =&(N-1)\Big(-f'\Big(\underline{u}\Big(\frac{d(x,\partial\Omega)}{\varepsilon}\Big)\Big)\psi_\varepsilon^* + \underline{u}'\Big(\frac{d(x,\partial\Omega)}{\varepsilon}\Big)\Big)H(\bar{x}) + O(\varepsilon) \\ =&f'\Big(\underline{u}\Big(\frac{d(x,\partial\Omega)}{\varepsilon}\Big)\Big)\psi_\varepsilon^{**} + (N-1)\underline{u}'\Big(\frac{d(x,\partial\Omega)}{\varepsilon}\Big)H(\bar{x}) + O(\varepsilon). \end{aligned}$$

Thus,

$$\begin{aligned} &-\varepsilon^2\Delta(\psi_\varepsilon^{**} - \psi_\varepsilon) - f'\Big(\underline{u}\Big(\frac{d(x,\partial\Omega)}{\varepsilon}\Big)\Big)(\psi_\varepsilon^{**} - \psi_\varepsilon) \\ =&\Big(f'\Big(\underline{u}\Big(\frac{d(x,\partial\Omega)}{\varepsilon}\Big)\Big) - c_\varepsilon\Big)\psi_\varepsilon + \underline{u}'\Big(\frac{d(x,\partial\Omega)}{\varepsilon}\Big)((N-1)H(\bar{x}) + \Delta d(x,\partial\Omega)) + O(\varepsilon) \\ =&O(\varepsilon^\sigma), \end{aligned}$$

since

$$\begin{aligned} &f'\Big(\underline{u}\Big(\frac{d(x,\partial\Omega)}{\varepsilon}\Big)\Big) - c_\varepsilon \\ =&\int_0^1 \Big(f'\Big(\underline{u}\Big(\frac{d(x,\partial\Omega)}{\varepsilon}\Big)\Big) - f'\Big(\underline{u}_\varepsilon + \tau(u_\varepsilon^{**} - \underline{u}_\varepsilon)\Big)\Big)\,d\tau \\ =&O\Big(\Big|f'\Big(\underline{u}\Big(\frac{d(x,\partial\Omega)}{\varepsilon}\Big)\Big) - f'\Big(\underline{u}_\varepsilon + O(\varepsilon)\Big)\Big|\Big) = O(\varepsilon^\alpha) \end{aligned}$$

and

$$\begin{aligned} &\underline{u}'\Big(\frac{d(x,\partial\Omega)}{\varepsilon}\Big)\Big(\Delta d(x,\partial\Omega) + (N-1)H(\bar{x})\Big) \\ =&O\Big(\underline{u}'\Big(\frac{d(x,\partial\Omega)}{\varepsilon}\Big)|x-\bar{x}|\Big) = O(\varepsilon^{1-\tau}), \quad \text{for any } \tau > 0. \end{aligned}$$

Here, in the last equality, we have used Lemma 14.17 in [14]. By Lemma 2.2, we obtain

$$|\psi_\varepsilon^{**} - \psi_\varepsilon| = O(\varepsilon^\sigma),$$

for some $\sigma > 0$.

□

3. Existence of mountain pass solution in half space

Let $\underline{u}(t)$ be a solution of (2.1). Consider

$$\begin{cases} -\Delta u - f'(\underline{u}(x_N))u = g_1(x_N, u), & \text{in } R_+^N, \\ u \in H_0^1(R_+^N), \end{cases} \tag{3.1}$$

where $g_1(x_N, t) = f(\underline{u}(x_N) + t) - f(\underline{u}(x_N)) - f'(\underline{u}(x_N))t$.

Define

$$J(u) = \frac{1}{2}\int_{R_+^N} (|Du|^2 - f'(\underline{u}(x_N))u^2) - \int_{R_+^N} G_1(x_N, u),$$

where

$$\begin{aligned} G_1(x_N, t) = & \int_0^t g_1(x_N, \tau)\, d\tau \\ = & F(\underline{u}(x_N) + t) - F(\underline{u}(x_N)) - f(\underline{u}(x_N))t - \frac{1}{2}f'(\underline{u}(x_N))t^2, \end{aligned}$$

and $F(t) = \int_0^t f(\tau)\, d\tau$.

We summarize the properties of $g_1(x_N, t)$ as follows:

$(g1)$ $g_1(x_N, t) \geq 0$ for $t \geq 0$.

In fact, by (f_3), we have

$$g_1(x_N, 0) = 0, \quad \frac{\partial}{\partial t} g_1(x_N, 0) = 0,$$

$$\frac{\partial^2}{\partial t^2} g_1(x_N, t) = f''(\underline{u}(x_N) + t) > 0.$$

$(g2)$ $\frac{g_1(x_N,t)}{t}$ is strictly increasing in $t > 0$.

In fact, we have

$$\frac{\partial}{\partial t}\Big(\frac{g_1(x_N, t)}{t}\Big) = \big(t\frac{\partial}{\partial t} g_1(x_N, t) - g_1(x_N, t)\big)/t^2.$$

But

$$\big(t\frac{\partial}{\partial t} g_1(x_N, t) - g_1(x_N, t)\big)\Big|_{t=0} = 0,$$

$$\frac{\partial}{\partial t}\big(t\frac{\partial}{\partial t} g_1(x_N, t) - g_1(x_N, t)\big) = t\frac{\partial^2}{\partial t^2} g_1(x_N, t) = tf''(\underline{u}(x_N) + t) > 0,$$

for all $t > 0$. Thus $t\frac{\partial}{\partial t} g_1(x_N, t) - g_1(x_N, t) > 0$ for all $t > 0$.

$(g3)$ There is a constant $\bar{\mu} > 1$, such that

$$\bar{\mu} g_1(x_N, t) \leq t\frac{\partial g_1(x_N, t)}{\partial t}, \quad \forall\, t \geq 0. \tag{3.2}$$

First, by (f_5), we see that there is a large $T > 0$ such that (3.2) holds for $t \geq T$.

Next, we claim that there is a $\theta > 0$, such that

$$tf''(\tau + t) \geq \theta\big(f'(\tau + t) - f'(\tau)\big), \quad \forall\, t \in (0, T],\ a \leq \tau \leq 0. \tag{3.3}$$

It follows from $f''(t) > 0$ that (3.3) holds if $t \geq t_0 > 0$ or $\tau - a > \tau_0 > 0$, where t_0 and τ_0 are fixed numbers. Thus it remains to prove that (3.3) holds for $t \in (0, t_0]$ and $\tau \in (a, a + \tau_0]$.

If $t \le \tau - a$, then it follows from (f_4) that

$$\begin{aligned} &tf''(\tau+t) - \theta\big(f'(\tau+t) - f'(\tau)\big) = t\big(f''(\tau+t) - \theta f''(\tau+\eta t)\big) \\ \ge& t\Big(\frac{c_0'}{(t+\tau-a)^{1-\alpha}} - \frac{\theta c_0''}{(\eta t+\tau-a)^{1-\alpha}}\Big) \ge t\Big(\frac{c_0'}{(t+\tau-a)^{1-\alpha}} - \frac{\theta c_0''}{(\tau-a)^{1-\alpha}}\Big) \\ \ge& t\Big(\frac{c_0'}{2^{1-\alpha}(\tau-a)^{1-\alpha}} - \frac{\theta c_0''}{(\tau-a)^{1-\alpha}}\Big) > 0 \end{aligned}$$

if $\theta > 0$ is small enough.

Suppose that $\tau - a \le t$. From (f_4), we can deduce

$$|f'(t_1) - f'(t_2)| \le C|t_1 - t_2|^\alpha, \quad t_1, t_2 \in [a, a+\eta].$$

Thus,

$$\begin{aligned} &tf''(\tau+t) - \theta\big(f'(\tau+t) - f'(\tau)\big) \ge tf''(\tau+t) - C\theta t^\alpha \\ \ge& t\Big(\frac{c_0'}{(t+\tau-a)^{1-\alpha}} - \frac{\theta C}{t^{1-\alpha}}\Big) \ge t\Big(\frac{c_0'}{2^{1-\alpha}t^{1-\alpha}} - \frac{\theta C}{t^{1-\alpha}}\Big) > 0 \end{aligned}$$

Thus we have proved (3.3).

Now take $\bar\mu = 1 + \theta > 1$. Let

$$\eta(x_N, t) = t\frac{\partial g_1(x_N, t)}{\partial t} - \bar\mu g_1(x_N, t).$$

Then $\eta(x_N, 0) = 0$. We have

$$\begin{aligned} \frac{\partial}{\partial t}\eta(x_N, t) =& t\frac{\partial^2}{\partial t^2} g_1(x_N, t) + (1-\bar\mu)\frac{\partial}{\partial t} g_1(x_N, t) \\ =& tf''(\underline{u}(x_N) + t) + (1-\bar\mu)\big(f'(\underline{u}(x_N) + t) - f'(\underline{u}(x_N))\big) > 0, \end{aligned}$$

if $t \in (0, T]$. As a result, $\eta(x_N, t) > 0$ for $t \in (0, T]$.

$(g4)$ We have $f_1(x_N, t) = O(t^2)$ as $t \to 0$, and $|f_1(x_N, t)| \le C(|t|^{p-1} + 1)$.

Define

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(u), \tag{3.4}$$

where $\Gamma = \{\gamma(t) \in C([0,1], H_0^1(R_+^N)), \gamma(0) = 0, \gamma(1) = e\}$, $e \in H_0^1(R_+^N)$ with $J(e) < 0$. By $(g2)$, it is easy to check that c is independent of e.

It is easy to see that (3.1) is translation invariant in the x_i direction, $i = 1, \cdots, N-1$. Since $\underline{u}(t) \to a$ as $t \to +\infty$, we see that the corresponding limit problem in the x_N direction is

$$\begin{cases} -\Delta w = f(w + a), & \text{in } R^N, \\ w \in H^1(R^N). \end{cases} \tag{3.5}$$

Let

$$A = \frac{1}{2}\int_{R^N} |Dw|^2 - \int_{R^N}\Big(\int_0^w f(\tau + a)\, d\tau\Big),$$

where w is the least energy solution of (3.5).

Using the standard concentration compactness argument [19], we can prove that (3.1) has a solution with critical value c once we prove the following lemma:

Lemma 3.1. *We have $c < A$.*

Proof. Let $x_l = (0, \cdots, 0, l)$, $w_l(x) = w(x - x_l)$. Let Pw_l be the solution of

$$\begin{cases} -\Delta v - f'(a)v = f(w_l + a) - f'(a)w_l, & \text{in } R^N, \\ v \in H_0^1(R_+^N). \end{cases} \tag{3.6}$$

By the maximum principle, we have $|Pw_l - w_l| \le \max_{x_N=0} w_l \le Ce^{-\sqrt{-f'(a)}l}$. By the definition of c (see (3.4), we have

$$c \le \max_{t\ge 0} J(tPw_l).$$

Now we estimate $J(tPw_l)$.

Step 1. The estimate of $J(Pw_l)$.

Denote $\bar{g}(t) = f(t+a) - f'(a)t$,

$$\bar{G}(t) = \int_0^t \bar{g}(\tau)\, d\tau = F(t+a) - F(a) - \frac{1}{2}f'(a)t^2.$$

Then

$$\begin{aligned}
J(Pw_l) =& \frac{1}{2}\int_{R^N_+} |DPw_l|^2 - f'(\underline{u}(x_N))|Pw_l|^2 - \int_{R^N_+} G_1(x_N, Pw_l)\\
=& \frac{1}{2}\int_{R^N_+} \Big(f'(a) - f'(\underline{u}(x_N))\Big)|Pw_l|^2 + \int_{R^N_+}\Big(\frac{1}{2}\bar{g}(w_l)Pw_l - G_1(x_N, Pw_l)\Big)\\
=& \frac{1}{2}\int_{R^N_+} \Big(f'(a) - f'(\underline{u}(x_N))\Big)|Pw_l|^2 + \int_{R^N_+}\Big(\frac{1}{2}\bar{g}(w_l)Pw_l - \bar{G}(Pw_l)\Big)\\
&+ \int_{R^N_+}\Big(\bar{G}(Pw_l) - G_1(x_N, Pw_l)\Big)\\
=& \int_{R^N_+}\Big(\frac{1}{2}\bar{g}(w_l)Pw_l - \bar{G}(Pw_l)\Big)\\
&+ \int_{R^N_+}\Big(F(a+Pw_l) - F(a) - \big(F(\underline{u}(x_N) + Pw_l) - F(\underline{u}(x_N)) - f(\underline{u}(x_N)Pw_l)\Big)\\
=:& J_1 + J_2,
\end{aligned} \tag{3.7}$$

where J_1 and J_2 are defined by the last equality of the above relation.

For J_1, we have

$$\begin{aligned}
J_1 =& \int_{R^N_+}\Big(\frac{1}{2}\bar{g}(w_l)w_l - \bar{G}(w_l)\Big) + \frac{1}{2}\int_{R^N_+} \bar{g}(w_l)(w_l - Pw_l)\\
&+ O(e^{-2\sqrt{-f'(a)}l})\\
=& \int_{R^N}\Big(\frac{1}{2}\bar{g}(w_l)w_l - \bar{G}(w_l)\Big) + O(e^{-(1+\sigma)\sqrt{-f'(a)}l})\\
&+ \frac{1}{2}\int_{R^N_+}(-\Delta w_l - f'(a)w_l)(w_l - Pw_l) + O(e^{-2\sqrt{-f'(a)}l})\\
=& A + \frac{1}{2}\int_{x_N=0}\Big(\frac{\partial w_l}{\partial x_N}(w_l - Pw_l) - w_l\frac{\partial(w_l - Pw_l)}{\partial x_N}\Big)\\
&+ O(e^{-(1+\sigma)\sqrt{-f'(a)}l})\\
=& A + O(e^{-(1+\sigma)\sqrt{-f'(a)}l}).
\end{aligned} \tag{3.8}$$

Now we estimate J_2. Let

$$\begin{aligned}
K(t_1, t_2) =& F(a + t_1 + t_2) - F(a + t_1) - f(a + t_1)t_2\\
&- (F(a + t_2) - F(a) - f(a)t_2)\\
&- (f(a + t_2)t_1 - f(a)t_1 - f'(a)t_1t_2).
\end{aligned}$$

We claim that there are $C > 0$ and $\sigma > 0$, such that

$$|K(t_1, t_2)| \le C|t_1|^{1+\sigma}|t_2|^{1+\sigma}.$$

In fact, if $|t_2| \le |t_1|$, then

$$
\begin{aligned}
K(t_1,t_2) =&\Big[\big(f(a+t_2+\xi t_1)-f(a+\xi t_1)-f'(a+\xi t_1)t_2\big)\\
&-\big(f(a+t_2)-f(a)-f'(a)t_2\big)\Big]t_1\\
=&\Big[\big(f'(a+\xi_1 t_2+\xi t_1)-f'(a+\xi t_1)\big)-\big(f'(a+\xi_1 t_2)-f'(a)\big)\Big]t_1t_2.
\end{aligned}
$$

By $(f4)$, we see $|f'(t_1)-f'(t_2)| \le C|t_1-t_2|^\alpha$. As a result,

$$
|K(t_1,t_2)| \le C|t_2|^\alpha|t_1t_2| \le C|t_1|^{1+\alpha/2}|t_2|^{1+\alpha/2}.
$$

If $|t_1| \le |t_2|$, then

$$
\begin{aligned}
K(t_1,t_2) =&\frac{1}{2}\big(f'(a+t_2+\xi t_1)-f'(a+\xi t_1)\big)t_1^2\\
&-\big(f'(a+\xi t_1)-f'(a)\big)t_1t_2.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
K(t_1,t_2) =&C|t_2|^\alpha t_1^2 + C|t_1|^\alpha|t_1t_2|\\
\le& C|t_1|^{1+\sigma}|t_2|^{1+\sigma}.
\end{aligned}
$$

We have

$$
\begin{aligned}
J_2 =& -\int_{R^N_+}\Big(f(a+Pw_l)-f(a)-f'(a)Pw_l\Big)(\underline{u}(x_N)-a) - \int_{R^N_+} K(\underline{u}(x_N)-a, Pw_l)\\
=& -\int_{R^N_+}\Big(f(a+Pw_l)-f(a)-f'(a)Pw_l\Big)(\underline{u}(x_N)-a) + O\Big(\int_{R^N_+}|\underline{u}(x_N)-a|^{1+\sigma}|Pw_l|^{1+\sigma}\Big)\\
=& -\int_{R^N_+}\Big(f(a+Pw_l)-f(a)-f'(a)Pw_l\Big)(\underline{u}(x_N)-a)\\
&+O\Big(\int_{R^N_+} e^{-(1+\sigma)\sqrt{-f'(a)}x_N}e^{-(1+\sigma)\sqrt{-f'(a)}|x-x_l|}\Big)\\
=& -\int_{R^N_+}\Big(f(a+Pw_l)-f(a)-f'(a)Pw_l\Big)(\underline{u}(x_N)-a) + O\Big(e^{-(1+\sigma)\sqrt{-f'(a)}l}\Big).
\end{aligned}
$$

So we have

$$
\begin{aligned}
J(Pw_l) =&A - \int_{R^N_+}\Big(f(a+Pw_l)-f(a)-f'(a)Pw_l\Big)(\underline{u}(x_N)-a)\\
&+O\Big(e^{-(1+\sigma)\sqrt{-f'(a)}l}\Big).
\end{aligned}
$$

Using the convexity of f, we see

$$
f(a+t)-f'(a)t > 0, \quad \forall\, t>0.
$$

Noting that $\underline{u}(x_N)-a>0$, we obtain

$$
\begin{aligned}
&\int_{R^N_+}\Big(f(a+Pw_l)-f(a)-f'(a)Pw_l\Big)(\underline{u}(x_N)-a)\\
\ge& \int_{B_1(x_l)}\Big(f(a+Pw_l)-f(a)-f'(a)Pw_l\Big)(\underline{u}(x_N)-a)\\
\ge& c_0 e^{-\sqrt{-f'(a)}l}
\end{aligned}
$$

for some $c_0>0$. So we have proved $J(Pw_l) < A$.

Step 2. The estimate of $\max_{t\ge0} J(tPw_l) - J(Pw_l)$.

Let t_l be the maximum point of $\max_{t\ge 0} J(tPw_l)$. Then

$$
\big\langle J'(t_lPw_l), Pw_l\big\rangle = 0. \tag{3.9}
$$

We claim that $t_l \to 1$ as $l \to +\infty$.

In fact, from $g_1(x_N,t) = O(t^2)$ as $t \to 0$, and $\frac{g_1(x_N,t)}{t} \to +\infty$ as $t \to +\infty$, we see $0 < t_0 \le t_l \le T$, $\forall\, l$. Suppose that $t_l \to t_\infty$. Letting $l \to +\infty$ in (3.9), we see

$$t_\infty \int_{R^N} |Dw|^2 = \int_{R^N} f(a + t_\infty w)w. \tag{3.10}$$

It is easy to see that from $(g2)$ there is exactly one $t > 0$ satisfies (3.10). On the other hand, $t = 1$ satisfies (3.10). Thus, $t_\infty = 1$.

But

$$\begin{aligned}\langle J'(Pw_l), Pw_l\rangle =& \int_{R^N_+} f'(a)Pw_l(Pw_l - w_l) + \int_{R^N_+} \big(f(\underline{u}(x_N)) - f(a)\big)Pw_l \\ &+ \int_{R^N_+} \big(f(w_l + a) - f(Pw_l + \underline{u}(x_N))\big)Pw_l = O(e^{-\sqrt{-f'(a)}l}),\end{aligned}$$

$$\begin{aligned}&J''(Pw_l)(Pw_l, Pw_l) \\ =& \int_{R^N_+} (|DPw_l|^2 - f'(\underline{u})|Pw_l|^2) - \int_{R^N_+} \frac{\partial g_1(x_N, Pw_l)}{\partial t}|Pw_l|^2 \\ \le& -c_0 < 0\end{aligned}$$

since $t^2 \frac{\partial g_1(x_N,t)}{\partial t} - tg_1(x_N,t) \ge (\bar{\mu} - 1)tg_1(x_N,t)$. Thus we see $t_l - 1 = O(e^{-\sqrt{-f'(a)}l})$. As a result,

$$\begin{aligned}J(t_lPw_l) - J(Pw_l) =&(t_l - 1)\langle J'(Pw_l), Pw_l\rangle + O(|t_l - 1|^2) \\ =&O(e^{-2\sqrt{-f'(a)}l}).\end{aligned}$$

Combining Step 1 and Step 2, we obtain

$$\begin{aligned}\max_{t\ge 0} J(tPw_l) =&J(t_lPw_l) = J(Pw_l) + O(e^{-2\sqrt{-f'(a)}l}) \\ \le& A - c_o e^{-\sqrt{-f'(a)}l} + O(e^{-(1+\sigma)\sqrt{-f'(a)}l}) < A.\end{aligned}$$

□

Using Lemma 3.1, we have

Theorem 3.2. *Problem* (3.1) *has a least energy solution u with $J(u) = c$.*

Proof. Let $u_m \in H^1_0(R^N_+)$ be a sequence with $J(u_m) \to c$ and $J'(u_m) \to 0$ as $m \to +\infty$. By Lemma 2.1, we see that

$$\int_{R^N_+} (|Du_m|^2 - f'(\underline{u})u_m^2) \ge c_0\|u_m\|^2$$

for some $c_0 > 0$. Then by $(g3)$, we see that u_m is bounded in $H^1_0(R^N_+)$. Using the concentration compactness argument [19] and Lemma 3.1, we can deduce that there are $x_m \in \{x_N = 0\}$, $m = 1, \cdots$, such that $u_m(x + x_m)$ is compact in $H^1_0(R^N_+)$. As a result, (3.1) has a solution with $J(u) = c$. On the other hand, it is easy to check that for any nontrivial solution v of (3.1), $J(v) \ge c$. So u is the least energy solution.

□

Remark 3.3. By Proposition A.1, $u(x) \le Ce^{-\sqrt{\lambda_0}|x|}$, for some $\lambda_0 > 0$. Since (3.1) does not depends on x', we can the moving plane method of [13] in the directions x_i, $i = 1, \cdots, N-1$ to prove that (3.1) has a least energy solution $u(x) = u(|x'|, x_N)$, $x' = (x_1, \cdots, x_{N-1})$.

Remark 3.4. Let $U = \underline{u}(x_N) + u$. Then U is a solution of

$$\begin{cases} -\Delta U = f(U), & \text{in } R^N_+, \\ U(x',0) = 0, & U(x',x_N) \to a, \text{uniformly in } x' \text{ as } x_N \to +\infty. \end{cases} \tag{3.11}$$

Clearly, $U \geq \underline{u}$, and $U < 0$ if $|x'|$ is sufficiently large and $x_N > 0$ is small enough. Suppose that U has fixed sign, then $\underline{u} \leq U \leq 0$. By Proposition 2.5 in [4], we have $U = \underline{u}$. This is a contradiction. So U is a changing sign solution. In next section, we shall give a direct and simple proof of this fact. See Remark 4.2.

Remark 3.5. To prove that the mountain pass type solution $u_\varepsilon = \underline{u}_\varepsilon + v_\varepsilon$ for (1.1) has exactly one positive local maximum point, it is important to show that U has exactly one positive local maximum point. It is easy to prove that this is true if $N = 1$ by using the relation $(U'(t))^2 - (U'(0))^2 = -2\int_0^{U(t)} f(\tau)d\tau$. It is an open problem whether U has exactly one positive local maximum point if $N \geq 2$.

Remark 3.6. It is possible to replace the convexity assumption on f by other conditions to obtain the existence result for (3.1). Under the condition that $\frac{f(t)}{t^p \ln^q t} \to a_\infty \in (0, +\infty)$, as $t \to +\infty$, where $1 < p < \frac{N+2}{N-2}$ and q is finite, or $0 < A < t^{-p} f(t) < B < +\infty$ for large t, where $1 < p < \frac{N+1}{N-1}$, we can prove that the Dirichlet problem on $\{|x'| < M, x_N \in (0, m)\}$ has a positive mountain pass solution $u_{M,m}(|x'|, x_N)$, which are uniformly bounded (by using a blow-up argument). The idea is to let $m \to +\infty$ first, and then let $M \to +\infty$ to obtain a decaying solution for (3.1). To make this work, we need to assume $f(a+t) > f'(a)t$ for $t > 0$ and $f(t)(t - \tilde{t}) > 2F(t)$ for $t > \tilde{t}$, where $\tilde{t} = \inf\{t : F(t) = 0\}$. Using the second condition, we can rewrite the energy of the solution to an integral, where the integrand is positive. Thus we can use the first condition to stop part of the solution moving to infinity in the x_N direction.

4. The location of the peak of the mountain pass solution

Let $\underline{u}_\varepsilon(x)$ be a solution of (1.1) such that $|\underline{u}_\varepsilon - a|$ is small on any $K \subset\subset \Omega$. Now we consider (1.2). Similar to the proof of Proposition 2 of [5], we can check that the first eigenvalue $\lambda_{\varepsilon,1}$ of $-\varepsilon^2\Delta - f'(\underline{u}_\varepsilon)I$ in $H_0^1(\Omega)$ satisfies $\lambda_{\varepsilon,1} \geq \lambda_0 > 0$. Thus by a similar argument to that in the proof of Lemma 2.1, we see

$$\int_\Omega (\varepsilon^2|Du|^2 - f'(\underline{u}_\varepsilon)u^2) \geq c' \int_\Omega (\varepsilon^2|Du|^2 + u^2), \quad \forall\, u \in H_0^1(\Omega),$$

for some $c' > 0$. So it is easy to check that (1.2) has a positive solution v_ε with $I_\varepsilon(v_\varepsilon) = c_\varepsilon$. In this section, we shall prove that all the local maximum points of v_ε tend to the same point x_0 on the boundary, at which the mean curvature $H(x)$ attains its global maximum.

First, we have an upper bound for c_ε.

Proposition 4.1. *Let $u \in H_0^1(R_+^N)$ be any solution of (3.1) with $J(u) = c$. Then we have*

$$c_\varepsilon \leq \varepsilon^N\big(c - \varepsilon B(u)H_M + O(\varepsilon^{1+\sigma})\big),$$

where $H_M = \max_{x\in\partial\Omega} H(x)$, and

$$B(u) = -\frac{1}{2}\int_{x_N=0} D_{x_N}u\underline{u}'(x_N)|x'|^2 - \frac{1}{4}\int_{x_N=0} |D_{x_N}u|^2|x'|^2.$$

Moreover, we have $B(u) > 0$.

Proof. Take any solution $u(r, x_N)$ of (2.1) with $J(u) = c$. Let $x_0 \in \partial\Omega$ be a point such that $H(x_0) = \max_{x\in\partial\Omega} H(x)$. After translation and rotation, we may assume that $x_0 = 0$ and

$$\Omega \cap B_\delta(0) = \{x : x_N > \varphi(x')\} \cap B_\delta(0), \tag{4.1}$$

$$\partial\Omega \cap B_\delta(0) = \{x : x_N = \varphi(x')\} \cap B_\delta(0), \tag{4.2}$$

where $\varphi(x') \in C^2(R^{N-1})$, $\varphi(0) = 0$, $D\varphi(0) = 0$ and

$$\varphi(x') = \frac{1}{2}\sum_{i=1}^{N-1} a_i x_i^2 + O(|x'|^2).$$

Let $\eta \in C_0^\infty(B_\delta(0))$, $\eta = 1$ for $x \in B_{\delta/2}(0)$, $0 \le \eta \le 1$. Define

$$w_\varepsilon = \eta(x) u\big(\varepsilon^{-1} r, \varepsilon^{-1}(x_N - \varphi(x'))\big).$$

Then $w_\varepsilon \in H_0^1(\Omega)$. So we have

$$c_\varepsilon \le \max_{t \ge 0} I_\varepsilon(t w_\varepsilon).$$

Now we estimate $I_\varepsilon(w_\varepsilon)$. We have

$$\begin{aligned} I_\varepsilon(w_\varepsilon) =& \frac{1}{2}\int_\Omega \varepsilon^2 |Dw_\varepsilon|^2 - \int_\Omega \Big(F(\underline{u}_\varepsilon + w_\varepsilon) - F(\underline{u}_\varepsilon) - f(\underline{u}_\varepsilon) w_\varepsilon\Big) \\ =& I_1 - I_2. \end{aligned} \tag{4.3}$$

Since $\underline{u}_\varepsilon = \underline{u}(\varepsilon^{-1} d(x, \partial\Omega)) + \varepsilon(N-1) H(\bar{x}) \psi(\varepsilon^{-1} d(x, \partial\Omega)) + O(\varepsilon^{1+\sigma})$, where $\bar{x} \in \partial\Omega$ with $|x - \bar{x}| = d(x, \partial\Omega)$, we have

$$\begin{aligned} I_2 =& \int_\Omega \Big(F(\underline{u}(\frac{d(x,\partial\Omega)}{\varepsilon}) + w_\varepsilon) - F\big(\underline{u}(\frac{d(x,\partial\Omega)}{\varepsilon})\big) - f\big(\underline{u}(\frac{d(x,\partial\Omega)}{\varepsilon})\big) w_\varepsilon\Big) \\ &+ \varepsilon(N-1)\int_\Omega \Big(f(\underline{u}(\frac{d(x,\partial\Omega)}{\varepsilon}) + w_\varepsilon) - f\big(\underline{u}(\frac{d(x,\partial\Omega)}{\varepsilon})\big) \\ &\qquad - f'\big(\underline{u}(\frac{d(x,\partial\Omega)}{\varepsilon})\big) w_\varepsilon\Big) \psi(\frac{d(x,\partial\Omega)}{\varepsilon}) H(\bar{x}) \\ &+ O(\varepsilon^{N+1+\sigma}) \\ =& \varepsilon^N \Big[\int_{\Omega_\varepsilon} \Big(F(\underline{u}(d(x,\partial\Omega_\varepsilon)) + \tilde{w}_\varepsilon) - F\big(\underline{u}(d(x,\partial\Omega_\varepsilon))\big) - f\big(\underline{u}(d(x,\partial\Omega_\varepsilon))\big)\tilde{w}_\varepsilon\Big) \\ &+ \varepsilon(N-1)\int_{\Omega_\varepsilon} \Big(f(\underline{u}(d(x,\partial\Omega_\varepsilon)) + \tilde{w}_\varepsilon) - f\big(\underline{u}(d(x,\partial\Omega_\varepsilon))\big) \\ &\qquad - f'\big(\underline{u}(d(x,\partial\Omega_\varepsilon))\big)\tilde{w}_\varepsilon\Big) \psi(d(x,\partial\Omega_\varepsilon)) H(\varepsilon\bar{x})\Big] \\ &+ O(\varepsilon^{N+1+\sigma}), \end{aligned} \tag{4.4}$$

where $\Omega_\varepsilon = \{x : \varepsilon x \in \Omega\}$, $\tilde{w}_\varepsilon = w_\varepsilon(\varepsilon x)$, $x \in \Omega_\varepsilon$.

Let $\varphi_\varepsilon(x') = \varepsilon^{-1}\varphi(\varepsilon x')$. Let $\bar{x}_\varepsilon \in \partial\Omega_\varepsilon$ be such that $|x - \bar{x}_\varepsilon| = d(x, \partial\Omega_\varepsilon)$. Then we have $x_i - \bar{x}_{\varepsilon,i} + (x_N - \varphi_\varepsilon(\bar{x}')) D_i \varphi_\varepsilon(\bar{x}') = 0$, $i = 1, \cdots, N-1$. Thus $|x' - \bar{x}'| = O(\varepsilon)$. As a result,

$$\begin{aligned} d(x, \partial\Omega_\varepsilon) &= (1 + |D\varphi_\varepsilon(\bar{x}')|^2)^{1/2} (x_N - \varphi_\varepsilon(\bar{x}')) \\ &= x_N - \varphi_\varepsilon(\bar{x}') + O(|D\varphi_\varepsilon(x')|^2) = x_N - \varphi_\varepsilon(x') + O(\varepsilon^2). \end{aligned}$$

Thus, making the change of variable $y' = x'$, $y_N = x_N - \varphi_\varepsilon(x')$, we obtain

$$\begin{aligned} I_2 =& \varepsilon^N \Big[\int_{R^N_+} \Big(F(\underline{u}(x_N) + u) - F(\underline{u}(x_N)) - f(\underline{u}(x_N)) u\Big) \\ &+ \varepsilon(N-1)\int_{R^N_+} \Big(f(\underline{u}(x_N) + u) - f(\underline{u}(x_N)) - f'(\underline{u}(x_N)) u\Big)\psi(x_N) H(0)\Big] \\ &+ O(\varepsilon^{N+1+\sigma}). \end{aligned} \tag{4.5}$$

Here, we have used the fact that u decays exponentially.

For the estimate of I_1, we have

$$
\begin{aligned}
&\varepsilon^2\int_\Omega |Dw_\varepsilon|^2=\varepsilon^2\int_{\Omega\cap B_{\delta/2}(0)}|Dw_\varepsilon|^2+O\big(\varepsilon^{-\delta'/\varepsilon}\big)\\
=&\varepsilon^2\int_{\Omega\cap B_{\delta/2}(0)}\Big(|Du(\frac{r}{\varepsilon},\frac{x_N-\varphi(x')}{\varepsilon})|^2\\
&\qquad -2D_{x_N}u(\frac{r}{\varepsilon},\frac{x_N-\varphi(x')}{\varepsilon})\sum_{i=1}^{N-1}D_{x_i}u(\frac{r}{\varepsilon},\frac{x_N-\varphi(x')}{\varepsilon})D_{x_i}\varphi(x')\\
&\qquad +|D_{x_N}u(\frac{r}{\varepsilon},\frac{x_N-\varphi(x')}{\varepsilon})|^2\sum_{i=1}^{N-1}|D_{x_i}\varphi(x')|^2\Big)\\
&+O(e^{-\delta'/\varepsilon})\\
=&\varepsilon^N\Big(\int_{R^N_+}|Du|^2-2\varepsilon\int_{R^N_+}D_{x_N}u\sum_{i=1}^{N-1}a_ix_iD_{x_i}u+O(\varepsilon^2)\Big).
\end{aligned}
\tag{4.6}
$$

In the last equality, we have used again the fact that u decays exponentially.

Since u is a solution of $-\Delta u=f(\underline{u}(x_N)+u)-f(\underline{u}(x_N))$, we have

$$
-\int_{R^N_+}\Delta u x_i^2D_{x_N}u=\int_{R^N_+}\Big(f(\underline{u}(x_N)+u)-f(\underline{u}(x_N))\Big)x_i^2D_{x_N}u
$$

for $i=1,\cdots,N-1$.

But for each fixed i, we have

$$
\begin{aligned}
&-\int_{R^N_+}\Delta u x_i^2D_{x_N}u\\
=&-\int_{x_N=0}\frac{\partial u}{\partial n}x_i^2D_{x_N}u+2\int_{R^N_+}D_{x_N}ux_iD_{x_i}u+\sum_{j=1}^N\int_{R^N_+}x_i^2D_{x_j}uD_{x_jx_N}u\\
=&\int_{x_N=0}D_{x_N}ux_i^2D_{x_N}u+2\int_{R^N_+}D_{x_N}ux_iD_{x_i}u+\frac{1}{2}\sum_{j=1}^N\int_{R^N_+}x_i^2D_{x_N}|D_{x_j}u|^2\\
=&\frac{1}{2}\int_{x_N=0}x_i^2|D_{x_N}u|^2+2\int_{R^N_+}D_{x_N}ux_iD_{x_i}u,
\end{aligned}
$$

since $D_{x_j}u = 0$ if $x_N = 0$, for $j = 1, \cdots, N-1$. Here we do not use the summation convention. So

$$
\begin{aligned}
&2\int_{R^N_+} D_{x_N}u x_i D_{x_i}u \\
&= \int_{R^N_+}\Big(f(\underline{u}(x_N)+u) - f(\underline{u}(x_N))\Big)x_i^2 D_{x_N}u - \frac{1}{2}\int_{x_N=0} x_i^2|D_{x_N}u|^2 \\
&= \int_{R^N_+}\frac{d}{dx_N}\Big(F(\underline{u}(x_N)+u) - F(\underline{u}(x_N)) - f(\underline{u}(x_N))u\Big)x_i^2 \\
&\quad - \int_{R^N_+}\Big(f(\underline{u}(x_N)+u) - f(\underline{u}(x_N)) - f'(\underline{u}(x_N))u\Big)\underline{u}'(x_N)x_i^2 \\
&\quad - \frac{1}{2}\int_{x_N=0} x_i^2|D_{x_N}u|^2 \\
&= -\int_{R^N_+}\Big(f(\underline{u}(x_N)+u) - f(\underline{u}(x_N)) - f'(\underline{u}(x_N))u\Big)\underline{u}'(x_N)x_i^2 \\
&\quad - \frac{1}{2}\int_{x_N=0} x_i^2|D_{x_N}u|^2 \\
&= -\frac{1}{N-1}\int_{R^N_+}\Big(f(\underline{u}(x_N)+u) - f(\underline{u}(x_N)) - f'(\underline{u}(x_N))u\Big)\underline{u}'(x_N)|x'|^2 \\
&\quad - \frac{1}{2(N-1)}\int_{x_N=0}|x'|^2|D_{x_N}u|^2.
\end{aligned}
\tag{4.7}
$$

Combining (4.6) and (4.7), we obtain

$$
\begin{aligned}
I_1 = \varepsilon^N\Big(&\frac{1}{2}\int_{R^N_+}|Du|^2 + \frac{\varepsilon H(0)}{4}\int_{x_N=0}|x'|^2|D_{x_N}u|^2 + O(\varepsilon^2) \\
&+ \frac{\varepsilon H(0)}{2}\int_{R^N_+}\Big(f(\underline{u}(x_N)+u) - f(\underline{u}(x_N)) - f'(\underline{u}(x_N))u\Big)\underline{u}'(x_N)|x'|^2\Big).
\end{aligned}
\tag{4.8}
$$

Putting (4.5) and (4.8) together, we see

$$
\begin{aligned}
I_\varepsilon(w_\varepsilon) = \varepsilon^N\Big(c &+ \frac{\varepsilon H(0)}{4}\int_{x_N=0}|x'|^2|D_{x_N}u|^2 + O(\varepsilon^{1+\sigma}) \\
&+ \frac{\varepsilon H(0)}{2}\int_{R^N_+}\Big(f(\underline{u}(x_N)+u) - f(\underline{u}(x_N)) - f'(\underline{u}(x_N))u\Big)\underline{u}'(x_N)|x'|^2\Big) \\
&- (N-1)\varepsilon H(0)\int_{R^N_+}\Big(f(\underline{u}(x_N)+u) - f(\underline{u}(x_N)) - f'(\underline{u}(x_N))u\Big)\psi(x_N)\Big).
\end{aligned}
\tag{4.9}
$$

Using the exponentially decay of the solution u, $u = \psi = 0$ if $x_N = 0$, and self adjointness of the Laplacian, we see

$$
\begin{aligned}
&\int_{R^N_+}\Big(f(\underline{u}(x_N)+u) - f(\underline{u}(x_N)) - f'(\underline{u}(x_N))u\Big)\psi(x_N)\Big) \\
&= \int_{R^N_+}\Big(-\Delta\psi - f'(\underline{u}(x_N))\psi\Big)u \\
&= -\int_{R^N_+}\underline{u}'(x_N)u.
\end{aligned}
\tag{4.10}
$$

Similarly, we have

$$
\begin{aligned}
&\int_{R^N_+}\Big(f(\underline{u}(x_N)+u)-f(\underline{u}(x_N))\Big)\underline{u}'(x_N)|x'|^2\\
=&-\int_{R^N_+}\Delta u\underline{u}'(x_N)|x'|^2\\
=&-\int_{x_N=0}\frac{\partial u}{\partial n}\underline{u}'(x_N)|x'|^2-\int_{R^N_+}u\Delta(\underline{u}'(x_N)|x'|^2)\\
=&\int_{x_N=0}D_{x_N}u\underline{u}'(x_N)|x'|^2-\int_{R^N_+}u(\underline{u}'''|x'|^2+2(N-1)\underline{u}'(x_N))\\
=&\int_{x_N=0}D_{x_N}u\underline{u}'(x_N)|x'|^2+\int_{R^N_+}u\Big(f'(\underline{u}(x_N))u'(x_N)|x'|^2-2(N-1)\underline{u}'(x_N)\Big).
\end{aligned}
$$

As a result,

$$
\begin{aligned}
&\int_{R^N_+}\Big(f(\underline{u}(x_N)+u)-f(\underline{u}(x_N))-f'(\underline{u}(x_N))u\Big)\underline{u}'(x_N)|x'|^2\\
=&\int_{x_N=0}D_{x_N}u\underline{u}'(x_N)|x'|^2-2(N-1)\int_{R^N_+}u\underline{u}'(x_N).
\end{aligned}
\tag{4.11}
$$

Combining (4.9), (4.10) and (4.11), we obtain

$$
I_\varepsilon(w_\varepsilon)=\varepsilon^N\big(c-\varepsilon B(u)H(0)+O(\varepsilon^{1+\sigma})\big),\tag{4.12}
$$

where

$$
B(u)=-\frac{1}{2}\int_{x_N=0}D_{x_N}u\underline{u}'(0)|x'|^2-\frac{1}{4}\int_{x_N=0}|D_{x_N}u|^2|x'|^2
$$

It is easy to check $\langle I'_\varepsilon(w_\varepsilon),w_\varepsilon\rangle=O(\varepsilon^{N+1})$, $I''_\varepsilon(w_\varepsilon)(w_\varepsilon,w_\varepsilon)\le -c_0\varepsilon^N<0$. Thus similar to the proof of step 2 of Lemma 3.1, we see that if t_ε achieves $\max_{t\ge 0}I_\varepsilon(tw_\varepsilon)$, then $t_\varepsilon=1+O(\varepsilon)$. So

$$
\begin{aligned}
I_\varepsilon(t_\varepsilon w_\varepsilon)=&I_\varepsilon(w_\varepsilon)+(t_\varepsilon-1)\langle I'_\varepsilon(w_\varepsilon),w_\varepsilon\rangle+O(\varepsilon^N|t_\varepsilon-1|^2)\\
=&I_\varepsilon(w_\varepsilon)+O(\varepsilon^{N+2}).
\end{aligned}
\tag{4.13}
$$

Thus, the upper bound follows from (4.12) and (4.13).

It remains to prove that $B(u)>0$. Let $U=\underline{u}+u$. Then U satisfies of $-\Delta U=f(U)$ on R^N_+, $U=0$ on $x_N=0$ and $U\to a<0$ as $x_N\to+\infty$. It is easy to check that

$$
B(u)=\frac{1}{4}\int_{x_N=0}(\underline{u}'(0)^2-|D_{x_N}U(x',0)|^2)|x'|^2.
$$

First we claim that

$$
\int_{x_N=0}(\underline{u}'(0)^2-|D_{x_N}U(x',0)|^2)=0.\tag{4.14}
$$

Multiplying $-\Delta U=f(U)$ by $D_{x_N}U$ and integrating over $B^{N-1}_R(0)\times[0,+\infty)$, where $B^{N-1}_R(0)$ is the ball in R^{N-1}, centered at the origin with radius R, we obtain

$$
\begin{aligned}
&-\sum_{i=1}^{N-1}\int_{B^{N-1}_R(0)\times[0,+\infty)}U_{x_ix_i}U_{x_N}+\frac{1}{2}\int_{B^{N-1}_R(0)}U^2_{x_N}(x',0)\\
=&\int_{B^{N-1}_R(0)}\int_0^{+\infty}\frac{d}{dx_N}F(U)dx_Ndx'=\int_{B^{N-1}_R(0)}F(a)=\int_{B^{N-1}_R(0)}\int_0^a f(\tau)d\tau.
\end{aligned}
$$

But $\frac{1}{2}|\underline{u}'(0)|^2 = \int_0^a f(\tau)d\tau$. So we have

$$\int_{B_R^{N-1}(0)} (\underline{u}'(0)^2 - |D_{x_N}U(x',0)|^2) = -2\sum_{i=1}^{N-1}\int_{B_R^{N-1}(0)\times[0,+\infty)} U_{x_ix_i}U_{x_N}.$$

On the other hand, we have

$$\int_{B_R^{N-1}(0)} U_{x_ix_i}U_{x_N}dx' = \int_{\partial B_R^{N-1}(0)} n_iU_{x_i}U_{x_N} - \int_{B_R^{N-1}(0)} U_{x_i}U_{x_Nx_i}dx'$$
$$= \int_{\partial B_R^{N-1}(0)} n_iU_{x_i}U_{x_N} - \frac{1}{2}\frac{d}{dx_N}\int_{B_R^{N-1}(0)} U_{x_i}^2dx'.$$

So

$$\int_{B_R^{N-1}(0)\times[0,+\infty)} U_{x_ix_i}U_{x_N} = \int_{\partial B_R^{N-1}(0)\times[0,+\infty)} n_iU_{x_i}U_{x_N} \to 0,$$

as $R \to +\infty$, since $U_{x_i} = u_{x_i}$ decays exponentially as $|x| \to +\infty$. Thus (4.14) follows.

Now we prove that $B(u) > 0$. Since $U(|x'|, x_N)$ is a decreasing function of $|x'|$, we see that $U_{x_N}(|x'|, 0)$ is nonincreasing in $|x'|$. Let r_0 be such that $\underline{u}'(0)^2 - |D_{x_N}U(x',0)|^2 \geq 0$ if $|x'| \geq r_0$, $\underline{u}'(0)^2 - |D_{x_N}U(x',0)|^2 \leq 0$ if $|x'| \leq r_0$. Since $u_{x_N}(x',0)$ tends to zero as $|x'| \to +\infty$ and $u_{x_N}(x',0) \geq 0$, we see that $\underline{u}'(0)^2 - |D_{x_N}U(x',0)|^2 > 0$ if $|x'|$ is large. So $r_0 < +\infty$. Hence, by (4.14), we obtain

$$B(u) = \int_{|x'|\leq r_0} (\underline{u}'(0)^2 - |D_{x_N}U(x',0)|^2)|x'|^2 + \int_{|x'|\geq r_0} (\underline{u}'(0)^2 - |D_{x_N}U(x',0)|^2)|x'|^2$$
$$\geq \int_{|x'|\leq r_0} (\underline{u}'(0)^2 - |D_{x_N}U(x',0)|^2)r_0^2 + \int_{|x'|\geq r_0} (\underline{u}'(0)^2 - |D_{x_N}U(x',0)|^2)|x'|^2$$
$$= \int_{|x'|\geq r_0} (\underline{u}'(0)^2 - |D_{x_N}U(x',0)|^2)(|x'|^2 - r_0^2) > 0.$$

□

Remark 4.2. Since $U_{x_N}(x',0) \geq \underline{u}'(0)$, it is easy to see from (4.14) that

$$U_{x_N}(0) = \max_{x_N=0} U_{x_N}(x',0) \geq -\underline{u}'(0) > 0.$$

Moreover, it is easy to see that $U_{x_N}(x',0)$ is negative if $|x'|$ is large. Thus U is positive near the origin and is negative if $|x'|$ is large and $x_N > 0$ is small. So we have proved that U is a changing sign solution.

Next, we shall obtain a lower bound for c_ε.

Lemma 4.3. *Suppose that v_ε is a solution of (1.2) with $I_\varepsilon(v_\varepsilon) = c_\varepsilon$. Let x_ε be a location maximum point of u_ε. Then*

(i) *We have $d(x_\varepsilon, \partial\Omega) \leq C\varepsilon$.*
(ii) *Suppose that u_ε has another local maximum point $x_\varepsilon^{(1)}$. Then $|x_\varepsilon^{(1)} - x_\varepsilon| \leq C\varepsilon$.*
(iii) *For any $\theta > 0$, there is a $\nu > 0$, such that $v_\varepsilon(x) \leq Ce^{-\nu|x-x_\varepsilon|/\varepsilon}$, $\forall\, x \in \Omega \setminus B_\theta(x_\varepsilon)$.*

Proof. Suppose that $\varepsilon^{-1}d(x_\varepsilon, \partial\Omega) \to +\infty$. Let $\bar{v}_\varepsilon(y) = v_\varepsilon(\varepsilon y + x_\varepsilon)$, $y \in \Omega_{\varepsilon,x} = \{y : \varepsilon y + x_\varepsilon \in \Omega\}$. Then, from the upper bound for c_ε, we see that $\bar{v}_\varepsilon$ is bounded in $H^1(R^N)$. So we may assume (up to a subsequence) that

$$\bar{v}_\varepsilon \to v_0, \text{ in } C^1_{loc}(R^N)$$

and

$$-\Delta v_0 = f(v_0 + a), \text{ in } R^N.$$

Thus, noting that $\frac{1}{2}g_\varepsilon(x,t)t - G_\varepsilon(x,t) \geq 0$, we see

$$
\begin{aligned}
I_\varepsilon(v_\varepsilon) =& \int_\Omega \big(\frac{1}{2} g_\varepsilon(x, v_\varepsilon) v_\varepsilon - G_\varepsilon(x, v_\varepsilon)\big) \\
\geq& \int_{B_{\varepsilon R}(x_\varepsilon)} \big(\frac{1}{2} g_\varepsilon(x, v_\varepsilon) v_\varepsilon - G_\varepsilon(x, v_\varepsilon)\big) \\
=&\varepsilon^N \int_{B_R(0)} \big(\frac{1}{2} g_\varepsilon(\varepsilon y + x_\varepsilon, \bar{v}_\varepsilon)\bar{v}_\varepsilon - G_\varepsilon(\varepsilon y + x_\varepsilon, \bar{v}_\varepsilon)\big) \\
=&\varepsilon^N \Big(\int_{B_R(0)} \big(\frac{1}{2} f(a + v_0) v_0 - (F(a + v_0) - F(a))\big) + o_\varepsilon(1) \Big) \\
\geq&\varepsilon^N (A + o_{\varepsilon,R}(1)),
\end{aligned}
$$

where $o_{\varepsilon,R} \to 0$ as $\varepsilon \to 0$ and $R \to +\infty$. This is a contradiction to Proposition 4.1.

Next, we prove (ii). Suppose that $\frac{|x_\varepsilon^{(1)} - x_\varepsilon|}{\varepsilon} \to +\infty$, then it is easy to check

$$
I_\varepsilon(v_\varepsilon) \geq \varepsilon^N (2c + o(1)),
$$

where $o(1) \to 0$ as $\varepsilon \to 0$. This is a contradiction.

It remains to prove (iii).

First, we prove that as $\varepsilon \to 0$, $v_\varepsilon \to 0$ uniformly in $\Omega \setminus B_\theta(x_\varepsilon)$, for any $\theta > 0$. In fact, suppose that there is $\bar{x}_\varepsilon \in \Omega \setminus B_\theta(x_\varepsilon)$, such that $v_\varepsilon(\bar{x}_\varepsilon) \geq c_0' > 0$. Then $v_\varepsilon(\varepsilon y + \bar{x}_\varepsilon) \rightharpoonup \bar{v}_0 \neq 0$. As a result,

$$
\begin{aligned}
I_\varepsilon(v_\varepsilon) =& \int_\Omega \big(\frac{1}{2} g_\varepsilon(x, v_\varepsilon) v_\varepsilon - G_\varepsilon(x, v_\varepsilon)\big) \\
\geq& \int_{B_{\varepsilon R}(x_\varepsilon)} \big(\frac{1}{2} g_\varepsilon(x, v_\varepsilon) v_\varepsilon - G_\varepsilon(x, v_\varepsilon)\big) + \int_{B_{\varepsilon R}(\bar{x}_\varepsilon)} \big(\frac{1}{2} g_\varepsilon(x, v_\varepsilon) v_\varepsilon - G_\varepsilon(x, v_\varepsilon)\big) \\
\geq&\varepsilon^N (2c + o_{\varepsilon,R}(1)).
\end{aligned}
$$

This is a contradiction.

Next, from $-f'(\underline{u}_\varepsilon) \geq 2\nu_0 > 0$, for $x \in \Omega_{\varepsilon R} =: \{x \in \Omega, d(x, \partial\Omega) \geq \varepsilon R\}$ if $R > 0$ is large enough, we can check easily by using the standard comparison theorem that

$$
v_\varepsilon(x) \leq C e^{-\sqrt{\nu_0}|x - x_\varepsilon|/\varepsilon}, \quad x \in \Omega_{\varepsilon R}.
$$

Suppose now that $x \in (\Omega \setminus \Omega_{\varepsilon R}) \setminus B_\theta(x_\varepsilon)$. Without loss of generality, we may assume $x_\varepsilon = 0$. Denote $\bar{x} \in \partial\Omega$ such that $d(x, \partial\Omega) = |x - \bar{x}|$. Let $\xi(t)$ be the positive function defined in Appendix A. Let

$$
w_\varepsilon(x) = \xi\Big(\frac{d(x, \partial\Omega)}{\varepsilon}\Big) e^{-\bar{\nu}|\bar{x}|/\varepsilon}, \quad x \in \Omega \setminus \Omega_{\varepsilon R},
$$

where $\bar{\nu} > 0$ is a small number. Noting that $\langle De^{-\bar{\nu}|\bar{x}|/\varepsilon}, Dd(x, \partial\Omega) = 0\rangle$, we have for $x \in \Omega \setminus \Omega_{\varepsilon R}$,

$$
\begin{aligned}
&-\varepsilon^2 \Delta w_\varepsilon \\
=&-\varepsilon^2 e^{-\bar{\nu}|\bar{x}|/\varepsilon} \Delta\xi\Big(\frac{d(x, \partial\Omega)}{\varepsilon}\Big) - 2\varepsilon^2 \langle De^{-\bar{\nu}|\bar{x}|/\varepsilon}, D\xi\Big(\frac{d(x, \partial\Omega)}{\varepsilon}\Big)\rangle \\
&-\varepsilon^2 \xi\Big(\frac{d(x, \partial\Omega)}{\varepsilon}\Big) \Delta e^{-\bar{\nu}|\bar{x}|/\varepsilon} \\
=&\Big(-\xi''\Big(\frac{d(x, \partial\Omega)}{\varepsilon}\Big) + O(\varepsilon)\Big) e^{-\bar{\nu}|\bar{x}|/\varepsilon} - \varepsilon^2 \xi\Big(\frac{d(x, \partial\Omega)}{\varepsilon}\Big) \Delta e^{-\bar{\nu}|\bar{x}|/\varepsilon} \\
=&\Big(-\xi''\Big(\frac{d(x, \partial\Omega)}{\varepsilon}\Big) + O(\varepsilon)\xi\Big(\frac{d(x, \partial\Omega)}{\varepsilon}\Big)\Big) e^{-\bar{\nu}|\bar{x}|/\varepsilon} - \varepsilon^2 \xi\Big(\frac{d(x, \partial\Omega)}{\varepsilon}\Big) \Delta e^{-\bar{\nu}|\bar{x}|/\varepsilon},
\end{aligned}
$$

since $\xi\Big(\frac{d(x,\partial\Omega)}{\varepsilon}\Big) \ge c_0' > 0$ for $x \in \Omega \setminus \Omega_{\varepsilon R}$. As a result,

$$\begin{aligned}
&- \varepsilon^2\Delta w_\varepsilon - \big(f'(\underline{u}_\varepsilon) - \tau\big) w_\varepsilon \\
=& - \varepsilon^2\xi\Big(\frac{d(x,\partial\Omega)}{\varepsilon}\Big)\Delta e^{-\bar\nu|\bar x|/\varepsilon} \\
&+ \Big(-\xi''\Big(\frac{d(x,\partial\Omega)}{\varepsilon}\Big) - \Big(f'\Big(\underline{u}\Big(\frac{d(x,\partial\Omega)}{\varepsilon}\Big)\Big) + O(\varepsilon) - \tau\Big)\xi\Big(\frac{d(x,\partial\Omega)}{\varepsilon}\Big)\Big)e^{-\bar\nu|\bar x|/\varepsilon} \\
=&\xi\Big(\frac{d(x,\partial\Omega)}{\varepsilon}\Big)\Big(-\varepsilon^2\Delta e^{-\bar\nu|\bar x|/\varepsilon} + (3\lambda_0 + O(\varepsilon) - \tau)e^{-\bar\nu|\bar x|/\varepsilon}\Big) > 0, \quad \forall\, x \in \Omega\setminus\Omega_{\varepsilon R}.
\end{aligned}$$

¿From $-\Delta v_\varepsilon - f'(\underline{u}_\varepsilon)v_\varepsilon = O(|v_\varepsilon|^2)$ and $v_\varepsilon \to 0$ as $\varepsilon \to 0$ uniformly for $x \in \Omega\setminus B_{\theta/2}(0)$, we see that $-\Delta v_\varepsilon - \big(f'(\underline{u}_\varepsilon) - \tau\big)v_\varepsilon \le 0$ if $x \in \Omega \setminus B_{\theta/2}(0)$. So by comparison in the domain $(\Omega\setminus\Omega_{\varepsilon R})\setminus B_{\theta/2}(0)$, we have

$$v_\varepsilon(x) \le C \max_{z\in\Omega\cap\partial B_{\theta/2}(0)}\big(w_\varepsilon(z)\big)^{-1}w_\varepsilon(x), \quad \forall\, x \in (\Omega\setminus\Omega_{\varepsilon R})\setminus B_{\theta/2}(0),$$

since

$$v_\varepsilon \le C \max_{z\in\Omega\cap\partial B_{\theta/2}(0)}\big(w_\varepsilon(z)\big)^{-1}w_\varepsilon(x) \quad \text{if } x \in \partial\big((\Omega\setminus\Omega_{\varepsilon R})\setminus B_{\theta/2}(0)\big).$$

Thus, if $x \in (\Omega\setminus\Omega_{\varepsilon R})\setminus B_{2\theta}(0)$, then

$$\begin{aligned}
v_\varepsilon(x) \le& C \max_{z\in\Omega\cap\partial B_{\theta/2}(0)}\big(w_\varepsilon(z)\big)^{-1}e^{-\bar\nu 3\theta/4\varepsilon}e^{-\bar\nu|\bar x|/4\varepsilon} \\
\le& C' \max_{z\in\Omega\cap\partial B_{\theta/2}(0)} e^{\nu|\bar z|}e^{-\bar\nu 3\theta/4\varepsilon}e^{-\bar\nu|\bar x|/4\varepsilon} \le C'e^{-\bar\nu|\bar x|/4\varepsilon}.
\end{aligned}$$

□

Proposition 4.4. *Let x_ε be a global maximum point of v_ε and $\bar x_\varepsilon \in \partial\Omega$ be such that $|x_\varepsilon - \bar x_\varepsilon| = d(x_\varepsilon, \partial\Omega)$. Then,*

$$c_\varepsilon \ge \varepsilon^N\big(c - \varepsilon H(\bar x_\varepsilon)B(u) + o(\varepsilon)\big),$$

where $u \in H_0^1(R_+^N)$ is a solution of $-\Delta u = f(\underline{u} + u) - f(\underline{u})$, $B(u)$ is defined as in Proposition 4.1.

Proof. Let v_ε be a solution of (1.2) with $I_\varepsilon(v_\varepsilon) = c_\varepsilon$. Let $\bar x_\varepsilon \in \partial\Omega$ be such that $|\bar x_\varepsilon - x_\varepsilon| = d(x_\varepsilon, \partial\Omega)$. We assume that $\bar x_\varepsilon = 0$.

Similar to Proposition 4.1, we define

$$\tilde v_\varepsilon(x) = \eta(x)v_\varepsilon(x', x_N + \varphi(x')) \in H_0^1(R_+^N),$$

where $\eta(x) \in C_0^\infty(B_\delta(0))$, with $0 \le \eta \le 1$, $\eta = 1$ for $x \in B_{\delta/2}(0)$.

Denote $\bar v_\varepsilon = \tilde v_\varepsilon(\varepsilon y)$. Then

$$\bar v_\varepsilon \to u, \quad \text{in } C^1_{loc}(R_+^N),$$

and

$$\begin{cases} -\Delta u = f(\underline{u}(x_N) + u) - f(\underline{u}(x_N)), & \text{in } R_+^N, \\ u \in H_0^1(R_+^N). \end{cases}$$

Because $\bar v_\varepsilon$ attains its maximum at $y' = 0$, we see that $u(y) = u(|y'|, y_N)$.

Since v_ε is a mountain pass solution, we have

$$\begin{aligned}
c_\varepsilon =& I_\varepsilon(v_\varepsilon) = \max_{t\ge0} I_\varepsilon(tv_\varepsilon) \\
=& \max_{t\ge0}\Big[\frac{1}{2}t^2\varepsilon^2\int_\Omega |Dv_\varepsilon|^2 - \int_\Omega\Big(F(\underline{u}_\varepsilon + tu_\varepsilon) - F(\underline{u}_\varepsilon) - f(\underline{u}_\varepsilon)tv_\varepsilon\Big)\Big].
\end{aligned}$$

Similar to the estimate of the upper bound for c_ε, using the fact that v_ε is exponentially small outside $B_{\delta/2}(0)$, we can prove that

$$\begin{aligned}
&\frac{1}{2}\varepsilon^2\int_\Omega |Dv_\varepsilon|^2 = \frac{1}{2}\varepsilon^2\int_{\Omega\cap B_{\delta/2}(x_\varepsilon)} |Dv_\varepsilon|^2 + O(e^{-\delta'/\varepsilon})\\
=&\varepsilon^N\Big[\frac{1}{2}\int_{R_+^N}|D\bar{v}_\varepsilon|^2 + \frac{\varepsilon H(\bar{x}_\varepsilon)}{4}\int_{x_N=0}|x'|^2|D_{x_N}u|^2 + o(\varepsilon)\\
&+\frac{\varepsilon H(\bar{x}_\varepsilon)}{2}\int_{R_+^N}|x'|^2\Big(f(\underline{u}(x_N)+u) - f(\underline{u}(x_N)) - f'(\underline{u}(x_N))u\Big)\underline{u}'(x_N)\Big].
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\int_\Omega\Big(F(\underline{u}_\varepsilon + tv_\varepsilon) - F(\underline{u}_\varepsilon) - f(\underline{u}_\varepsilon)tv_\varepsilon\Big)\\
=&\varepsilon^N\Big[\int_{R_+^N}\Big(F(\underline{u}(x_N)+t\bar{v}_\varepsilon) - F(\underline{u}(x_N)) - f(\underline{u}(x_N))t\bar{v}_\varepsilon\Big)\\
&+(N-1)\varepsilon\int_{R_+^N}\Big(f(\underline{u}(x_N)+tu) - f(\underline{u}(x_N)) - f'(\underline{u}(x_N))tu\Big)\psi(x_N)H(\bar{x}_\varepsilon) + o(\varepsilon)\Big]
\end{aligned}$$

Thus

$$\begin{aligned}
&I_\varepsilon(tv_\varepsilon)\\
=&\varepsilon^N\Big[\frac{t^2}{2}\int_{R_+^N}|D\bar{v}_\varepsilon|^2 - \int_{R_+^N}\Big(F(\underline{u}(x_N)+t\bar{v}_\varepsilon) - F(\underline{u}(x_N)) - f(\underline{u}(x_N))t\bar{v}_\varepsilon\Big)\Big]\\
&+\varepsilon^{N+1}H(\bar{x}_\varepsilon)\Big[\frac{1}{4}t^2\int_{x_N=0}|x'|^2|D_{x_N}u|^2\\
&+\frac{t^2}{2}\int_{R_+^N}|x'|^2\Big(f(\underline{u}(x_N)+u) - f(\underline{u}(x_N)) - f'(\underline{u}(x_N))u\Big)\underline{u}_\varepsilon'(x_N)\\
&-(N-1)\int_{R_+^N}\Big(f(\underline{u}(x_N)+tu) - f(\underline{u}(x_N)) - f'(\underline{u}(x_N))tu\Big)\psi(x_N)\Big] + o(\varepsilon^{N+1})\\
=&I_{\varepsilon 4}(t) + I_{\varepsilon 5}(t) + o(\varepsilon^{N+1}).
\end{aligned}$$

Choose $\tilde{t}_\varepsilon$ such that $I_{\varepsilon 4}(\tilde{t}_\varepsilon) = \max_{t\geq 0} I_{\varepsilon 4}(t)$. Since $\bar{v}_\varepsilon \in H_0^1(R_+^N)$, we see that

$$I_{\varepsilon 4}(\tilde{t}_\varepsilon) \geq c.$$

On the other hand, from $\bar{v}_\varepsilon \to u$ in $H_0^1(R_+^N)$, and u is a solution of $-\Delta u = f(\underline{u}+u) - f(\underline{u})$, we can deduce easily that $\tilde{t}_\varepsilon \to 1$ as $\varepsilon \to 0$. Thus, $I_{\varepsilon 5}(\tilde{t}_\varepsilon) = I_{\varepsilon 5}(1) + o(\varepsilon^{N+1})$. As a result

$$c_\varepsilon = \max_{t\geq 0} I_\varepsilon(tv_\varepsilon) \geq I_\varepsilon(\tilde{t}_\varepsilon v_\varepsilon) \geq \varepsilon^N\big(c - \varepsilon H(\bar{x}_\varepsilon)B(u) + o(\varepsilon)\big),$$

where $u \in H_0^1(R_+^N)$ is a solution of $-\Delta u = f(\underline{u}+u) - f(\underline{u})$. So we have completed the proof of this proposition. □

Now we are ready to prove the following results.

Theorem 4.5. *Let v_ε be a positive mountain pass solution of* (1.2). *We have*

(i) *For any local maximum point x_ε of v_ε, we have $d(x_\varepsilon, \partial\Omega) \leq C\varepsilon$. If v_ε has another local maximum point $x_\varepsilon^{(1)}$, then $|x_\varepsilon - x_\varepsilon^{(1)}| \leq C\varepsilon$. Moreover, for any $\theta > 0$, there is a $\nu > 0$, such that $|v_\varepsilon(x)| \leq Ce^{-\nu|x-x_\varepsilon|/\varepsilon}$ for $x \in \Omega \setminus B_\theta(x_\varepsilon)$.*

(ii) *For any sequence of ε, there is a subsequence $\varepsilon_j \to 0$, such that $x_j \to x_0 \in \partial\Omega$ with $H(x_0) = \max_{x\in\partial\Omega} H(x)$, where x_j is any local maximum point of v_ε, $H(x)$ is the mean curvature of $\partial\Omega$ at x.*

Proof. It is easy to see (i) follows from Lemma 4.3. To prove (ii), we can combine Propositions 4.1 and 4.4 to obtain

$$-H(x_0) \le -\max_{x\in\partial\Omega} H(x) + o(1).$$

As a result, $H(x_0) = \max_{x\in\partial\Omega} H(x)$.

□

Proof of Theorem 1.1. Let z_ε be a global maximum point of v_ε. Then $u_\varepsilon(\varepsilon y + z_\varepsilon)$ converges in $C^1_{\text{loc}}(R^N_+)$ to $U(y) = \underline{u}(y_N) + u$, where u is the mountain pass solution of (3.1). By Remark 4.2, we know that $\max_{y\in R^N_+} U(y) > 0$. Thus (i) follows. It also follows from Remark 4.2 that $U_{x_N}(0) > 0$. So we see that $\frac{\partial}{\partial\nu} u_\varepsilon(\varepsilon y + z_\varepsilon) > 0$ at $\bar{z}_\varepsilon$, where ν is the inward unit normal of $\partial\Omega$ at $\bar{z}_\varepsilon$, $\bar{z}_\varepsilon \in \partial\Omega_\varepsilon$ satisfies $|z_\varepsilon - \bar{z}_\varepsilon| = d(z_\varepsilon, \partial\Omega_\varepsilon)$, $\Omega_\varepsilon = \{y : \varepsilon y + z_\varepsilon \in \Omega\}$. As a result, $u_\varepsilon(\varepsilon y + z_\varepsilon) > 0$ in a neighbourhood of $\bar{z}_\varepsilon$ and (vi) follows. On the other hand, (ii) follows from (i) of Theorem 4.5.

To prove (iii), we claim that for any local maximum point x_ε of u_ε with $u_\varepsilon(x_\varepsilon) \ge \bar{c}_0 > 0$, we have $|x_\varepsilon - z_\varepsilon| \le C\varepsilon$, where z_ε is a local maximum point of v_ε. In fact, suppose that there is a sequence of x_ε, such that $\varepsilon^{-1}|x_\varepsilon - z_\varepsilon| \to +\infty$ as $\varepsilon \to 0$ for any local maximum point z_ε of v_ε. Then $v_\varepsilon \ge \bar{c}_0$ in a small neighbourhood of x_ε. So we can blow up v_ε at x_ε and see the energy of v_ε is strictly larger than c_ε. So we get a contradiction. See the proof of Lemma 4.3. Thus (iii) follows from (ii) of Theorem 4.5. □

5. Existence of interior peak solutions

In this section, we shall briefly prove the existence of interior peak solutions for (1.2) and estimate the number of such solutions. For simplicity, we only discuss the case $f(t) = (t-a)^{p-1} - (t-a)$. So the mountain pass solution of (3.5) is nondegenerate. See [23].

As we see in Section 4, the main contribution to the energy of $I_\varepsilon(v_\varepsilon)$ comes from the error term in the expansion of $\underline{u}_\varepsilon$ near the boundary of Ω. To construct the interior peak solution for (1.2), we need the following proposition.

Proposition 5.1. *We have*

$$\underline{u}_\varepsilon(x) = a + e^{-\sqrt{-f'(a)}(d(x,\partial\Omega)+o(1))/\varepsilon},$$

for any $x \in \Omega$ with $\varepsilon^{-1}d(x,\partial\Omega)$ large. Here, $o(1) \to 0$ as $\varepsilon \to 0$.

Proof. See the proof of Theorem 2.1 in [7]. □

Let w be the unique positive solution of (3.5), which is nondegenerate. Define

$$w_{\varepsilon,x}(y) = w\Big(\frac{y-x}{\varepsilon}\Big).$$

Let $P_{\varepsilon,\Omega} w_{\varepsilon,x} \in H^1_0(\Omega)$ be the solution of

$$-\Delta v - f'(a)v = f(a + w_{\varepsilon,x}) - f'(a)w_{\varepsilon,x}.$$

Using Proposition 5.1, we can prove

Proposition 5.2.

$$I(P_{\varepsilon,\Omega} w_{\varepsilon,x}) = \varepsilon^N A - \tau_{\varepsilon,x} + O\big(e^{-(1+\sigma)\sqrt{-f'(a)}d(x,\partial\Omega)/\varepsilon}\big),$$

where $\tau_{\varepsilon,x} = \int_\Omega \big(f(a + w_{\varepsilon,x}) - f'(a)w_{\varepsilon,x}\big)(\underline{u}_\varepsilon - a)$. Moreover, $\tau_{\varepsilon,x}$ satisfies

$$c_0 e^{-(1+\theta)\sqrt{-f'(a)}d(x,\partial\Omega)/\varepsilon} \le \tau_{\varepsilon,x} \le c_1 e^{-(1-\theta)\sqrt{-f'(a)}d(x,\partial\Omega)/\varepsilon}.,$$

for any $\theta > 0$, and $c_0 > 0$ and $c_1 > 0$ are some constants.

Proof. The proof of this proposition is similar to that of Step 1 of Lemma 3.1 and thus we omit it.

□

By direct calculation, it is not difficult to prove the following proposition (see for example [9]):

Proposition 5.3.

$$I\Big(\sum_{j=1}^{k} P_{\varepsilon,\Omega} w_{\varepsilon,x_j}\Big)$$
$$=\sum_{j=1}^{k} I\Big(P_{\varepsilon,\Omega} w_{\varepsilon,x_j}\Big) - \int_\Omega \Big(F_a\big(\sum_{j=1}^{k} w_{\varepsilon,x_j}\big) - \sum_{j=1}^{k} F_a\big(w_{\varepsilon,x_j}\big) - \sum_{i<j} f_a(w_{\varepsilon,x_i}) w_{\varepsilon,x_j}\Big)$$
$$+O\Big(\sum_{j=1}^{k} e^{-(1+\sigma)\sqrt{-f'(a)}d(x_j,\partial\Omega)/\varepsilon} + \sum_{i\neq j} e^{-(1+\sigma)\sqrt{-f'(a)}|x_j-x_i|/\varepsilon}\Big),$$

where $f_a(t) = f(a+t)$, $F_a(t) = \int_0^t f_a(\tau)\,d\tau$.

Propositions 5.2 and 5.3 shows that the energy of the approximate multipeak solution $\sum_{j=1}^k P_{\varepsilon,\Omega} w_{\varepsilon,x_j}$ will become larger if the peak x_j moves away from the boundary of Ω, or if a pair of peak (x_i, x_j) moves away from each other. Thus this estimate is similar to that for the interior peak solution of Neumann problem [28]. But one should note that in Proposition 5.2, the first small term is $e^{-\sqrt{-f'(a)}d(x,\partial\Omega)/\varepsilon}$, instead of $e^{-2\sqrt{-f'(a)}d(x,\partial\Omega)/\varepsilon}$ as in the Neumann problem.

Arguing in exactly the same way as in [6, 28], we have

Theorem 5.4. *For any positive integer k, there is an $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0]$,* (1.2) *has a solution of the form*

$$v_\varepsilon = \sum_{j=1}^{k} P_{\varepsilon,\Omega} w_{\varepsilon,x_{\varepsilon,j}} + \varphi_\varepsilon, \tag{5.1}$$

where φ_ε satisfies

$$\int_\Omega |D\varphi_\varepsilon|^2 + \varphi_\varepsilon^2 = O\Big(\sum_{j=1}^{k} e^{-(1+\sigma)\sqrt{-f'(a)}d(x_{\varepsilon,j},\partial\Omega)/\varepsilon} + \sum_{i\neq j} e^{-(1+\sigma)\sqrt{-f'(a)}|x_{\varepsilon,j}-x_{\varepsilon,i}|/\varepsilon}\Big).$$

Furthermore, we have

(i) *if $k = 1$, the number of the interior single peak solution is at least $Cat(\Omega)$;*
(ii) *if $k \geq 2$, the number of the interior k peak solution is at least N.*

Remark 5.5. By Propositions 5.2 and 5.3, we see easily that for $\varepsilon > 0$ small, (1.2) has an interior peak solution of the form (5.1) such that $(x_{\varepsilon,1}, \cdots, x_{\varepsilon,k}) \to (x_{0,1}, \cdots, x_{0,k})$ and $(x_{0,1}, \cdots, x_{0,k})$ is a maximum point of the function:

$$\min\big(d(x_j, \partial\Omega), |x_i - x_j|, i, j = 1, \cdots, k, i \neq j\big).$$

Thus we see that the locations of the peaks of the positive interior peaks solutions of (1.2) are different from those for the Neumann problem, where there is a positive interior peak solution, whose peaks are near a maximum point of the function

$$\min\big(2d(x_j, \partial\Omega), |x_i - x_j|, i, j = 1, \cdots, k, i \neq j\big).$$

Appendix A.

Let u be a positive solution of (3.1). In this section, we shall prove that u decays exponentially as $|x| \to +\infty$. Since we do not assume that $f'(t)$ is negative in $[a, 0]$, we can not use the comparison theorem as usual to obtain the decay estimate. More work is needed.

Proposition A.1. *Suppose that u is a positive solution of (3.1). Then there are $C > 0$ and $\lambda_0 > 0$, such that $u(x) \le Ce^{-\sqrt{\lambda_0}|x|}$.*

Proof. Choose $R > 0$ large enough, such that $f'(\underline{u}(x_N)) < -2\mu_0 < 0$ for $x_N \ge R$. Here $\mu_0 > 0$ is a small constant. By standard comparison argument, we can get easily that

$$u(x) \le Ce^{-\sqrt{\mu_0}|x|}, \quad \forall\, x_N \ge R. \tag{A.1}$$

On the other hand, by Lemma 2.1, we have

$$\int_0^{+\infty} \left(|\xi'|^2 - f'(\underline{u}(t))\xi^2\right) dt \ge 4\lambda_0 \int_0^{+\infty} \left(|\xi'|^2 + \xi^2\right) dt, \quad \forall \xi \in H_0^1((0, +\infty)),$$

for some $\lambda_0 > 0$. So we see that the following problem has a positive solution $\xi(t)$:

$$\begin{cases} -\xi'' - f'(\underline{u}(t))\xi = 3\lambda_0\xi, & t \in (0, R), \\ \xi(0) = \xi(R) = 1. \end{cases} \tag{A.2}$$

In fact, let $\bar{\xi}(t) = \xi(t) - 1$. By Lax-Milgram theorem, we see that the following problem has a unique solution:

$$\begin{cases} -\bar{\xi}'' - f'(\underline{u}(t))\bar{\xi} - 3\lambda_0\bar{\xi} = -f'(\underline{u}(t)) - 3\lambda_0, & t \in (0, R), \\ \xi(0) = \xi(R) = 0. \end{cases}$$

Thus (A.2) has a unique solution. Besides, using $\varphi = \xi^- = \min(0, \xi) \in H_0^1((0, 1))$ as a test function, we see $\xi^- = 0$. Hence, ξ is nonnegative. Suppose that the minimum of ξ equals to 0, then $\xi' = 0$ at the minimum point. By the uniqueness of the ordinary differential equation, $\xi = 0$. This is a contradiction. So ξ is positive.

Let $v = Ce^{-\sqrt{\lambda_0}|x'|}\xi(x_N)$, $x_N \in [0, R]$. Then

$$\begin{aligned} & -\Delta v - \left(f'(\underline{u}(t)) + \lambda_0\right)v \\ =& \xi(x_N)\Big(-C\Delta e^{-\sqrt{\lambda_0}|x'|} + Ce^{-\sqrt{\lambda_0}|x'|}\frac{-\xi''(x_N) - \left(f'(\underline{u}(t)) + \lambda_0\right)\xi(x_N)}{\xi(x_N)}\Big) \\ =& \xi(x_N)\left(-C\Delta e^{-\sqrt{\lambda_0}|x'|} + 2\lambda_0 Ce^{-\sqrt{\lambda_0}|x'|}\right) > 0, \end{aligned}$$

if $|x'| \ge R_1 > 0$ is large enough.

Since $u \in H_0^1(R_+^N)$, we see that $u(x) \to 0$ as $|x| \to +\infty$. Thus we may choose $R_1 > 0$ large enough, such that $\frac{g_1(x,u(x))}{u(x)} \le \lambda_0$ for $|x'| \ge R_1$. As a result,

$$-\Delta u - \left(f'(\underline{u}(x_N)) + \lambda_0\right)u < 0, \quad \text{if}\, |x'| \ge R_1.$$

Thus we see

$$-\Delta(u - v) - \left(f'(\underline{u}(t)) + \lambda_0\right)(u - v) < 0, \quad \text{if}\, |x'| \ge R_1 \text{ and } x_N \in [0, R].$$

In view of (A.1), we see we can choose $C > 0$ large enough, such that $u(x) \le v(x)$ if $|x'| \le R_1$ and $x_N \in [0, R]$, or $x_N = 0$, or $x_N = R$ (We can always choose $\lambda_0 < \mu_0$). Let $\eta = (u - v)^+$ if $x_N \in [0, R]$, $\eta = 0$ if $x_N \ge R$. Because $u \le v$ for $x_N = R$, we see $\eta \in H_0^1(R_+^N)$. Thus

$$\begin{aligned} & \int_{R_+^N} \left(|D(u - v)^+|^2 - \left(f'(\underline{u}(t)) + \lambda_0\right)|(u - v)^+|^2\right) \\ =& \int_{\{|x'| \ge R_1\} \cap \{x_N \in [0, R]\}} \left(D(u - v)^+ D\eta - \left(f'(\underline{u}(t)) + \lambda_0\right)(u - v)^+\eta\right) \le 0, \end{aligned}$$

which implies $(u-v)^+ = 0$. That is $u \leq v$. Here we have used the natural generalization of Lemma 2.1 to the half space. □

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