VANISHING THEOREMS AND CHARACTER FORMULAS FOR THE HILBERT SCHEME OF POINTS IN THE PLANE

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ABSTRACT. In an earlier paper [13], we showed that the Hilbert scheme of points in the plane $H_n = \mathrm{Hilb}^n(\mathbb{C}^2)$ can be identified with the Hilbert scheme of regular orbits $\mathbb{C}^{2n}/\!/S_n$. Using this result, together with a recent theorem of Bridgeland, King and Reid [4] on the generalized McKay correspondence, we prove vanishing theorems for tensor powers of tautological bundles on the Hilbert scheme. We apply the vanishing theorems to establish (among other things) the character formula for diagonal harmonics conjectured by Garsia and the author in [9]. In particular we prove that the dimension of the space of diagonal harmonics is equal to $(n+1)^{n-1}$.

1. INTRODUCTION

In this article we continue the investigation begun in [13] of the geometry of the Hilbert scheme of points in the plane and its algebraic and combinatorial implications. In the earlier article, we showed that the isospectral Hilbert scheme has Gorenstein singularities, and used this to prove the "$n!$ conjecture" of Garsia and the author, and the positivity conjecture for Macdonald polynomials. Here we extend these results by proving vanishing theorems for tensor products of tautological vector bundles over the Hilbert scheme $H_n = \mathrm{Hilb}^n(\mathbb{C}^2)$ and over its zero fiber Z_n (the fiber over $\underline{0}$ of the Chow morphism $\sigma\colon H_n \to S^n\mathbb{C}^2$).

The algebraic-combinatorial consequence of the new results is a collection of character formulas for the spaces of global sections of the vector bundles in question. As a special case, we obtain the character formula for the space of diagonal harmonics, or equivalently, for the ring of coinvariants of the diagonal action of the symmetric group S_n on $\mathbb{C}^{2n}$. This character formula had been conjectured by Garsia and the author in [9], where we proved that it in turn implies a series of earlier conjectures in [10] relating the character of the diagonal harmonics to q-Lagrange inversion, q-Catalan numbers, and q-enumeration of rooted forests and parking functions. The formula implies that the dimension of the space of diagonal harmonics is

$$\dim DH_n = (n+1)^{n-1}. \tag{1}$$

It also implies that the Hilbert series of the doubly-graded space $(DH_n)^\epsilon$ of S_n-alternating diagonal harmonics is given by the q,t-Catalan polynomial $C_n(q,t)$

Date: May 1, 2001.

2000 *Mathematics Subject Classification.* Primary 14C05; Secondary 05E05, 14F17.

Key words and phrases. Macdonald polynomials, diagonal harmonics, coinvariants, Hilbert scheme, sheaf cohomology, vanishing theorem, McKay correspondence.

Research supported in part by N.S.F. grants DMS-9701218 and DMS-0070772 and the Isaac Newton Institute.

from [9, 11]. Hence $C_n(q,t)$ has positive integer coefficients. Recently, Garsia and Haglund [8] gave a different proof of this fact, based on a combinatorial interpretation of the coefficients. In [10] we also conjectured that the space of diagonal harmonics is generated by certain S_n-invariant polarization operators applied to the space of classical harmonics. We prove this "operator conjecture" here, using our identification of the coinvariant ring with the space of global sections of a vector bundle on Z_n.

To describe our results further, we first need to recall from [13] that H_n is isomorphic to the Hilbert scheme of orbits $\mathbb{C}^{2n}/\!/S_n$ for the diagonal action of S_n on $\mathbb{C}^{2n}$. Full definitions are in Section 2; for now we merely fix notation to announce our main theorems. On the Hilbert scheme H_n we have a natural *tautological vector bundle* B of rank n, while on $\mathbb{C}^{2n}/\!/S_n$ we have a tautological bundle P of rank $n!$, with an S_n action in which each fiber affords the regular representation. We can view both B and P as bundles on H_n via the isomorphism $H_n \cong \mathbb{C}^{2n}/\!/S_n$. The usual tautological bundle B is the pushdown to H_n of the sheaf $\mathcal{O}_{F_n}$ of regular functions on the universal family F_n over H_n. The "unusual" tautological bundle P may similarly be identified with the pushdown of the sheaf $\mathcal{O}_{X_n}$ of regular functions on the isospectral Hilbert scheme X_n, which is actually the universal family over $\mathbb{C}^{2n}/\!/S_n$.

Our first main result, Theorem 2.1, is a vanishing theorem for the higher cohomology groups $H^i(H_n, P \otimes B^{\otimes l})$, $i > 0$ of the tensor product of P with any tensor power of B. We also identify the space of global sections $H^0(H_n, P \otimes B^{\otimes l})$. The latter turns out to be the coordinate ring $R(n,l)$ of the polygraph, a subspace arrangement defined in [13], which plays an important technical role there and again here. This identification of $R(n,l)$ with $H^0(H_n, P \otimes B^{\otimes l})$ explains why the polygraph carries geometric information about the Hilbert scheme, an explanation which we were only able to hint at in [13]. Our theorem extends vanishing theorems of Danila [5] for the tautological bundle B and of Kumar and Thomsen [16] for the natural ample line bundles $\mathcal{O}_{H_n}(k)$, $k > 0$. Indeed, it implies the vanishing of the higher cohomology groups $H^i(H_n, \mathcal{O}(k) \otimes B^{\otimes l})$ for all $k, l \geq 0$. This is an immediate corollary, since the trivial bundle $\mathcal{O}_{H_n}$ is a direct summand of P, and the line bundle $\mathcal{O}_{H_n}(1)$ is the highest exterior power of B.

Our second main result, Theorem 2.2, is a vanishing theorem for the same vector bundles on the zero fiber Z_n. The vanishing part of this second theorem follows immediately from the first theorem, applied to an explicit locally free resolution of $\mathcal{O}_{Z_n}$ described in [11] and reviewed in detail in Section 2, below. By examining the resolution more closely, we can also identify the space of global sections $H^0(Z_n, P \otimes B^{\otimes l})$. When $l = 0$ it turns out that $H^0(Z_n, P)$ coincides with the coinvariant ring for the diagonal S_n action on $\mathbb{C}^{2n}$, yielding the applications to diagonal harmonics.

Character formulas for the spaces of global sections, and in particular for the diagonal harmonics, follow from our vanishing theorems by an application of the Atiyah–Bott Lefschetz formula [1]. The calculation completes a program proposed by Procesi, who was the first to suggest that the character of the diagonal harmonics might be determined this way. To carry out the calculation, we need to know the characters of the fibers of P at distinguished torus-fixed points I_μ on H_n. By our results in [13], these characters are given by the Macdonald polynomials. The character formulas we obtain here are therefore also expressed in terms of Macdonald polynomials. Specifically, they are symmetric functions with coefficients depending on two parameters q, t. By virtue of being characters, these symmetric functions are necessarily *q, t-Schur positive*, that is, they are linear combinations of Schur functions by polynomials or power series in q and t with positive integer coefficients. This partially establishes a positivity conjecture in [2]. The full conjecture in [2] is slightly stronger than what we obtain here. Its proof using the methods of this paper would require an improved vanishing theorem, which we offer as a conjecture at the end of Section 3.

Among our character formulas is one for the polygraph coordinate ring $R(n,l)$ as a doubly graded algebra. Specializing this, we get a formula for its Hilbert series $\mathcal{H}_{R(n,l)}(q,t)$ in terms of symmetric function operators whose eigenfunctions are Macdonald polynomials. A combinatorial interpretation of $\mathcal{H}_{R(n,l)}(q,t)$ is implicit in the basis construction for $R(n,l)$ in [13]. It can be made explicit (although we will not do so here), yielding an identity between a combinatorial generating function and the expression involving Macdonald operators in Corollary 3.9, below. This is one of only two combinatorial interpretations known at present for q, t-(Schur) positive expressions arising from our character formulas. The other is the Garsia–Haglund interpretation of $C_n(q,t)$ alluded to above. An important problem that remains open is to combinatorialize all the character formulas, and eventually the Kostka-Macdonald coefficients $K_{\lambda\mu}(q,t)$ as well.

In Section 2, after giving the relevant definitions, we state our two main theorems in full and then apply Theorem 2.1 to deduce Theorem 2.2. The character formulas and the operator conjecture follow from the vanishing theorems, as explained in Sections 3 and 4. For the proof of Theorem 2.1, we combine results from [13] with a recent general theorem of Bridgeland, King and Reid [4]. This is done in Section 5. To complete this introduction, we preview the proof of Theorem 2.1.

The Bridgeland–King–Reid theorem concerns the Hilbert scheme of orbits $V/\!/G$, for a finite subgroup $G \subseteq \mathrm{SL}(V)$. The theorem has two parts. The first part (which we will not use) is a criterion for $V/\!/G$ to be a *crepant resolution of singularities* of V/G, meaning that $V/\!/G$ is non-singular and its canonical sheaf is trivial. The second (and for us, crucial) part says that when the criterion holds there is an equivalence of categories $\Phi\colon D(V/\!/G) \to D^G(V)$. Here $D(V/\!/G)$ is the derived category of complexes of sheaves on $V/\!/G$ with bounded, coherent cohology, and $D^G(V)$ is the similar derived category of G-equivariant sheaves on V.

Our identification of $\mathbb{C}^{2n}/\!/S_n$ with H_n shows that the Bridgeland–King–Reid criterion holds for $V = \mathbb{C}^{2n}$, $G = S_n$. It is well-known that H_n is a crepant resolution of $\mathbb{C}^{2n}/S_n = S^n\mathbb{C}^2$, which is why we don't need the first part of their theorem. By the second part, however, we have an equivalence Φ between the derived category $D(H_n)$ of sheaves on the Hilbert scheme and the derived category $D^{S_n}(\mathbb{C}^{2n})$ of finitely generated S_n-equivariant modules over the polynomial ring $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ in $2n$ variables. In this notation, Theorem 2.1 reduces to an identity $\Phi B^{\otimes l} = R(n,l)$. Denoting the inverse equivalence by Ψ, we may rewrite this as $\Psi R(n,l) = B^{\otimes l}$, which is the form in which we prove it. The advantage of this form is that there is no sheaf cohomology involved in the calculation of Ψ, only commutative algebra. Conveniently, the commutative algebraic fact we need is precisely the freeness theorem for the polygraph ring $R(n,l)$, which was the key technical theorem in [13]. Thus we use here both the geometric results from [13] and the main algebraic ingredient in their proof.

In closing, let us remark that a number of important problems relating to this circle of ideas remain open. We have already mentioned the problem of combinatorializing the rest of the character formulas. Another set of problems involves phenomena in three or more sets of variables. We expect, for example, that the analog of the operator conjecture should continue to hold in additional sets of variables $\mathbf{x}, \mathbf{y}, \ldots, \mathbf{z}$. For exactly three sets of variables, we remind the reader of the empirical conjecture in [10] that the dimension of the space of "triagonal" harmonics should be

$$2^n(n+1)^{n-2}, \tag{2}$$

and that of its S_n-alternating subspace should be

$$(3n+3)(3n+4)\cdots(4n+1)/3\cdot 4\cdots(n+1). \tag{3}$$

Our present methods do not readily apply to these problems, as we make heavy use of special properties of the Hilbert scheme $\operatorname{Hilb}^n(\mathbb{C}^2)$ that do not hold for $\operatorname{Hilb}^n(\mathbb{C}^d)$ with $d \geq 3$. Another open problem is to generalize from S_n to other Weyl groups or complex reflection groups. Such a generalization will not be entirely straightforward, as shown by some obstacles discussed in [13] and [10]. Finally, despite the strength of the vanishing theorems proven here, they surely are not the strongest possible. The conjecture at the end of Section 3 suggests one possible improvement.

2. Definitions and main theorems

We denote by H_n the Hilbert scheme of points $\operatorname{Hilb}^n(\mathbb{C}^2)$ parametrizing 0-dimensional subschemes of length n in the affine plane over $\mathbb{C}$. By Fogarty's theorem [7], H_n is irreducible and non-singular, of dimension $2n$. As a matter of notation, if $V(I) \subseteq \mathbb{C}^2$ is the subscheme corresponding to a (closed) point of H_n, we refer to this point by its defining ideal $I \subseteq \mathbb{C}[x,y]$. Thus H_n is identified with the set of ideals I such that $\mathbb{C}[x,y]/I$ has dimension n as a complex vector space.

The *multiplicity* of a point $P \in V(I)$ is the length of the Artin local ring $(\mathbb{C}[x,y]/I)_P$. The multiplicities of all points in $V(I)$ sum to n, giving rise to a 0-dimensional algebraic cycle $\sum_i m_i P_i$ of weight $\sum_i m_i = n$. We may view this cycle as an unordered n-tuple $[\![P_1,\ldots,P_n]\!] \in S^n\mathbb{C}^2$, in which each point is repeated according to its multiplicity. The *Chow morphism*

$$\sigma\colon H_n \to S^n\mathbb{C}^2 = \mathbb{C}^{2n}/S_n \tag{4}$$

is the projective and birational morphism mapping each $I \in H_n$ to the corresponding algebraic cycle $\sigma(I) = [\![P_1,\ldots,P_n]\!]$.

We denote by F_n the *universal family* over the Hilbert scheme,

$$\begin{array}{c} F_n \subseteq H_n \times \mathbb{C}^2 \\ \pi \big\downarrow \quad\quad\quad \\ H_n, \quad\quad\quad \end{array} \tag{5}$$

whose fiber over a point $I \in H_n$ is the subscheme $V(I) \subseteq \mathbb{C}^2$. The universal family is flat and finite of degree n over H_n, and hence is given by $F_n = \operatorname{Spec} B$, where $B = \pi_* \mathcal{O}_{F_n}$ is a locally free sheaf of $\mathcal{O}_{H_n}$-algebras of rank n. Here and elsewhere we identify any locally free sheaf of rank r with the rank r algebraic vector bundle whose sheaf of sections it is. Then B is the *tautological vector bundle*, the quotient of the trivial bundle $\mathbb{C}[x,y] \otimes \mathcal{O}_{H_n}$ with fiber $\mathbb{C}[x,y]/I$ at each point $I \in H_n$.

If G is a finite subgroup of $\mathrm{GL}(V)$, where $V = \mathbb{C}^d$ is a finite-dimensional complex vector space, we denote by $V/\!/G$ the Hilbert scheme of regular G-orbits in V, as defined by Ito and Nakamura [14, 15]. Specifically, if $v \in V$ has trivial stabilizer (as is true for all v in a Zariski open set), then its orbit Gv is a point of $\mathrm{Hilb}^{|G|}(V)$, and $V/\!/G$ is the closure in $\mathrm{Hilb}^{|G|}(V)$ of the locus of all such points. By definition, $V/\!/G$ is irreducible. The universal family over $\mathrm{Hilb}^{|G|}(V)$ restricts to a universal family

$$\begin{array}{ccc} X & \subseteq & (V/\!/G) \times V \\ {\scriptstyle\rho}\big\downarrow & & \\ V/\!/G. & & \end{array} \tag{6}$$

The group G acts on X and on the tautological bundle $P = \rho_* \mathcal{O}_X$. This action makes P a vector bundle of rank $|G|$ whose fibers afford the regular representation of G. There is a canonical Chow morphism $V/\!/G \to V/G$, which can be conveniently defined as follows. Since P is a sheaf of $\mathcal{O}_{V/\!/G}$-algebras, it comes equipped with a homomorphism $\mathcal{O}_{V/\!/G} \to P$. This homomorphism is an isomorphism of $\mathcal{O}_{V/\!/G}$ onto the sheaf of invariants P^G. Geometrically, this means that the map $X/G \to V/\!/G$ induced by ρ is an isomorphism. The canonical projection $X \to V$ induces a morphism $X/G \to V/G$ whose composite with the isomorphism $V/\!/G \cong X/G$ yields the Chow morphism. The Chow morphism is projective and birational, restricting to an isomorphism on the open locus consisting of orbits Gv for v with trivial stabilizer.

The case of interest to us is $V = \mathbb{C}^{2n}$, $G = S_n$, where S_n acts on $\mathbb{C}^{2n} = (\mathbb{C}^2)^n$ by permuting the cartesian factors. This is the same as the diagonal action of S_n on the direct sum of two copies of its natural representation $\mathbb{C}^n$. Coordinates on $\mathbb{C}^{2n}$ will be denoted

$$\mathbf{x}, \mathbf{y} = x_1, y_1, \ldots, x_n, y_n; \tag{7}$$

then S_n acts by permuting the x variables and the y variables simultaneously. In [12] we constructed a canonical morphism $\mathbb{C}^{2n}/\!/S_n \to H_n$ such that the composite

$$\mathbb{C}^{2n}/\!/S_n \to H_n \xrightarrow{\sigma} S^n\mathbb{C}^2 \tag{8}$$

is the Chow morphism for $\mathbb{C}^{2n}/\!/S_n$. By Theorem 5.1 of [13], the canonical morphism is an isomorphism $\mathbb{C}^{2n}/\!/S_n \cong H_n$.

The universal family over $\mathbb{C}^{2n}/\!/S_n$ will be denoted X_n. We identify $\mathbb{C}^{2n}/\!/S_n$ with H_n by means of the canonical isomorphism, so that the projection ρ of the universal family onto $\mathbb{C}^{2n}/\!/S_n$ becomes a morphism from X_n to H_n. We have a commutative square

$$\begin{array}{ccc} X_n & \xrightarrow{f} & \mathbb{C}^{2n} \\ {\scriptstyle\rho}\big\downarrow & & \big\downarrow \\ H_n & \xrightarrow{\sigma} & S^n\mathbb{C}^2, \end{array} \tag{9}$$

in which $X_n \subseteq H_n \times \mathbb{C}^{2n}$ is the set-theoretic fiber product, with its induced reduced scheme structure. In other words, X_n is the isospectral Hilbert scheme, as defined in [13]. We again write $P = \rho_* \mathcal{O}_{X_n}$, as we did above for a general $V//G$. Now we regard P as a bundle on H_n rather than on $\mathbb{C}^{2n}//S_n$. Thus H_n has two different "tautological" bundles, the usual one B and the *unusual* one P. The unusual tautological bundle P has rank $n!$, with an S_n action affording the regular representation on every fiber. Our notation for the various schemes, bundles and morphisms just described is identical to that in [13].

The two-dimensional torus group

$$\mathbb{T}^2 = (\mathbb{C}^*)^2 \tag{10}$$

acts linearly on $\mathbb{C}^2$ as the group of 2×2 diagonal matrices. We write

$$\tau_{t,q} = \begin{bmatrix} t^{-1} & 0 \\ 0 & q^{-1} \end{bmatrix} \tag{11}$$

for its elements. Note that when a group G acts on a scheme V, elements $g \in G$ act on regular functions $f \in \mathcal{O}(V)$ as $gf = f \circ g^{-1}$. The inverses in (11) serve to make $\mathbb{T}^2$ act on the coordinate ring $\mathbb{C}[x, y]$ of $\mathbb{C}^2$ by the convenient rule

$$\tau_{t,q} x = tx; \quad \tau_{t,q} y = qy. \tag{12}$$

The action of $\mathbb{T}^2$ on $\mathbb{C}^2$ induces an action on the Hilbert scheme H_n and all other schemes under consideration. In particular, $\mathbb{T}^2$ acts on the universal family F_n and the isospectral Hilbert scheme X_n, so that the projections $\pi\colon F_n \to H_n$ and $\sigma\colon X_n \to H_n$ are equivariant. Hence $\mathbb{T}^2$ acts equivariantly on the vector bundles B and P. There are induced $\mathbb{T}^2$ actions on various algebraic spaces, such as the coordinate ring $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ of $\mathbb{C}^{2n}$, the space of global sections of any $\mathbb{T}^2$-equivariant vector bundle, or the fiber of such a bundle at a torus-fixed point in H_n. In these spaces, the $\mathbb{T}^2$ action is equivalently described by a $\mathbb{Z}^2$-grading. Namely, an element f is homogeneous of degree (r, s) if and only if it is a simultaneous eigenvector of the $\mathbb{T}^2$ action with weight $\tau_{t,q} f = t^r q^s f$. Where there is an obvious natural double grading, as in $\mathbb{C}[\mathbf{x}, \mathbf{y}]$, it coincides with the weight grading for the torus action.

We have now defined the bundles whose tensor products will be the subject of our vanishing theorems. The theorems also specify the spaces of global sections of the bundles in question. To identify these spaces, we first need to recall the definition of the *polygraph* $Z(n, l)$ from [13]. There, $Z(n, l)$ was defined as a certain union of linear subspaces in $\mathbb{C}^{2n+2l}$, but it is better here to describe it first from a Hilbert scheme point of view. Let

$$W = X_n \times F_n^l \,/\, H_n \tag{13}$$

be the fiber product over H_n of X_n with l copies of the universal family F_n. The scheme W is thus a closed subscheme of $H_n \times \mathbb{C}^{2n+2l}$, since we have $X_n \subseteq H_n \times \mathbb{C}^{2n}$ and $F_n \subseteq H_n \times \mathbb{C}^2$. We now define $Z(n, l) \subseteq \mathbb{C}^{2n+2l}$ to be the image of the projection of W on $\mathbb{C}^{2n+2l}$.

To see that this agrees with the original definition in [13], let us identify the set $Z(n, l)$ more directly. From (9), we see that a point of X_n is an ordered tuple $(I, P_1, \ldots, P_n) \in H_n \times \mathbb{C}^{2n}$ such that $\sigma(I) = [\![P_1, \ldots, P_n]\!]$. In particular, this implies $V(I) = \{P_1, \ldots, P_n\}$ as a set. A point of F is a pair $(I, Q) \in H_n \times \mathbb{C}^2$ such that $Q \in V(I)$. Hence a point of W is a tuple

$$(I, P_1, \ldots, P_n, Q_1, \ldots, Q_l) \tag{14}$$

such that $\sigma(I) = [\![P_1, \ldots, P_n]\!]$ and $Q_i \in \{P_1, \ldots, P_n\}$ for all $1 \le i \le l$. Projecting on $\mathbb{C}^{2n+2l}$, we see that

$$Z(n,l) = \{(P_1, \ldots, P_n, Q_1, \ldots, Q_l) \in \mathbb{C}^{2n+2l} : Q_i \in \{P_1, \ldots, P_n\}\ \forall i\}. \tag{15}$$

This is equivalent to the definition in [13]. The scheme W is flat over H_n and reduced over the generic locus (the open set in H_n where the P_i are all distinct). Hence W is reduced. The set-theoretic description we have just given of the projection of W on $Z(n,l)$ therefore also describes a morphism of schemes $W \to Z(n,l)$, in which we regard $Z(n,l)$ as a reduced closed subscheme of $\mathbb{C}^{2n+2l}$.

As in [13], the coordinate ring of the polygraph $Z(n,l)$ will be denoted $R(n,l)$. Writing

$$\mathbf{x}, \mathbf{y}, \mathbf{a}, \mathbf{b} = x_1, y_1, \ldots, x_n, y_n, a_1, b_1, \ldots, a_l, b_l \tag{16}$$

for the coordinates on $\mathbb{C}^{2n+2l}$, we see that $R(n,l)$ is the quotient of the polynomial ring $\mathbb{C}[\mathbf{x}, \mathbf{y}, \mathbf{a}, \mathbf{b}]$ by a suitable ideal $I(n,l)$. Given a global regular function on $Z(n,l)$, we may compose it with the projection $W \to Z(n,l)$ to get a global regular function on W, which is the same thing as a global section of $P \otimes B^{\otimes l}$ on H_n. Hence we have a canonical injective ring homomorphism

$$\psi \colon R(n,l) \hookrightarrow H^0(H_n, P \otimes B^{\otimes l}). \tag{17}$$

We can now state our first vanishing theorem, which will be proven in Section 5.

Theorem 2.1. *For all l we have*

$$H^i(H_n, P \otimes B^{\otimes l}) = 0 \quad \text{for } i > 0, \text{ and} \tag{18}$$

$$H^0(H_n, P \otimes B^{\otimes l}) = R(n,l), \tag{19}$$

where $R(n,l)$ is the coordinate ring of the polygraph $Z(n,l) \subseteq \mathbb{C}^{2n+2l}$.

The equal sign in (19) is to be understood as signifying that the homomorphism ψ in (17) is an isomorphism.

Our second vanishing theorem is the analog of Theorem 2.1 for the restriction of the tautological bundles to the *zero fiber* $Z_n = \sigma^{-1}(\{\underline{0}\}) \subseteq H_n$. In [11] we showed that the scheme-theoretic zero fiber is reduced, so there is no ambiguity as to the scheme structure of Z_n. The ideal of the origin $\{\underline{0}\} \subseteq S^n\mathbb{C}^2 = \mathbb{C}^{2n}/S_n$ is the homogeneous maximal ideal $\mathfrak{m} = \mathbb{C}[\mathbf{x}, \mathbf{y}]_+^{S_n}$ in the ring of invariants $\mathbb{C}[\mathbf{x}, \mathbf{y}]^{S_n}$. Pulled back to H_n via σ, the elements of $\mathfrak{m}$ represent global functions on H_n that vanish on Z_n. The bundle $P \otimes B^{\otimes l}$ is a sheaf of $\mathcal{O}_{H_n}$-algebras, so we have a canonical inclusion

$$H^0(H_n, \mathcal{O}_{H_n}) \subseteq H^0(H_n, P \otimes B^{\otimes l}). \tag{20}$$

Our choice of coordinates $\mathbf{x}, \mathbf{y}, \mathbf{a}, \mathbf{b}$ on $Z(n,l)$ identifies $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ and $\mathbb{C}[\mathbf{x}, \mathbf{y}]^{S_n}$ with subrings of $R(n,l)$, in such a way that the diagram

$$\begin{array}{ccc} H^0(H_n, \mathcal{O}_{H_n}) & \hookrightarrow & H^0(H_n, P \otimes B^{\otimes l}) \\ {\scriptstyle \sigma^*}\big\uparrow & & {\scriptstyle \psi}\big\uparrow \\ \mathbb{C}[\mathbf{x}, \mathbf{y}]^{S_n} & \hookrightarrow & R(n,l) \end{array} \tag{21}$$

commutes. It follows immediately that ψ maps every element of the ideal $\mathfrak{m}R(n,l)$ to a section of $P \otimes B^{\otimes l}$ that vanishes on Z_n. Composing ψ with restriction of sections to the zero fiber, we get a well-defined homomorphism

$$\psi_1\colon R(n,l)/\mathfrak{m}R(n,l) \to H^0(Z_n, P\otimes B^{\otimes l}). \tag{22}$$

A priori, ψ_1 need neither be injective nor surjective, but according to our next theorem, it is both.

Theorem 2.2. *For all l we have*

$$H^i(Z_n, P\otimes B^{\otimes l}) = 0 \quad \text{for } i>0, \text{ and} \tag{23}$$

$$H^0(Z_n, P\otimes B^{\otimes l}) = R(n,l)/\mathfrak{m}R(n,l), \tag{24}$$

where $R(n,l)$ is the polygraph coordinate ring and $\mathfrak{m}$ is the homogeneous maximal ideal in the subring $\mathbb{C}[\mathbf{x},\mathbf{y}]^{S_n} \subseteq R(n,l)$.

Again, the equal sign in (24) signifies that the homomorphism ψ_1 in (22) is an isomorphism.

In a sense, Theorem 2.2 is a corollary to Theorem 2.1. Its proof uses an $\mathcal{O}_{H_n}$-locally free resolution of $\mathcal{O}_{Z_n}$, which we now describe. Afterwards, we will prove that Theorem 2.1 implies Theorem 2.2. The resolution we construct will be $\mathbb{T}^2$-equivariant. To write it down we need a bit more notation. Let $\mathbb{C}_t$ and $\mathbb{C}_q$ denote the 1-dimensional representations of $\mathbb{T}^2$ on which $\tau_{t,q}\in\mathbb{T}^2$ acts by t and q, respectively. We write

$$\mathcal{O}_t = \mathbb{C}_t\otimes\mathcal{O}_{H_n}, \quad \mathcal{O}_q = \mathbb{C}_q\otimes\mathcal{O}_{H_n} \tag{25}$$

for $\mathcal{O}_{H_n}$ with its natural $\mathbb{T}^2$ action twisted by these 1-dimensional characters. The $\mathbb{T}^2$-equivariant sheaves $\mathcal{O}_t$ and $\mathcal{O}_q$ may be thought of as copies of $\mathcal{O}_{H_n}$ with respective degree shifts of $(1,0)$ and $(0,1)$.

There is a *trace homomorphism* of $\mathcal{O}_{H_n}$-modules

$$\operatorname{tr}\colon B\to\mathcal{O}_{H_n} \tag{26}$$

defined as follows. Let $\alpha\in B(U)$ be a section of B on some open set U. Since B is a sheaf of $\mathcal{O}_{H_n}$-algebras and also a vector bundle, there is a regular function $\operatorname{tr}(\alpha)\in\mathcal{O}_{H_n}(U)$ whose value at I is the trace of multiplication by α on the fiber $B(I)$. The sheaf B is a quotient of $\mathbb{C}[x,y]\otimes\mathcal{O}_{H_n}$, so it is generated by its global sections x^ry^s (*i.e.*, they span every fiber). The trace map is given on these sections by

$$\operatorname{tr}(x^ry^s) \;=\; p_{r,s}(\mathbf{x},\mathbf{y}) \underset{\text{def}}{=} \sum_{i=1}^n x_i^r y_i^s. \tag{27}$$

Here we regard the symmetric function $p_{r,s}\in\mathbb{C}[\mathbf{x},\mathbf{y}]^{S_n}$, called a *polarized power-sum*, as a global regular function on H_n pulled back from $S^n\mathbb{C}^2$ via the Chow morphism. To verify (27) we need only check it on points I in the generic locus, where the fiber $B(I)=\mathbb{C}[x,y]/I$ is the coordinate ring of a set of n distinct points $\{(x_1,y_1),\ldots,(x_n,y_n)\}\subseteq\mathbb{C}^2$. There it is clear that the eigenvalues of multiplication by x^ry^s in $B(I)$ are just $x_1^ry_1^s,\ldots,x_n^ry_n^s$. In particular, $\frac{1}{n}\operatorname{tr}(1)=1$, so

$$\frac{1}{n}\operatorname{tr}\colon B\to\mathcal{O}_{H_n} \tag{28}$$

is left inverse to the canonical inclusion $\mathcal{O}_{H_n}\hookrightarrow B$. Thus we have a direct-sum decomposition of $\mathcal{O}_{H_n}$-module sheaves, or of vector bundles,

$$B=\mathcal{O}_{H_n}\oplus B', \quad \text{where} \quad B'=\ker(\operatorname{tr}). \tag{29}$$

The projection of B on its summand B' is given by $\mathrm{id} - \frac{1}{n}\,\mathrm{tr}$, so from (27), we see that B' is generated by its global sections

$$x^r y^s - \frac{1}{n} p_{r,s}(\mathbf{x}, \mathbf{y}). \tag{30}$$

Here we can omit the section corresponding to $r = s = 0$, which is identically zero.

Let J be the sheaf of ideals in B generated by the global sections x and y and the subsheaf B'. An alternative way to describe J is as follows. There are $\mathbb{T}^2$-equivariant sheaf homomorphisms $\mathcal{O}_t \to B$ and $\mathcal{O}_q \to B$ sending the generating section 1 in $\mathcal{O}_t$ and $\mathcal{O}_q$ to x and y, respectively. Combining these with the inclusion $B' \hookrightarrow B$, we get a homomorphism of sheaves of $\mathcal{O}_{H_n}$-modules

$$\nu \colon B' \oplus \mathcal{O}_t \oplus \mathcal{O}_q \to B. \tag{31}$$

Now composing $1 \otimes \nu \colon B \otimes (B' \oplus \mathcal{O}_t \oplus \mathcal{O}_q) \to B \otimes B$ with the multiplication map $\mu \colon B \otimes B \to B$, we get a homomorphism of sheaves of B-modules

$$\xi \colon B \otimes (B' \oplus \mathcal{O}_t \oplus \mathcal{O}_q) \to B, \tag{32}$$

whose image is exactly J. Note that since x and y generate B as a sheaf of $\mathcal{O}_{H_n}$-algebras, the canonical homomorphism $\mathcal{O}_{H_n} \to B/J$ is surjective. Thus B/J is identified with a quotient of $\mathcal{O}_{H_n}$, which turns out to be $\mathcal{O}_{Z_n}$.

Proposition 2.3. *Let J be the sheaf of ideals in B generated by x, y and B'. Then B/J is isomorphic as a sheaf of $\mathcal{O}_{H_n}$-algebras to $\mathcal{O}_{Z_n}$.*

Let us recall the proof from [11, 12], skipping some details. Denote by Z_n' the set-theoretic preimage $\pi^{-1}(Z_n)$, regarded as a reduced closed subscheme of the universal family F_n. Clearly the regular functions x, y and $p_{r,s}(\mathbf{x}, \mathbf{y})$ for $r + s > 0$ vanish on Z_n'. By an old theorem of Weyl [25], the $p_{r,s}$ generate $\mathbb{C}[\mathbf{x}, \mathbf{y}]^{S_n}$, so their vanishing defines Z_n as a subscheme of H_n. Hence Z_n' is defined set-theoretically by the vanishing of x, y and all $p_{r,s}$, or equivalently of x, y, and every $x^r y^s - \frac{1}{n} p_{r,s}$. But these sections generate J, so the subscheme of F_n defined by the ideal sheaf $J \subseteq B$ coincides set-theoretically with Z_n'. We already know that $B/J \cong \mathcal{O}_Z$ for some subscheme $Z \subseteq H_n$, and this shows that Z coincides set-theoretically with Z_n. Now F_n is flat and finite over the non-singular scheme H_n, hence Cohen-Macaulay. Since Z_n' projects bijectively on Z_n, it has codimension $n+1$ in F_n. But $B' \oplus \mathcal{O}_t \oplus \mathcal{O}_q$ is locally free of rank $n + 1$, so J is everywhere locally generated by $n + 1$ elements. It follows that $\operatorname{Spec} B/J$ is a local complete intersection in F_n. Finally, one shows that $\operatorname{Spec} B/J$ is generically reduced, hence reduced, which implies $B/J \cong \mathcal{O}_{Z_n}$.

The point of reviewing this is to note that J is locally a complete intersection ideal in B generated by the image under ξ of any local basis of $B' \oplus \mathcal{O}_t \oplus \mathcal{O}_q$. Hence the Koszul complex on the map ξ in (32) is a resolution of $B/J \cong \mathcal{O}_{Z_n}$. Since everything in the construction is $\mathbb{T}^2$-equivariant we deduce the following result.

Proposition 2.4. *We have a $\mathbb{T}^2$-equivariant locally $\mathcal{O}_{H_n}$-free resolution*

$$\cdots \to B \otimes \wedge^k (B' \oplus \mathcal{O}_t \oplus \mathcal{O}_q) \to \cdots \to B \otimes (B' \oplus \mathcal{O}_t \oplus \mathcal{O}_q) \underset{\xi}{\to} B \to \mathcal{O}_{Z_n} \to 0, \tag{33}$$

where ξ is the sheaf homomorphism in (32).

As in [11], it follows as a corollary that the scheme-theoretic zero fiber is equal to the reduced zero fiber, and that it is Cohen-Macaulay.

Proof that Theorem 2.1 implies Theorem 2.2. Let $V_{\bullet}$ denote the complex in (33) with the final term $\mathcal{O}_{Z_n}$ deleted. The fact that (33) is a resolution means that $V_{\bullet}$ and $\mathcal{O}_{Z_n}$ are isomorphic as objects in the derived category $D(H_n)$. Here and below we work in the derived category of complexes of sheaves of $\mathcal{O}_{H_n}$-modules with bounded, coherent cohomology. Note that $V_{\bullet}$ is a complex of locally free sheaves, each of which is a sum of direct summands of tensor powers of B. It follows from Theorem 2.1 that $P \otimes B^{\otimes l} \otimes V_{\bullet}$ is a complex of acyclic objects for the global section functor Γ on H_n, so we have

$$R\Gamma(P \otimes B^{\otimes l} \otimes V_{\bullet}) = \Gamma(P \otimes B^{\otimes l} \otimes V_{\bullet}). \tag{34}$$

Now $P \otimes B^{\otimes l} \otimes V_{\bullet}$ is isomorphic to $P \otimes B^{\otimes l} \otimes \mathcal{O}_{Z_n}$ in $D(H_n)$, so $H^i(Z_n, P \otimes B^{\otimes l}) = R^i\Gamma(P \otimes B^{\otimes l} \otimes V_{\bullet})$ is the i-th cohomology of the complex in (34). This complex is zero in positive degrees, so we deduce that $H^i(Z_n, P \otimes B^{\otimes l}) = 0$ for $i > 0$, which is the first part of Theorem 2.2. This is just the standard argument for the higher cohomology vanishing of a sheaf with an acyclic left resolution. Since $H^i(Z_n, P \otimes B^{\otimes l})$ is zero in negative degrees, we also deduce that the complex in (34) is a resolution of $H^0(Z_n, P \otimes B^{\otimes l})$.

Consider the last terms in this resolution:

$$\Gamma(P \otimes B^{\otimes l+1} \otimes (B' \oplus \mathcal{O}_t \oplus \mathcal{O}_q)) \underset{\Gamma(1\otimes\xi)}{\rightarrow} R(n, l+1) \rightarrow H^0(Z_n, P \otimes B^{\otimes l}) \rightarrow 0. \tag{35}$$

Here we have identified $\Gamma(P \otimes B^{\otimes l} \otimes B)$ with $R(n, l+1)$ using Theorem 2.1. To keep the notation consistent, we denote the coordinates in $R(n, l+1)$ corresponding to the tensor factor B coming from $V_{\bullet}$ by x, y instead of the usual a_{l+1}, b_{l+1}. The subring of $R(n, l+1)$ generated by the remaining coordinates $\mathbf{x}, \mathbf{y}, \mathbf{a}, \mathbf{b}$ is just $R(n, l)$, since the projection of $Z(n, l+1)$ on these coordinates is $Z(n, l)$. The homomorphism $R(n, l+1) \rightarrow H^0(Z_n, P \otimes B^{\otimes l})$ sends x and y to zero and coincides on $R(n, l)$ with ψ_1 composed with the canonical map $R(n, l) \rightarrow R(n, l)/\mathfrak{m}R(n, l)$. Using Theorem 2.1 we can also identify $\Gamma(P \otimes B^{\otimes l+1} \otimes (B' \oplus \mathcal{O}_t \oplus \mathcal{O}_q))$ with

$$R(n, l+2)' \oplus R(n, l+1) \oplus R(n, l+1), \tag{36}$$

where $R(n, l+2)'$ is the direct summand $\Gamma(P \otimes B^{\otimes l+1} \otimes B')$ of $R(n, l+2) = \Gamma(P \otimes B^{\otimes l+1} \otimes B)$. In $R(n, l+2)$ we write x, y, x', y' for a_{l+1}, b_{l+1}, a_{l+2}, b_{l+2}. By (30), $R(n, l+2)'$ is the $R(n, l+1)$-submodule of $R(n, l+2)$ generated by all

$$(x')^r(y')^s - \frac{1}{n}p_{r,s}(\mathbf{x}, \mathbf{y}). \tag{37}$$

More precisely, $R(n, l+2)$ is generated as an $R(n, l+1)$-module by the monomials $(x')^r(y')^s$, and the projection on the summand $R(n, l+2)'$ is the homomorphism of $R(n, l+1)$ modules mapping $(x')^r(y')^s$ to the expression in (37). Although we are implicitly relying on Theorem 2.1 to guarantee that this is well defined, it can also be shown directly.

The map $\Gamma(1 \otimes \xi)$ in (35) now becomes the $R(n, l+1)$-module homomorphism

$$R(n, l+2)' \oplus R(n, l+1) \oplus R(n, l+1) \rightarrow R(n, l+1) \tag{38}$$

given on the first summand by $(x')^r(y')^s \mapsto x^r y^s$ and on the second and third summands by multiplication by x and y, respectively. Its image is therefore the ideal in $R(n, l+1)$ generated by x, y and all $x^r y^s - \frac{1}{n}p_{r,s}(\mathbf{x}, \mathbf{y})$, or equivalently, the ideal

$$J = (x, y) + \mathfrak{m}R(n, l+1). \tag{39}$$

Since x and y generate $R(n,l+1)$ as an $R(n,l)$-module, the inclusion $R(n,l) \subseteq R(n,l+1)$ induces a surjective ring homomorphism

$$R(n,l) \to R(n,l+1)/J \tag{40}$$

with kernel

$$I = R(n,l) \cap J. \tag{41}$$

By (35), we have $R(n,l)/I \cong R(n,l+1)/J \cong H^0(Z_n, P \otimes B^{\otimes l})$. The isomorphism here is induced by ψ_1. Thus it only remains to show that $I = \mathfrak{m}R(n,l)$.

Clearly, I contains $\mathfrak{m}R(n,l)$, so we are to show that the homomorphism

$$\zeta\colon R(n,l)/\mathfrak{m}R(n,l) \to R(n,l)/I \cong R(n,l+1)/J \tag{42}$$

is injective. For this we construct its left inverse. From the equation $R(n,l+1) = \Gamma(P \otimes B^{\otimes l+1})$ and the decomposition $B = \mathcal{O}_{H_n} \oplus B'$, taken in the last tensor factor B, we see that $R(n,l)$ is a direct summand of $R(n,l{+}1)$ as an $R(n,l)$-module. Using (27) and (28), we obtain the formula

$$\theta(x^r y^s) = \frac{1}{n} p_{r,s}(\mathbf{x}, \mathbf{y}) \tag{43}$$

for the projection $\theta\colon R(n,l+1) \to R(n,l)$. Now, θ is a homomorphism of $R(n,l)$-modules and $\mathfrak{m}$ is generated by a subset of $R(n,l)$, so θ carries $\mathfrak{m}R(n,l+1)$ into $\mathfrak{m}R(n,l)$. The monomials $x^r y^s$ with $r+s>0$ generate $(x,y)R(n,l+1)$ as an $R(n,l)$-module, so (43) shows that θ also carries $(x,y)R(n,l+1)$ into $\mathfrak{m}R(n,l)$. Hence θ induces a map

$$\overline{\theta}\colon R(n,l+1)/J \to R(n,l)/\mathfrak{m}R(n,l). \tag{44}$$

The endomorphism $\overline{\theta} \circ \zeta$ of $R(n,l)/\mathfrak{m}R(n,l)$ is a homomorphism of $R(n,l)$-modules, so it is the identity, and $\overline{\theta}$ is the required left inverse of ζ. $\square$

3. Character formulas

Theorems 2.1 and 2.2 allow us to identify the ring of diagonal coinvariants and the polygraph coordinate ring $R(n,l)$, among other things, with spaces of global sections of $\mathbb{T}^2$-equivariant coherent sheaves on H_n. When the higher cohomology vanishes, we can calculate the $\mathbb{T}^2$ character of the space of global sections, or what is the same, its Hilbert series as a doubly graded module, using the Lefschetz formula of Atiyah and Bott [1]. We will apply this method to obtain explicit character formulas for spaces of interest in the Hilbert scheme context. As we shall see, the resulting formulas are naturally expressed in terms of operators arising in the theory of Macdonald polynomials.

Let $M = \bigoplus M_{r,s}$ be a finitely-generated doubly graded module over $\mathbb{C}[\mathbf{x},\mathbf{y}]$ or $\mathbb{C}[\mathbf{x},\mathbf{y}]^{S_n}$. The *Hilbert series* of M is the Laurent series in two variables

$$\mathcal{H}_M(q,t) = \sum_{r,s} t^r q^s \dim(M_{r,s}). \tag{45}$$

If M is finite-dimensional as a vector space over $\mathbb{C}$, then $\mathcal{H}_M(q,t) = \operatorname{tr}(M, \tau_{t,q})$ is the character of M as a $\mathbb{T}^2$-module in the strict sense. In general, it is a good idea to think of $\mathcal{H}_M(q,t)$ as a formal $\mathbb{T}^2$ character, for reasons that will become apparent below. The Laurent series $\mathcal{H}_M(q,t)$ is a rational function of q and t. When M is a $\mathbb{C}[\mathbf{x},\mathbf{y}]$-module this is well-known and can be shown easily by calculating the Hilbert series using a finite graded free resolution of M. When M is a $\mathbb{C}[\mathbf{x},\mathbf{y}]^{S_n}$-module, one obtains the same result by regarding M as a module over $\mathbb{C}[p_1(\mathbf{x}),p_1(\mathbf{y}),\dots,p_n(\mathbf{x}),p_n(\mathbf{y})]$, since the power sums $p_k(\mathbf{x})$, $p_k(\mathbf{y})$ form a doubly homogeneous system of parameters in $\mathbb{C}[\mathbf{x},\mathbf{y}]^{S_n}$. Now let A be a $\mathbb{T}^2$-equivariant coherent sheaf on H_n. The Chow morphism $\sigma\colon H_n \to S^n\mathbb{C}^2$ is projective, and $S^n\mathbb{C}^2$ is affine, so the sheaf cohomology modules $H^i(H_n,A)$ are finitely-generated $\mathbb{T}^2$-equivariant—which is to say, doubly graded—$\mathbb{C}[\mathbf{x},\mathbf{y}]^{S_n}$-modules. We denote their Hilbert series by

$$\mathcal{H}^i_A(q,t) = \mathcal{H}_{H^i(H_n,A)}(q,t). \tag{46}$$

The Atiyah–Bott formula expresses the Euler characteristic

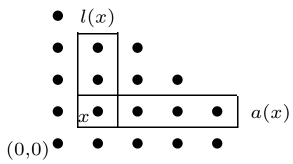
$$\chi_A(q,t) \underset{\text{def}}{=} \sum_i (-1)^i \mathcal{H}^i_A(q,t) \tag{47}$$

as a sum of local contributions from the $\mathbb{T}^2$-fixed points of H_n. These local contributions are described by data associated with partitions of n. Let us fix some notation. We write a partition of n as $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_l > 0)$, with the understanding that $\mu_i = 0$ for $i > l$. The *Ferrers diagram* of μ is the set of lattice points

$$d(\mu) = \{(i,j) \in \mathbb{N}\times\mathbb{N} : j < \mu_{i+1}\}. \tag{48}$$

The *arm* $a(x)$ and *leg* $l(x)$ of a point $x \in d(\mu)$ denote the number of points strictly to the right of x and above x, respectively, as indicated in this example:

(49) $\mu = (5,5,4,3,1)$



$a(x) = 3, \quad l(x) = 2.$

To each partition μ is associated a monomial ideal

$$I_\mu = \mathbb{C}\cdot\{x^r y^s : (r,s) \notin d(\mu)\} \subseteq \mathbb{C}[x,y]. \tag{50}$$

A $\mathbb{C}$-basis of $\mathbb{C}[x,y]/I_\mu$ is given by the set of monomials not in I_μ,

$$\mathcal{B}_\mu = \{x^r y^s : (r,s) \in d(\mu)\}. \tag{51}$$

In particular, $\dim_{\mathbb{C}} \mathbb{C}[x,y]/I_\mu = n$, so I_μ is a point of H_n.

Proposition 3.1. *The $\mathbb{T}^2$-fixed points of H_n are the ideals I_μ for all partitions μ of n. The cotangent space of H_n at I_μ has a basis of $\mathbb{T}^2$-eigenvectors $\{u_x, d_x : x \in d(\mu)\}$ with eigenvalues*

$$\tau_{t,q} d_x = t^{1+l(x)} q^{-a(x)} d_x, \quad \tau_{t,q} u_x = t^{-l(x)} q^{1+a(x)} u_x. \tag{52}$$

Proof. An ideal $I \subseteq \mathbb{C}[x,y]$ is $\mathbb{T}^2$-fixed if and only if it is doubly homogeneous, or equivalently, a monomial ideal. This establishes the first part. The eigenvalues (expressed somewhat differently) were determined by Ellingsrud and Strömme [6]. The basis elements u_x, d_x are given explicitly in terms of local coordinates in [11, Corollary 2.5]. □

Now we give the Atiyah–Bott formula as it applies in our context. For simplicity we only state it for vector bundles, *i.e.*, locally free sheaves, which is all we need.

Proposition 3.2. *Let A be a $\mathbb{T}^2$-equivariant locally free sheaf of finite rank on H_n. Then*

$$\chi_A(q,t) = \sum_{|\mu|=n} \frac{\mathcal{H}_{A(I_\mu)}(q,t)}{\prod_{x\in d(\mu)}(1-t^{1+l(x)}q^{-a(x)})(1-t^{-l(x)}q^{1+a(x)})}. \tag{53}$$

Proof. What we have written is the classical formula in Theorem 2 of [1], evaluated on the data in Proposition 3.1. Since H_n is not a projective variety, however, and the left-hand side in (53) is only a formal $\mathbb{T}^2$ character, some further justification is required. Various authors have extended the classical formula to more general contexts and given algebraic proofs. We will use the following corollary to a very general theorem of Thomason [24, Théorème 3.5].

Proposition 3.3. *Let $T = \mathbb{T}^d = \operatorname{Spec}\mathbb{C}[t_1,t_1^{-1},\dots,t_d,t_d^{-1}]$ be an algebraic torus, X and Y separated schemes of finite type over $\mathbb{C}$ on which T acts, and $f\colon X\to Y$ a T-equivariant proper morphism. Assume X is non-singular. Let $K_0(T,X)$, $K_0(T,Y)$, etc. denote the Grothendieck groups of T-equivariant coherent sheaves, and $K^0(T,X)$, etc. the Grothendieck rings of T-equivariant algebraic vector bundles. Recall that (for any X) $K_0(T,X)$ is a $K^0(T,X)$-module and $K^0(T,X)$ is an algebra over the representation ring $R(T)$, which we identify with $\mathbb{Z}[\mathbf{t},\mathbf{t}^{-1}] = \mathbb{Z}[t_1,t_1^{-1},\dots,t_d,t_d^{-1}]$. Define*

$$K_0(T,X)_{(0)} = \mathbb{Q}(\mathbf{t})\otimes_{\mathbb{Z}[\mathbf{t},\mathbf{t}^{-1}]} K_0(T,X), \tag{54}$$

and similarly for Y, etc.. Then the following hold.

(1) Let N be the conormal bundle of the fixed-point locus X^T in X, and set $\wedge N = \sum_i(-1)^i[\wedge^i N]\in K^0(T,X^T)$. Then $\wedge N$ is invertible in $K^0(T,X^T)_{(0)}$.

(2) Let $f_*\colon K_0(T,X)_{(0)}\to K_0(T,Y)_{(0)}$ be the homomorphism induced by the derived pushforward, that is, $f_*[A]=\sum_i(-1)^i[R^if_*A]$, and let $f_*^T\colon K_0(T,X^T)_{(0)}\to K_0(T,Y^T)_{(0)}$ denote the same for the fixed-point loci. Then

$$f_*[A] = i_*f_*^T\left((\wedge N)^{-1}\cdot\sum_k(-1)^k[\operatorname{Tor}_k^{\mathcal{O}_X}(\mathcal{O}_{X^T},A)]\right), \tag{55}$$

where $i_*\colon K_0(T,Y^T)_{(0)}\to K_0(T,Y)_{(0)}$ is induced by $i\colon Y^T\hookrightarrow Y$.

To obtain (53), we apply Thomason's theorem with $T=\mathbb{T}^2$ and $f\colon X\to Y$ equal to the Chow morphism $\sigma\colon H_n\to S^n\mathbb{C}^2$. The group $K_0(T,Y)$ is identified with the Grothendieck group of finitely-generated doubly graded $\mathbb{C}[\mathbf{x},\mathbf{y}]^{S_n}$-modules. The Hilbert series $\mathcal{H}_M(q,t)$ only depends on the class $[M]\in K_0(T,Y)$ of M, and so induces a $\mathbb{Z}[q,q^{-1},t,t^{-1}]$-linear map

$$\mathcal{H}\colon K_0(T,Y)_{(0)}\to\mathbb{Q}(q,t). \tag{56}$$

The fixed-point locus Y^T is a point, so $K_0(T,Y^T)_{(0)}=\mathbb{Q}(q,t)$, and $\mathcal{H}\circ i_*$ is the identity map on $\mathbb{Q}(q,t)$. Similarly, X^T is the finite set $\{I_\mu : |\mu|=n\}$ and $K_0(T,X^T)_{(0)}$ is the direct sum of copies of $\mathbb{Q}(q,t)$, one for each μ. With these identifications, f_*^T is just summation over μ. Applying $\mathcal{H}$ to both sides in (55) yields (53). ☐

Some of our sheaves and spaces have S_n actions, so we need to sharpen our notation a bit to keep track of it. Recall that the *Frobenius characteristic map* from S_n characters to symmetric functions is defined by

$$\phi\chi = \frac{1}{n!}\sum_{w\in S_n} \chi(w) p_{\tau(w)}(z), \tag{57}$$

where $\tau(w)$ is the partition of n given by the disjoint cycle lengths of the permutation w, and $p_\lambda(z) = p_{\lambda_1}\cdots p_{\lambda_l}(z)$ denotes the power-sum symmetric function. The irreducible characters of S_n are then given by the identity

$$\phi\chi^\lambda = s_\lambda(z), \tag{58}$$

where $s_\lambda(z)$ is a Schur function. Here and below we always work in the algebra

$$\Lambda = \Lambda_{\mathbb{Q}(q,t)}(z) \tag{59}$$

of symmetric function in infinitely many variables $z = z_1, z_2, \ldots$ with coefficients in $\mathbb{Q}(q,t)$. As λ runs over partitions of n, the power-sums $p_\lambda(z)$, Schur functions $s_\lambda(z)$, Macdonald polynomials $P_\lambda(z;q,t)$, and so forth are bases of the homogeneous subspace Λ_n of degree n in Λ. Occasionally below we will use *plethystic substitution*, also known as λ-ring notation. Let A be an algebra of polynomials or formal series in some indeterminates U with coefficients in $\mathbb{Q}(q,t)$. Given $Y \in A$, we define $p_k[Y]$ to be the result of replacing each indeterminate in Y, including q and t, with its k-th power. The algebra Λ is freely generated over $\mathbb{Q}(q,t)$ by the power-sums $p_k(z)$, so there is a unique $\mathbb{Q}(q,t)$-linear homomorphism

$$\mathrm{ev}_Y\colon \Lambda \to A, \quad \mathrm{ev}_Y\, p_k(z) = p_k[Y]. \tag{60}$$

We now define for all $f \in \Lambda$, $Y \in A$:

$$f[Y] = \mathrm{ev}_Y\, f(z). \tag{61}$$

We will specifically need the following instances of this construction.

- Setting (here and throughout) $Z = z_1 + z_2 + \cdots$, we recover $f(z) = f[Z]$.
- $f\left[\frac{Z}{1-t}\right]$ is the image of f under the automorphism of Λ sending $p_k(z)$ to $p_k(z)/(1-t^k)$. We can equate $f\left[\frac{Z}{1-t}\right]$ with $f(z, tz, t^2z, \ldots)$, provided we interpret the coefficients of the latter expression, which are rational Laurent series in t, with rational functions. The same holds with q in place of t.
- If $Y = a_1 + \cdots + a_k$ is a sum of monomials a_i in the indeterminates, each with coefficient 1, then $f[Y] = f(a_1, \ldots, a_k)$.

Now let M be a finitely-generated doubly graded $\mathbb{C}[\mathbf{x},\mathbf{y}]^{S_n}$-module with an S_n action that respects the grading, that is, commutes with the $\mathbb{T}^2$ action. For instance, M might be a doubly graded $\mathbb{C}[\mathbf{x},\mathbf{y}]$ module with an equivariant S_n action, regarded as a $\mathbb{C}[\mathbf{x},\mathbf{y}]^{S_n}$-module. We denote by V^λ the irreducible representation of S_n with character χ^λ. Then M has a canonical direct-sum decomposition

$$M = \bigoplus_{|\lambda|=n} V^\lambda \otimes M_\lambda, \quad M_\lambda \underset{\mathrm{def}}{=} \mathrm{Hom}^{S_n}(V^\lambda, M), \tag{62}$$

in which each M_λ is a doubly graded $\mathbb{C}[\mathbf{x},\mathbf{y}]^{S_n}$-module. We define the *Frobenius series* of M to be

$$\mathcal{F}_M(z;q,t) \underset{\mathrm{def}}{=} \sum_{|\lambda|=n} \mathcal{H}_{M_\lambda}(q,t) s_\lambda(z) = \sum_{r,s} t^r q^s \phi\, \mathrm{char}(M_{r,s}). \tag{63}$$

The last expression follows from (58) and shows that the Frobenius series is a generating function for the characters $\mathrm{char}(M_{r,s})$ in the same way that the Hilbert series is a generating function for the dimensions. The Hilbert series can be recovered from the Frobenius series by the formula

$$\mathcal{H}_M(q,t) = \langle s_1^n, \mathcal{F}_M(z;q,t)\rangle, \tag{64}$$

where $\langle \cdot,\cdot\rangle$ is the usual Hall inner product on symmetric functions.

If A is a $\mathbb{T}^2$-equivariant coherent sheaf on H_n with an S_n action commuting with the $\mathbb{T}^2$ action, then A has a decomposition

$$A = \bigoplus_{|\lambda|=n} V^\lambda \otimes_{\mathbb{C}} A_\lambda \tag{65}$$

as in (62), inducing the decomposition (62) for the cohomology modules $M = H^i(H_n, A)$. We set

$$\mathcal{F}^i_A(z;q,t) = \mathcal{F}_{H^i(H_n,A)}(z;q,t), \quad \chi\mathcal{F}_A(z;q,t) = \sum_i (-1)^i \mathcal{F}^i_A(q,t). \tag{66}$$

Then for A locally free we immediately obtain the Frobenius series version of (53):

$$\chi\mathcal{F}_A(q,t) = \sum_{|\mu|=n} \frac{\mathcal{F}_{A(I_\mu)}(q,t)}{\prod_{x\in d(\mu)}(1-t^{1+l(x)}q^{-a(x)})(1-t^{-l(x)}q^{1+a(x)})}. \tag{67}$$

Let us now evaluate this in some specific cases.

Character formula for $R(n,l)$. Taking $A = P\otimes B^{\otimes l}$, the S_n action on A is induced by that on P. By Theorem 2.1, we have

$$\mathcal{F}_{R(n,l)}(z;q,t) = \chi\mathcal{F}_A(z;q,t). \tag{68}$$

To calculate this using (67) we must evaluate

$$\mathcal{F}_{(P\otimes B^{\otimes l})(I_\mu)}(z;q,t) = \mathcal{F}_{P(I_\mu)}(z;q,t)\mathcal{H}_{B(I_\mu)}(q,t)^l. \tag{69}$$

The set $\mathcal{B}_\mu$ in (51) is a doubly homogeneous basis of $B(I_\mu) = \mathbb{C}[x,y]/I_\mu$, so we have

$$\mathcal{H}_{B(I_\mu)}(q,t) = B_\mu(q,t) \underset{\text{def}}{=} \sum_{(r,s)\in d(\mu)} t^r q^s. \tag{70}$$

The Frobenius series of $P(I_\mu)$ is given by the *transformed Macdonald polynomial*

$$\tilde{H}_\mu(z;q,t) \underset{\text{def}}{=} t^{n(\mu)} J_\mu\left[\tfrac{Z}{1-t^{-1}}; q, t^{-1}\right], \tag{71}$$

where J_μ is the *integral form* Macdonald polynomial defined in [17, VI, eq. (8.3)], and $n(\mu) = \sum_i (i-1)\mu_i$. Equivalently,

$$\tilde{H}_\mu(z;q,t) = \sum_\lambda \tilde{K}_{\lambda\mu}(q,t)s_\lambda(z), \quad \tilde{K}_{\lambda\mu}(q,t) = t^{n(\mu)}K_{\lambda\mu}(q,t^{-1}), \tag{72}$$

where $K_{\lambda\mu}(q,t)$ is the Kostka–Macdonald coefficient [17, VI, eq. (8.11)].

Proposition 3.4 ([13])**.** *We have $\mathcal{F}_{P(I_\mu)}(z;q,t) = \tilde{H}_\mu(z;q,t)$.*

The identity in the proposition is equivalent to $\tilde{K}_{\lambda\mu}(q,t) = \mathcal{H}_{P_\lambda(I_\mu)}(q,t)$, where $P = \bigoplus_\lambda V^\lambda \otimes P_\lambda$ is the decomposition in (65). As a corollary, we have $\tilde{K}_{\lambda\mu}(q,t) \in \mathbb{N}[q,t]$, the proof of which was the main combinatorial objective in [13]. The bundles P_λ are called *character sheaves*. We have established the following result.

Theorem 3.5. *The Frobenius series of $R(n,l)$ is given by*

$$\mathcal{F}_{R(n,l)}(z;q,t) = \sum_{|\mu|=n} \frac{B_\mu(q,t)^l \tilde{H}_\mu(z;q,t)}{\prod_{x\in d(\mu)}(1-t^{1+l(x)}q^{-a(x)})(1-t^{-l(x)}q^{1+a(x)})}. \tag{73}$$

We can express this more succinctly in terms of the linear operator Δ on Λ defined by

$$\Delta \tilde{H}_\mu(z;q,t) = B_\mu(q,t)\tilde{H}_\mu(z;q,t). \tag{74}$$

This operator was introduced in [9], where we gave a direct plethystic expression for it [*op. cit.*, Theorem 2.2].

Lemma 3.6. *Let M be a finitely-generated doubly graded $\mathbb{C}[\mathbf{x},\mathbf{y}]$-module with an equivariant S_n action. If the $\mathbf{x}$ variables $x_1,\dots,x_n$ form an M-regular sequence, then*

$$\mathcal{F}_M(z;q,t) = \mathcal{F}_{M/(\mathbf{x})M}\left[\tfrac{Z}{1-t};q,t\right], \tag{75}$$

and similarly with $\mathbf{y}$ and q in place of $\mathbf{x}$ and t.

Proof. For a module over a local ring, this was proven in [12, Proposition 5.3]. The same proof applies in the graded setting essentially without change. □

Lemma 3.7. *The Frobenius series of $\mathbb{C}[\mathbf{x},\mathbf{y}]$ is given by*

$$\mathcal{F}_{\mathbb{C}[\mathbf{x},\mathbf{y}]}(z;q,t) = h_n\left[\tfrac{Z}{(1-q)(1-t)}\right]. \tag{76}$$

Proof. Apply Lemma 3.6 first to the regular sequence $\mathbf{x}$ in $\mathbb{C}[\mathbf{x},\mathbf{y}]$, then to $\mathbf{y}$ in $\mathbb{C}[\mathbf{y}]$. This reduces (76) to $\mathcal{F}_{\mathbb{C}}(z;q,t) = h_n(z) = s_{(n)}(z)$, which is correct since $\mathbb{C}$ is the trivial representation in degree $(0,0)$. □

Corollary 3.8. *The formula* (73) *in Theorem 3.5 is equivalent to*

$$\mathcal{F}_{R(n,l)}(z;q,t) = \Delta^l h_n\left[\tfrac{Z}{(1-q)(1-t)}\right]. \tag{77}$$

Proof. From (73) it is clear that $\mathcal{F}_{R(n,l)}(z;q,t) = \Delta^l \mathcal{F}_{R(n,0)}(z;q,t)$. But $R(n,0) = \mathbb{C}[\mathbf{x},\mathbf{y}]$. □

Note that the case $l=0$ gives a geometric interpretation and proof of one of the basic identities in the theory of Macdonald polynomials [9, Theorem 2.8]:

$$h_n\left[\tfrac{Z}{(1-q)(1-t)}\right] = \sum_{|\mu|=n} \frac{\tilde{H}_\mu(z;q,t)}{\prod_{x\in d(\mu)}(1-t^{1+l(x)}q^{-a(x)})(1-t^{-l(x)}q^{1+a(x)})}. \tag{78}$$

From the preceding corollary we obtain a formula for the Hilbert series of $R(n,l)$.

Corollary 3.9. *We have*

$$\mathcal{H}_{R(n,l)} = \langle s_1^n(z), \Delta^l h_n\left[\tfrac{Z}{(1-q)(1-t)}\right]\rangle \tag{79}$$

$$= \frac{1}{(1-q)^n(1-t)^n}\langle e_n(z), \Delta^l s_1^n(z)\rangle, \tag{80}$$

where $e_n(z)$ is the n-th elementary symmetric function.

Proof. The first equation is immediate from Corollary 3.8. For the second, recall from [9] that the transformed Macdonald polynomials are orthogonal with respect to the inner product

$$\langle f, g\rangle_* \underset{\text{def}}{=} \langle \omega f[Z(1-q)(1-t)], g\rangle, \tag{81}$$

where ω is the usual involution on Λ defined by $\omega e_k(z) = h_k(z)$. Any operator with the $\tilde{H}_\mu(z;q,t)$ as eigenfunctions, and Δ in particular, is therefore self-adjoint with respect to $\langle\cdot,\cdot\rangle_*$. Hence

$$\begin{aligned}
\langle s_1^n(z), \Delta^l h_n\left[\tfrac{Z}{(1-q)(1-t)}\right]\rangle &= \langle \omega s_1^n\left[\tfrac{Z}{(1-q)(1-t)}\right], \Delta^l h_n\left[\tfrac{Z}{(1-q)(1-t)}\right]\rangle_* \\
&= \frac{1}{(1-q)^n(1-t)^n}\langle \Delta^l s_1^n(z), h_n\left[\tfrac{Z}{(1-q)(1-t)}\right]\rangle_* \\
&= \frac{1}{(1-q)^n(1-t)^n}\langle e_n(z), \Delta^l s_1^n(z)\rangle.
\end{aligned} \tag{82}$$

□

Character formula for diagonal coinvariants. The *ring of coinvariants* for the diagonal action of S_n on $\mathbb{C}^{2n}$ is, by definition,

$$R_n = \mathbb{C}[\mathbf{x},\mathbf{y}]/\mathfrak{m}\mathbb{C}[\mathbf{x},\mathbf{y}], \tag{83}$$

where $\mathfrak{m}$ is the homogeneous maximal ideal in $\mathbb{C}[\mathbf{x},\mathbf{y}]^{S_n}$. Ignoring its ring structure, R_n is isomorphic as a doubly graded S_n-module to the space of *diagonal harmonics*

$$DH_n = \{f \in \mathbb{C}[\mathbf{x},\mathbf{y}] : p(\partial\mathbf{x},\partial\mathbf{y})f = 0\ \forall p \in \mathfrak{m}\}. \tag{84}$$

Its Frobenius series was the subject of a series of combinatorial conjectures by the author and others in [10]. Later, in [9], Garsia and the author showed that these conjectures would follow from a conjectured master formula giving $\mathcal{F}_{R_n}(z;q,t)$ in terms of Macdonald polynomials, which we will now prove.

From Theorem 2.2, with $l = 0$, we obtain

$$\mathcal{F}_{R_n}(z;q,t) = \chi\mathcal{F}_{P\otimes\mathcal{O}_{Z_n}}(z;q,t). \tag{85}$$

To calculate this using (67), we replace $\mathcal{O}_{Z_n}$ with the resolution $V_\bullet$ given by the complex in (33) with the final term deleted. This gives

$$\chi\mathcal{F}_{P\otimes\mathcal{O}_{Z_n}}(z;q,t) = \sum_{k=0}^{n+1}(-1)^k\chi\mathcal{F}_{P\otimes V_k}(z;q,t), \tag{86}$$

where $V_k = B\otimes\wedge^k(B'\oplus\mathcal{O}_t\oplus\mathcal{O}_q)$. The eigenvalues of $\tau_{t,q}\in\mathbb{T}^2$ on the fiber $(B'\oplus\mathcal{O}_t\oplus\mathcal{O}_q)(I_\mu)$ are q and t, from the summand $\mathcal{O}_t\oplus\mathcal{O}_q$, and $\{t^rq^s : (r,s)\in d(\mu)\setminus\{(0,0)\}\}$, from the basis $\mathcal{B}_\mu\setminus\{1\}$ of $B'(I_\mu)$. The Hilbert series of $\wedge^k(B'\oplus\mathcal{O}_t\oplus\mathcal{O}_q)(I_\mu)$ is the k-th elementary symmetric function of these eigenvalues, and its alternating sum over k is therefore $(1-q)(1-t)\Pi_\mu(q,t)$, where

$$\Pi_\mu(q,t) \underset{\text{def}}{=} \prod_{\substack{(r,s)\in d(\mu)\\(r,s)\neq(0,0)}}(1-t^rq^s). \tag{87}$$

Hence we have

$$\sum_{k=0}^{n+1}(-1)^k\mathcal{F}_{(P\otimes V_k)(I_\mu)}(z;q,t) = (1-q)(1-t)\Pi_\mu(q,t)B_\mu(q,t)\tilde{H}_\mu(z;q,t), \tag{88}$$

and (67) yields the following character formula for the diagonal coinvariants.

Theorem 3.10. *The Frobenius series of the coinvariant ring R_n, or of the diagonal harmonics DH_n, is given by*

$$\mathcal{F}_{R_n}(z;q,t) = \sum_{|\mu|=n} \frac{(1-q)(1-t)\Pi_\mu(q,t)B_\mu(q,t)\tilde{H}_\mu(z;q,t)}{\prod_{x\in d(\mu)}(1-t^{1+l(x)}q^{-a(x)})(1-t^{-l(x)}q^{1+a(x)})}. \tag{89}$$

We briefly review some of the consequences of this formula, as developed in [9]. First, there is reformulation of (89) along the lines of (77). Let ∇ be the linear operator on Λ defined by

$$\nabla \tilde{H}_\mu(z;q,t) = t^{n(\mu)}q^{n(\mu')}\tilde{H}_\mu(z;q,t), \tag{90}$$

with $n(\mu)$ as in (71) and μ' denoting the conjugate partition.

Proposition 3.11. *The formula* (89) *may be simply expressed as*

$$\mathcal{F}_{R_n}(z;q,t) = \nabla e_n(z). \tag{91}$$

Next, making use of the known specializations of $\tilde{H}_\mu(z;q,t)$ at $t=q^{-1}$ and $t=1$, we were able to determine the corresponding specializations of (91).

Proposition 3.12. *For $t=q^{-1}$ we have*

$$\begin{aligned} q^{\binom{n}{2}}\mathcal{F}_{R_n}(z;q,q^{-1}) &= \frac{1}{1+q+\cdots+q^n} h_n\left[Z\tfrac{1-q^{n+1}}{1-q}\right] \\ &= \sum_{|\lambda|=n} \frac{s_\lambda(1,q,\ldots,q^n)}{1+q+\cdots+q^n} s_\lambda(z) \end{aligned} \tag{92}$$

and hence

$$q^{\binom{n}{2}}\mathcal{H}_{R_n}(q,q^{-1}) = (1+q+\cdots+q^n)^{n-1}. \tag{93}$$

In particular, setting $q=1$, we have

$$\dim R_n = (n+1)^{n-1}. \tag{94}$$

The specialization at $t=1$ is most conveniently expressed combinatorially, in terms of parking functions. A function $f\colon \{1,\ldots,n\}\to\{1,\ldots,n\}$ is called a *parking function* if $|f^{-1}(\{1,\ldots,k\})|\geq k$, for all $1\leq k\leq n$. To understand the name, picture a one-way street with n parking spaces numbered 1 through n. Suppose that n cars arrive in succession, each with a preferred parking space given by $f(i)$ for the i-th car. Each driver proceeds directly to his or her preferred space and parks there, or in the next available space, if the desired space is already taken. The necessary and sufficient condition for everyone to park without being forced to the end of the street is that f is a parking function. The *weight* of f is the quantity $w(f)=\sum_{i=1}^n f(i)-i$. It measures the quantity of frustration experienced by the drivers in having to pass up occupied parking spaces. The symmetric group acts on the set PF_n of parking functions by permuting the cars (that is, the domain of f) and this action preserves the weight. Let $\mathbb{C}PF_n=\bigoplus_d \mathbb{C}PF_{n,d}$ be the permutation representation on parking functions, graded by weight, *i.e.*, $PF_{n,d}=\{f\in PF_n : w(f)=d\}$.

Proposition 3.13. *For $t=1$, we have*

$$\mathcal{F}R_n(z;q,1) = \sum_d q^d \phi\, \mathrm{char}(\varepsilon\otimes \mathbb{C}PF_{n,d}), \tag{95}$$

where ε is the sign representation. In other words, R_n and $\varepsilon\otimes\mathbb{C}PF_n$ are isomorphic as singly graded S_n-modules when we consider only the y-degree in R_n and ignore the x-degree.

Since it is known that $|PF_n| = (n+1)^{n-1}$, we again recover the dimension formula (94). Of particular interest is the subspace R_n^ε of S_n-alternating coinvariants, whose Hilbert series is given by

$$\mathcal{H}_{R_n^\varepsilon}(q,t) = \langle e_n(z), \mathcal{F}_{R_n}(z;q,t)\rangle. \tag{96}$$

We can expand this by substituting into (89) the known identity

$$\langle e_n(z), \tilde{H}_\mu(z;q,t)\rangle = \tilde{K}_{(1^n),\mu}(q,t) = t^{n(\mu)}q^{n(\mu')}, \tag{97}$$

obtaining the following result.

Corollary 3.14. *The Hilbert series of the S_n-alternating diagonal coinvariants is given by*

$$\mathcal{H}_{R_n^\varepsilon}(q,t) = C_n(q,t) \underset{\text{def}}{=} \sum_{|\mu|=n} \frac{t^{n(\mu)}q^{n(\mu')}(1-q)(1-t)\Pi_\mu(q,t)B_\mu(q,t)}{\prod_{x\in d(\mu)}(1-t^{1+l(x)}q^{-a(x)})(1-t^{-l(x)}q^{1+a(x)})}. \tag{98}$$

The quantity $C_n(q,t)$, studied in [9, 11], is called the *q,t-Catalan polynomial*. From either Proposition 3.12 or 3.13, we see that $C_n(q,t)$ is a q,t-analog of the Catalan number

$$C_n(1,1) = \frac{1}{n+1}\binom{2n}{n} \tag{99}$$

By the corollary above, we have $C_n(q,t)\in\mathbb{N}[q,t]$. Recently, Garsia and Haglund also proved this by establishing the following combinatorial interpretation.

Proposition 3.15 ([8])**.** *Let D_n be the set of non-negative integer sequences $(e_1 = 0, e_2, \ldots, e_n)\in\mathbb{N}^n$ satisfying $e_{k+1}\le e_k+1$ for all k. Put $|e| = \sum_i e_i$ and let $i(e)$ be the number of index pairs $i<j$ such that $e_j = e_i$ or $e_j = e_i - 1$. Then*

$$C_n(q,t) = \sum_{e\in D_n} t^{|e|}q^{i(e)}. \tag{100}$$

We remark that (97) has a direct geometric interpretation. The bundle P is a quotient of $B^{\otimes n}$ (see [13, Section 3.7]), so we have an equivariant isomorphism of line bundles

$$P_{(1^n)} = P^\varepsilon \cong \wedge^n B \cong \mathcal{O}(1). \tag{101}$$

Hence $\tilde{K}_{(1^n),\mu}(q,t)$, which is the $\mathbb{T}^2$ character of the fiber $P_{(1^n)}(I_\mu) = \wedge^n B(I_\mu)$, is equal to $\prod_{(r,s)\in d(\mu)} t^r q^s = t^{n(\mu)}q^{n(\mu')}$. The notation $\mathcal{O}(1)$ here refers to the very ample line bundle coming from the projective embedding of H_n over $S^n\mathbb{C}^2$ constructed in [11, Proposition 2.6]. The identity $\wedge^n B\cong\mathcal{O}(1)$ is [*op. cit.*, Proposition 2.12]. See also Proposition 5.4, below.

Other character formulas. The ring $R(n,l)$ and its quotient $R(n,l)/\mathfrak{m}R(n,l)$ have S_l actions permuting the coordinates $a_1, b_1, \ldots, a_l, b_l$, and commuting with the S_n action. Under our identification of these rings with the spaces of global sections $H^0(H_n, P\otimes B^{\otimes l})$ and $H^0(Z_n, P\otimes B^{\otimes l})$, the S_l action corresponds to permutation of the tensor factors in $B^{\otimes l}$.

Recall that the *Schur functor* S^ν for ν a partition of l is defined by

$$S^\nu(W) = (W^{\otimes l})_\nu = \operatorname{Hom}^{S_n}(V^\nu, W^{\otimes l}). \tag{102}$$

It makes sense as a functor on vector spaces and also on vector bundles. The following classical result of Schur [22] can be viewed as a formulation of Schur-Weyl duality.

Proposition 3.16. *If $\alpha \in \operatorname{End}(W)$ has eigenvalues $t_1, \ldots, t_d$, then the trace of $S^\nu(\alpha) \in \operatorname{End} S^\nu(W)$ is given by the Schur function*

$$s_\nu(t_1, \ldots, t_d). \tag{103}$$

Corollary 3.17. *The Hilbert series of $S^\nu(B(I_\mu))$ is given by*

$$\mathcal{H}_{S^\nu(B(I_\mu))} = s_\nu[B_\mu(q,t)] \tag{104}$$

in the notation of (61)*.*

Proceeding as in the derivation of Theorems 3.5 and 3.10, one obtains the following refinement, which takes account of the S_l action.

Theorem 3.18. *The Frobenius series of $R(n,l)_\nu = \operatorname{Hom}^{S_l}(V^\nu, R(n,l))$ is given by*

$$\mathcal{F}_{R(n,l)_\nu}(z;q,t) = \sum_{|\mu|=n} \frac{s_\nu[B_\mu(q,t)]\tilde{H}_\mu(z;q,t)}{\prod_{x\in d(\mu)}(1-t^{1+l(x)}q^{-a(x)})(1-t^{-l(x)}q^{1+a(x)})}. \tag{105}$$

Setting $S(n,l,\nu) = (R(n,l)/\mathfrak{m}R(n,l))_\nu$, its Frobenius series is given by

$$\mathcal{F}_{S(n,l,\nu)}(z;q,t) = \sum_{|\mu|=n} \frac{(1-q)(1-t)\Pi_\mu(q,t)B_\mu(q,t)s_\nu[B_\mu(q,t)]\tilde{H}_\mu(z;q,t)}{\prod_{x\in d(\mu)}(1-t^{1+l(x)}q^{-a(x)})(1-t^{-l(x)}q^{1+a(x)})}. \tag{106}$$

It is convenient to express these identities with the aid of operators ∇_f defined for any symmetric function f by

$$\nabla_f \tilde{H}_\mu(z;q,t) = f[B_\mu(q,t)]\tilde{H}(z;q,t). \tag{107}$$

In this notation, the operator Δ in (74) is ∇_{e_1}, and ∇ in (90) is the operator which coincides with ∇_{e_n} in degree n, for each n. From the expressions for the $l=0$ cases of (105) and (106) in Lemma 3.7 and Proposition 3.11, we get the following corollary.

Corollary 3.19. *The two Frobenius series in* (105) *and* (106) *may be simply expressed as*

$$\mathcal{F}_{R(n,l)_\nu}(z;q,t) = \nabla_{s_\nu} h_n\left[\tfrac{Z}{(1-q)(1-t)}\right], \tag{108}$$

$$\mathcal{F}_{S(n,l,\nu)}(z;q,t) = \nabla_{s_\nu}\nabla e_n(z) = \nabla_{e_n s_\nu} e_n(z). \tag{109}$$

In particular, the expression on the right-hand side is a q,t-Schur positive formal power series in (108) *and polynomial in* (109)*.*

The operators ∇_{s_ν} were studied in [2], where we made the following conjecture.

Conjecture 3.20. *The quantity $\nabla_{s_\nu} e_n(z)$ is a q,t-Schur positive polynomial for all ν and n.*

This statement is stronger than the positivity of the expression in (109), because ∇_f is linear in f, and $e_n s_\nu$ is a positive linear combination of Schur functions.

Proposition 3.21. *We have*

$$\chi\mathcal{F}_{\mathcal{O}_{Z_n}\otimes P^*\otimes S^\nu(B)} = \nabla_{s_\nu} e_n(z). \tag{110}$$

Proof. Equation (109) gives $\chi\mathcal{F}_{\mathcal{O}_{Z_n}\otimes P\otimes S^\nu(B)} = \nabla_{s_\nu}\nabla e_n(z)$. To remove the extra factor ∇, we should divide the numerator in (106) by $t^{n(\mu)}q^{n(\mu')}$. By the remarks following (101), this is achieved if we replace P with $\mathcal{O}(-1)\otimes P$. The latter is isomorphic to the dual bundle P^* [13, eq. (45)]. □

From this we see that $\nabla_{s_\nu} e_n(z)$ is at least a polynomial and that Conjecture 3.20 would be a consequence of the following strengthening of Theorems 2.1 and 2.2.

Conjecture 3.22. *We have $H^i(H_n, P^*\otimes B^{\otimes l}) = 0$ for all $i > 0$, and hence also $H^i(Z_n, P^*\otimes B^{\otimes l}) = 0$ for all $i > 0$.*

Note that the "hence also" part follows precisely as in the derivation of Theorem 2.2 from Theorem 2.1. The identification of the spaces of global sections seems rather difficult, and will not be addressed here.

4. The operator conjecture

In [10], we proved the following proposition and conjectured that the theorem stated below holds. This theorem was called the *operator conjecture* there.

Proposition 4.1. *The space DH_n of diagonal harmonics defined in* (84) *is closed under the action of the* polarization operators

$$E_k = \sum_{i=1}^{n} y_i \partial x_i^k, \quad k > 0. \tag{111}$$

Theorem 4.2. *The Vandermonde determinant $\Delta(\mathbf{x})$ generates DH_n as a module for the algebra of operators $\mathbb{C}[\partial x_1,\dots,\partial x_n, E_1,\dots,E_{n-1}]$.*

Note that the operators ∂x_j and E_k all commute, and that we need not go past E_{n-1}, as $E_k\Delta(\mathbf{x}) = 0$ for $k \geq n$. We will prove the theorem using the isomorphism

$$\psi_1\colon R_n \to H^0(Z_n, P) \tag{112}$$

given by the case $l = 0$ of Theorem 2.2, where R_n is the ring of diagonal coinvariants. The first step is to recast Theorem 4.2 in ideal-theoretic terms. There is a symmetric inner product $(\cdot,\cdot)$ on $\mathbb{C}[\mathbf{x},\mathbf{y}]$ defined by

$$(f, g) = g(\partial\mathbf{x}, \partial\mathbf{y}) f(\mathbf{x},\mathbf{y})|_{\mathbf{x},\mathbf{y}\mapsto 0}\,. \tag{113}$$

The set of all monomials $\mathbf{x}^h\mathbf{y}^k$ is an orthogonal basis, with $(\mathbf{x}^h\mathbf{y}^k, \mathbf{x}^h\mathbf{y}^k) = \prod_{i=1}^n (h_i)!(k_i)!$. In particular, this verifies that $(\cdot,\cdot)$ is in fact symmetric. The inner product is compatible with the grading and non-degenerate. Since $\mathbb{C}[\mathbf{x},\mathbf{y}]_d$ is finite-dimensional in each degree d, we have $I^{\perp\perp} = I$ for any homogeneous subspace $I \subseteq \mathbb{C}[\mathbf{x},\mathbf{y}]$. One sees easily from (113) that the operator ∂x_j is adjoint to multiplication by x_j, and likewise for y_j. A polynomial f is orthogonal to an ideal $(g_1,\dots,g_k)$ if and only if

$$p(\partial\mathbf{x},\partial\mathbf{y}) g_i(\partial\mathbf{x},\partial\mathbf{y}) f(\mathbf{x},\mathbf{y}) = 0 \tag{114}$$

for all i and all $p \in \mathbb{C}[\mathbf{x},\mathbf{y}]$. By Taylor's theorem, this is equivalent to $g_i(\partial\mathbf{x},\partial\mathbf{y}) f(\mathbf{x},\mathbf{y}) = 0$ for all i. Setting

$$I = \mathfrak{m}\mathbb{C}[\mathbf{x},\mathbf{y}], \tag{115}$$

we therefore see that $DH_n = I^\perp$, or $I = DH_n^\perp$. The following version of Theorem 4.2 in one set of variables is classical.

Proposition 4.3 ([23]). *Let $I_0 \subseteq \mathbb{C}[\mathbf{x}]$ be the ideal generated by the homogeneous maximal ideal in $\mathbb{C}[\mathbf{x}]^{S_n}$, or equivalently by the elementary symmetric functions $e_1(\mathbf{x}), \ldots, e_n(\mathbf{x})$, so that $I_0^\perp$ is the space of harmonics for the usual action of S_n on $\mathbb{C}^n$. Then the Vandermonde determinant $\Delta(\mathbf{x})$ generates $I_0^\perp$ as a $\mathbb{C}[\partial\mathbf{x}]$-module.*

Returning to the diagonal situation, set

$$OP_n = \mathbb{C}[\partial\mathbf{x}, E_1, \ldots, E_{n-1}]\Delta(\mathbf{x}). \tag{116}$$

We have $OP_n \subseteq DH_n$, and hence

$$I \subseteq OP_n^\perp, \tag{117}$$

and we are to prove that equality holds here.

Proposition 4.4. *We have $f(\mathbf{x}, \mathbf{y}) \in OP_n^\perp$ if and only if*

$$f(\mathbf{x}, \phi_\lambda(\mathbf{x})) \in (e_1(\mathbf{x}), \ldots, e_n(\mathbf{x})), \tag{118}$$

identically in λ, where $\phi_\lambda(z) = \lambda_{n-1}z^{n-1} + \cdots + \lambda_1 z$ is the polynomial of degree $n-1$ in one variable with zero constant term and generic coefficients.

Proof. Since the adjoint of ∂x_j is x_j, and the adjoint of E_k is $E_k^* = \sum_i x_i^k \partial y_i$, it follows that we have $f \in OP_n$ if and only if

$$\Delta(\mathbf{x}) \perp \mathbb{C}[\mathbf{x}, E_1^*, \ldots, E_{n-1}^*]f \tag{119}$$

We can regard the expression $\exp(\lambda_{n-1}E_{n-1}^* + \cdots + \lambda_1 E_1^*)$ as a generating function in the indeterminates λ_k for all monomials in the operators E_k^*. Condition (119) is then equivalent to

$$\exp(\textstyle\sum_k \lambda_k E_k^*)f \subseteq (\mathbb{C}[\partial\mathbf{x}]\Delta(\mathbf{x}))^\perp \tag{120}$$

holding identically in λ. This last condition depends only on the y-degree zero part of $\exp(\sum_k \lambda_k E_k^*)f$, so from Proposition 4.3 we see that it is in turn equivalent to

$$(\exp(\textstyle\sum_k \lambda_k E_k^*)f)|_{\mathbf{y}\mapsto 0} \in (e_1(\mathbf{x}), \ldots, e_n(\mathbf{x})). \tag{121}$$

By Taylor's theorem, $\exp(\lambda_k E_k^*)f$ is equal to the result of substituting $y_j + \lambda_k x_j^k$ for y_j in f, for all j. Hence $(\exp(\sum_k \lambda_k E_k^*)f)|_{\mathbf{y}\mapsto 0} = f(\mathbf{x}, \phi_\lambda(\mathbf{x}))$, and the proposition is proved. ☐

Theorem 4.2 is a corollary to the preceding propostion and the next.

Proposition 4.5. *If $f \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ satisfies $f(\mathbf{x}, \phi_\lambda(\mathbf{x})) \in (e_1(\mathbf{x}), \ldots, e_n(\mathbf{x}))$, with ϕ_λ as in Proposition 4.4, then $f(\mathbf{x}, \mathbf{y}) \in \mathfrak{m}\mathbb{C}[\mathbf{x}, \mathbf{y}]$, where $\mathfrak{m} = \mathbb{C}[\mathbf{x}, \mathbf{y}]_+^{S_n}$.*

Proof. Using Theorem 2.2, it suffices to show that the global section $\psi f(\mathbf{x}, \mathbf{y}) \in H^0(H_n, P)$ restricts to zero on Z_n. Equivalently, we are to show that the function $f(\mathbf{x}, \mathbf{y})$ on X_n belongs to the ideal of the scheme-theoretic preimage $\rho^{-1}(Z_n)$.

Let $U_x \subseteq H_n$ be the open set consisting of ideals I such that x generates the tautological fiber $B(I) = \mathbb{C}[x, y]/I$ as a $\mathbb{C}$-algebra, that is,

$$U_x = \{I \in H_n : \{1, x, \ldots, x^{n-1}\} \text{ is a basis of } B(I)\}. \tag{122}$$

As shown in [13, Section 3.6], U_x is an affine cell with coordinates $e_1, \ldots, e_n$, $\gamma_0, \ldots, \gamma_{n-1}$ such that the equations of the universal family over U_x are given in terms of these and the coordinates x, y on $\mathbb{C}^2$ by

$$\begin{aligned} x^n - e_1 x^{n-1} + \cdots + (-1)^n e_n &= 0 \\ y = \gamma_{n-1} x^{n-1} + \cdots + \gamma_1 x + \gamma_0&. \end{aligned} \tag{123}$$

The preimage $U'_x = \rho^{-1}(U_x)$ of U_x in X_n is an affine cell with coordinates $x_1, \ldots, x_n, \gamma_0, \ldots, \gamma_n$. The morphism $\rho \colon X_n \to H_n$ is given on the coordinate level by the identification of e_i with the i-th elementary symmetric function $e_i(\mathbf{x})$. Each coordinate pair x_j, y_j on X_n satisfies equations (123), so the coordinates y_j are given in terms of $\mathbf{x}, \boldsymbol{\gamma}$ by $y_j = \phi_\gamma(x_j)$, where $\phi_\gamma(z)$ is the polynomial $\gamma_{n-1} z^{n-1} + \cdots + \gamma_1 z + \gamma_0$ with coefficients $\boldsymbol{\gamma}$.

The zero fiber Z_n is irreducible [3], so $U_x \cap Z_n$ is dense in Z_n, and it suffices to check that the section represented by f is zero there. In terms of the coordinates $\mathbf{e}, \boldsymbol{\gamma}$ on U_x, the ideal of $U_x \cap Z_n$ is $(\gamma_0, \mathbf{e})$, so the coordinate ring of the scheme-theoretic preimage $U'_x \cap \rho^{-1}(Z_n)$ is

$$\mathbb{C}[\mathbf{x}, \gamma_1, \ldots, \gamma_{n-1}]/(e_1(\mathbf{x}), \ldots, e_n(\mathbf{x})). \tag{124}$$

In terms of the coordinates $\mathbf{x}, \boldsymbol{\gamma}$, the given function $f(\mathbf{x}, \mathbf{y})$ becomes $f(\mathbf{x}, \phi_\gamma(\mathbf{x}))$, which belongs to $(e_1(\mathbf{x}), \ldots, e_n(\mathbf{x}))$ by hypothesis. $\square$

5. Proof of the main theorem

We will prove Theorem 2.1 by combining two results from [13]—the isomorphism $\mathbb{C}^{2n}//S_n \cong H_n$ and the theorem that $R(n, l)$ is a free $\mathbb{C}[\mathbf{y}]$-module—with the theorem of Bridgeland, King and Reid mentioned in the introduction. We begin by reviewing these results.

Let $V = \mathbb{C}^m$ be a complex vector space and G a finite subgroup of $SL(V)$. As in Section 2, we have a diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & V \\ {\scriptstyle\rho}\downarrow & & \downarrow \\ V//G & \xrightarrow{\sigma} & V/G, \end{array} \tag{125}$$

whose special case for $V = \mathbb{C}^{2n}$, $G = S_n$ is (9). Let $D(V//G)$ be the derived category of complexes of sheaves of $\mathcal{O}_{V//G}$-modules with bounded, coherent cohomology, and $D^G(V)$ the derived category of complexes of G-equivariant sheaves of $\mathcal{O}_V$-modules, again with bounded, coherent cohomology. Bridgeland, King and Reid define a functor

$$\Phi \colon D(V//G) \to D^G(V) \tag{126}$$

by the formula

$$\Phi = Rf_* \circ \rho^*. \tag{127}$$

Note that ρ is flat, so we can write ρ^* instead of $L\rho^*$ here.

Theorem 5.1 ([4]). *Suppose that the Chow morphism $V//G \to V/G$ satisfies the following smallness criterion: for every d, the locus of points $x \in V/G$ such that $\dim \sigma^{-1}(x) \geq d$ has codimension at least $2d - 1$. Then*

(1) *$V//G$ is a crepant resolution of singularities of V/G, i.e., it is non-singular and its canonical line bundle is trivial, and*
(2) *the functor Φ is an equivalence of categories.*

We apply the theorem with $V = \mathbb{C}^{2n}$ and $G = S_n$. Note that S_n, acting diagonally, is a subgroup of $\mathrm{SL}(\mathbb{C}^{2n})$. It is known [13, 18] that $\omega_{H_n} \cong \mathcal{O}_{H_n}$, so $\mathbb{C}^{2n}//S_n \cong H_n$ is a crepant resolution of $\mathbb{C}^{2n}/S_n = S^n\mathbb{C}^2$. Moreover, the smallness criterion in Theorem 5.1 holds. This follows either from the description of the fibers of the Chow morphism due to Briançon [3], or from the observation in [4] that, conversely to Theorem 5.1, the criterion holds whenever G preserves a symplectic form on V and $V//G$ is a crepant resolution. We identify $D^{S_n}(\mathbb{C}^{2n})$ with the derived category of bounded complexes of finitely-generated S_n-equivariant $\mathbb{C}[\mathbf{x}, \mathbf{y}]$-modules. The functor Rf_* is thereby identified with $R\Gamma_{X_n}$. Since ρ is finite and therefore affine, and $P = \rho_* \mathcal{O}_{X_n}$, the functor $R\Gamma_{X_n} \circ \rho^*$ is naturally isomorphic to $R\Gamma_{H_n}(P \otimes -)$.

Corollary 5.2. *The functor $\Phi = R\Gamma(P \otimes -)$ is an equivalence of categories $\Phi\colon D(H_n) \to D^{S_n}(\mathbb{C}^{2n})$.*

Using this we can reformulate our main theorem.

Proposition 5.3. *Theorem 2.1 is equivalent to the identity in $D^{S_n}(\mathbb{C}^{2n})$*

$$\Phi B^{\otimes l} \cong R(n, l), \tag{128}$$

where the isomorphism is given by the map $R(n, l) \to \Phi B^{\otimes l}$ obtained by composing the canonical natural transformation $\Gamma \to R\Gamma$ with the homomorphism ψ in (17).

We will prove identity (128), and thus Theorem 2.1, by using the *inverse* Bridgeland–King–Reid functor $\Psi\colon D^{S_n}(\mathbb{C}^{2n}) \to D(H_n)$, which also has a simple description in our case. In general, as observed in [4], the inverse functor Ψ can be calculated using Grothendieck duality as the right adjoint of Φ, given by the formula

$$\Psi = (\rho_*(\omega_X \overset{L}{\otimes} Lf^* -))^G. \tag{129}$$

To simplify this, we use the following result from [13].

Proposition 5.4. *The line bundle $\mathcal{O}(1) = \wedge^n B$ is the Serre twisting sheaf induced by a natural embedding of H_n as a scheme projective over $S^n\mathbb{C}^2$. Writing $\mathcal{O}(1)$ also for its pullback to X_n, we have that X_n is Gorenstein with canonical sheaf $\omega_{X_n} \cong \mathcal{O}(-1)$.*

We need an extra bit of information not contained in the proposition. There are two possible equivariant S_n actions on $\mathcal{O}_{X_n}(1)$: the trivial action coming from the definition of $\mathcal{O}_{X_n}(1)$ as $\rho^*\mathcal{O}_{H_n}(1)$, or its twist by the sign character of S_n. The latter action is the correct one, in the sense that the isomorphism $\omega_{X_n} \cong \mathcal{O}(-1)$ is S_n-equivariant for this action, as can be seen from the proof in [13]. Taking this into account, and using the fact that $\mathcal{O}_{X_n}(-1)$ is pulled back from H_n, we have the following description of the inverse functor.

Proposition 5.5. *The inverse of the functor Φ in Corollary 5.2 is given by*

$$\Psi = \mathcal{O}(-1) \otimes (\rho_* \circ Lf^*)^\epsilon. \tag{130}$$

Here $(-)^\epsilon$ denotes the functor of S_n-alternants, i.e., $A^\epsilon = \mathrm{Hom}^{S_n}(\varepsilon, A)$, where ε is the sign representation.

Now we recall the algebraic result that was the key technical tool in [13].

Theorem 5.6. *The polygraph coordinate ring $R(n,l)$ is a free $\mathbb{C}[\mathbf{y}]$-module.*

We need to strengthen this in two ways. Any automorphism of $\mathbb{C}^2$ induces an automorphism of $\mathbb{C}^{2n+2l}$, and the corresponding automorphism of $\mathbb{C}[\mathbf{x},\mathbf{y},\mathbf{a},\mathbf{b}]$ leaves invariant the defining ideal $I(n,l)$ of $Z(n,l)$. In particular, this is so for translations in the x-direction, which also leave invariant the ideal $(\mathbf{y})$ and hence $I(n,l)+(\mathbf{y})$. This implies that any of the coordinates x_i, a_i is a non-zero-divisor in $R(n,l)/(\mathbf{y})$, yielding the following two corollaries.

Corollary 5.7. *The coordinate ring $R(n,l)$ is a free $\mathbb{C}[x_1,\mathbf{y}]$-module.*

Corollary 5.8. *The coordinate ring $R(n,l)$ has a free resolution of length $n-1$ as a $\mathbb{C}[\mathbf{x},\mathbf{y}]$-module.*

As in [13, Definition 4.1.1], the polygraph $Z(n,l)$ is the union of linear subspaces $W_f \subseteq \mathbb{C}^{2n+2l}$ defined by

$$W_f = V(I_f), \quad I_f = (a_i - x_{f(i)}, b_i - y_{f(i)} : 1 \le i \le l) \tag{131}$$

for all functions $f\colon \{1,\ldots,l\} \to \{1,\ldots,n\}$. The polygraph ring can be defined with any ground ring S in place of $\mathbb{C}$ as

$$R(n,l) = S[\mathbf{x},\mathbf{y},\mathbf{a},\mathbf{b}]/I(n,l), \quad I(n,l) = \bigcap_f I_f, \tag{132}$$

with I_f as above. Theorem 5.6 holds in this more general setting [13, Theorem 4.3]. If θ is an automorphism of $S[x,y]$ as an S-algebra, then the automorphism $\theta^{\otimes(n+l)}$ of $S[\mathbf{x},\mathbf{y},\mathbf{a},\mathbf{b}] \cong S[x,y]^{\otimes(n+l)}$ leaves $I(n,l)$ invariant, inducing an automorphism of $R(n,l)$. Hence we have the following corollary.

Corollary 5.9. *Let S be a $\mathbb{C}$-algebra and let y' denote the image of y under some automorphism of $S[x,y]$ as an S-algebra. Then $S \otimes_{\mathbb{C}} R(n,l)$ is a free $S[y'_1,\ldots,y'_n]$-module.*

In addition to the results on polygraphs we need the following local structure theorem for X_n. It allows us to assume by induction on n that a desired geometric result holds locally over the open locus consisting of points $I \in H_n$ such that $V(I)$ is not concentrated at a single point of $\mathbb{C}^2$.

Proposition 5.10. *Let $U_k \subseteq X_n$ be the open set consisting of points $(I,P_1,\ldots,P_n)$ for which $\{P_1,\ldots,P_k\}$ and $\{P_{k+1},\ldots,P_n\}$ are disjoint. Then U_k is isomorphic to an open set in $X_k \times X_{n-k}$. More precisely, the morphism $f\colon X_n \to \mathbb{C}^{2n}$ restricted to U_k corresponds to the restriction of $f_k \times f_{n-k}\colon X_k \times X_{n-k} \to \mathbb{C}^{2k} \times \mathbb{C}^{2(n-k)} = \mathbb{C}^{2n}$.*

The pullback $F'_n = F_n \times X_n / H_n$ of the universal family to X_n decomposes over U_k as the disjoint union $F'_n = F'_k \times X_{n-k} \cup X_k \times F'_{n-k}$ of the pullbacks of the universal families from H_k and H_{n-k}. Hence the tautological sheaf ρ^*B decomposes as $\rho^*B = \eta_k^*\rho_k^*B_k \oplus \eta_{n-k}^*\rho_{n-k}^*B_{n-k}$, where η_k and η_{n-k} are the projections of $X_k \times X_{n-k}$ on the factors.

The final piece of our puzzle will be supplied by a fundamental result of commutative algebra known as the new intersection theorem.

Theorem 5.11 ([19, 20, 21])**.** *Let $0 \to C_n \to \cdots \to C_1 \to C_0 \to 0$ be a bounded complex of locally free coherent sheaves on a Noetherian scheme X. Denote by*

$\operatorname{Supp}(C_\bullet)$ *the union of the supports of the homology sheaves* $H_i(C_\bullet)$*. Then every component of* $\operatorname{Supp}(C_\bullet)$ *has codimension at most* n *in* X*. In particular, if* $C_\bullet$ *is exact on an open set* $U \subseteq X$ *whose complement has codimension exceeding* n*, then* $C_\bullet$ *is exact.*

Proof of Theorem 2.1. By Proposition 5.3, we have a map

$$R(n,l) \to \Phi B^{\otimes l} \tag{133}$$

in the derived category $D^{S_n}(\mathbb{C}^{2n})$, and it suffices to show that it is an isomorphism. Applying the inverse functor Ψ yields a map

$$\Psi R(n,l) \to B^{\otimes l} \tag{134}$$

in $D(H_n)$, and we can equally well show that this is an isomorphism. Let C be the third vertex of a distinguished triangle

$$C[-1] \to \Psi R(n,l) \to B^{\otimes l} \to C. \tag{135}$$

We are to show that $C = 0$.

We may compute $\Psi R(n,l)$ as follows. By Corollary 5.8, the $\mathbb{C}[\mathbf{x},\mathbf{y}]$-algebra $R(n,l)$ has a free resolution of length $n-1$. We can assume that the resolution is S_n-equivariant, for instance by taking a graded minimal free resolution. In derived category terminology, we have an S_n-equivariant complex of free $\mathbb{C}[\mathbf{x},\mathbf{y}]$-modules

$$A_\bullet = \cdots \to 0 \to A_{n-1} \to \cdots \to A_1 \to A_0 \to 0 \to \cdots \tag{136}$$

quasi-isomorphic to $R(n,l)$. Using the formula for Ψ from Proposition 5.5, we have $\Psi R(n,l) = \mathcal{O}(-1) \otimes (\rho_* f^* A_\bullet)^\epsilon$. Moreover, since ρ is flat, and since the functor $(-)^\epsilon$ is a direct summand of the identity functor, $\mathcal{O}(-1) \otimes (\rho_* f^* A_\bullet)^\epsilon$ is a complex of locally free sheaves. Since $B^{\otimes l}$ is a sheaf, the map $\Psi R(n,l) \to B^{\otimes l}$ in (134) is represented by an honest homomorphism of complexes, and not merely by a quasi-isomorphism. The object C is represented by the mapping cone of this homomorphism, namely, the complex of locally free sheaves

$$0 \to C_n \to \cdots \to C_2 \to C_1 \to B^{\otimes l} \to 0, \tag{137}$$

where $C_i = \mathcal{O}(-1) \otimes (\rho_* f^* A_{i-1})^\epsilon$. We are to prove that this complex is exact.

Let $U \subseteq H_n$ be the open set of points I such that $V(I)$ contains at least two distinct points of $\mathbb{C}^2$. Let U_k be the open subset in X_n on which $\{P_1,\ldots,P_k\}$ is disjoint from $\{P_{k+1},\ldots,P_n\}$. Clearly the open set $\rho^{-1}(U) \subseteq X_n$ is the union of open sets conjugate by some permutation $w \in S_n$ to U_k for some $0 < k < n$. On U_k, the decomposition of the tautological sheaf $\rho^* B$ from Proposition 5.10 induces a decomposition of $\rho^* B^{\otimes l}$ as a a direct sum

$$\rho^* B^{\otimes l} \cong \bigoplus_{j=0}^{l} \binom{l}{j} \cdot (\eta_k^* \rho_k^* B_k)^{\otimes j} \otimes (\eta_{n-k}^* \rho_{n-k}^* B_{n-k})^{\otimes l-j}. \tag{138}$$

Let $R(n,l)^\sim$ be the sheaf of $\mathcal{O}_{\mathbb{C}^{2n}}$ modules corresponding to the $\mathbb{C}[\mathbf{x},\mathbf{y}]$-module $R(n,l)$. We partition the set $\{1,\ldots,n\}$ into two subsets $S_1 = \{1,\ldots,k\}$ and $S_2 = \{k+1,\ldots,n\}$, and define $\alpha\colon \{1,\ldots,n\} \to \{1,2\}$ to be the function mapping the elements of S_i to i. Let U_k' be the open subset consisting of points $(P_1,\ldots,P_n) \in \mathbb{C}^{2n}$ satisfying the same condition that defines U_k, namely that $\{P_1,\ldots,P_k\}$ and $\{P_{k+1},\ldots,P_n\}$ are disjoint. Over U_k', components W_f, W_g of the polygraph $Z(n,l)$ are disjoint if $\alpha \circ f \neq \alpha \circ g$. Hence $Z(n,l)$ is a union of 2^l disjoint closed subschemes Z_h, indexed by functions $h : \{1,\ldots,l\} \to \{1,2\}$, where Z_h is the union of the components W_f for which $\alpha \circ f = h$. Each subscheme Z_h is isomorphic over U'_k to $Z(k,j) \times Z(n-k, l-j)$, where $j = |h^{-1}(\{1\})|$. The number of Z_h that occur for a given value of j is $\binom{l}{j}$. This decomposition of $Z(n,l)$ gives a direct sum decomposition of $R(n,l)^\sim$ on U'_k as

$$R(n,l)^\sim \cong \bigoplus_{j=0}^{l} \binom{l}{j} \cdot R(k,j)^\sim \otimes R(n-k, l-j)^\sim. \tag{139}$$

The decompositions (138) and (139) are compatible with the map $\psi\colon R(n,l) \to H^0(X_n, \rho^* B^{\otimes l})$ in (17).

Now assume by induction that Theorem 2.1 holds for smaller values of n, the base case $n = 1$ being trivial. The preceding remarks then show that the map $R(n,l) \to \Phi B^{\otimes l}$ in (133) restricts to an isomorphism on the open set $U' \subseteq \mathbb{C}^{2n}$ of points $(P_1, \dots, P_n)$ with $P_1, \dots, P_n$ not all equal. The functors Φ and Ψ are defined locally with respect to $S^n\mathbb{C}^2$, so we conclude that the map $\Psi R(n,l) \to B^{\otimes l}$ in (134) is an isomorphism on U, and hence the complex C in (137) is exact on U. The complement of U in H_n is isomorphic to $\mathbb{C}^2 \times Z_n$, so it has dimension $n+1$ and codimension $n-1$. Before applying Theorem 5.11, we first need to enlarge U to an open set whose complement has codimension $n+1$. The desired open set will be $U \cup U_x \cup U_y$, where U_x is as in (122), and U_y is defined in the obvious analogous way. Its complement is isomorphic to $\mathbb{C}^2 \times (Z_n \setminus (U_x \cup U_y))$, which has codimension $n+1$ by the following lemma.

Lemma 5.12. *The complement $Z_n \setminus (U_x \cup U_y)$ of $U_x \cup U_y$ in the zero fiber has dimension $n-3$.*

Proof. Let $V = Z_n \setminus (U_x \cup U_y)$. Interpreting $\dim V < 0$ to mean that V is empty, the lemma holds trivially for $n = 1$, so we can assume $n \geq 2$. We consider the decomposition of Z_n into affine cells as in [3, 6], and show that each cell intersects V in a locus of dimension at most $n-3$. There is one open cell, of dimension $n-1$. This cell is actually $U_x \cap Z_n$, so it is disjoint from V. There is also one cell of dimension $n-2$. It has non-empty intersection with U_y, so its intersection with V has dimension at most $n-3$. In fact this intersection has dimension exactly $n-3$, since the complement of U_y is the zero locus of a section of the line bundle $\wedge^n B = \mathcal{O}(1)$. All remaining cells have dimension less than or equal to $n-3$. $\square$

We digress briefly to point out the geometric meaning of this lemma. For I in the zero fiber, the fiber $B(I)$ is an Artin local $\mathbb{C}$-algebra with maximal ideal (x,y). The point I belongs to $U_x \cup U_y$ if and only if the maximal ideal is principal, that is, $B(I)$ has embedding dimension one, or equivalently, the corresponding closed subscheme $V(I)$ is a subscheme of some smooth curve through the origin in $\mathbb{C}^2$. In this case I is said to be *curvilinear*. The lemma says that the non-curvilinear locus has codimension two in the zero fiber.

The proof of Theorem 2.1 is now completed by the following lemma and its symmetric partner with U_y in place of U_x.

Lemma 5.13. *The map $\Psi R(n,l) \to B^{\otimes l}$ restricts to an isomorphism on U_x.*

Proof. Recall the description in the proof of Proposition 4.5 of the coordinates on U_x and its preimage $U'_x = \rho^{-1}(U_x)$ in X_n. The coordinates on U'_x are $\mathbf{x}, \boldsymbol{\gamma}$, with y_j equal to $\phi_\gamma(x_j)$, where $\phi_\gamma(z) = \gamma_{n-1}z^{n-1} + \cdots + \gamma_1 z + \gamma_0$. The coordinates on U_x are $\mathbf{e}, \boldsymbol{\gamma}$, where $e_i = e_i(\mathbf{x})$ is the i-th elementary symmetric function, so $\mathbb{C}[\mathbf{e}, \boldsymbol{\gamma}] = \mathbb{C}[\mathbf{x}, \boldsymbol{\gamma}]^{S_n}$.

We have a trivial isomorphism

$$\mathbb{C}[\mathbf{x}, \boldsymbol{\gamma}] \cong \mathbb{C}[\mathbf{x}, \mathbf{y}, \boldsymbol{\gamma}]/(y_j - \phi_\gamma(x_j) \colon 1 \leq j \leq n) \tag{140}$$

which is nonetheless useful because it describes $\mathbb{C}[\mathbf{x}, \boldsymbol{\gamma}]$ as a $\mathbb{C}[\mathbf{x}, \mathbf{y}]$-module. Since $\mathbb{C}[\mathbf{x}, \boldsymbol{\gamma}]$ and $\mathbb{C}[\mathbf{x}, \mathbf{y}, \boldsymbol{\gamma}]$ are polynomial rings of dimension $2n$ and $3n$, respectively, the ideal in (140) is a complete intersection ideal. Hence the Koszul complex $K_\bullet(\mathbf{y} - \phi_\gamma(\mathbf{x}))$ over $\mathbb{C}[\mathbf{x}, \mathbf{y}, \boldsymbol{\gamma}]$ is a free resolution of $\mathbb{C}[\mathbf{x}, \boldsymbol{\gamma}]$ as a $\mathbb{C}[\mathbf{x}, \mathbf{y}, \boldsymbol{\gamma}]$-module, and therefore as a $\mathbb{C}[\mathbf{x}, \mathbf{y}]$-module.

The restriction of $Lf^*R(n,l)$ to the affine open set U'_x is the complex of sheaves associated to the complex of modules $\mathbb{C}[\mathbf{x}, \boldsymbol{\gamma}] \overset{L}{\underset{\mathbb{C}[\mathbf{x},\mathbf{y}]}{\otimes}} R(n,l)$. This can be computed by tensoring $R(n,l)$ with the above free resolution of $\mathbb{C}[\mathbf{x}, \boldsymbol{\gamma}]$, and the result is the Koszul complex $K_\bullet(\mathbf{y} - \phi_\gamma(\mathbf{x}))$ over $\mathbb{C}[\boldsymbol{\gamma}] \otimes_{\mathbb{C}} R(n,l)$. It follows from Corollary 5.9 that this Koszul complex is a free resolution of $\mathbb{C}[\boldsymbol{\gamma}] \otimes_{\mathbb{C}} R(n,l)/(\mathbf{y} - \phi_\gamma(\mathbf{x}))$. In other words, on U'_x we have $Lf^*R(n,l) = f^*R(n,l)$, and we have a description of this object as the sheaf associated to the $\mathbb{C}[\mathbf{x}, \boldsymbol{\gamma}]$-algebra $\mathbb{C}[\boldsymbol{\gamma}] \otimes_{\mathbb{C}} R(n,l)/(\mathbf{y} - \phi_\gamma(\mathbf{x}))$. It follows that $\Psi R(n,l) = \mathcal{O}(-1) \otimes (\rho_* Lf^*R(n,l))^\epsilon$ is described on U_x as the sheaf associated to the S_n-alternating part of this algebra, regarded as a module over $\mathbb{C}[\mathbf{e}, \boldsymbol{\gamma}] = \mathbb{C}[\mathbf{x}, \boldsymbol{\gamma}]^{S_n}$.

The equations of the universal family in (123) give us the description of the tautological bundle B as a sheaf of algebras on U_x, from which we can get a description of $B^{\otimes l}$. To make the variable names match the ones in $R(n,l)$, we should replace x, y with variables a_i, b_i standing for the generators of the i-th tensor factor in $B^{\otimes l}$. In this notation, $B^{\otimes l}$ is the sheaf associated to the $\mathbb{C}[\mathbf{e}, \boldsymbol{\gamma}]$-algebra

$$\mathbb{C}[\mathbf{e}, \boldsymbol{\gamma}, \mathbf{a}, \mathbf{b}]/\sum_{i=1}^{l}(b_i - \phi_\gamma(a_i), \prod_{j=1}^{n}(a_i - x_j)). \tag{141}$$

Note that the products $\prod_{j=1}^n (a_i - x_j)$ written here really only depend on $\mathbf{a}$ and the elementary symmetric functions $e_i = e_i(\mathbf{x})$.

The map $\Psi R(n,l) \to B^{\otimes l}$ is now expressed in local coordinates as a homomorphism from

$$(\mathbb{C}[\boldsymbol{\gamma}] \otimes_{\mathbb{C}} R(n,l)/(\mathbf{y} - \phi_\gamma(\mathbf{x})))^\epsilon \tag{142}$$

to the algebra in (141). The algebra $\mathbb{C}[\boldsymbol{\gamma}] \otimes_{\mathbb{C}} R(n,l)/(\mathbf{y} - \phi_\gamma(\mathbf{x}))$ is generated by the variables $\mathbf{x}, \boldsymbol{\gamma}, \mathbf{a}, \mathbf{b}$, all of which are S_n-invariant except $\mathbf{x}$. It follows that all its S_n-alternating elements are multiples of the Vandermonde determinant $\Delta(\mathbf{x})$ by polynomials in $\boldsymbol{\gamma}, \mathbf{a}, \mathbf{b}$ and the elementary symmetric functions $e_i(\mathbf{x})$. Written out explicitly, the homomorphism in question sends an element $\Delta(\mathbf{x})p(\mathbf{e}, \boldsymbol{\gamma}, \mathbf{a}, \mathbf{b})$ to $p(\mathbf{e}, \boldsymbol{\gamma}, \mathbf{a}, \mathbf{b})$. The division by $\Delta(\mathbf{x})$ here reflects the presence of the factor $\mathcal{O}(-1)$ in the formula for $\Psi R(n,l)$. The space of global sections of $\mathcal{O}(1)$ on X_n can be identified with the ideal $J \subseteq \mathbb{C}[\mathbf{x}, \mathbf{y}]$ generated by $\mathbb{C}[\mathbf{x}, \mathbf{y}]^\epsilon$, in such a way that $\Delta(\mathbf{x})$ represents the essentially unique section which vanishes nowhere on U_x.

Let us denote the above-described homomorphism by ξ. Since the algebra in (141) is generated by the variables $\mathbf{e}, \boldsymbol{\gamma}, \mathbf{a}, \mathbf{b}$, it is clear that ξ is surjective. The injectivity of ξ amounts to saying that the expressions $b_i - \phi_\gamma(a_i)$ and $\prod_{j=1}^n (a_i - x_j)$ are annihilated by $\Delta(\mathbf{x})$ in $\mathbb{C}[\boldsymbol{\gamma}] \otimes_{\mathbb{C}} R(n,l)/(\mathbf{y} - \phi_\gamma(\mathbf{x}))$. This condition is clearly necessary, and it is sufficient since it makes multiplication by $\Delta(\mathbf{x})$ a well-defined left inverse to ξ. The products $\prod_{j=1}^{n}(a_i - x_j)$ are zero in $R(n,l)$ and thus present no difficulty. The expressions $b_i - \phi_\gamma(a_i)$ are more subtle, as they do not vanish in $\mathbb{C}[\boldsymbol{\gamma}] \otimes_{\mathbb{C}} R(n,l)/(\mathbf{y} - \phi_\gamma(\mathbf{x}))$.

For $x_1, \ldots, x_n$ distinct, the Lagrange interpolation problem

$$y_j = \sum_{k=0}^{n-1} \beta_k x_j^k, \quad 1 \leq j \leq n, \tag{143}$$

is solved by a formula giving the coefficients β_k as rational functions of the form $\beta_k = \Delta_k(\mathbf{x}, \mathbf{y})/\Delta(\mathbf{x})$, where Δ_k is a certain determinant involving the variables $\mathbf{x}$, $\mathbf{y}$. Multiplying through by $\Delta(\mathbf{x})$ yields the identity of polynomials

$$y_j \Delta(\mathbf{x}) = \sum_{k=0}^{n-1} \Delta_k(\mathbf{x}, \mathbf{y}) x_j^k, \quad 1 \leq j \leq n. \tag{144}$$

On each component W_f of $Z(n,l)$ we have $a_i = x_{f(i)}$, $b_i = y_{f(i)}$, and therefore

$$b_i \Delta(\mathbf{x}) = \sum_{k=0}^{n-1} \Delta_k(\mathbf{x}, \mathbf{y}) a_i^k, \quad 1 \leq i \leq l. \tag{145}$$

Since these equations hold on every component of $Z(n,l)$, they hold identically in $R(n,l)$. Similarly, for arbitrary values of the parameters $\boldsymbol{\gamma}$, we may substitute $\phi_\gamma(\mathbf{x})$ for $\mathbf{y}$ in (144) and then let $x_j = a_i$, to obtain the identity

$$\phi_\gamma(a_i) \Delta(\mathbf{x}) = \sum_{k=0}^{n-1} \Delta_k(\mathbf{x}, \phi_\gamma(\mathbf{x})) a_i^k, \quad 1 \leq i \leq l, \tag{146}$$

valid when a_i is equal to any of the x_j. Again this holds on every component of $Z(n,l)$ and hence as an identity in $R(n,l)$. Subtracting (146) from (145), we see that $\Delta(\mathbf{x})$ annihilates $b_i - \phi_\gamma(a_i)$ in $\mathbb{C}[\boldsymbol{\gamma}] \otimes_{\mathbb{C}} R(n,l)/(\mathbf{y} - \phi_\gamma(\mathbf{x}))$, which was the only thing left to prove. □

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